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Higher Harmonics of Flapping on the Helicopter Rotor

By

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SUMMARY

The amplitudes of the flapping harmonics up to the sixth harmonic have been calculated for a range of blade pitch and blade inertia number for tip speed ratios up to 0.6. The calculations are for a straight (infinitely rigid) blade and blade stalling is not taken into account.

The results are given in the form of generalised curves. For the ordinary helicopter, the amplitude of any harmonic is about 1/12th of the preceding harmonic, for a tip speed ratio about 0.3.

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1 Introduction

In many forms of helicopter work, particularly in relation to vibration, it is necessary to have some knowledge of the magnitude of the higher harmonics of the blade flapping motion. It had always been appreciated qualitatively that the amplitudes diminish rapidly with higher frequency but no quantitative assessment had been made. This was due, in part, to the complicated nature of the evaluation and also to the fact that other parameters, such as blade bending and twisting, could have a large influence in the final effects on the helicopter.

In order to make a start to this work, this report deals with the evaluation of the flapping motion for a straight blade. The effects of blade bending on the flapping are not yet known, but it is thought that these effects can be treated separately, in which case the present calculations would give the flapping for the line through the root and 0.75 radius position on the blade. However, at the present stage, the calculations are strictly valid only for the infinitely rigid blade. Blade stalling is not taken into account in the calculations.

When the helicopter is in forward flight, the velocity of the air over the advancing blade is increased while that over the retreating blade is decreased. In order to deal with this distribution of velocity and to maintain trim, the appropriate cyclic pitch is applied to the blades. This is, on ordinary helicopters, simply a first order harmonic application and the blade is free to flap at any other frequencies. Thus, the flapping motion of the blades can be expressed as an infinite Fourier series, in which the constant term is the coning angle, the first harmonics are trimmed by the appropriate control angles (longitudinal and lateral cyclic pitch) and flapping at higher harmonics is free to take place according to the aerodynamic, centrifugal and inertia forces involved. This report deals with the evaluation of the magnitude of these harmonics.

2 Theory

A rotor of radius R is assumed to have a forward velocity V and a rotational velocity Ω. The axes of reference are taken through the rotor centre, parallel and perpendicular to the mean tip path plane. This definition is similar to that used in general helicopter work, Ref.1, except that a mean tip path plane must be taken to allow for the introduction of higher harmonics of flapping.

The component of the forward velocity parallel to the mean tip path plane is

$$\mu \, \Omega \, R$$

The velocity through the disc, perpendicular to the mean tip path plane is

$$\lambda \, \Omega \, R$$

At the present stage of the theory, the usual assumption is made that λ is constant over the rotor disc.

The velocities of the air with respect to a blade element distant $r = x\,R$ from the rotor axis are as follows:-

Velocity parallel to mean tip path plane and perpendicular to the blade (chordwise) is

$$(x + \mu \sin\psi) \, \Omega R \qquad (1)$$

Velocity perpendicular to the mean tip path plane and to the blade (through the disc) is

$$\left(\lambda + \beta\,\mu \cos\psi + \frac{\dot{\beta}}{\Omega}\,x\right)\Omega R \qquad (2)$$

where β is assumed to be small, so that $\sin\beta$ can be replaced by β and $\cos\beta$ by unity.

The velocity along the blade (spanwise) is neglected for the reasons already discussed in Ref.1.

Defining flapping angle in the usual way

$$\beta = a_0 - a_1 \cos\psi - b_1 \sin\psi - a_2 \cos 2\psi - b_2 \sin 2\psi - - a_3 \cos 3\psi - b_3 \sin 3\psi \qquad (3)$$

When the axes are taken relative to the rotor disc, the values of a_1 and b_1 are zero. However, the higher harmonics of flapping are not dependent on this definition and in order to make the equations more general, the a_1 and b_1 have been retained. In the subsequent evaluations of the higher harmonics, the first order effects are eliminated in the generalised forms $B_1 + a_1$ longitudinally and $-A_1 + b_1$ laterally. The definition of λ perpendicular to the rotor disc is required for the evaluation of a_0 and of the control angles if required.

Differentiating equation (3)

$$\frac{\dot{\beta}}{\Omega} = +a_1 \sin\psi - b_1 \cos\psi + 2\,a_2 \sin 2\psi - 2\,b_2 \cos 2\psi + + 3\,a_3 \sin 3\psi - 3\,b_3 \cos 3\psi \ \ldots\ldots. \qquad (4)$$

$$\frac{\ddot{\beta}}{\Omega^2} = a_1 \cos\psi + b_1 \sin\psi + 4\,a_2 \cos 2\psi + 4\,b_2 \sin 2\psi + + 9\,a_3 \cos 3\psi + 9\,b_3 \sin 3\psi \ \ldots\ldots.. \qquad (5)$$

Hence, the velocity perpendicular to the blade axis (through the disc) by combining equations (2), (3) and (4), is

$$[\lambda + \mu \cos\psi\,(a_0 - a_1 \cos\psi - b_1 \sin\psi - a_2 \cos 2\psi - b_2 \sin 2\psi - - a_3 \cos 3\psi - b_3 \sin 3\psi \ldots\ldots..) + x\,(a_1 \sin\psi - b_1 \cos\psi + 2\,a_2 \sin 2\psi - 2\,b_2 \cos 2\psi + 3\,a_3 \sin 3\psi - - 3\,b_3 \cos 3\psi \ldots\ldots..)]\ \Omega R \qquad (6)$$

Assuming the usual expression for feathering, the blade pitch at any azimuth position is given by

$$\vartheta = \vartheta_0 - A_1 \cos\psi - B_1 \sin\psi \qquad (7)$$

From equations (1), (2) and (7) the effective incidence of the blade element is

$$\alpha = \vartheta_0 - A_1 \cos\psi - B_1 \sin\psi - \frac{\lambda + \beta\mu\cos\psi + \frac{\dot\beta}{\Omega}x}{x + \mu\sin\psi} \tag{8}$$

or, using the expansion of equation (2) as given by equation (6)

$$\begin{aligned}\alpha &= \vartheta_0 - A_1\cos\psi - B_1\sin\psi \\ &- [\lambda + \mu\cos\psi\,(a_0 - a_1\cos\psi - b_1\sin\psi - a_2\cos 2\psi - \\ &\qquad - b_2\sin 2\psi - a_3\cos 3\psi\,\ldots\ldots..) \\ &\qquad + x\,(a_1\sin\psi - b_1\cos\psi + 2a_2\sin 2\psi - 2b_2\cos 2\psi + \\ &\qquad + 3a_3\sin 3\psi\ldots.)]\Big/[x + \mu\sin\psi]\end{aligned} \tag{9}$$

The aerodynamic force acting on the blade element $c\,R\,dx$ is

$$dF = \tfrac{1}{2}\rho\,a\,c\,\Omega^2 R^3 (x + \mu\sin\psi)^2\,\alpha\,dx \tag{10}$$

The corresponding moment with respect to the flapping hinge is

$$\begin{aligned}dM &= x\,R\,dF \\ &= \tfrac{1}{2}\rho\,a\,c\,\Omega^2 R^4 (x + \mu\sin\psi)^2\,\alpha\,x\,dx\end{aligned} \tag{11}$$

The aerodynamic moment is then obtained by integrating equation (11) along the blade. To allow for the tip loss, the integration is taken from $x = 0$ to B

$$\therefore M = \int_0^B \tfrac{1}{2}\rho\,a\,c\,\Omega^2 R^4 x\,(x + \mu\sin\psi)^2\,\alpha\,dx \tag{12}$$

$$\frac{M}{\tfrac{1}{2}\rho\,a\,c\,\Omega^2 R^4} = \int_0^B (x^3 + 2x^2\mu\sin\psi + x\mu^2\sin^2\psi)\,\alpha\,dx \tag{13}$$

Using the evaluation of α from equation (9) the above equation becomes

$$\frac{M}{\frac{1}{2}\rho a c \Omega^2 R^4} = \int_0^B [(x^3 + 2x^2 \mu \sin\psi + x\mu^2 \sin^2\psi)\,\vartheta_o$$

$$- (x^3 + 2x^2 \mu \sin\psi + x\mu^2 \sin^2\psi)(A_1 \cos\psi)$$

$$- (x^3 + 2x^2 \mu \sin\psi + x\mu^2 \sin^2\psi)(B_1 \sin\psi)$$

$$- (x^2 + x\mu \sin\psi)\,\lambda$$

$$- (x^2 + x\mu \sin\psi)(\mu \cos\psi)(a_o - a_1 \cos\psi - b_1 \sin\psi \ldots.)$$

$$- (x^3 + x^2 \mu \sin\psi)(a_1 \sin\psi - b_1 \cos\psi + 2 a_2 \sin 2\psi -$$

$$- 2 b_2 \cos 2\psi + \ldots\ldots\ldots)]\; dx \qquad (14)$$

Integrating equation (14), expressing trignometrical products in harmonic angles and evaluating the expansions up to coefficients a_6 and b_6, the aerodynamic moment becomes

