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by

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Crossflow in Turbulent Boundary Layers

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Summary

Two well known representations for the crossflow velocity profile, due to Mager and Johnston, are discussed, and limitations to their applicability are outlined. A number of ideas relating to the Johnston triangular model are discussed and a method for extending its usefulness is presented. Finally an approach which should lead to a prediction of the form of cross-over crossflow profiles (where the sign of the crossflow changes through the depth of the layer) is described. The predictions are shown to be consistent with the available measured profiles.

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1. Introduction

When the direction of the mean velocity changes through the depth of the boundary layer it is generally expedient to resolve the flow into two components parallel to the plane of the surface, one in the direction of the flow outside the boundary layer and the other normal to this. These components are generally referred to as the streamwise flow and crossflow respectively, and in the present notation their mean velocities are denoted by u and v. The corresponding curvilinear coordinates in the streamwise and crossflow directions are denoted by s and n and the coordinate normal to the surface by ζ.

Crossflow can arise from a number of causes but this paper is restricted to considering the profiles produced when the external flow follows streamlines which are curved in the plane of the surface. Turning the initial boundary-layer vorticity (which is normal to the external flow velocity) produces a component of vorticity parallel to the external flow, and consequently a velocity component perpendicular to the streamwise direction. More simply, the centrifugal acceleration of the fluid outside the boundary layer, which is a consequence of following the curved path, must be exactly balanced by a pressure gradient normal to the external streamlines. But because the velocity within the boundary layer is less than that of the external flow, its centrifugal acceleration is correspondingly reduced. The excess transverse pressure gradient therefore produces an acceleration of the slower moving fluid normal to the streamwise direction, and consequently leads to the development of crossflow.

The generation of crossflow can radically alter the boundary-layer development and calculations neglecting it are therefore liable to be considerably in error. It is thus of great practical interest to be able to specify the form of the crossflow profile.

There have been two representations proposed for the turbulent crossflow profile which have either found wide application or aroused great interest. The first was proposed by Mager[1] based on the measurements of Gruschwitz[2] and is

$$\frac{v}{u} = (1 - \zeta/\delta)^2 \tan\beta ,$$

where δ is the boundary-layer thickness and β is the angle between the streamwise direction and limiting direction of the flow very close to the surface. More recently Johnston[3] proposed a hodograph model of triangular form, Figure 1, which divides the crossflow profile into two regions; one close to the surface where

$$v = u.\tan\beta$$

and another in the outer part where

$$v = A(U_s - u).$$

It is interesting to note that Gruschwitz had plotted his results as a hodograph and obtained the triangular form. It was Mager[4], however, who drew attention to the applicability of inviscid analysis to the outer part of the turbulent boundary layer and showed, for zero streamwise pressure gradient, the

proportionality/

proportionality of the crossflow to the streamwise velocity defect. Subsequently Johnston showed the generality of the triangular form.

It can be readily shown (Cumpsty and Head[5]) that Mager's profile, used with a good streamwise profile representation, produces closely triangular hodograph plots. The Mager representation can therefore be treated as a special case of the triangular representation with one instead of two independent parameters. This is largely attributable to the fact that Mager based his profile on Gruschwitz's data which, it has been noted, fits the triangular form.

Neither Johnston's nor Mager's representations are able to describe the crossflow profile when the sign of the crossflow changes through the depth of the layer. Such profiles, in fact, occur quite frequently in cases of practical interest, and in the last part of this paper a method for predicting the profiles is outlined. Until then the discussion refers to crossflows of one sign only.

2. The Mager representation

Mager based his profile on the crossflows measured by Gruschwitz in a comparatively gently curved duct and the form may therefore be expected to be fairly satisfactory for crossflows which have developed gradually in a fairly gentle pressure field. Figure 2 compares a number of crossflow profiles with the Mager expression. The measurements by Francis and Pierce[6]* were made in a curved duct, while those of Hornung and Joubert[7] were made in the rapidly disturbed region upstream of an obstacle. As would be expected, the former measurements are comparatively well represented whilst the latter are represented rather badly. One profile measured near to the separation line on a swept wing by Cumpsty and Head[8] is also shown, and the agreement in this case is very satisfactory, the crossflow having developed comparatively gradually.

