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by

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ATMOSPHERIC TURBULENCE AND AIRCRAFT HEIGHT-KEEPING ACCURACY

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SUMMARY

As a contribution to the study of vertical separation standards for use in air traffic control, an examination is made of the possibility of applying the spectral methods already used in gust load evaluations to the problem of determining the height-keeping errors caused by atmospheric turbulence. Although it is found that the data available on the low frequency components of atmospheric turbulence and on the nature of the control applied by the pilot, whether human or automatic, are not sufficient to allow an accurate estimation of these errors, it is concluded that they do not make a significant contribution to the total errors experienced. It is noted, however, that certain atmospheric phenomena lie outside the scope of the theory used here.

* Replaces R.A.E. Technical Report 67195 - A.R.C. 29841

CONTENTS

1 INTRODUCTION

The magnitude and frequency of the height-keeping errors of an aircraft attempting to fly steadily along a fixed flight path are of interest in the study of vertical separation standards. The experimental approach to their estimation raises severe difficulties of measurement and evaluation of results, but some work has been reported[1]. A theoretical attack on the problem would involve the study of each of the many factors which contribute to height deviations. In the present work we examine a possible method of assessing the contribution made by one such factor, atmospheric turbulence.

The response to atmospheric turbulence of an aircraft under the control of an autopilot is represented as that of a linear system subjected to a continuous random disturbance. The techniques of spectral analysis are applied in order to determine the characteristics of the resulting motion. The problem is thus resolved into three parts, as follows.

First, the representation of the characteristics of atmospheric turbulence. The object of this work has been to test the applicability in this context of the spectral techniques already used successfully in gust load evaluations[2]. A brief discussion of the method and its scope is given, with reference to the available data.

Second, the derivation of the equations of motion of an aircraft flying through turbulence. The standard longitudinal equations of motion are used, with additional terms to allow for the variations, due to turbulence, in the velocity of the relative wind. A simple form is assumed for the elevator control equation, which represents the actions of an autopilot. An appendix summarises the derivation of these equations.

Third, the derivation of the characteristics of the height errors. An expression is given for the standard deviation of height error of an aircraft in turbulence of a fixed intensity. A method is also proposed for obtaining the distribution of height errors in routine operations, when turbulence of varying intensity is encountered.

The results obtained from this approach are illustrated by some numerical examples. Comparison with the experimental results shows the calculated height errors to be much smaller than the total errors experienced in practice, and the reasons for this are discussed.

2 ATMOSPHERIC TURBULENCE

An extensive discussion of the nature of atmospheric turbulence is beyond the scope of this Report. Lumley and Panofsky[3] treat the subject fully and some more recent work was described at a meeting in 1966 organised jointly by the Institute of Navigation and the Society of Automotive Engineers[4]. An earlier meeting at R.A.E.[5] was also devoted to this topic.

Briefly, we assume atmospheric turbulence to be a random process characterised by certain spectral density functions[6] and probability distributions. Although it is not truly stationary or homogeneous, we assume that it can be regarded as such over moderate periods of time and large horizontal regions; its dependence on height, however, cannot be neglected. Not all phenomena which might be classed under the general heading of "atmospheric turbulence" can be included in this treatment[2]; notable examples which require separate consideration are waves, the single large up-draughts associated with cumulonimbus, and vertical wind-shears[8].

We describe the variation of each component of the turbulent velocity by means of a spectral density function containing parameters which are assumed to vary slowly with time. This variation is described by means of observed probability distributions, where these are available; otherwise the parameters are assigned values consistent with observed spectra.

For the purposes of this study, we require expressions for the spectral density functions of the longitudinal and vertical components of atmospheric turbulence. The expressions[9]

$$G_{uu}(\Omega) = \frac{2\,\sigma_g^2\,L}{\pi(1 + L^2\,\Omega^2)} \tag{1}$$

and

$$G_{ww}(\Omega) = \frac{\sigma_g^2\,L\,(1 + 3L^2\Omega^2)}{\pi(1 + L^2\Omega^2)^2} \tag{2}$$

respectively, where

$$\Omega = \omega/U, \tag{3}$$

ω is the frequency of the turbulence component as observed from an aircraft, U is the steady speed of the aircraft, L is the scale of the turbulence (a measure of the mean eddy size) and σ_g is the root mean square gust velocity (indicating the intensity of the turbulence), are frequently used in aeronautical work and we shall adopt them as standard spectra. For comparison, we shall also use the "minus five-thirds law" spectrum[3] in the form

$$G_{uu}(\Omega) = G_{ww}(\Omega) = \begin{cases} \sigma_g^2\lambda/(5\pi) & (\Omega < 2\pi/\lambda) \\ 0.4\,\sigma_g^2\,(2\pi/\lambda)^{2/3}\,\Omega^{-5/3} & (\Omega > 2\pi/\lambda), \end{cases} \tag{4}$$

where λ is the wavelength up to which the law is assumed to hold.

Of the parameters L, σ_g and λ, only the root mean square gust velocity σ_g has been studied in sufficient detail for probability distributions relating to its behaviour in routine operations to be available. Press, Meadows and Hadlock[10] have given the three following formulae, corresponding to the altitude ranges 0–10000 ft, 10000–30000 ft and 30000–50000 ft respectively:

$$f(\sigma_g) = \begin{cases} \dfrac{0.99}{1.48}\exp(-\sigma_g/1.48) + \dfrac{0.01}{2.84}\exp(-\sigma_g/2.84) \\[2ex] \dfrac{1}{2(0.32)^2}\exp(-\sigma_g^{1/2}/0.32) \\[2ex] \dfrac{1}{2(0.29)^2}\exp(-\sigma_g^{1/2}/0.29), \end{cases} \tag{5}$$

where σ_g is measured in ft/sec; the corresponding cumulative probability distributions are shown in Fig.1. For L and λ, the values 1000 ft and 5000 ft respectively give spectra in good agreement with observations[10]. However, as pointed out in Ref.10, at low frequencies (gust wavelengths $2\pi/\Omega$ greater than 3000 ft) the spectra are not yet adequately defined, although the available measurements suggest a flattening of $G_{ww}(\Omega)$ in this region.

3 EQUATIONS OF MOTION OF AN AIRCRAFT

We consider an aircraft which, in still air, would be flying steadily along a straight flight path. Axes are fixed in the aircraft, with origin at the centre of gravity. In the steady flight condition, Ox is directed forwards in the direction of the relative wind, making an angle γ above the horizontal; Oy is horizontal and to starboard, and Oz is downwards in the vertical plane containing Ox (Fig.2(a)). It is assumed that there is no coupling between the motions of the aircraft in and normal to the vertical plane containing Ox, and that the latter do not make any contribution to the variations in height; thus only forward, vertical and pitching motions need be considered, together with changes in the elevator angle η, which is controlled by the autopilot. A control equation representing an idealised autopilot height lock is used, no account being taken of time-lags or non-linearities, or of errors in the signal fed to it[11]. We also assume that the aircraft is rigid; Dobrolenskii[12] has shown that this may lead to underestimation of the effects of turbulence.