/Equation (15)

$$
\begin{aligned}
\frac{M}{\frac{1}{2}\rho\, a\, c\, \Omega^2 R^4} = & \left(\frac{B^4}{4} + \frac{2}{3} B^3 \mu \sin\psi + \frac{B^2\mu^2}{4} - \frac{B^2\mu^2}{4}\cos 2\psi\right)\vartheta_o \\
& - \left(\frac{B^4}{4}\cos\psi + \frac{1}{3} B^3\mu \sin 2\psi + \frac{B^2\mu^2}{4}\cos\psi - \frac{B^2\mu^2}{8}\cos 3\psi - \frac{B^2\mu^2}{8}\cos\psi\right)A_1 \\
& - \left(\frac{B^4}{4}\sin\psi + \frac{1}{3} B^3\mu - \frac{1}{3} B^3\mu \cos 2\psi + \frac{3}{8} B^2\mu^2 \sin\psi - \frac{B^2\mu^2}{8}\sin 3\psi\right)B_1 \\
& - \left(\frac{B^3}{3} + \frac{B^2\mu}{2}\sin\psi\right)\lambda \\
& - \frac{B^3\mu}{3}\Big(a_o \cos\psi - \frac{a_1}{2}\cos 2\psi - \frac{a_1}{2} - \frac{b_1}{2}\sin 2\psi - \frac{a_2}{2}\cos 3\psi - \frac{a_2}{2}\cos\psi \\
& \quad - \frac{b_2}{2}\sin 3\psi - \frac{b_2}{2}\sin\psi - \frac{a_3}{2}\cos 4\psi - \frac{a_3}{2}\cos 2\psi - \frac{b_3}{2}\sin 4\psi - \frac{b_3}{2}\sin 2\psi \\
& \quad - \frac{a_4}{2}\cos 5\psi - \frac{a_4}{2}\cos 3\psi - \frac{b_4}{2}\sin 5\psi - \frac{b_4}{2}\sin 3\psi - \frac{a_5}{2}\cos 6\psi - \frac{a_5}{2}\cos 4\psi \\
& \quad - \frac{b_5}{2}\sin 6\psi - \frac{b_5}{2}\sin 4\psi - \frac{a_6}{2}\cos 7\psi - \frac{a_6}{2}\cos 5\psi - \frac{b_6}{2}\sin 7\psi - \frac{b_6}{2}\sin 5\psi\Big) \\
& - \frac{B^2\mu^2}{4}\Big(a_o \sin 2\psi - \frac{a_1}{2}\sin 3\psi - \frac{a_1}{2}\sin\psi + \frac{b_1}{2}\cos 3\psi - \frac{b_1}{2}\cos\psi - \frac{a_2}{2}\sin 4\psi \\
& \quad - \frac{b_2}{2} + \frac{b_2}{2}\cos 4\psi - \frac{a_3}{2}\sin 5\psi + \frac{a_3}{2}\sin\psi + \frac{b_3}{2}\cos 5\psi - \frac{b_3}{2}\cos\psi \\
& \quad - \frac{a_4}{2}\sin 6\psi + \frac{a_4}{2}\sin 2\psi + \frac{b_4}{2}\cos 6\psi - \frac{b_4}{2}\cos 2\psi - \frac{a_5}{2}\sin 7\psi + \frac{a_5}{2}\sin 3\psi \\
& \quad + \frac{b_5}{2}\cos 7\psi - \frac{b_5}{2}\cos 3\psi - \frac{a_6}{2}\sin 8\psi + \frac{a_6}{2}\sin 4\psi + \frac{b_6}{2}\cos 8\psi - \frac{b_6}{2}\cos 4\psi\Big) \\
& - \frac{B^4}{4}\big(a_1 \sin\psi - b_1 \cos\psi + 2a_2 \sin 2\psi - 2b_2 \cos 2\psi + 3a_3 \sin 3\psi - 3b_3 \cos 3\psi \\
& \quad + 4a_4 \sin 4\psi - 4b_4 \cos 4\psi + 5a_5 \sin 5\psi - 5b_5 \cos 5\psi + 6a_6 \sin 6\psi - 6b_6 \cos 6\psi\big) \\
& - \frac{B^3\mu}{3}\Big(\frac{a_1}{2} - \frac{a_1}{2}\cos 2\psi - \frac{b_1}{2}\sin 2\psi - a_2 \cos 3\psi + a_2 \cos\psi \\
& \quad - b_2 \sin 3\psi + b_2 \sin\psi - \frac{3}{2} a_3 \cos 4\psi + \frac{3}{2} a_3 \cos 2\psi - \frac{3}{2} b_3 \sin 4\psi + \frac{3}{2} b_3 \sin 2\psi \\
& \quad - 2a_4 \cos 5\psi + 2a_4 \cos 3\psi - 2b_4 \sin 5\psi + 2b_4 \sin 3\psi - \frac{5}{2} a_5 \cos 6\psi + \frac{5}{2} a_5 \cos 4\psi \\
& \quad - \frac{5}{2} b_5 \sin 6\psi + \frac{5}{2} b_5 \sin 4\psi - 3a_6 \cos 7\psi + 3a_6 \cos 5\psi - 3b_6 \sin 7\psi + 3b_6 \sin 5\psi\Big)
\end{aligned}
\tag{15}
$$

For a blade element dx of mass m, distant xR from the rotor axis, the increment in centrifugal force and the corresponding moment about the flapping hinge are as follows

$$d\ C.F. = m\,\Omega^2\,xR\,dx \qquad (16)$$

$$d\ (C.F.\ Mom) = m\,\Omega^2\,x^2\,R^2\beta\ dx \qquad (17)$$

$$\therefore\ C.F.\ Mom. = \int_0^1 m\,x^2\,R^2\,\Omega^2\,\beta\,dx$$

$$= I_1\,\Omega^2\,\beta \qquad (18)$$

Taking moments about the flapping hinge and neglecting the moment due to the weight of the blade

$$M = I_1\,\Omega^2\,\beta + I_1\,\ddot{\beta}$$

$$= I_1\,\Omega^2\left(\beta + \frac{\ddot{\beta}}{\Omega^2}\right) \qquad (19)$$

Re-writing in the form of the L.H.S. of equation (15)

$$\frac{M}{\frac{1}{2}\,\rho\ a\ \sigma\,\Omega^2 R^4} = \frac{I_1\,\Omega^2}{\frac{1}{2}\,\rho\ a\ c\ \Omega^2\,R^4}\left(\beta + \frac{\ddot{\beta}}{\Omega^2}\right)$$

$$= \frac{2}{\gamma}\left(\beta + \frac{\ddot{\beta}}{\Omega^2}\right) \qquad (20)$$

where γ is Lock's Inertia Number $= \dfrac{\rho\ a\ c\ R^4}{I_1}$

Evaluating by substitution from equations (3) and (5)

$$\frac{M}{\frac{1}{2}\rho\ a\ c\Omega^2 R^4} = \frac{2}{\gamma}\,(a_o + 3a_2\cos 2\psi + 3b_2\sin 2\psi + 8a_3\cos 3\psi + 8b_3\sin 3\psi$$

$$+ 15a_4\cos 4\psi + 15b_4\sin 4\psi + 24a_5\cos 5\psi + 24b_5\sin 5\psi$$

$$+ 35a_6\cos 6\psi + 35b_6\sin 6\psi\ \ldots\ldots..) \qquad (21)$$

Comparing equations (15) and (21) and equating the corresponding coefficients, the following set of equations are obtained.

$$\frac{B^4}{4}(B^2+\mu^2)\vartheta_0 - \frac{B^3}{3}\lambda - \frac{B^3\mu}{3}B_1 - \frac{2}{\gamma}a_0 + \frac{B^2\mu^2}{8}b_2 = 0 \qquad (22)$$

$$-\frac{B^2}{4}(B^2+\tfrac{1}{2}\mu^2)A_1 - \frac{B^3\mu}{3}a_0 + \frac{B^2}{4}(B^2+\tfrac{1}{2}\mu^2)b_1 - \frac{B^3\mu}{6}a_2 + \frac{B^2\mu^2}{8}b_3 = 0 \qquad (23)$$

$$\frac{2}{3}B^3\mu\,\vartheta_0 - \frac{B^2\mu}{2}\lambda - \frac{B^2}{4}(B^2+\tfrac{3}{2}\mu^2)B_1 - \frac{B^2}{4}(B^2-\tfrac{1}{2}\mu^2)a_1 - \frac{B^3\mu}{6}b_2 - \frac{B^2\mu^2}{8}a_3 = 0 \qquad (24)$$

$$-\frac{B^2\mu^2}{4}\vartheta_0 + \frac{B^3\mu}{3}B_1 + \frac{B^3\mu}{3}a_1 - \frac{6}{\gamma}a_2 + \frac{B^4}{2}b_2 - \frac{B^3\mu}{3}a_3 + \frac{B^2\mu^2}{8}b_4 = 0 \qquad (25)$$

$$-\frac{B^3\mu}{3}A_1 - \frac{B^2\mu^2}{4}a_0 + \frac{B^3\mu}{3}b_1 - \frac{B^4}{2}a_2 - \frac{6}{\gamma}b_2 - \frac{B^3\mu}{3}b_3 - \frac{B^2\mu^2}{8}a_4 = 0 \qquad (26)$$

$$\frac{B^2\mu^2}{8}A_1 - \frac{B^2\mu^2}{8}b_1 + \frac{B^3\mu}{2}a_2 - \frac{16}{\gamma}a_3 + \frac{3}{4}B^4 b_3 - \frac{B^3\mu}{2}a_4 + \frac{B^2\mu^2}{8}b_5 = 0 \qquad (27)$$

$$\frac{B^2\mu^2}{8}B_1 + \frac{B^2\mu^2}{8}a_1 + \frac{B^3\mu}{2}b_2 - \frac{3}{4}B^4 a_3 - \frac{16}{\gamma}b_3 - \frac{B^3\mu}{2}b_4 - \frac{B^2\mu^2}{8}a_5 = 0 \qquad (28)$$

$$-\frac{B^2\mu^2}{8}b_2 + \frac{2}{3}B^3\mu\,a_3 - \frac{30}{\gamma}a_4 + B^4 b_4 - \frac{2}{3}B^3\mu\,a_5 + \frac{B^2\mu^2}{8}b_6 = 0 \qquad (29)$$

$$\frac{B^2\mu^2}{8}a_2 + \frac{2}{3}B^3\mu\,b_3 - B^4 a_4 - \frac{30}{\gamma}b_4 - \frac{2}{3}B^3\mu\,b_5 - \frac{B^2\mu^2}{8}a_6 = 0 \qquad (30)$$

$$-\frac{B^2\mu^2}{8}b_3 + \frac{5}{6}B^3\mu\,a_4 - \frac{48}{\gamma}a_5 + \frac{5}{4}B^4 b_5 - \frac{5}{6}B^3\mu\,a_6 = 0 \qquad (31)$$

$$\frac{B^2\mu^2}{8}a_3 + \frac{5}{6}B^3\mu\,b_4 - \frac{5}{4}B^4 a_5 - \frac{48}{\gamma}b_5 - \frac{5}{6}B^3\mu\,b_6 = 0 \qquad (32)$$

$$-\frac{B^2\mu^2}{8}b_4 + B^3\mu\,a_5 - \frac{70}{\gamma}a_6 + \frac{3}{2}B^4 b_6 = 0 \qquad (33)$$

$$\frac{B^2\mu^2}{8}a_4 + B^3\mu\,b_5 - \frac{3}{2}B^4 a_6 - \frac{70}{\gamma}b_6 = 0 \qquad (34)$$

Equations (22) to (34) represent, in generalised form, the flapping motion of the blades and the evaluation of the various coefficients follows from the simultaneous solution of these equations. However, the solution can be made fairly easily by successive approximation methods since it can be shown that the magnitude of each coefficient is much less than that of the preceding harmonic. Equations (22) to (24) are dealt with first and then the others can be taken in successive pairs.