No comparison is shown in Figure 2 for crossflows measured in regions where the crossflow was decreasing or where the streamline curvature had changed sign. In these cases much worse agreement would be expected, particularly when the crossflow changes sign.

The inherent weakness of the Mager representation becomes apparent from the discussion of the Johnston representation. It can be shown that for most of the boundary-layer thickness the crossflow is proportional to the streamwise velocity defect and a function of the external flow. The Mager representation, however, implicitly assumes that the crossflow is everywhere determined by the angle β, which must depend upon the streamwise boundary-layer development, the shear stresses and the gradients of crossflow, as well as the turning of the external flow.

3. The Johnston representation

There is a very large body of experiment, in a wide variety of flow geometries, to support the triangular representation. For example, both the rapidly disturbed flows measured by Johnston and by Hornung and Joubert and the comparatively gently disturbed flows measured by Gruschwitz and by Francis and Pierce support this form, and show very good agreement with the model.

The/

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* The influence of the side walls restricts the useful range of this data to the first few profiles.

The significance of Johnston's representation is two-fold. First there is the division of the profile into two regions, one effectively inviscid (so far as the development or crossflow is concerned) and the other in which shear stresses are dominant. Second there is the triangular plot which provides a simple and graphic way of describing this, but is incidental to the more important concept of two regions.

It is possible, by several different approaches, to use a first-order analysis (ignoring shear stresses) to find the ratio of the crossflow velocity to the streamwise velocity defect. If there is no streamline convergence or divergence (i.e., the flow outside the boundary layer does not have a component of velocity normal to the surface) it can be shown that the slope of the outer part of the triangle is given by

$$A = v/(U_s - u) = U_s^2 \int_0^{\alpha} \frac{d\alpha'}{U_s^2}$$

where α is the angle turned through by the external flow. If there is flow convergence or divergence the equation for A has a simple form (Cumpsty[9]), but not one which is directly integrable. Whether or not there is flow convergence or divergence, in the outer part of the layer the ratio of the crossflow velocity to the streamwise velocity defect is independent of the boundary layer properties and is a function only of the external flow.

Because the crossflow in the outer part of the layer is proportional to the streamwise velocity defect, the hodograph plot is ideally suited to representing the crossflow. Where the crossflow velocity is found to be proportional to streamwise velocity defect it is assumed that the crossflow development has been effectively inviscid, at least for some considerable distance. Figure 3 compares a crossflow profile measured by Johnston plotted conventionally (i.e., against distance from the surface) and as a hodograph. This shows just how much of the boundary layer is in the outer region and whose development can therefore be described by the inviscid analysis. It may well be that in some circumstances the region in which the inviscid analysis is valid is very small, but even then the inclination of the polar plot at the boundary layer edge is, at least in theory, known from the behaviour of external flow.

The existence of viscosity requires a deviation from the inviscid relation to satisfy the no-slip condition at the surface, and the simplest form, which was proposed by Johnston, is to assume that the inner region can be described by a straight line, $v = u.\tan\beta$, through the origin of the hodograph. There is a basic inaccuracy in this linear inner region, however, for it can be shown that in general unless the transverse pressure gradient, $\frac{\partial p}{\partial n}$, vanishes, the flow direction varies continuously as the surface is approached. (For example, see Cham[10]). Very careful measurements are required to find just how significant these changes in flow direction may be, but present results suggest that the overall effect of this inaccuracy on the crossflow momentum is small, although the position predicted for the separation may be appreciably affected.

The apex of the triangle represents the hypothetical point at which the inviscid region changes to a region in which shear stresses are dominant. If the position of the apex is known or can be estimated the whole crossflow profile is in principle determined (except for the region around the apex where a fairing curve should be adequate) since the slope of the outer part can be

obtained/

obtained from the external flow behaviour. Attempts have been made to specify the position of the apex using such parameters as u_a/u_τ or $\zeta_a u_\tau/\nu$, where u_a and ζ_a are the streamwise velocity and distance from the surface corresponding to the position of the apex, and u_τ is the friction velocity given by $u_\tau = \sqrt{\tau/\rho}$. Although this may be adequate for some applications such approaches are inherently unsatisfactory since the development of crossflow is complex (except in the outer region) and depends upon the behaviour of the streamwise component of the boundary layer, upon the crossflow development along adjacent streamlines and upon the behaviour of the external flow, as well as upon the skin friction.