Let the disturbed velocity of the aircraft relative to the steady wind have components U+u along Ox and w along Oz (Fig.2(b)), the undisturbed values being U and o, and the angle between Ox and the steady relative wind being θ. We assume that the only effect of atmospheric turbulence on the motion of the aircraft is to add a gust velocity with components u_g and w_g (Fig.2(c)) to the otherwise steady wind; variations of this velocity over the length and span of the aircraft are neglected. The components u_g and w_g and hence u and w, are assumed small compared to U and the equations of motion are linearised on this basis. It is shown in an appendix that the resulting equations, written in terms of non-dimensional variables, take the form

$$(D-x_u)\hat{u} \quad -x_w\hat{w} \quad +k\theta = x_u\hat{u}_g + x_w\hat{w}_g \qquad (6)$$

$$-z_u\hat{u} + (D-z_w)\hat{w} + (k_1 - D)\theta = z_u\hat{u}_g + z_w\hat{w}_g \qquad (7)$$

$$\kappa\hat{u} + (\chi D+\tilde{\omega})\hat{w} + (D^2+\nu D)\theta + \delta\eta = -\kappa\hat{u}_g - \tilde{\omega}\hat{w}_g \qquad (8)$$

$$-G_\theta D\theta + D\eta - (\hat{G}_h D + \hat{G}_{\bar{h}})\hat{h} = 0 \qquad (9)$$

$$(\cos\gamma)\hat{w} - (\cos\gamma)\theta \quad +D\hat{h} = 0 \qquad (10)$$

where $\hat{u} = u/U, \quad \hat{w} = w/U, \quad \hat{u}_g = u_g/U, \quad \hat{w}_g = w_g/U$, (11)

$D = \frac{d}{d\tau}, \quad \tau =$ time in airsecs

G_θ, $\hat{G}_h$ and $\hat{G}_{\bar{h}}$ are constants of the autopilot,

x_u, x_w, z_u, z_w, k, k_1, κ, $\tilde{\omega}$, χ, ν and δ are constants of the aircraft, (dimensionless aerodynamic derivatives), and $\hat{h}$ is the dimensionless vertical deviation of the aircraft from its undisturbed flight path.

From these equations we also derive in the appendix the aircraft's transfer functions $Y_{long}(p)$ and $Y_{lat}(p)$ for response to the longitudinal and lateral components of atmospheric turbulence.

4 CALCULATION OF THE HEIGHT ERRORS

We have obtained the transfer functions for an aircraft's height response to the longitudinal and lateral components of the turbulent gust velocity. For simplicity we shall now assume the flight path angle γ to be small, so that these components can to first order be identified with the longitudinal and vertical components of atmospheric turbulence referred to in section 2; for large values of γ the transfer functions relating to the latter components would be

$$Y_{long}(p)\ \cos\gamma + Y_{lat}(p)\ \sin\gamma$$

and (12)

$$Y_{long}(p)\ \sin\gamma + Y_{lat}(p)\ \cos\gamma$$

respectively.

The dimensionless frequency corresponding to the units introduced in the appendix is

$$\hat{\omega} = \frac{m}{\rho S}\,\Omega \tag{13}$$

and, from the spectral density function $G(\Omega)$ of a component of atmospheric turbulence, we can likewise derive the corresponding dimensionless form

$$\hat{G}(\hat{\omega}) = \left(\frac{\rho S}{mU^2}\right) G(\Omega)\ . \tag{14}$$

The dimensionless spectral density function of the height response of an aircraft to this component is then given[6] by

$$|Y\ (i\hat{\omega})|^2\ \hat{G}(\hat{\omega})\ , \tag{15}$$

where $Y(p)$ is the appropriate transfer function; the variance of the height response is

$$\int_0^\infty |Y\ (i\hat{\omega})|^2\ \hat{G}(\hat{\omega})\ d\hat{\omega}\ . \tag{16}$$

Assuming the two components of the turbulence to be independent, we obtain for the standard deviation of the height response

$$\sigma = \frac{m}{\rho S}\left[\int_0^\infty \left\{|Y_{long}\ (i\hat{\omega})|^2\ \hat{G}_{uu}(\hat{\omega}) + |Y_{lat}\ (i\omega)|^2\ \hat{G}_{ww}(\hat{\omega})\right\} d\hat{\omega}\right]^{\frac{1}{2}}\ ft\ . \tag{17}$$

Since the turbulence spectra given in section 2 are proportional to σ_g^2, their use in conjunction with equation (17) will make σ proportional to σ_g.

On the further assumption that the turbulence-induced height errors of an aircraft flying in turbulence of a given intensity (i.e. fixed root mean square gust velocity) have a Gaussian distribution with mean zero and standard deviation σ given by equation (17), the probability of the height error being between x_1 and x_2 feet during routine operations is

$$\text{prob}\ (x_1 < |h| < x_2) \quad = \quad 2\int_0^\infty \hat{f}(\sigma_g) \int_{x_1}^{x_2} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}x^2/\sigma^2\right) dx\, d\sigma_g \quad . \quad (18)$$

We may identify this probability with the proportion of flight time which the aircraft spends between x_1 and x_2 feet above or below its planned flight path, assuming atmospheric turbulence to be the only source of error.

5 NUMERICAL EXAMPLES

We use equation (17) to calculate the root mean square height response corresponding to unit root mean square gust velocity in a number of cases. Results for different intensities of turbulence may be calculated by multiplying the value of σ thus obtained by the appropriate value of σ_g. Except where otherwise stated, we use the standard turbulence spectra, equations (1) and (2), with $L = 1000$ feet.

We consider first a medium bomber cruising at 40000 feet. The aerodynamic data are given in Table 1; Figs.3 and 4 show the longitudinal and lateral transfer functions together with the corresponding turbulence spectra. From equation (17) we obtain in this case $\sigma = 2.43$ feet (this and subsequent results were calculated on the R.A.E. Mercury computer). Setting $Y_{long} = 0$, we obtain $\sigma = 2.37$ feet, showing that the response to the vertical component of the turbulence dominates that due to the longitudinal component; therefore in all subsequent calculations we neglect the latter.

We now examine how the computed value of σ depends on the accuracy of the data used. Table 2 shows the effect on σ of varying the aerodynamic derivatives x_u, x_w, z_u, z_w, κ, $\tilde{\omega}$, χ, ν and δ; we conclude that x_w, κ, $\tilde{\omega}$, χ and ν may be neglected altogether and that only z_w has a great effect on σ. For subsequent cases, we set $x_w = \kappa = \tilde{\omega} = \chi = \nu = 0$ and $\delta = 100$; with the remaining data as in Table 1, we obtain $\sigma = 2.35$ feet.

Table 3 shows the effect of varying the autopilot constants; σ is insensitive to changes in $G_{\dot{h}}$, but changes by 5% for a 10% change in G_θ or G_h. From equation (9), it can be seen that multiplying δ by a given factor is equivalent to multiplying all the autopilot constants by the same factor; hence the insensitivity of σ to δ could be deduced from the results given in Table 3.