If we neglect the small terms introduced by the second and third order harmonics in equations (22), (23) and (24), these equations become identical with the control angle work of Ref.1. Equation (22) leads to the solution for the coning angle (a_o); equation (23) gives the lateral trim of the rotor in terms of the feathering amplitude (A_1) or the lateral tilt of the disc from the no-feathering axis (b_1); equation (24) gives the longitudinal trim of the rotor in terms of the feathering amplitude (B_1) or the longitudinal tilt of the disc from the no-feathering axis (a_1). Ref.1 also deals in detail with the equivalence of feathering and flapping and with the evaluation of the corresponding coefficients in relation to the definition of the flow through the disc (λ) for the appropriate axis. λ in the present report is taken perpendicular to the rotor disc.

Hence,

$$a_o = \frac{\gamma}{2}\left[\frac{B^2}{4}(B^2+\mu^2)\,\vartheta_o - \frac{B^2}{3}\lambda - \frac{B^3\mu}{3}B_1 + \frac{B^2\mu^2}{8}b_2\right] \tag{35}$$

$$B_1 = \frac{2\mu}{B^2+\frac{3}{2}\mu^2}\left[\frac{4}{3}B\,\vartheta_o - \lambda\right] - \frac{2}{3}\,\frac{B\mu}{B^2+\frac{3}{2}\mu^2}\,b_2 - \frac{1}{2}\,\frac{\mu^2}{B^2+\frac{3}{2}\mu^2}\,a_3 \tag{36}$$

$$A_1 = -\frac{4}{3}\cdot\frac{B\mu}{B^2+\frac{1}{2}\mu^2}\,a_o - \frac{2}{3}\cdot\frac{B\mu}{B^2+\frac{1}{2}\mu^2}\,a_2 + \frac{1}{2}\,\frac{\mu^2}{B^2+\frac{1}{2}\mu^2}\,b_3 \tag{37}$$

If we substitute for B_1 from equation (36) in (35), a_o becomes

$$a_o = \frac{\gamma}{2}\left[\frac{B^2}{4}\,\frac{\left(B^4 - \frac{19}{18}B^2\mu^2 + \frac{3}{2}\mu^4\right)}{B^2+\frac{3}{2}\mu^2}\,\vartheta_o - \frac{B^3}{3}\cdot\frac{B^2-\frac{1}{2}\mu^2}{B^2+\frac{3}{2}\mu^2}\,\lambda + B^2\mu^2\,\frac{\left(\frac{25}{72}B^2 + \frac{3}{16}\mu^2\right)}{B^2+\frac{3}{2}\mu^2}\,b_2 + \frac{1}{6}\cdot\frac{B^3\mu^3}{B^2+\frac{3}{2}\mu^2}\,a_3\right] \tag{38}$$

If we neglect the second and third harmonic terms and omit the tip loss i.e. $B = 1$, these equations (36), (37) and (38) are identical with equations (15), (16) and (13) of Ref.1.

Rewriting equation (24) by substituting for λ from equation (22) we obtain

$$\frac{B\mu}{24}(7B^2 - 9\mu^2)\,\vartheta_o + \frac{3\mu}{B\gamma}a_o - \frac{B^2}{4}(B^2 - \tfrac{1}{2}\mu^2)\,B_1 - \frac{B^2}{4}(B^2 - \tfrac{1}{2}\mu^2)\,a_1$$

$$- B\mu\left(\frac{B^2}{6} + \frac{3\mu^2}{16}\right) b_2 - \frac{B^2\mu^2}{8}a_3 = 0 \qquad (39)$$

From this, the value of $B_1 + a_1$ is

$$B_1 + a_1 = \frac{\mu}{6B}\left(\frac{7B^2 - 9\mu^2}{B^2 - \frac{1}{2}\mu^2}\right)\vartheta_o + \frac{12\mu\,a_o}{B^3\gamma\,(B^2 - \frac{1}{2}\mu^2)} - \frac{\mu}{B}\left(\frac{\frac{2}{3}B^2 + \frac{3}{4}\mu^2}{B^2 - \frac{1}{2}\mu^2}\right)$$

$$- \frac{\mu^2\ a_3}{2(B^2 - \frac{1}{2}\mu^2)} \qquad (40)$$

From equation (23), the value of $-A_1+b_1$ is obtained

$$-A_1+b_1 = \frac{4}{B^2 + \frac{1}{2}\mu^2}\left(\frac{B\mu}{3}a_o + \frac{B\mu}{6}a_2 - \frac{\mu^2}{8}b_3\right) \qquad (41)$$

The value of B_1+a_1 is then substituted in equations (25) and (28) and the value of $-A_1+b_1$ in equations (26) and (27). By using this generalisation of B_1+a_1 and $-A_1+b_1$, it means that, as far as the evaluation of the higher harmonics is concerned, we do not have to consider any further the effect of different definitions of the flow through the disc.

Equations (25) to (28) can then be evaluated in the following form

$$\frac{B^2\mu^2}{B^2 - \frac{1}{2}\mu^2}\left(\frac{5}{36}B^2 - \frac{3}{8}\mu^2\right)\vartheta_o + \frac{4\mu^2}{\gamma(B^2 - \frac{1}{2}\mu^2)}a_o - \frac{6}{\gamma}a_2$$

$$+ \frac{B^2}{2}\ \frac{\left(B^4 - \frac{17}{18}B^2\mu^2 - \frac{1}{2}\mu^4\right)}{B^2 - \frac{1}{2}\mu^2}b_2 - \frac{B^5\mu}{3(B^2 - \frac{1}{2}\mu^2)}a_3 + \frac{B^2\mu^2}{8}b_4 = 0 \qquad (42)$$

$$\frac{B^2\mu^2}{B^2 + \frac{1}{2}\mu^2}\left(\frac{7}{36}B^2 - \frac{\mu^2}{8}\right)a_o - \frac{B^4\left(B^2 + \frac{\mu^2}{18}\right)}{2(B^2 + \frac{1}{2}\mu^2)}a_2 - \frac{6}{\gamma}b_2$$

$$- \frac{B^3\mu\,(B^2 + \mu^2)}{3(B^2 + \frac{1}{2}\mu^2)}b_3 - \frac{B^2\mu^2}{8}a_4 = 0 \qquad (43)$$

$$- \frac{B^3\mu^3}{6(B^2 + \frac{1}{2}\mu^2)} a_0 + \frac{B^3\mu\left(B^2 + \frac{1}{3}\mu^2\right)}{2(B^2 + \frac{1}{2}\mu^2)} a_2 - \frac{16}{\gamma} a_3 + \frac{B^2\left(\frac{3}{4}B^4 + \frac{3}{8}B^2\mu^2 + \frac{1}{16}\mu^4\right)}{B^2 + \frac{1}{2}\mu^2} b_3$$

$$- \frac{B^3\mu}{2} a_4 + \frac{B^2\mu^2}{8} b_5 = 0 \qquad (44)$$

$$\frac{B\mu^3}{48}\left(\frac{7B^2 - 9\mu^2}{B^2 - \frac{1}{2}\mu^2}\right) \vartheta_0 + \frac{3}{2}\frac{\mu^3}{B\gamma(B^2 - \frac{1}{2}\mu^2)} a_0 - \frac{B^3\mu}{2}\left(\frac{B^2 - \frac{2}{3}\mu^2}{B^2 - \frac{1}{2}\mu^2}\right) b_2$$

$$- \frac{3}{4}B^4 a_3 - \frac{16}{\gamma} b_3 - \frac{B^3\mu}{2} b_4 - \frac{B^2\mu^2}{8} a_5 = 0 \qquad (45)$$

Equations (42) and (43) can be solved simultaneously for a_2 and b_2. Using these values, equations (44) and (45) can be solved simultaneously for a_3 and b_3. More accurate values can then be obtained by successive approximation. The other coefficients are obtained in a similar way from equations (29) to (34). In order to produce a more generalised form, we may divide throughout each of these equations by a_0. The evaluation of the harmonic coefficients/a_0 can then be made in terms of only three parameters, viz. μ, γ and $\frac{\vartheta_0}{a_0}$.

3 Effect of Induced Velocity Distribution

In the theory of the preceding section, it has been assumed that the flow through the disc is constant over the disc area. However, it is known that a variation in the flow through the disc does exist. In some flight tests[2], it was found that the distribution of induced velocity was almost linear from front to rear of the disc in forward flight conditions. This effect is also referred to in paragraph 5.2 of Ref.1.

The induced velocity at any point in the disc can be expressed as

$$v_i = v_{i_0}(1 + x K \cos\psi) \qquad (46)$$

where v_{i_0} is the mean induced velocity and the value of K can be selected for the appropriate forward flight conditions.

The flow through the disc consists of the component of the forward velocity together with the induced velocity

$$\lambda = \frac{V \sin i}{\Omega R} + \frac{v_i}{\Omega R} \qquad (47)$$

Using the expression for induced velocity from equation (46)

$$\lambda = \frac{V \sin i}{\Omega R} + \frac{v_{i_0}}{\Omega R} + \frac{v_{i_0}\, x K \cos\psi}{\Omega R} \qquad (48)$$

Hence, the flow through the disc at any point can then be expressed as

$$\lambda = \lambda_o + \frac{v_{1_o} x K}{\Omega R} \cos \psi \qquad (49)$$

where λ_o is the mean flow through the disc.

Using the momentum equation

$$T = 2\pi R^2 \rho \; v_{1_o} \Omega R \sqrt{\mu^2 + \lambda_o^2} \qquad (50)$$

and substituting for v_{1_o} in equation (49)

$$\lambda = \lambda_o + \frac{x K \sigma t_c}{2\sqrt{\mu^2 + \lambda_o^2}} \cos \psi \qquad (51)$$

This value of λ can be introduced into equation (14). On integrating the moment equation and comparing the coefficients, the effect of the induced velocity distribution is to give the following additional terms

$$- \frac{B^4 K \sigma t_c}{8 \sqrt{\mu^2 + \lambda_o^2}} \qquad \text{in equation (23)}$$

and

$$- \frac{B^3 \mu \; K \sigma t_c}{12 \sqrt{\mu^2 + \lambda_o^2}} \qquad \text{in equation (26)}$$

4 Numerical Evaluation

The flapping harmonics have been evaluated in terms of μ, γ and $\frac{\vartheta_o}{a_o}$ for a range of μ up to 0.6, for $\frac{\vartheta_o}{a_o} = 2$, 1 and 0.5 with $\gamma = 12$ and for $\frac{\vartheta_o}{a_o} = 2$ with $\gamma = 8$. The results are given in Figs.1 - 12. The first harmonics are given on a linear scale but it was found more convenient to use a log scale for the higher harmonics.