There is, however, a possible solution to the problem. The crossflow must satisfy the boundary-layer momentum integral equations; the crossflow is dominant in the crosswise momentum integral equation just as the streamwise flow is dominant in the streamwise momentum integral equation. (The momentum integral equations include, of course, all the effects listed above). Thus, having obtained the crossflow in the outer part of the boundary layer in terms of the external flow and the streamwise velocity defect, the momentum integral equations can be used to determine where the deviation from the inviscid solution must occur. For this the crosswise equation can be solved in terms of β, using the triangular representation, but (as an improvement) possibly using a fairing curve to replace the apex of the triangle.

4. The crossflow decay process

When the streamlines outside the boundary layer become straight the crossflow must decay, and for this case it is possible to describe qualitatively the changes that must occur in the crossflow velocity profile. The existence of straight external streamlines implies that A will remain constant, and as Lowrie[11] suggested, the only way for the crossflow to decay is for the inviscid region to shrink while the viscous region extends outwards. The idealised process is shown in Figure 4. When the apex reaches the boundary layer edge the crossflow velocity will, of course, be zero. In fact, as the apex becomes very close to the boundary layer edge, A will merely represent the tangent of the curve at $u/U_s = 1.0$. The process may be observed in the results of Francis and Pierce, where decaying crossflow profiles were measured in a straight duct. Unfortunately the constraint introduced by the duct walls produced some distortion of the crossflow profiles and the demonstration is not as clear as it might otherwise be.

The conventional plot of crossflow against distance from the surface, as shown in Figure 3, helps to explain the crossflow decay process occurring in the absence of transverse pressure gradients. Although it is difficult to assign actual magnitudes to shear stress, it seems clear that the sign of the shear stress in the crossflow direction will depend on the sign of the crossflow velocity gradient. Now between the surface and the maximum crossflow the shear stresses act so as to decrease the crossflow above them and increase the crossflow below them. Above the maximum the velocity gradient is of opposite sign and the direction of the shear stresses is then such that they must act to decrease the crossflow below them and increase the crossflow above them. On reflection it can be seen that, in the absence of transverse pressure gradients, this moves the maximum crossflow outwards while decreasing the overall magnitude; the same process as was deduced by considering the triangular representation. Since decaying crossflows occur in cases of practical interest, as well as forming part of the process in the formation of cross-over crossflow profiles (discussed below), there is a need for definitive measurements of crossflow decay. Both the form of the crossflow profile and the rate of decay are topics which require clarification.

5. Cross-over crossflow profiles

When the direction of the external streamline curvature is reversed it is usual to obtain crossflow profiles in which the direction of crossflow changes through the depth of the layer, these being commonly referred to as cross-over profiles. Even where the sign of the crossflow does not change through the layer, the reversal of the curvature leads to a profile of unusual form which is not accommodated by the simple triangle, and certainly not by the Mager representation. The occurrence of cross-over profiles has led to considerable scepticism regarding, for example, the feasibility of carrying out full-chord calculations on swept wings. (It may be added, however, that the recent measurements on a swept wing by Cumpsty and Head suggest that, because the magnitudes of the crossflow in the cross-over profiles are generally very small, inaccuracies in their representation may be less serious than had been imagined.)

It is convenient to consider the crossflow profile in terms of inner and outer regions, and consequently the triangular representation is ideally suited. Now suppose that at the point of inflection of the external streamlines (where the direction of curvature and transverse pressure gradient changes sign), the crossflow profile is described by $v_1(\zeta)$, where v_1 is assumed small compared with U_s. Corresponding to v_1, which we shall call the initial crossflow, are the parameters β_1 and A_1, shown in Figure 5a. In the outer part of the boundary layer the turning of the streamwise flow produces new vorticity in the streamwise direction which is independent of the initial crossflow. At a small distance downstream of the inflection let us assume that the new curvature produces secondary flow $A_2(U_s - u)$, so that in the outer part of the boundary layer the net crossflow is given by $v_2 = A_2(U_s - u) - v_1$ in the new direction. The net crossflow must vanish at the wall and consequently there must be a deviation from the $A_2(U_s - u) - v_1$ relation at some point.