Table 4 shows the effect of varying the scale L used in defining the turbulence spectrum and also results obtained using the minus five-thirds law spectrum, equations (4), with various values of the cut-off wavelength λ. We see that use of the minus five-thirds law, with $\lambda = 5000$ feet, gives a result differing little from that obtained with the standard spectrum. However, σ is shown to be strongly dependent on the value chosen for L or λ and this underlines the need for more information on the low frequency part of the turbulence spectrum; Figs. 3 and 4 show that the major part of the response corresponds to gust wavelengths greater than 3000 feet, where the shape of the spectrum is not well established.

As a second example we consider a large turboprop aircraft in a range of flight configurations: cruise, climb, loiter and approach. Table 5 shows the data used and the results obtained, which indicate that the aircraft is least susceptible to height errors when in the cruise configuration and most susceptible during approach. It must be noted, however, that the same values of the autopilot constants were used in all these cases. The values are typical of those which would apply to a cruising aircraft under automatic control; in other circumstances the aircraft would be controlled differently.

Thirdly, we consider subsonic and supersonic jet transport aircraft (see Table 6). The standard deviations of height error calculated for the subsonic jet are slightly smaller than those obtained for the turboprop. The supersonic jet, however, shows a markedly smaller response in the cruise configuration, though it may be doubted whether the assumptions made about the aerodynamic derivatives hold in this case.

Finally, equation (18) is used to calculate the distribution of height-keeping errors of the subsonic jet, referred to above, in the cruise configuration at 40000 feet. The resulting histogram is shown in Fig.5, together with the experimental results of Gracey and Shipp[1]. It is apparent that, according to the present calculation, height errors of more than 100 feet due to turbulence occur far less frequently than do observed errors of the same magnitude.

6 CONCLUSION

We have described an aircraft flying under an autopilot height-lock as a linear system with a random input representing the effect of atmospheric turbulence and with the resulting height variation as output. In this way a formula has been developed for the standard deviation of height errors due to atmospheric turbulence and some numerical examples have been computed. An expression has also been given for the distribution of these height errors, and a comparison made between the calculated distribution for a subsonic jet transport aircraft and the distribution of observed height errors reported by Gracey and Shipp[1]. This comparison is illustrated in Fig.5, which shows the calculated errors due to atmospheric turbulence to be much smaller than the observed errors.

A major difficulty in the procedure described above is the representation of the autopilot; further information is needed on this and on the still more difficult problem of representing the behaviour of a human pilot. The numerical examples given in section 5 indicate that another drawback of the method is the absence of adequate data on the longer-wavelength components of atmospheric turbulence; these are comparatively unimportant for the purpose of gust load evaluations, but are almost entirely responsible for the height errors caused by the turbulence.

Several other factors contribute to the difference between the calculated and observed height errors. Firstly, the variation in the velocity of the relative wind is probably not the only way in which turbulence affects height-keeping - e.g. the static pressure altimeter reading used by the autopilot may be affected. Secondly, some atmospheric phenomena which cause considerable height-keeping errors cannot be described by the spectral methods used in the present work. Thirdly, there are the effects of the various approximations, such as the linearisation of the aircraft equations of motion and representation of atmospheric turbulence as a homogeneous, isotropic, stationary random process, which were necessary in order to set up a tractable mathematical model. Finally, the errors described by Gracey and Shipp are not entirely due to atmospheric turbulence; indeed we may conclude that those components of the turbulence which can be represented by the spectral theory make no appreciable contribution to these errors.

Thus the conclusion of this Report is that atmospheric turbulence, excluding such phenomena as waves, single large up-draughts, and wind-shears, which cannot be treated by the spectral method used here, does not make a significant contribution to aircraft height-keeping errors, but that the accuracy to which this contribution can be calculated is limited by the lack of

(i) sufficiently accurate data on the characteristics of atmospheric turbulence, particularly as regards the longer-wavelength components;

(ii) adequate representation in the aircraft equations of motion of the influence of control by the pilot, whether human or automatic; and,

(iii) consideration of the effects of turbulence on the equations of motion of the aircraft other than those resulting from the variation in the velocity of the relative wind.

Appendix A

DERIVATION OF THE AIRCRAFT EQUATIONS OF MOTION[13,14]
(see section 3)

Consider a rigid aircraft which, in still air, would be flying steadily with speed U along a straight flight path making an angle γ above the horizontal. Axes are fixed relative to the aircraft, with origin O at the centre of gravity, such that in the steady flight condition (Fig.2(a)) Ox is directed forwards along the flight path, Oy is horizontal and to starboard, and Oz is downwards in the vertical plane containing Ox. Only forward, vertical, and pitching motions of the aircraft are considered; thus Oy remains horizontal, while Ox in general makes an angle θ above the steady flight path, where θ is the pitching angle (Fig.2(b)). Writing U+u and w for the components of velocity along Ox and Oz respectively, and assuming u/U, w/U and θ to be small, we have the following equations of motion:

$$m\frac{du}{dt} = X - mg\sin(\gamma+\theta) \quad , \tag{A-1}$$

$$m\left(\frac{dw}{dt} - U\frac{d\theta}{dt}\right) = Z + mg\cos(\gamma+\theta) \quad , \tag{A-2}$$

$$B\frac{d^2\theta}{dt^2} = M \, , \tag{A-3}$$

where m = mass of the aircraft,

B = moment of inertia about Oy,

(X,Z) = components along Ox and Oz respectively of the aerodynamic force (in which we include the engine thrust),

and M = moment of the aerodynamic force about Oy.

The vertical deviation h from the steady flight path is given by

$$\frac{dh}{dt} = (U+u)\sin(\gamma+\theta) - w\cos(\gamma+\theta) - U\sin\gamma \quad , \tag{A-4}$$

and the elevator angle η by

$$\eta = G_\theta\theta + G_h h + G_{\bar{h}}\int h\,dt \, , \tag{A-5}$$

where G_θ, G_h and $G_{\bar{h}}$ are the autopilot constants.

The variation of the relative wind (Fig.2(c)) is composed of the variation of the velocity of the aircraft and of the wind itself. The components along Ox and Oz of the latter variation are the longitudinal and lateral gust velocities u_g and w_g respectively; thus the components of the relative wind are $U + u + u_g$ and $w + w_g$. We assume that if u_g and w_g are small compared to U, then u, w, θ and η remain small and the variation of the aerodynamic force can be represented by the linear expressions

$$X - X_o = (u + u_g) X_u + (w + w_g) X_w \tag{A-6}$$

$$Z - Z_o = (u + u_g) Z_u + (w + w_g) Z_w \tag{A-7}$$

$$M - M_o = (u + u_g) M_u + (w + w_g) M_w + \frac{dw}{dt} M_{\dot{w}} + \frac{d\theta}{dt} M_q + \eta M_\eta \quad , \tag{A-8}$$

where X_u, X_w, Z_u, Z_w, M_u, M_w, $M_{\dot{w}}$, M_q and M_η are constants (aerodynamic derivatives of the aircraft) and X_o, Z_o and M_o are the values of X, Z and M respectively in steady flight. From equations (A-1) to (A-3), we obtain in the steady flight condition

$$X_o = mg \sin \gamma \tag{A-9}$$

$$Z_o = - mg \cos \gamma \tag{A-10}$$

$$M_o = 0 \ . \tag{A-11}$$

Substituting the values of X, Z and M given by equations (A-6) to (A-11) into equations (A-1) to (A-4) and neglecting squares and products of small quantities, we obtain