When the assumed distribution of induced velocity is taken into account in the calculations, b_1 is affected to a large extent, a_2 and b_2 to a small extent and the effect on higher harmonics is negligible. The results of the induced velocity effect on b_1, a_2 and b_2 are included in Figs.2, 3 and 4 respectively

5 Discussion

Figs.1 - 12 can be used to obtain the amplitudes of the various flapping harmonics. It is only necessary to know the tip speed ratio, blade inertia number and collective pitch setting and to evaluate the coning angle (a_o) from equation (38).

It will be seen that, as the frequency of the harmonic increases, the amplitudes decrease rapidly. To obtain a general picture, the amplitudes of the a-coefficients and the b-coefficients are plotted in

Figs.13 and 14 respectively, for the conditions $\frac{\vartheta_0}{a_0} = 1$ and $\gamma = 12$. On the log scale used, the amplitudes of the harmonics decrease almost linearly and the decrease is more rapid at low tip speed ratios. Thus at $\mu = 0.1$ each harmonic is about 1/20th of the preceding harmonic, at $\mu = 0.3$ about 1/12th and at $\mu = 0.5$ about 1/10th.

Other workers on earlier autogyro work, such as Lock[3] and Wheatley[4,5], have evaluated the second harmonic of flapping and give similar results to this report. However, the earlier work does not take into account the tip loss, induced velocity is assumed to be uniform, values of μ^2 are neglected compared with unity and higher harmonics are not included. The equations are therefore much simpler. When these simplifying assumptions have been applied to the present report, identical comparison is obtained with the earlier work. No evaluations for third or higher harmonics are available from known sources.

Some flight tests in America on a helicopter[6] and in Britain on an autogyro[7], with a camera located on the rotor head, have enabled measurements of the second harmonic of flapping to be made with reasonable accuracy. In Ref.6 and in Ref.5, a comparison with theory is made and good agreement is obtained. General agreement is obtained in comparison of the various flight tests with the theory of this report but in most cases the experimental accuracy does not warrant too detailed a comparison. In Ref.6 and 7, measurements of the third harmonic of flapping are also given but these are of a very small order and, while in general agreement with the estimated values, no detailed comparison is made.

It must be remembered that blade stalling has not been taken into account in these estimates. At high tip speed ratio and large values of ϑ_0, where blade stalling can occur on the outer sections of the blade, the amplitude of some of the flapping harmonics may be increased considerably. Bending of the blade has also been omitted from the calculations. In many cases, the flapping and bending of the blade could be considered separately but further work is required to investigate bending effects. If the natural bending frequency happens to be close to the frequency of one of the flapping harmonics, this form of resonance could give a considerable increase in the amplitude of the bending or flapping. It is hoped that further work will be done to investigate the effects of blade stalling and blade bending. Some experimental work is now proceeding with a camera located on the head of a Bristol 171 helicopter.

LIST OF SYMBOLS

$a = \frac{dC_L}{d\alpha}$	-	lift slope of blade section
a_o	-	coning angle
a_1, b_1, a_2 etc	-	coefficients in Fourier series for flapping
A_1, B_1	-	coefficients for feathering
B	-	factor to allow for tip loss, usually taken as 0.97
c	-	blade chord
i	-	disc incidence
I_1	-	blade moment of inertia about flapping hinge
K	-	constant allowing for distribution of induced velocity
$r = xR$	-	radius of given blade section
R	-	rotor radius
T	-	rotor thrust
t_c	-	thrust coefficient $\frac{T}{b\ c\ R\ \rho\ (\Omega R)^2}$
V	-	velocity of steady flight
v_i	-	induced velocity at given position on disc
v_{i_o}	-	mean induced velocity
α	-	incidence of blade section
β	-	flapping angle
γ	-	Lock's Inertia Number $\frac{\rho\ a\ c\ R^4}{I_1}$
ϑ	-	instantaneous pitch of blade
ϑ_o	-	collective pitch of blade
λ	-	coefficient of flow through disc, perpendicular to the mean tip path plane
μ	-	tip speed ratio
ρ	-	air density
σ	-	blade solidity
ψ	-	blade azimuth position measured from downwind position in direction of rotation
Ω	-	angular velocity of rotor

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3	Lock	Further Development of Autogyro Theory. R & M No. 1127. March, 1927
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7	Hufton, etc.	General investigation into the characteristics of the C-30 autogyro. R & M No. 1859. 1939.