The argument is now restricted to cases where β changes sign before $A_2 > A_1$ The momentum integral equations must continue to be satisfied and the new value of β can be found from the crossflow momentum equation. A profile of this type is shown diagrammatically in Figure 5b. This has assumed that the only changes to have taken place in the outer region during this process were inviscid. Whereas in practice the decay of crossflow appears slower than the growth in all but the weakest pressure gradients, a better assumption should be possible after some definitive experiments to determine the rate of crossflow decay. Now if the initial crossflow is decaying before the point at which the curvature changes sign the initial triangular plot will show the apex towards the edge of the boundary layer. This is to be expected on, for example, swept wings, where the external streamlines tend to straighten around the region of the minimum pressure. The new crossflow will then be produced with its apex nearer to the surface and this is the idealised case shown in Figure 5b. Cumpsty and Head[8] observed this form on a swept wing and some results are shown in Figure 6. The initial crossflow apex is in the outer part of the layer and remains in virtually the same place in the cross-over profiles. It is also discernible as an inflection at approximately the same position for a considerable distance downstream. Although the general form of the profiles is quite well predicted, it is clear that the magnitude of the initial crossflow in the outer region does show some decrease which cannot be attributed to crossflow developing in the new direction. Hall and Dickens[12] obtained similar results in a supersonic flow in a curved duct intended to simulate the flow over swept wings.

If the crossflow were not decaying before the change in curvature, and if the transverse pressure gradient were larger before the change than after it, a different form of cross-over profile would be expected. Experience suggests that strong pressure gradients lead to the apex being close to the surface, and consequently a cross-over profile of the form sketched in Figure 5c is likely. Such profiles have been measured by Eichelbrenner[13] on an ellipsoid at incidence. Whereas in the former type of cross-over profile the position of the new apex could be found by applying the crossflow momentum equation knowing that the crossflow must vanish at the wall, for this second type of cross-over profile this condition does not suffice to specify the form. Furthermore for this second type of cross-over profile it seems, a priori, that because of the large shear stresses close to the wall it is not justifiable to ignore the decay of crossflow in the first direction.

6. Conclusions

1. The crossflow representation due to Mager can give a moderately or even very good description of the crossflow when the transverse pressure gradient is modest, and in one direction for a considerable streamwise distance. It is inaccurate when the crossflow is growing rapidly, decreasing, or where the direction of the flow curvature is reversed.

2. The two-region model for the crossflow appears to have very wide application and provides an improved insight into the behaviour of the flow. In the outer region the development of crossflow is effectively inviscid and it can be predicted that in this region the crossflow is proportional to the streamwise velocity defect with a coefficient which is only a function of the external flow. For flows in which the crossflow is growing, the outer region includes by far the greater part of the boundary layer thickness.

3. The triangular plot of Johnston gives a very convenient representation of the two regions. There is a large body of experimental evidence to support the two straight sides, although it can be predicted that the inner region is not in general exactly linear.

4. The slope of the outer part of the triangle can be obtained from an inviscid analysis. The slope of the inner part must be such that with the outer region fixed by the behaviour of the external flow, the momentum integral equations are satisfied.

5. A convenient model for the decay of crossflow is obtained using the triangular representation by recognising that the outer slope will remain constant while the apex moves towards the boundary layer edge. This behaviour can be predicted by considering the sign of crosswise shear stress which is expected to change at the position of maximum crossflow. Measurements to determine the rate of decay are badly needed.

6. The two-region approach makes it possible to predict, in qualitative terms, the development of cross-over crossflow profiles in an important case, namely that occurring when the crossflow has been decreasing prior to the imposition of the transverse pressure gradient in the new direction. Information regarding the decay of crossflow is, however, required before this can be put on a quantitative basis.