$$m \frac{du}{dt} + g\theta \cos \gamma = (u + u_g) X_u + (w + w_g) X_w \tag{A-12}$$

$$m \frac{dw}{dt} + g\theta \sin \gamma - U \frac{d\theta}{dt} = (u + u_g) Z_u + (w + w_g) Z_w \tag{A-13}$$

$$B \frac{d^2\theta}{dt^2} = (u + u_g) M_u + (w + w_g) M_w + \frac{dw}{dt} M_{\dot{w}} + \frac{d\theta}{dt} M_q + \eta M_\eta \tag{A-14}$$

$$\frac{dh}{dt} = (U\theta - w) \cos \gamma \ . \tag{A-15}$$

It is usual and convenient to rewrite these equations in non-dimensional form, using the following units of mass, speed and time:

the mass of the aircraft	m lb,	
the speed of steady motion	U ft/sec,	
the airsec	$\hat{t} = m/(\rho SU)$ sec,	

where ρ is the local air density in slugs/ft^3 and S is the aircraft's wing area in ft^2. We therefore write

$$\hat{u} = u/U, \text{ etc} \tag{A-16}$$

$$x_u = X_u/(\rho SU), \text{ etc} \tag{A-17}$$

$$k = (mg \cos\gamma)/(\rho SU^2) = \tfrac{1}{2} C_L, \tag{A-18}$$

where C_L is the lift coefficient of the aircraft,

$$k_1 = k \tan\gamma \tag{A-19}$$

$$\mu_1 = m/(\rho S\ell), \quad i_B = B/(m\ell^2), \tag{A-20}$$

where ℓ is the tail arm of the aircraft,

$$\hat{G}_h = mG_h/(\rho S), \qquad \hat{G}_{\dot{h}} = m^2 G_{\dot{h}}/(\rho^2 S^2 U) \tag{A-21}$$

$$m_u = M_u/(\rho SU\ell), \qquad \kappa = -\mu_1 m_u/i_B \tag{A-22}$$

$$m_w = M_w/(\rho SU\ell), \qquad \tilde{\omega} = -\mu_1 m_w/i_B \tag{A-23}$$

$$m_{\dot{w}} = M_{\dot{w}}/(m\ell), \qquad \chi = -\mu_1 m_{\dot{w}}/i_B \tag{A-24}$$

$$m_q = M_q/(\rho SU\ell^2), \qquad \nu = -m_q/i_B \tag{A-25}$$

$$m_\eta = mM_\eta/(\rho^2 S^2 U^2 \ell^2), \qquad \delta = -m_\eta/i_B \tag{A-26}$$

$$\tau = t/\hat{t}, \quad D \equiv \frac{d}{d\tau} = m/(\rho SU)\frac{d}{dt}, \quad \hat{h} = (\rho S/m)h \; . \tag{A-27}$$

Equations (A-12) to (A-15) and (A-5) can then be written in the form

$$(D - x_u)\hat{u} \quad - x_w\hat{w} \quad + k\theta \quad = x_u\hat{u}_g + x_w\hat{w}_g \tag{A-28}$$

$$-z_u\hat{u} + (D - z_w)\hat{w} + (k_1 - D)\theta \quad = z_u\hat{u}_g + z_w\hat{w}_g \tag{A-29}$$

$$\kappa\hat{u} + (\chi D + \tilde{\omega})\hat{w} + (D^2 + \nu D)\theta + \delta\eta \quad = -\kappa\hat{u}_g - \tilde{\omega}\hat{w}_g \tag{A-30}$$

$$(\cos\gamma)\hat{w} - (\cos\gamma)\,\theta \quad + D\hat{h} = 0 \tag{A-31}$$

$$G_\theta D\theta - D\eta + (\hat{G}_h D + \hat{G}_{\dot{h}})\hat{h} = 0 \tag{A-32}$$

Solving these equations for $\hat{h}$, we obtain

$$F(D)\hat{h} = F_{long}(D)\ \hat{u}_g + F_{lat}(D)\ \hat{w}_g \quad , \tag{A-33}$$

where

$$F(D) = D^6 + a_3 D^5 + a_2 D^4 + a_1 D^3 + a_o D^2 + a_{-1} D + a_{-2} \quad , \tag{A-34}$$

$$F_{long}(D) = -(D^4 + x_3 D^3 + x_2 D^2)\ z_u \cos\gamma \quad , \tag{A-35}$$

$$F_{lat}(D) = -(D^4 + z_3 D^3 + z_2 D^2 + z_1 D)\ z_w \cos\gamma \quad , \tag{A-36}$$

$$a_3 = K_3 \tag{A-37}$$

$$a_2 = K_2 + G_\theta \delta \tag{A-38}$$

$$a_1 = K_1 + N_1 G_\theta \delta \tag{A-39}$$

$$a_o = K_o + P_1 G_\theta \delta + (N_1 - Q_1)\hat{G}_h \delta \cos\gamma \tag{A-40}$$

$$a_{-1} = (P_1 - R_1)\hat{G}_h \delta \cos\gamma + (N_1 - Q_1)\hat{G}_{\bar{h}} \delta \cos\gamma \tag{A-41}$$

$$a_{-2} = (P_1 - R_1)G_{\bar{h}} o \cos\gamma \tag{A-42}$$

$$K_3 = N_1 + \nu + \chi \tag{A-43}$$

$$K_2 = P_1 + \nu N_1 + \chi Q_1 + \tilde{\omega} \tag{A-44}$$

$$K_1 = \nu P_1 + \chi R_1 + \tilde{\omega} Q_1 - \kappa S_1 \tag{A-45}$$

$$K_o = \tilde{\omega} R_1 - \kappa T_1 \tag{A-46}$$

$$N_1 = -(x_u + z_w) \tag{A-47}$$

$$P_1 = x_u z_w - x_w z_u \tag{A-48}$$

$$Q_1 = -(k_1 + x_u) \tag{A-49}$$

$$R_1 = k_1 x_u - k z_u \tag{A-50}$$

$$S_1 = k - x_w \tag{A-51}$$

$$T_1 = k_1 x_w - k z_w \tag{A-52}$$

$$x_3 = \chi + \nu \tag{A-53}$$

$$x_2 = \tilde{\omega} + G_\theta \delta + \kappa (N_1 - Q_1)/z_u \tag{A-54}$$

$$z_3 = \chi + \nu - P_1/z_w \tag{A-55}$$

$$z_2 = G_\theta \delta + \{\tilde{\omega} k_1 - P_1\ (\chi + \nu)\}/z_w \tag{A-56}$$

$$z_1 = (\kappa T_1 - \tilde{\omega} R_1 - P_1 G_\theta \delta)/z_w\ . \tag{A-57}$$