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FIG.1

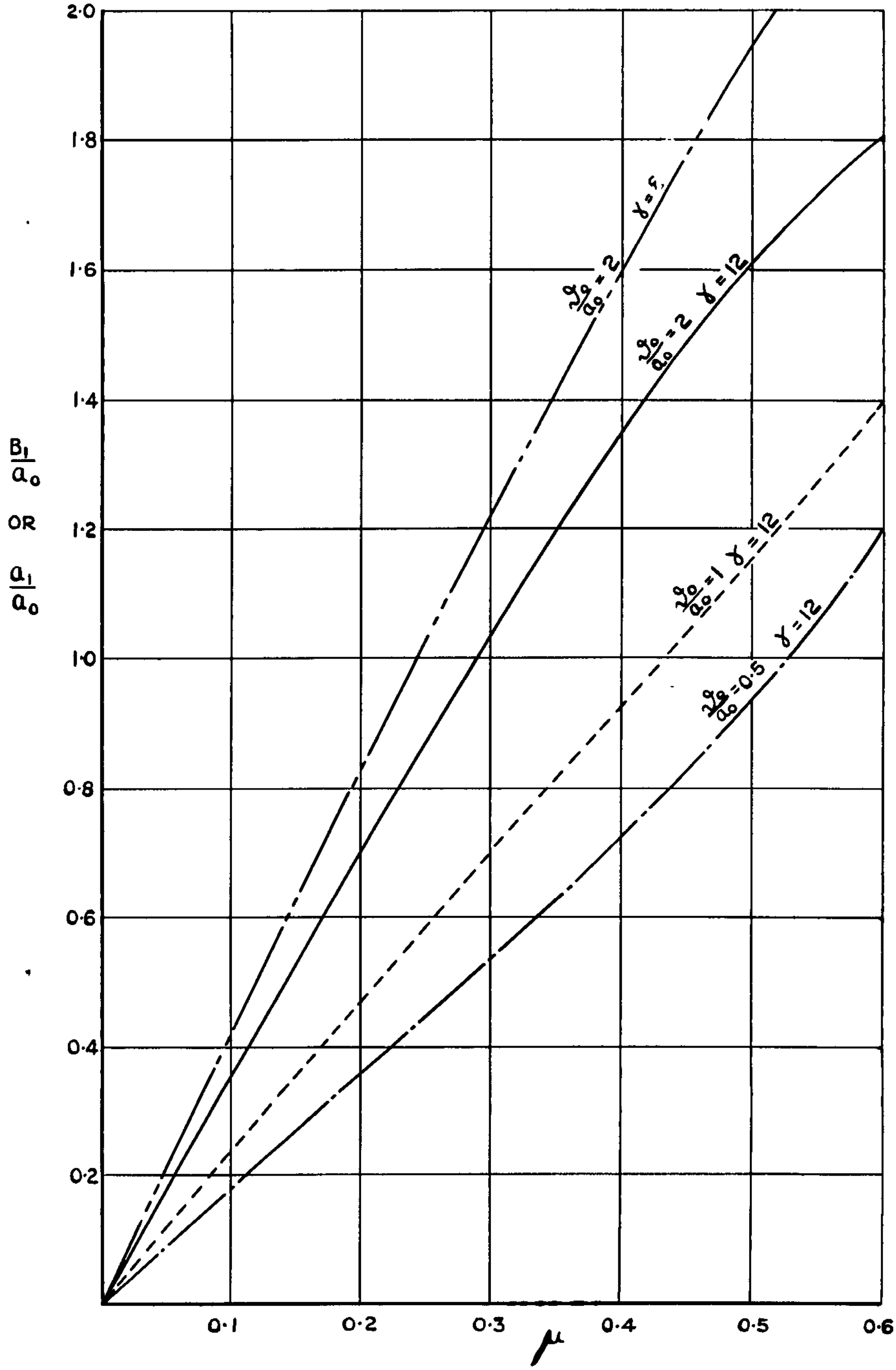

FIG.1 $\frac{B_1}{a_0}$ OR $\frac{a_1}{a_0}$ V. TIP SPEED RATIO

FIG.2

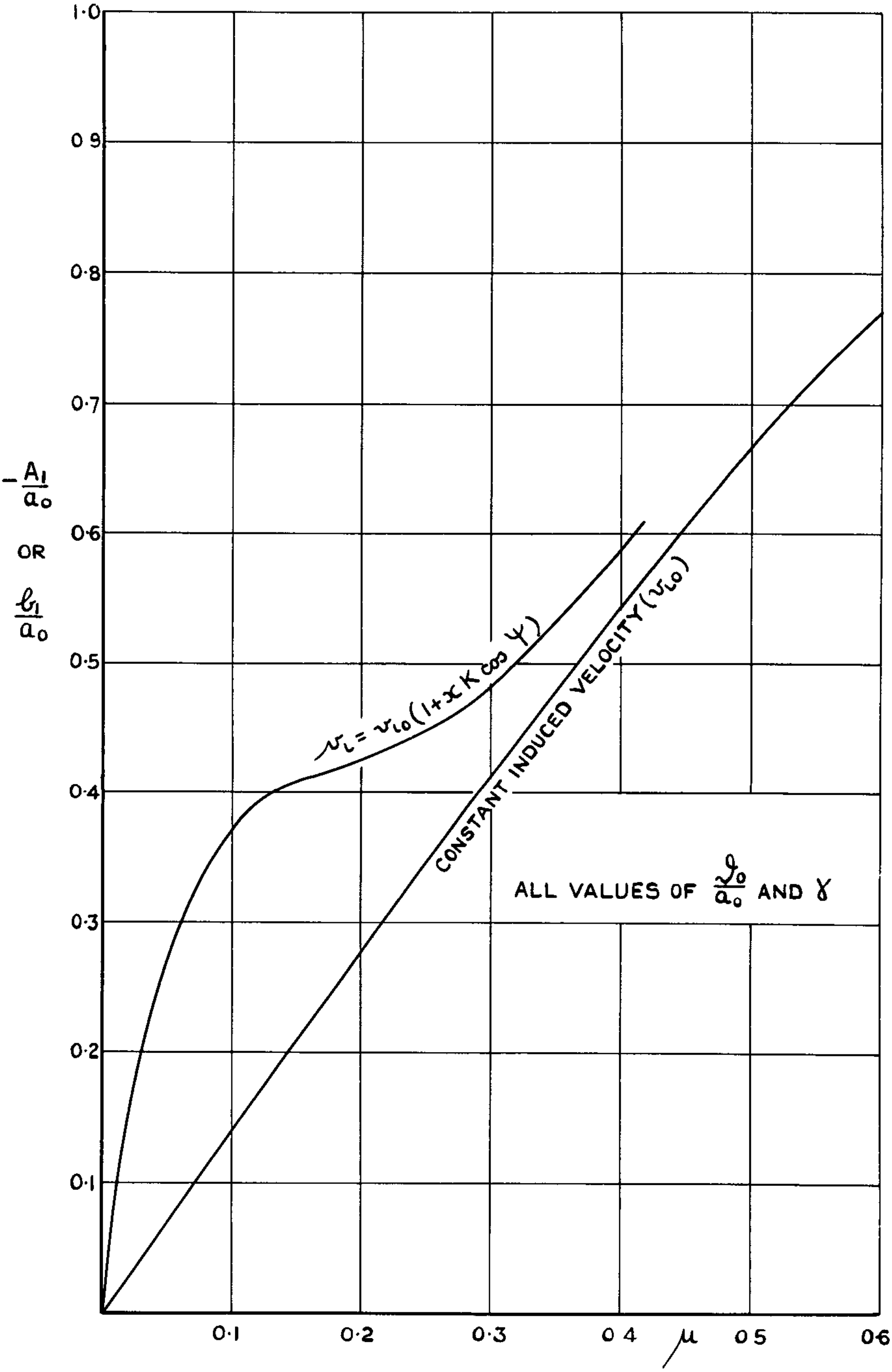

FIG.2 $-\frac{A_1}{a_0}$ OR $\frac{b_1}{a_0}$ V. TIP SPEED RATIO

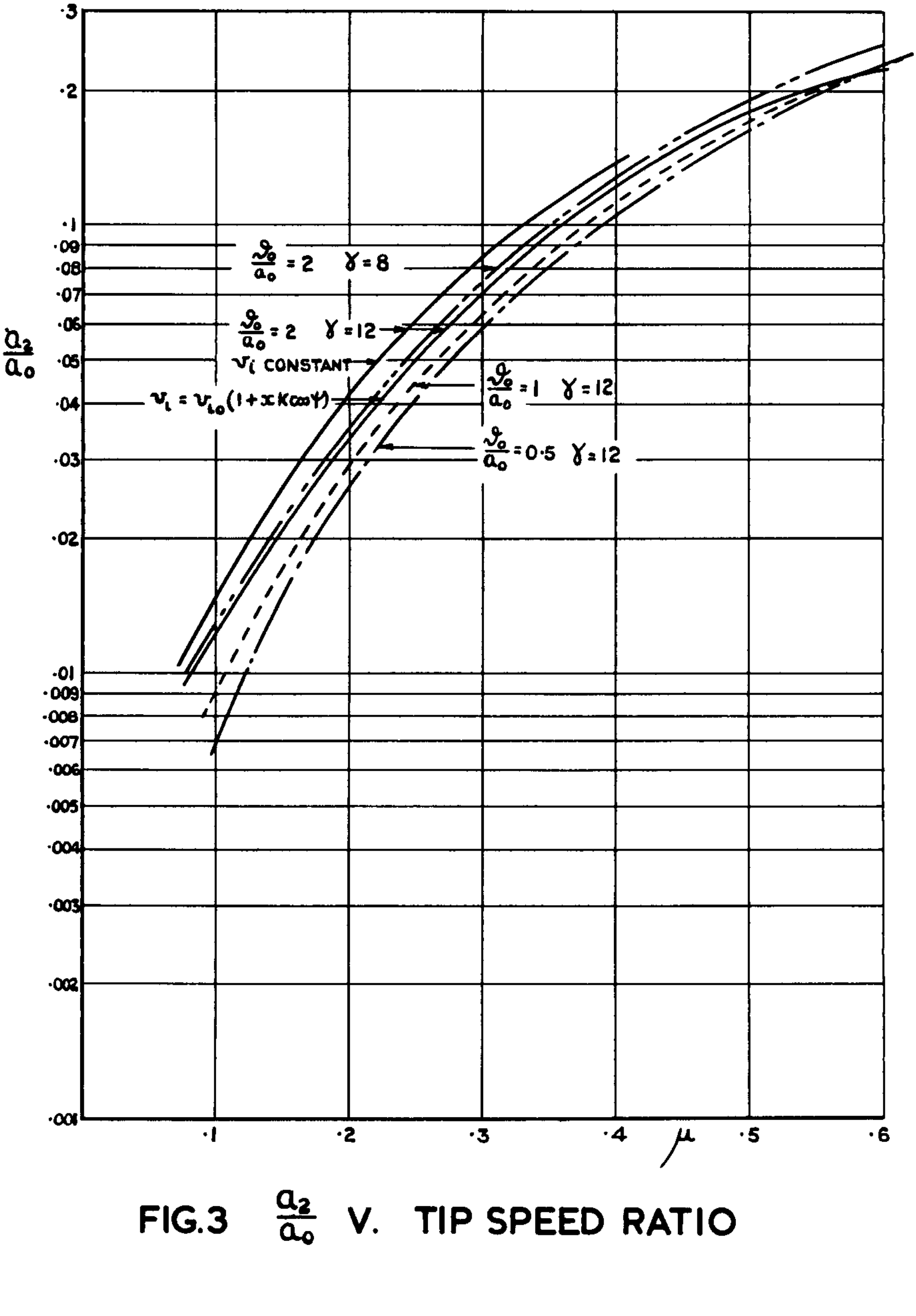

FIG.3 $\frac{a_2}{a_0}$ V. TIP SPEED RATIO

FIG. 4