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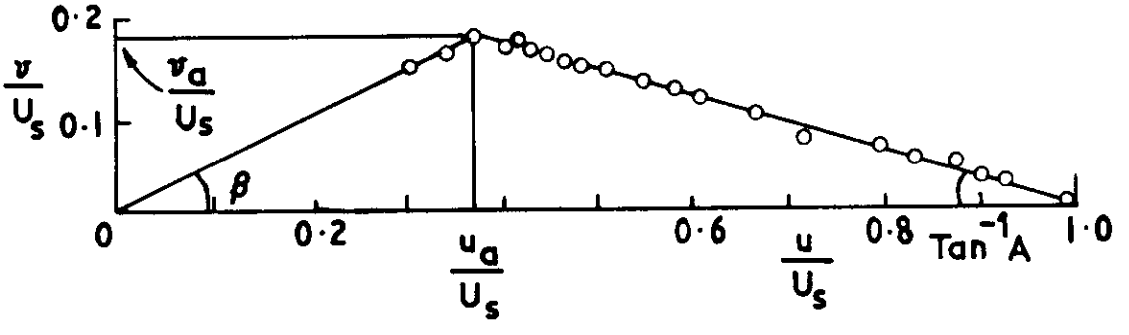


FIG. 1 Polar plot of profile G-X5 measured by Johnston also showing notation

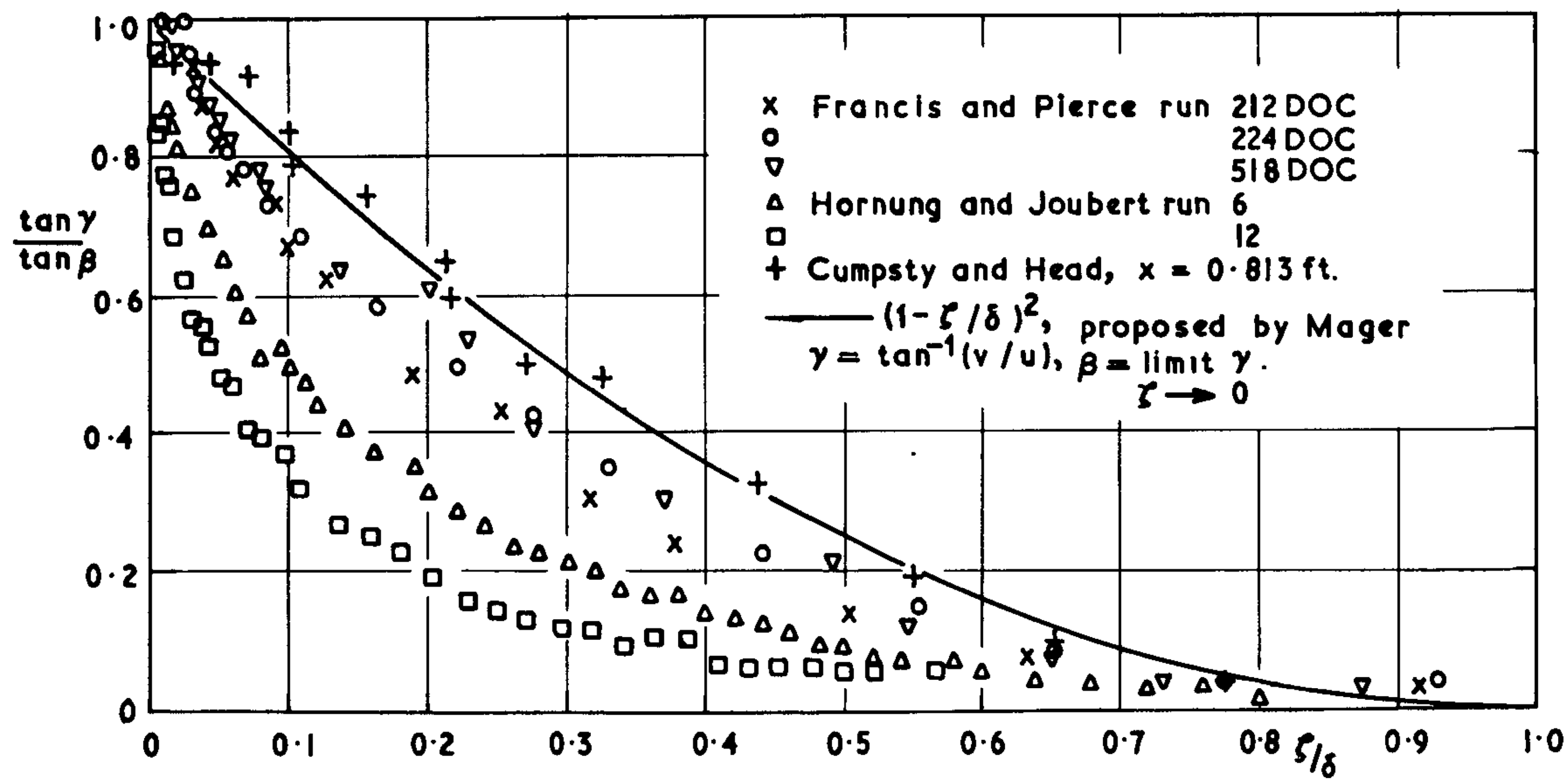


FIG. 2. Some measured crossflow profiles compared with the representation proposed by Mager