The longitudinal and lateral transfer functions are

$$Y_{long}(p) = F_{long}(p)/F(p) \tag{A-58}$$

$$Y_{lat}(p) = F_{lat}(p)/F(p) \quad . \tag{A-59}$$

Table 1

DATA FOR A MEDIUM BOMBER CRUISING AT 40000 FT

x_u	-0.02	κ	-0.849
x_w	0.011	$\tilde{\omega}$	19.5
z_u	-0.365	χ	3.15
z_w	-2.56	ν	4.50
C_L	0.274	δ	165.6
S	960 ft^2	G_θ	1.0
W	40620 lb	G_h	0.01 deg ft^{-1}
U	726 ft sec^{-1}	$G_{\dot{h}}$	0.0002 deg ft^{-1} sec^{-1}

Table 2

SENSITIVITY OF COMPUTED HEIGHT ERROR TO CHANGES IN THE AERODYNAMIC DERIVATIVES

Aerodynamic derivative with value different from that in Table 1		σ (ft)
none		2.37
x_u	-0.02×10^{-2}	2.34
x_w	0.011×10^{-2}	2.37
z_u	-0.365×10^{-2}	2.43
z_w	-2.56×10^{-2}	0.32
κ	-0.849×10^{-2}	2.37
$\tilde{\omega}$	19.5×10^{-2}	2.37
χ	3.15×10^{-2}	2.36
ν	4.50×10^{-2}	2.36
δ	50	2.42
δ	100	2.39
δ	200	2.37
δ	500	2.36

Table 3

SENSITIVITY OF COMPUTED HEIGHT ERROR TO CHANGES IN THE AUTOPILOT CONSTANTS

<table>
<tr><th>G_θ</th><th>G_h
deg ft^{-1}</th><th>$G_{\bar{h}}$
deg ft^{-1} sec^{-1}</th><th>σ
ft</th></tr>
<tr><td>1.0</td><td>0.01</td><td>0.0002</td><td>2.35</td></tr>
<tr><td>0.9</td><td rowspan="2">0.01</td><td rowspan="2">0.0002</td><td>2.23</td></tr>
<tr><td>1.1</td><td>2.46</td></tr>
<tr><td rowspan="2">1.0</td><td>0.009</td><td rowspan="2">0.0002</td><td>2.47</td></tr>
<tr><td>0.011</td><td>2.24</td></tr>
<tr><td rowspan="2">1.0</td><td rowspan="2">0.01</td><td>0.00018</td><td>2.35</td></tr>
<tr><td>0.00022</td><td>2.35</td></tr>
</table>

Table 4

SENSITIVITY OF COMPUTED HEIGHT ERROR TO THE FORM OF THE TURBULENCE SPECTRUM

Standard spectrum		Minus five-thirds law	
L ft	σ ft	λ ft	σ ft
500	1.65	2000	1.46
1000	2.35	5000	2.30
2000	3.29	10000	3.20

Table 5

STANDARD DEVIATION OF HEIGHT ERROR OF A LARGE TURBOPROP AIRCRAFT IN VARIOUS FLIGHT CONFIGURATIONS

Configuration	Height (ft)	x_u	z_u	z_w	C_L	W (lb)	U (ft sec^{-1})	σ (ft)
Cruise	20000	−0.030	−0.327	−3.71	0.327	120 000	615	2.8
	30000	−0.038	−0.540	−3.71	0.540	120 000	571	3.0
Climb	0	−0.040	−0.575	−3.71	0.575	120 000	338	4.8
	10000	−0.040	−0.575	−3.71	0.575	120 000	394	4.2
	20000	−0.040	−0.575	−3.71	0.575	120 000	463	3.6
Loiter	0	−0.040	−0.590	−3.71	0.587	110 000	321	5.0
	10000	−0.040	−0.590	−3.71	0.587	110 000	374	4.4
Approach	0	−0.150	−1.30	−2.87	1.30	105 000	211	5.6

In all configurations	$S = 1529$ ft^2 $\delta = 100$ $G_\theta = 1$ $G_h = 0.01$ deg ft^{-1} $G_{\dot{h}} = 1.667 \times 10^{-4}$ deg ft^{-1} sec^{-1}

Table 6

STANDARD DEVIATION OF HEIGHT ERROR OF A SUBSONIC AND A SUPERSONIC JET TRANSPORT

Aircraft type	Configuration	Height (ft)	x_u	z_u	z_w	C_L	W (lb)	U (ft sec^{-1})	S (ft^2)	σ (ft)
Subsonic	Cruise	40000	−0.019	−0.478	−2.36	0.478	200 000	767	2430	2.2
	Loiter	20000	−0.019	−0.3	−2.36	0.301	180 000	623	2430	2.7
Supersonic	Cruise	60000	0.00653	0.034	−0.813	0.105	270 000	2130	5040	0.8
	Climb	30000	−0.012	−0.142	−0.969	0.143	230 000	848	5040	2.0

In all cases	$\delta = 100$ $G_\theta = 1$ $G_h = 0.01$ deg ft^{-1} $G_{\dot{h}} = 1.667 \times 10^{-4}$ deg ft^{-1} sec^{-1}

SYMBOLS

$a_{-2}, a_{-1}, \ldots, a_3$	coefficients in F(D), see (A-34) and (A-37-42)
B	moment of inertia of aircraft about Oy
C_L	lift coefficient of aircraft
D	$d/d\tau$
$\hat{f}$	probability distribution of σ_g for routine operations, see (5)
F, F_{lat}, F_{long}	operators in height response equation, see (A-33-36)
g	acceleration due to gravity 32.2 ft/sec^2
G	power spectral density function of a component of atmospheric turbulence
$\hat{G}$	dimensionless form of G, see (14)
G_{uu}, G_{ww}	power spectral density functions of longitudinal and vertical components of atmospheric turbulence
$G_\theta, G_h, G_{\bar{h}}$	autopilot constants, see (A-5)
$\hat{G}_h, \hat{G}_{\bar{h}}$	dimensionless autopilot constants, see (A-21)
h	vertical deviation of aircraft from steady flight path
$\hat{h}$	dimensionless form of h, see (A-27)
i_B	$B/(m\ell^2)$
k	$\frac{1}{2}C_L$
k_1	$k \tan \gamma$
K_o, K_1, K_2, K_3	constants, see (A-43-46)
ℓ	tail arm of aircraft
L	scale of turbulence
m	mass of the aircraft
$m_u, m_w, m_{\dot{w}}, m_q, m_\eta$	dimensionless aerodynamic derivatives, see (A-22-26)
M	moment of aerodynamic force about Oy
M_o	value of M in steady flight
$M_u, M_w, M_{\dot{w}}, M_q, M_\eta$	aerodynamic derivatives, see (A-8)
N_1	$-(x_u + z_w)$
O	centre of gravity of aircraft
P_1	$x_u z_w - x_w z_u$
Q_1	$-(k_1 + x_u)$
R_1	$k_1 x_u - k z_u$
S	aircraft wing area
S_1	$k - x_w$
t	time in seconds

SYMBOLS (Contd)