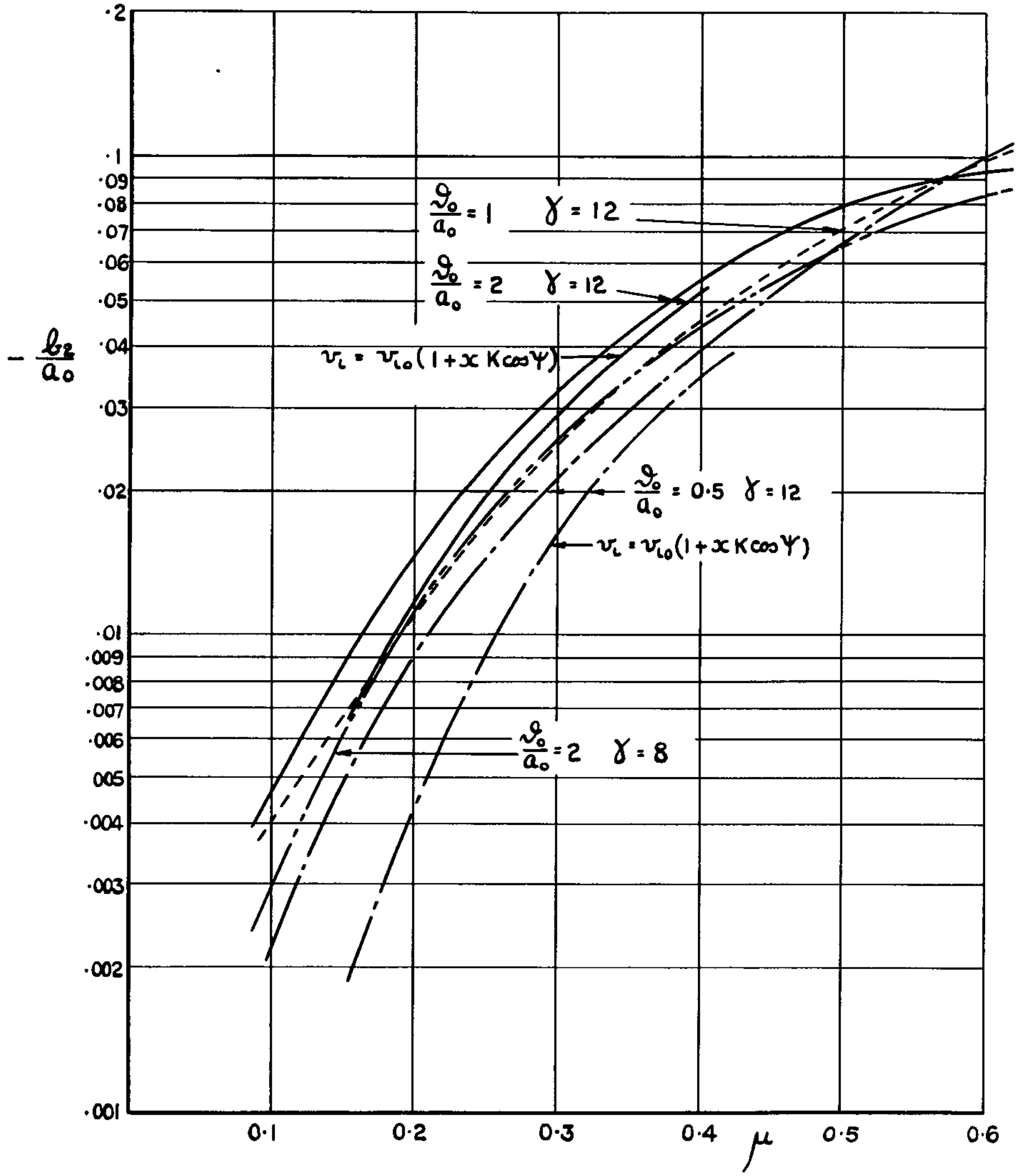

FIG. 4 $-\frac{b_2}{a_0}$ v. TIP SPEED RATIO

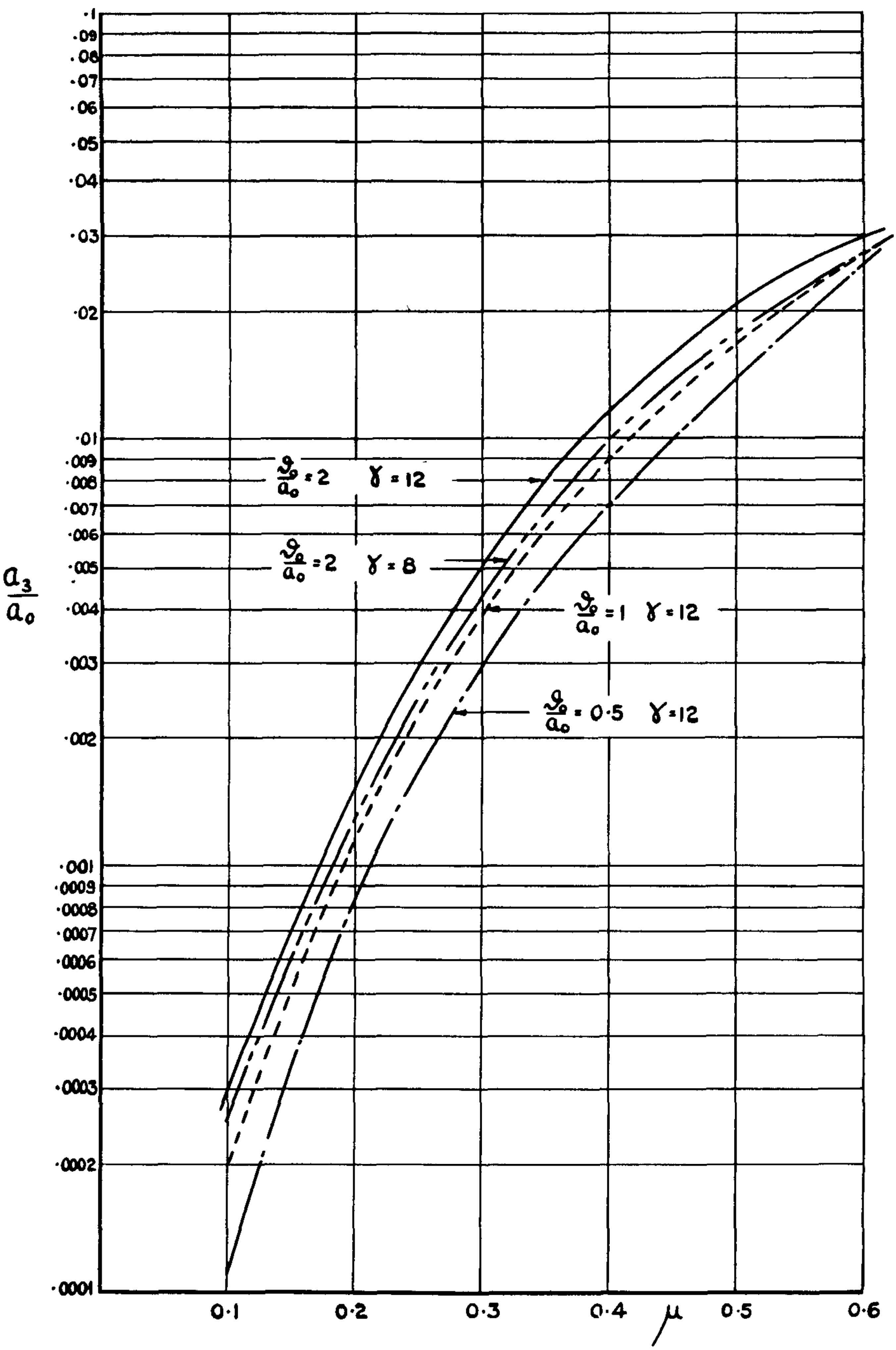

FIG. 5 $\frac{a_3}{a_0}$ v. TIP SPEED RATIO

FIG. 6

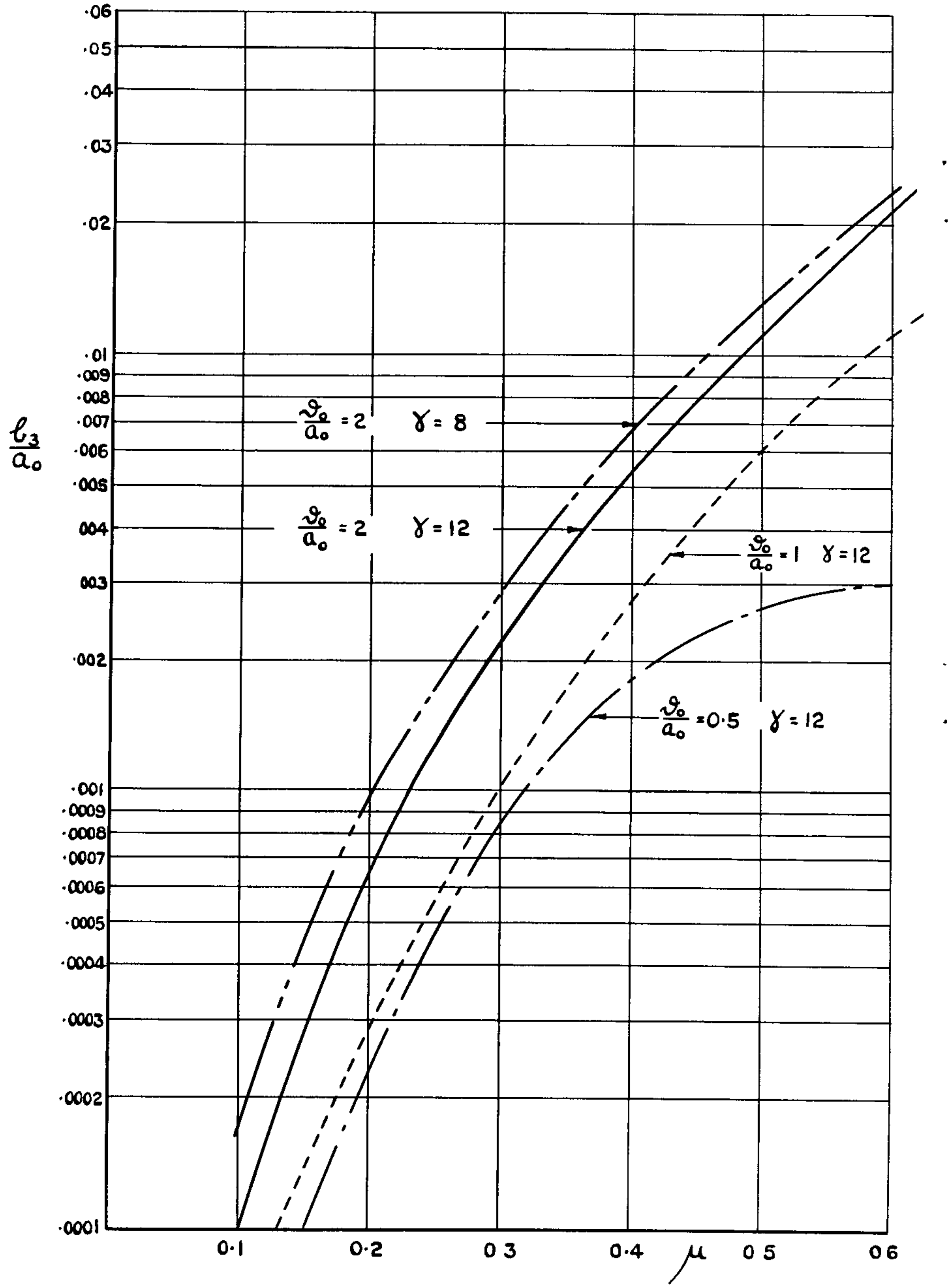

FIG.6 $\frac{b_3}{a_0}$ V. TIP SPEED RATIO

FIG.7

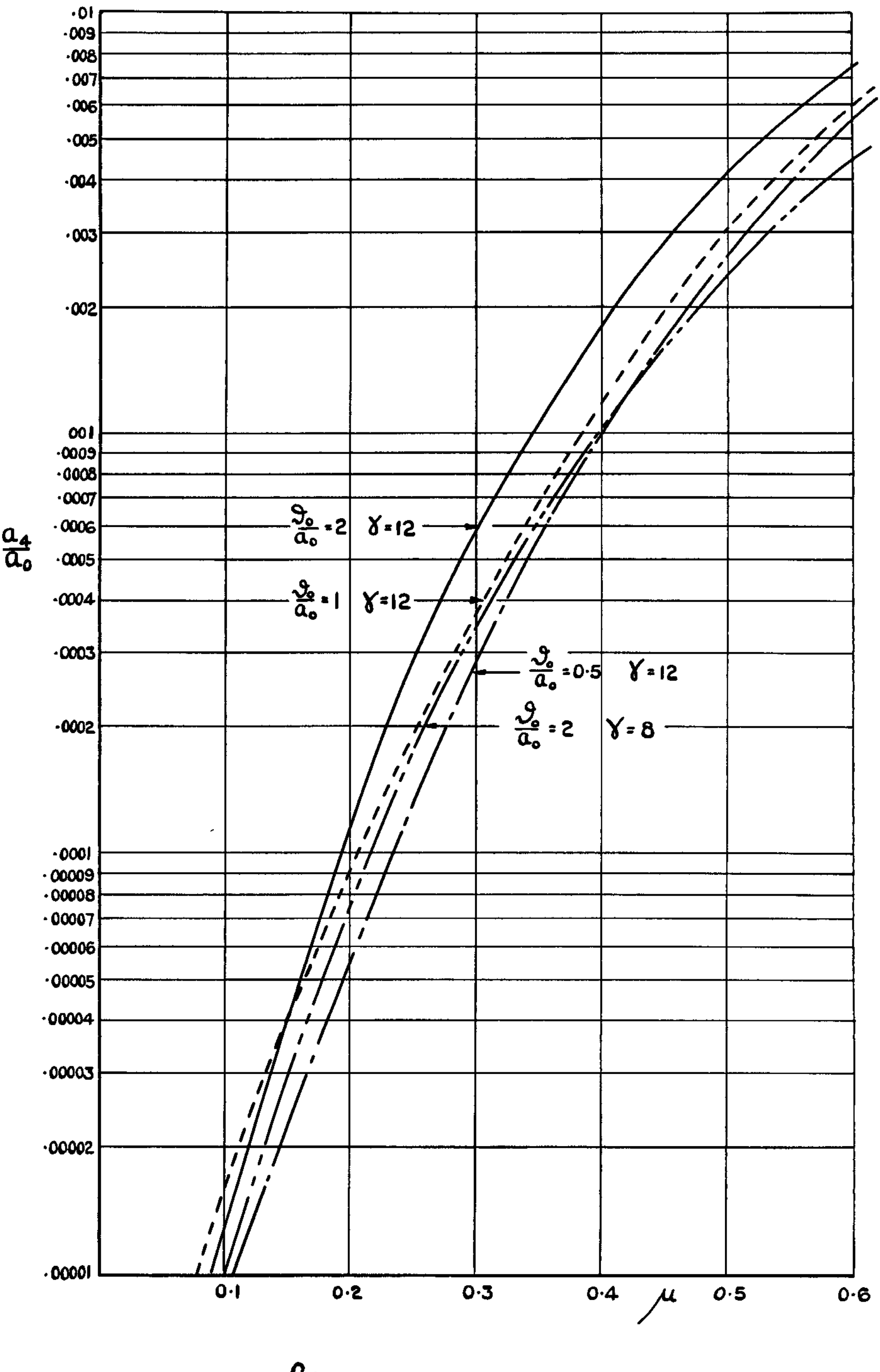

FIG.7 $\frac{a_4}{a_0}$ V. TIP SPEED RATIO

FIG. 8.