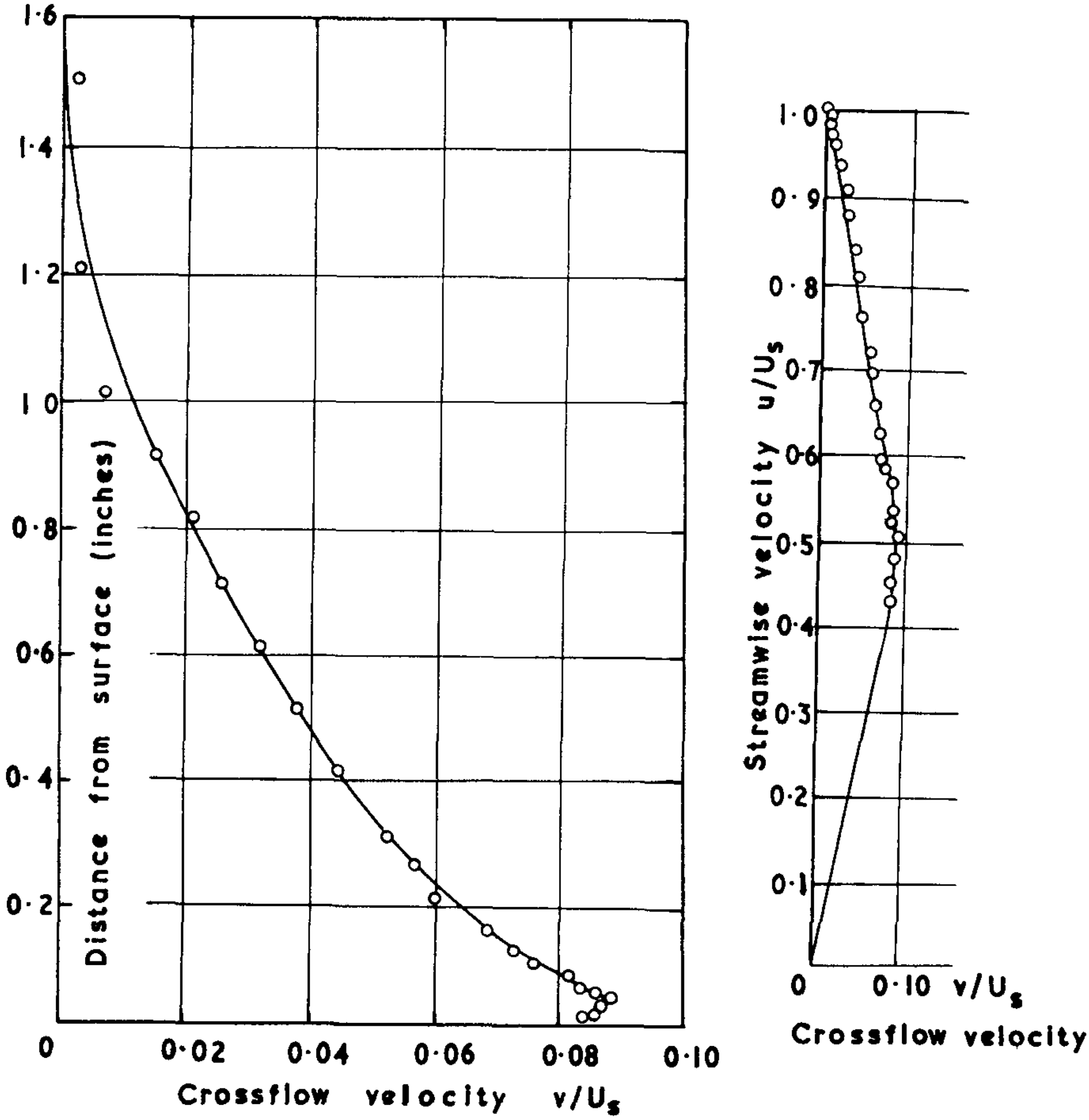


FIG. 3. The crossflow of Johnston's profile A-X 6 plotted in two different ways for comparison

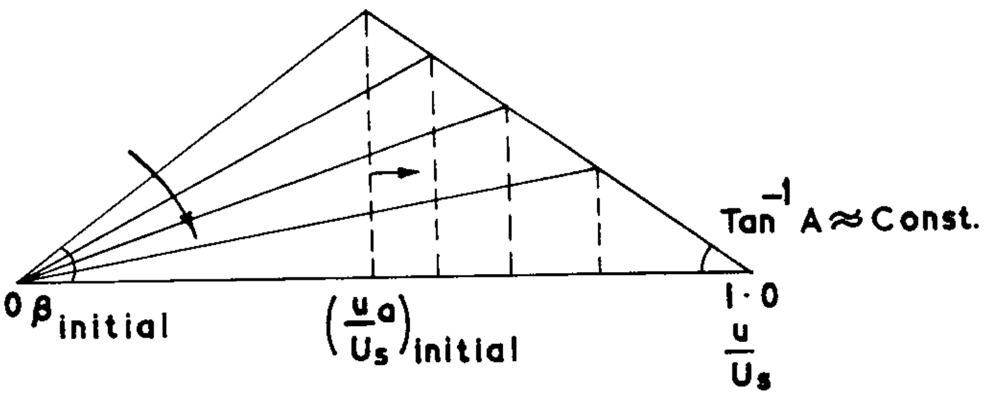


FIG. 4. The idealised crossflow decay process: β decreases, u_a/U_s increases and A remains constant in the flow direction.