$\hat{t}$	$m/(\rho SU)$ sec = 1 airsec
T_1	$k_1 x_w - kz_w$
u	component along Ox of difference between steady and unsteady velocity of aircraft
$\hat{u}$	u/U
u_g	longitudinal gust velocity
$\hat{u}_g$	u_g/U
U	steady relative wind speed
w	component of aircraft velocity along Oz
$\hat{w}$	w/U
w_g	lateral gust velocity
$\hat{w}_g$	w_g/U
W	mg = weight of aircraft
x	co-ordinate in direction fixed in aircraft and, in steady flight, directed forwards along flight path
x_2, x_3	coefficients in $F_{long}(D)$, see (A-35, 53, 54)
x_u, x_w	dimensionless aerodynamic derivatives, see (A-17)
X	component of aerodynamic force along Ox
X_o	value of X in steady flight
X_u, X_w	aerodynamic derivatives, see (A-6)
y	coordinate in direction fixed in aircraft and, in steady flight, horizontal and to starboard
Y_{lat}, Y_{long}	transfer functions of height response to lateral and longitudinal gust velocities respectively, see (A-58, 59)
z	co-ordinate in direction fixed in aircraft and, in steady flight, directed downwards, perpendicular to flight path
z_1, z_2, z_3	coefficients in $F_{lat}(D)$, see (A-36, 55-57)
Z	component of aerodynamic force along Oz
Z_o	value of Z in steady flight
Z_u, Z_w	aerodynamic derivatives, see (A-7)
γ	angle of steady flight path above horizontal
δ	$-m_\eta/i_B$
η	elevator angle
θ	pitching angle, see Fig. 2(b)

SYMBOLS (Contd)

κ	$-\mu_1 m_u / i_B$
λ	cut-off wavelength in minus five-thirds law spectrum
Λ	$m/(\sqrt{3}\rho SL)$
μ_1	$m/(\rho SL)$
ν	$-m_q / i_B$
ρ	local air density
σ	rms height deviation due to turbulence
σ_g	intensity of turbulence = rms gust velocity
τ	$t/\hat{t}$ = time in airsecs
χ	$-\mu_1 m_{\dot{w}} / i_B$
ω	frequency of turbulence observed from aircraft
$\hat{\omega}$	$(m/\rho S)\Omega$
$\tilde{\omega}$	$-\mu_1 m_w / i_B$
Ω	ω/U

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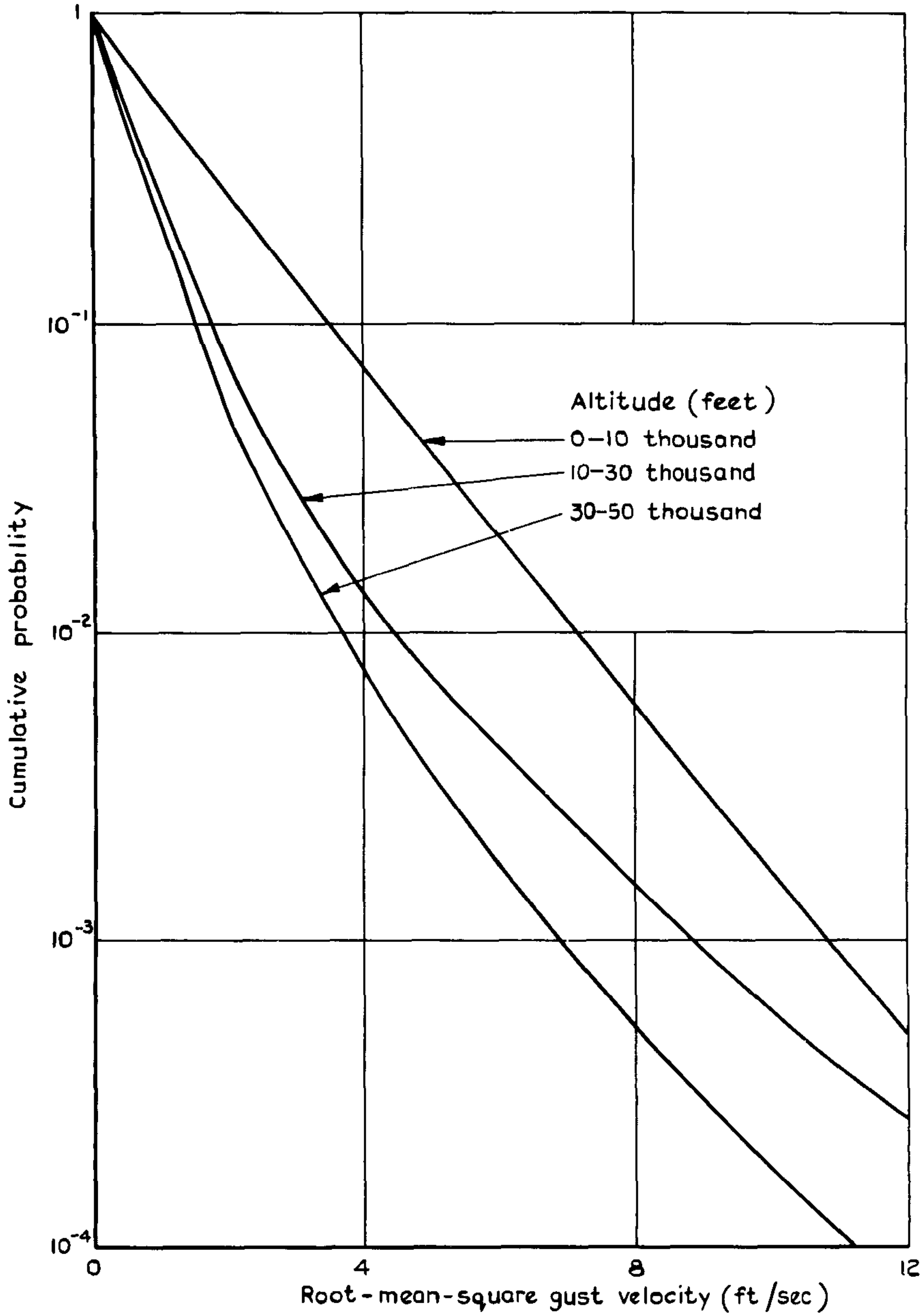
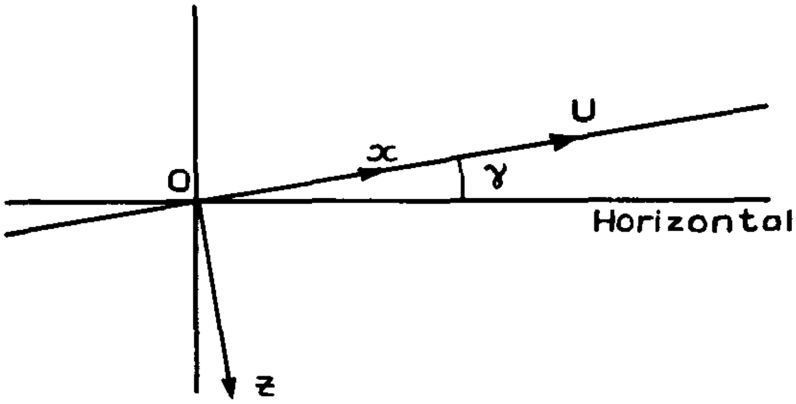
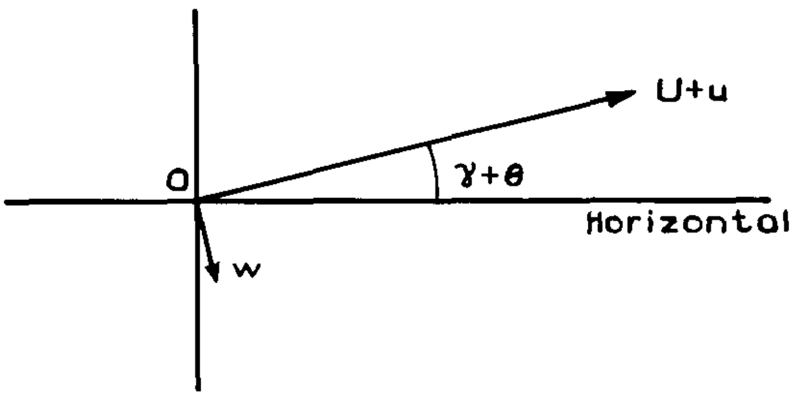