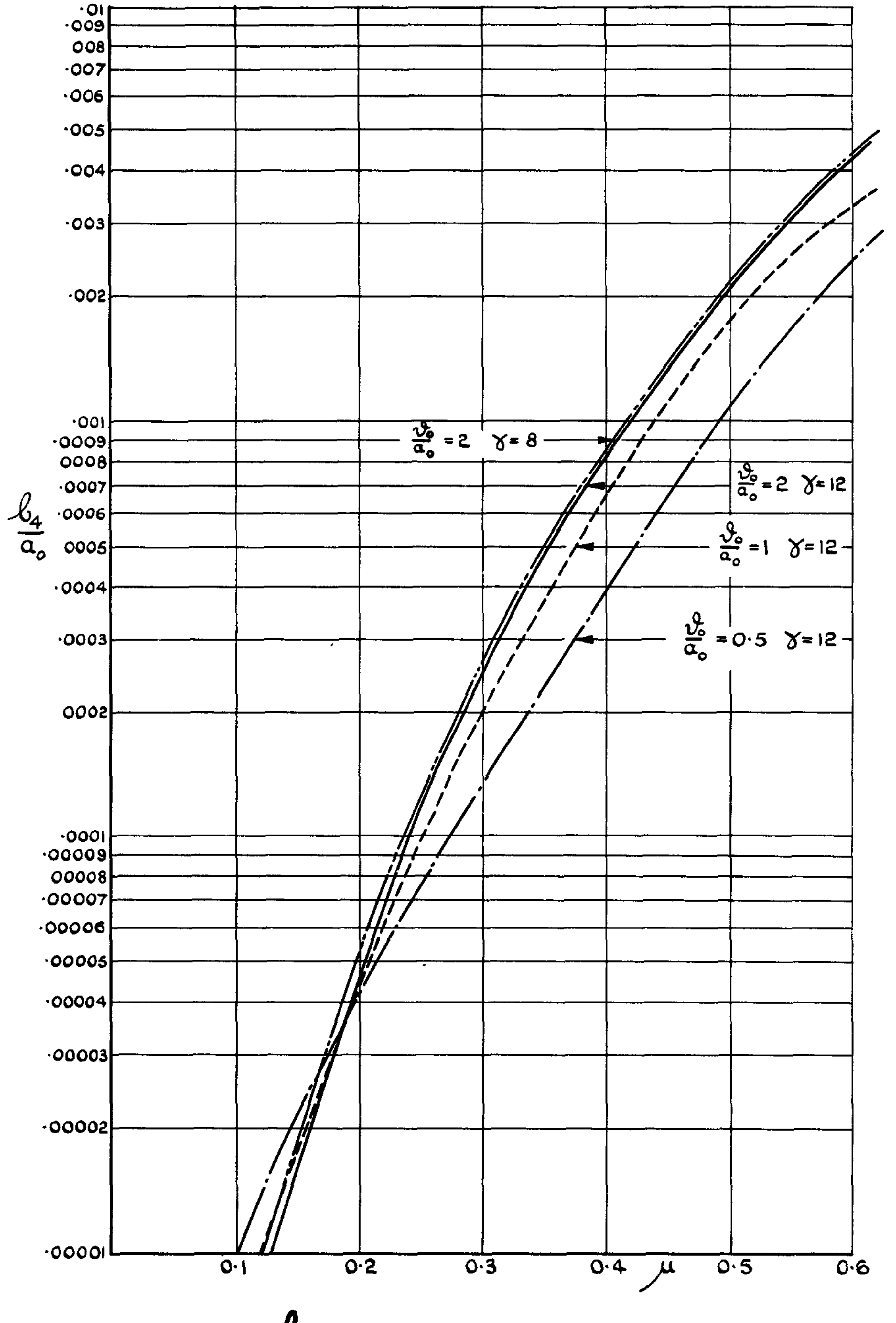

FIG. 8. $\frac{b_4}{a_0}$ V TIP SPEED RATIO

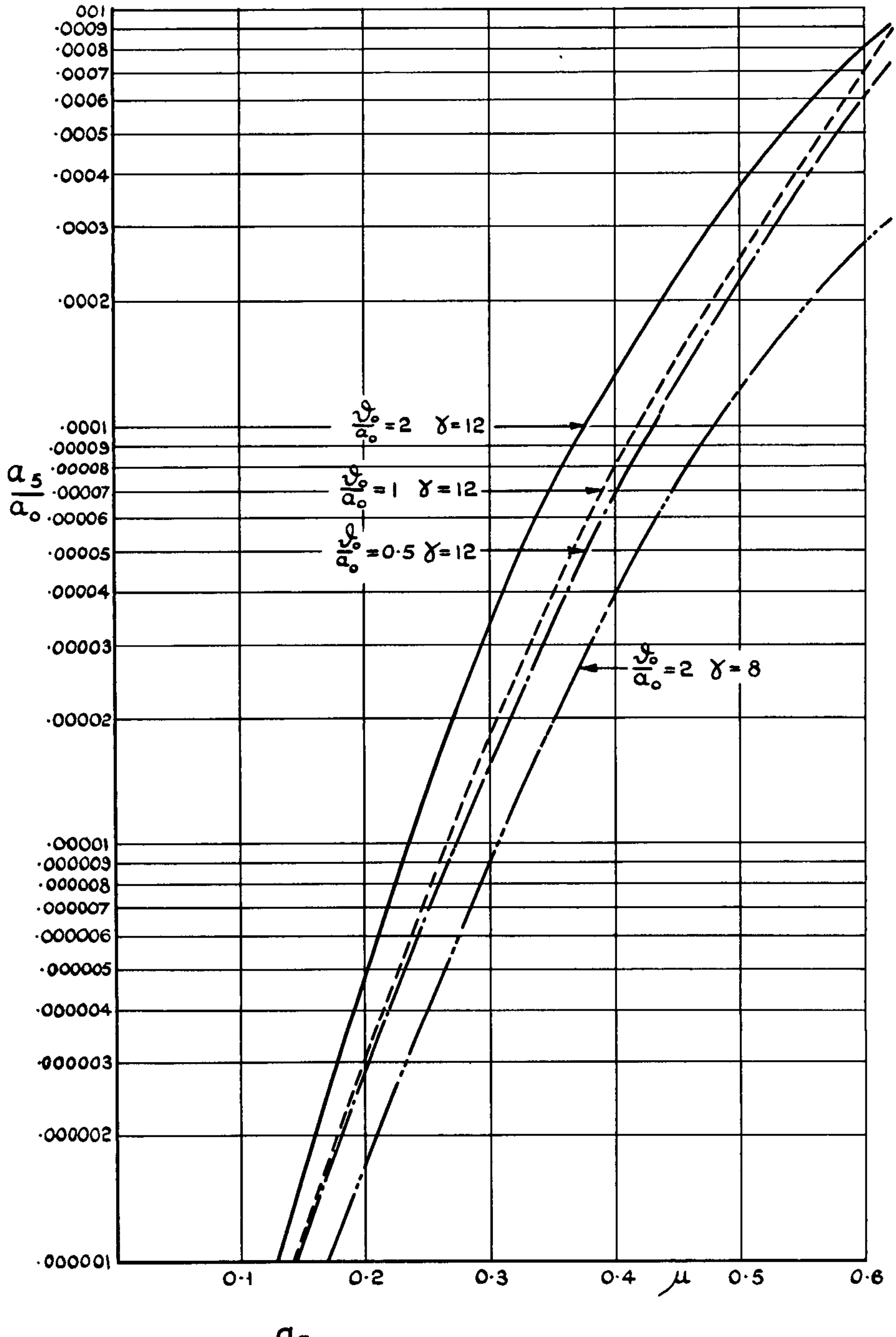

FIG. 9. $\frac{a_5}{a_0}$ V. TIP SPEED RATIO.

FIG. 10.