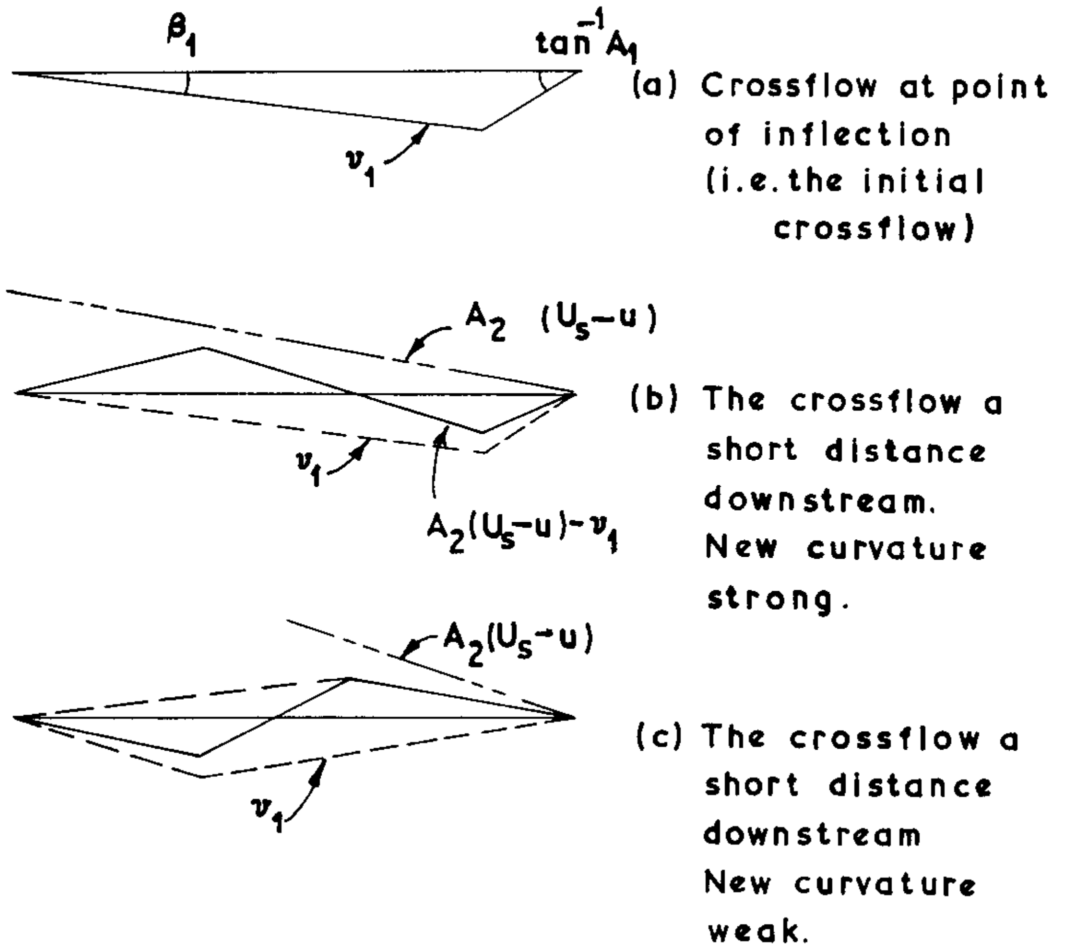
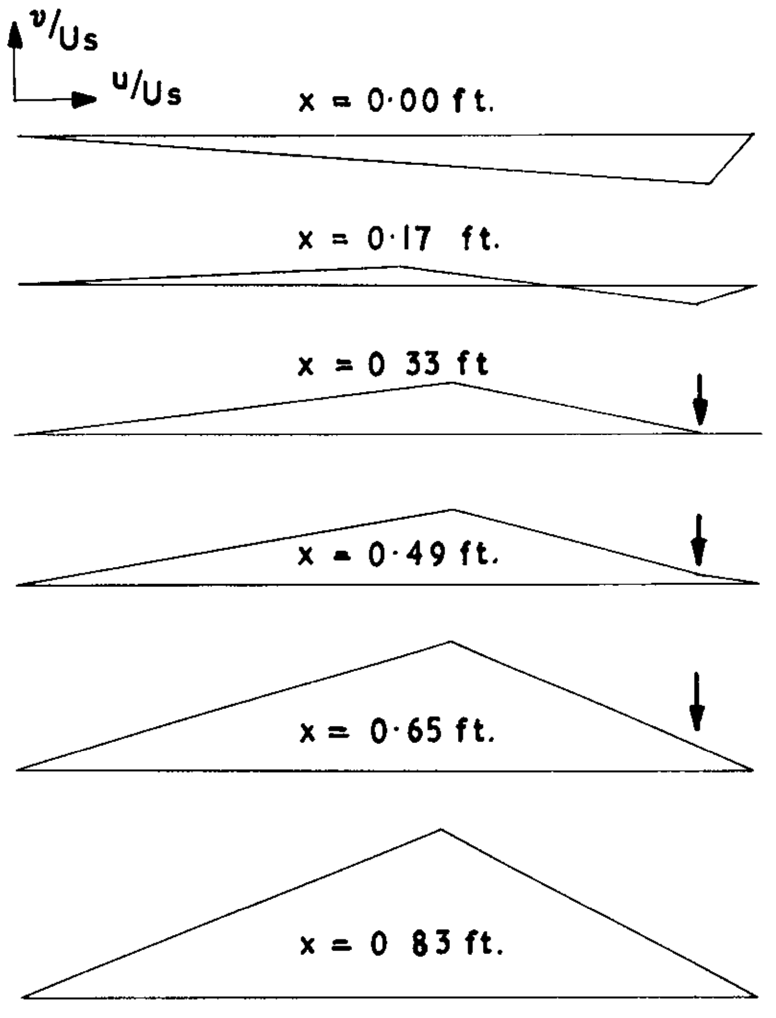


FIG. 5 **Polar plots showing formation of cross-over crossflow profiles in two alternative cases**



x measures distance from pressure minimum in plane of surface and in direction normal to the leading edge. Inflection indicated by an arrow.

FIG 6 Polar plots of velocity profiles measured by Cumpsty and Head on the rear of a swept wing.

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