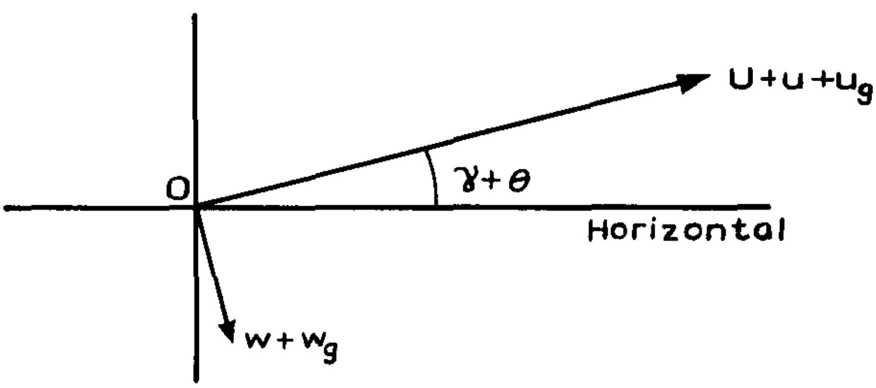
Fig.1 Cumulative probability distribution of rms gust velocity for routine operations



a Axes in steady flight condition



b Velocity components relative to steady wind



c Velocity components relative to wind

Fig. 2 ab&c Axes and aircraft velocity components

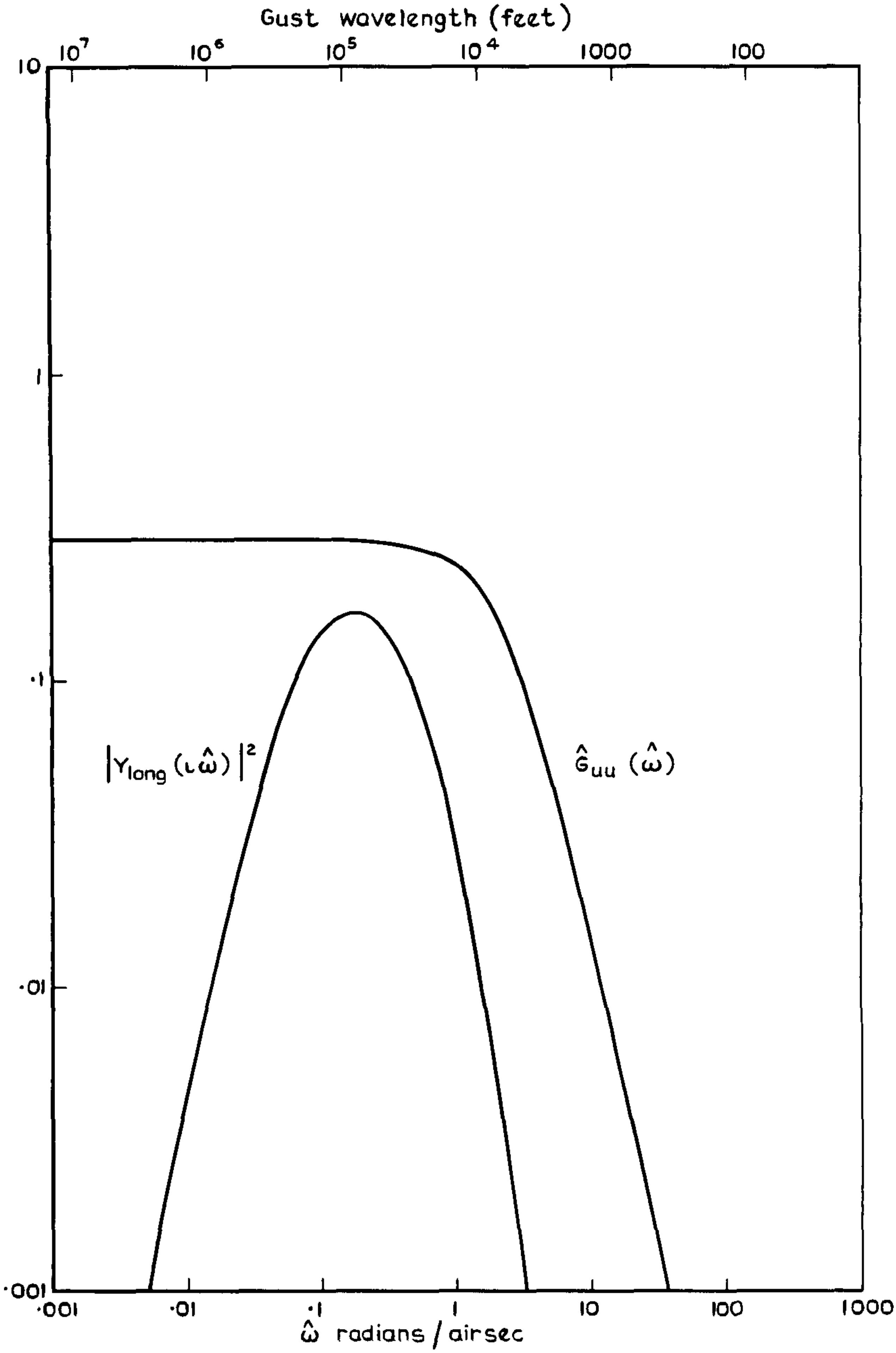


Fig.3 Longitudinal transfer function of medium bomber and longitudinal turbulence spectrum

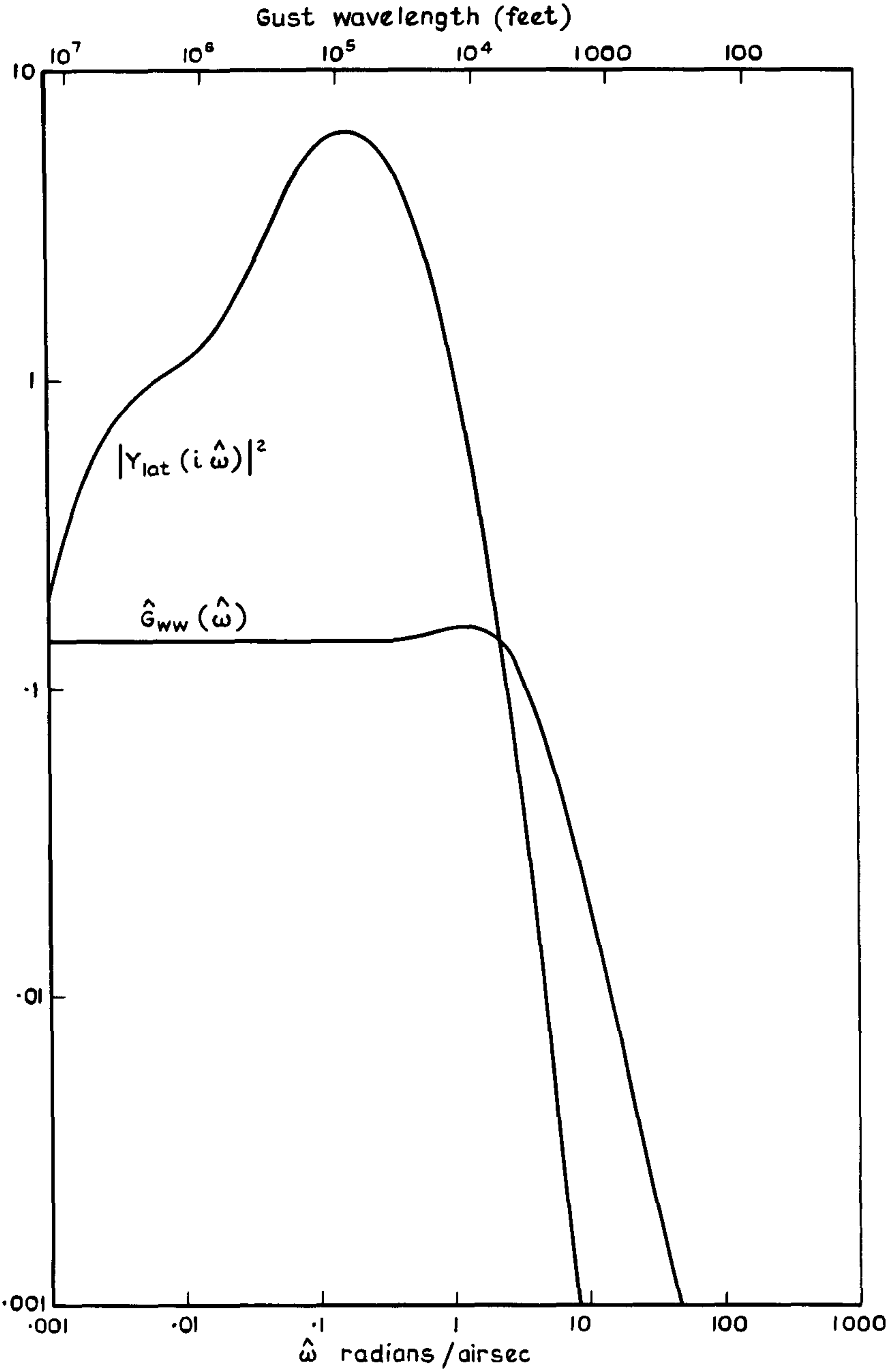


Fig.4 Lateral transfer function of medium bomber and vertical turbulence spectrum

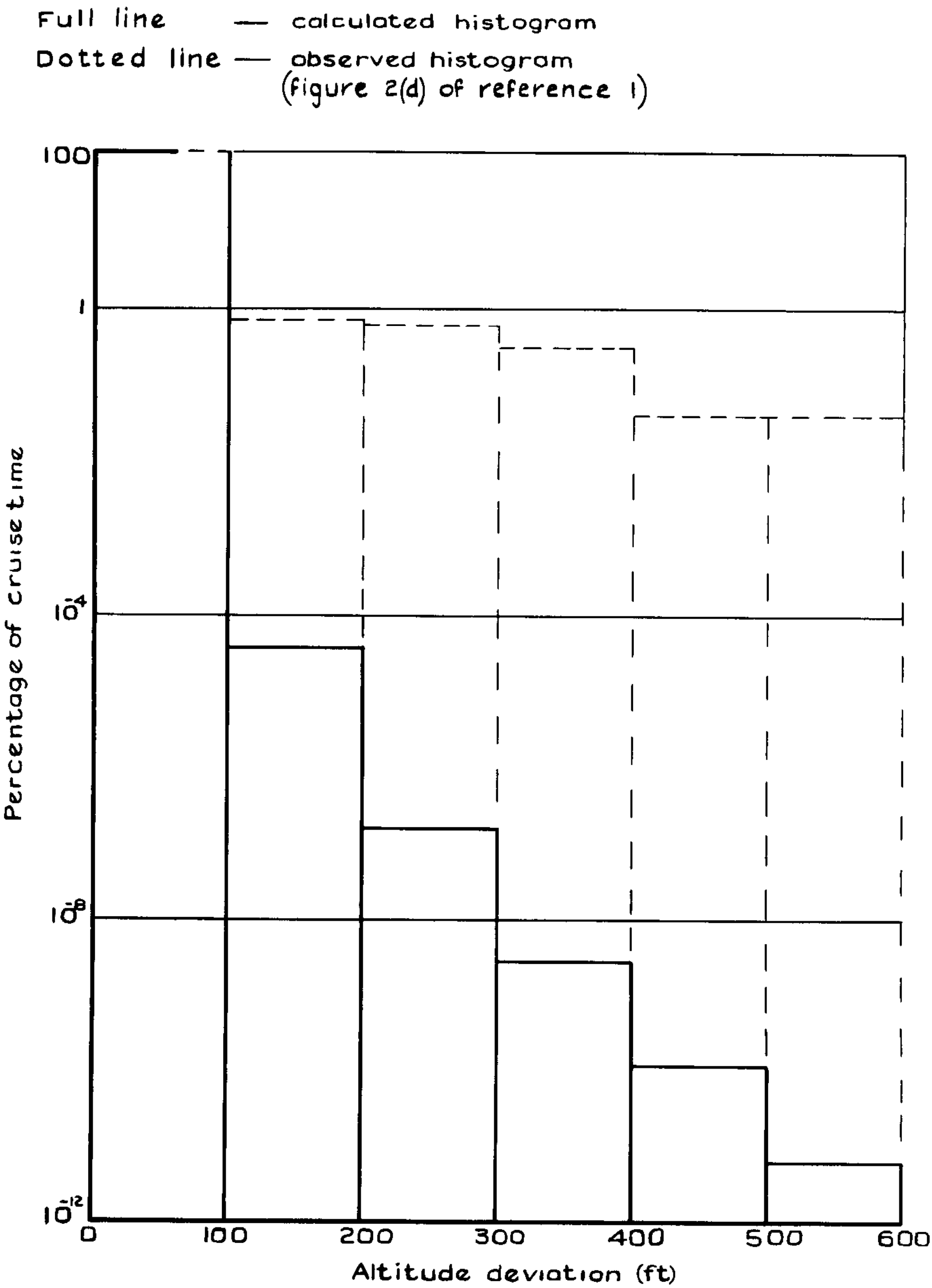


Fig.5 Percentage of cruise time spent by a subsonic jet within each altitude increment from cruise altitude, 40000 ft

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