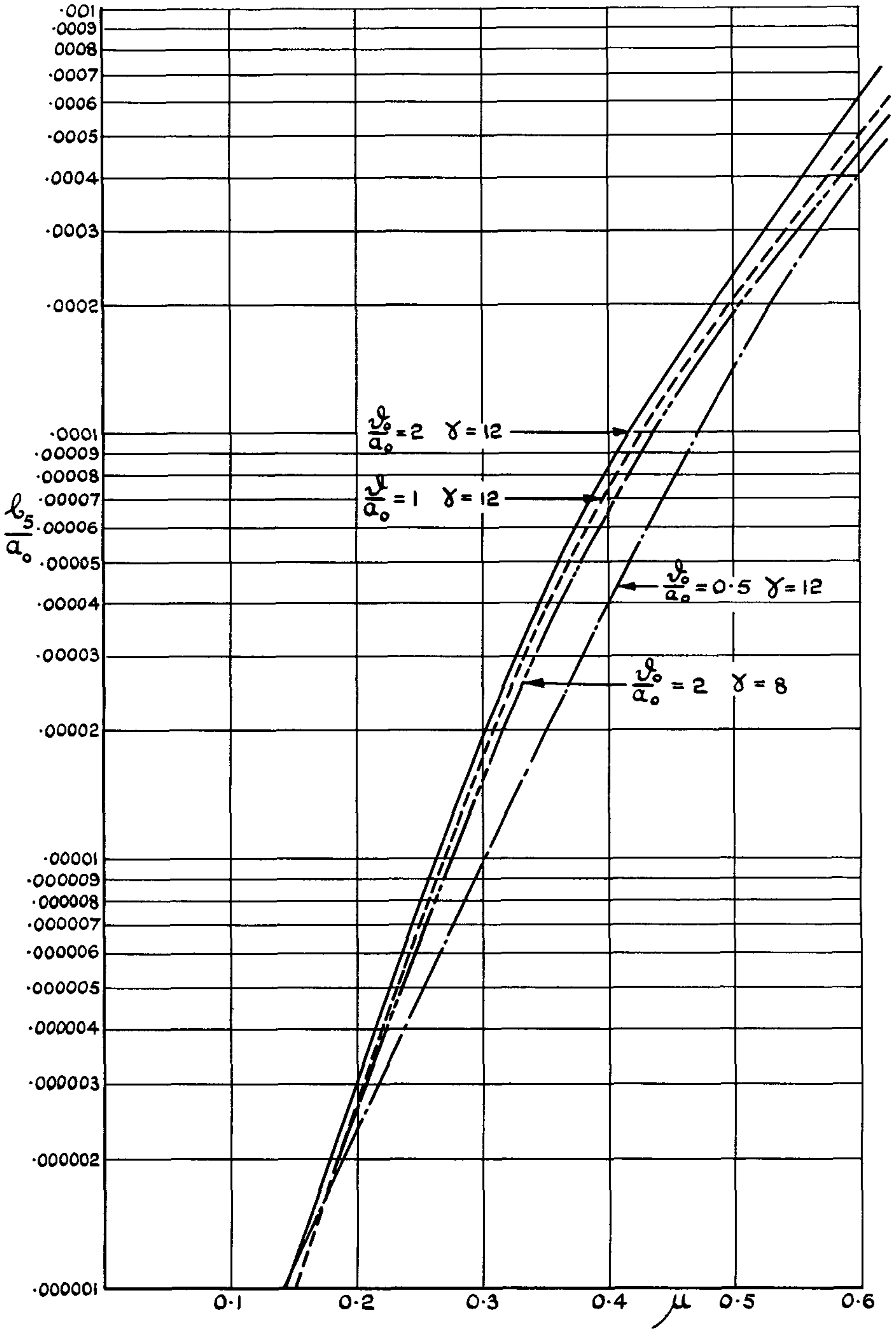

FIG. 10. $\frac{b_5}{a_0}$ V. TIP SPEED RATIO.

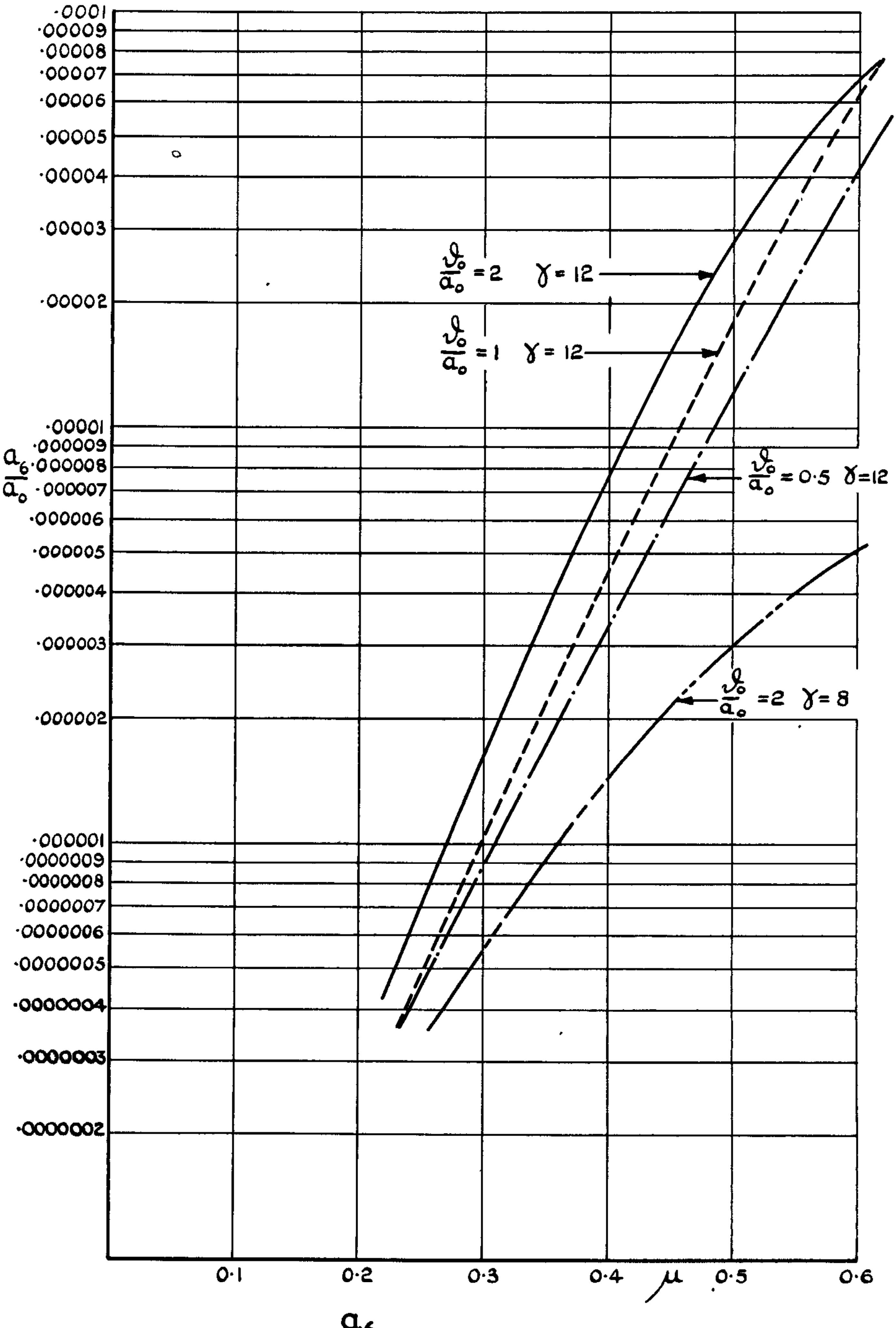

FIG. 11. $\frac{a_6}{a_0}$ V. TIP SPEED RATIO.

FIG. 12.

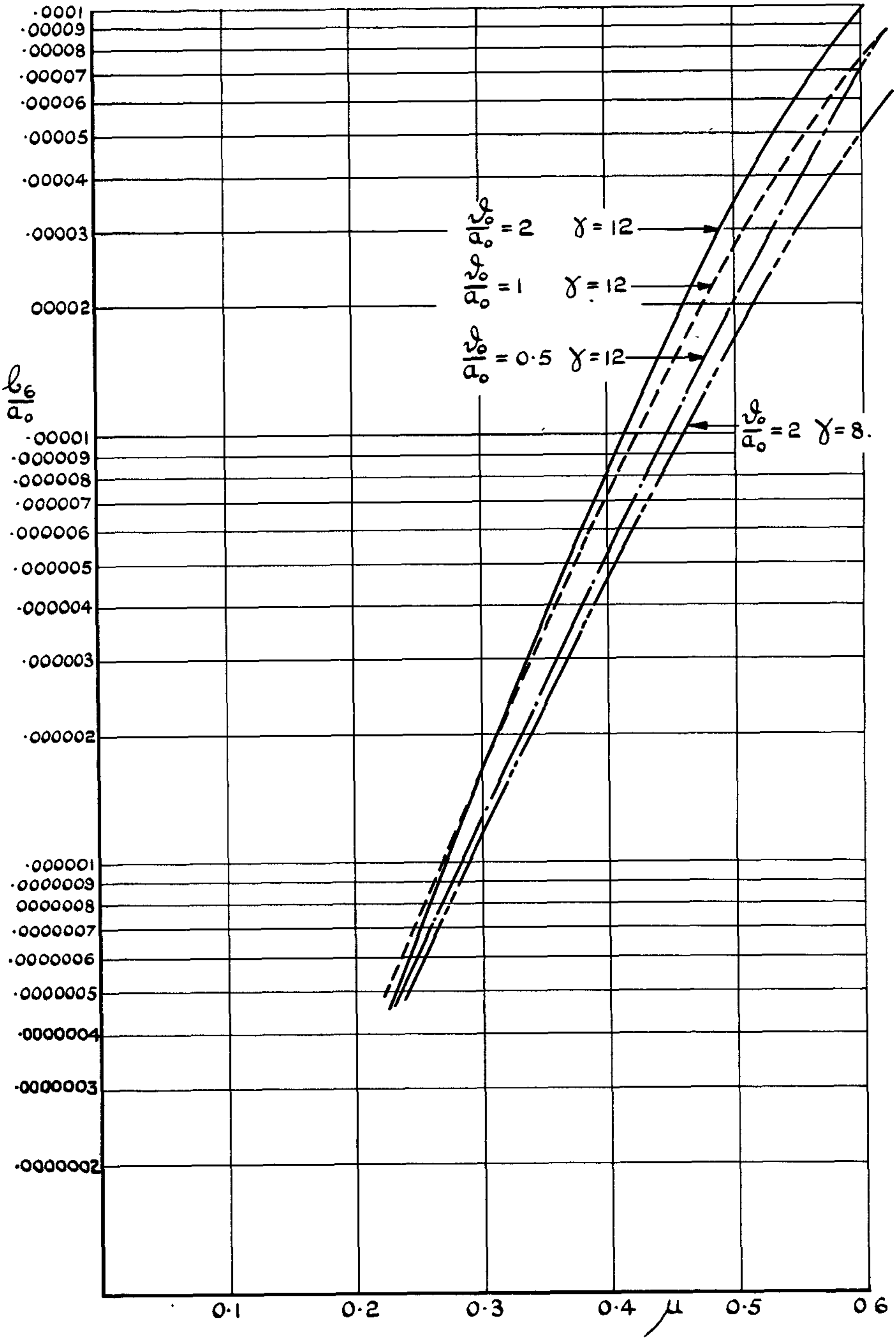

FIG. 12. $\frac{b_6}{a_o}$ V. TIP SPEED RATIO.

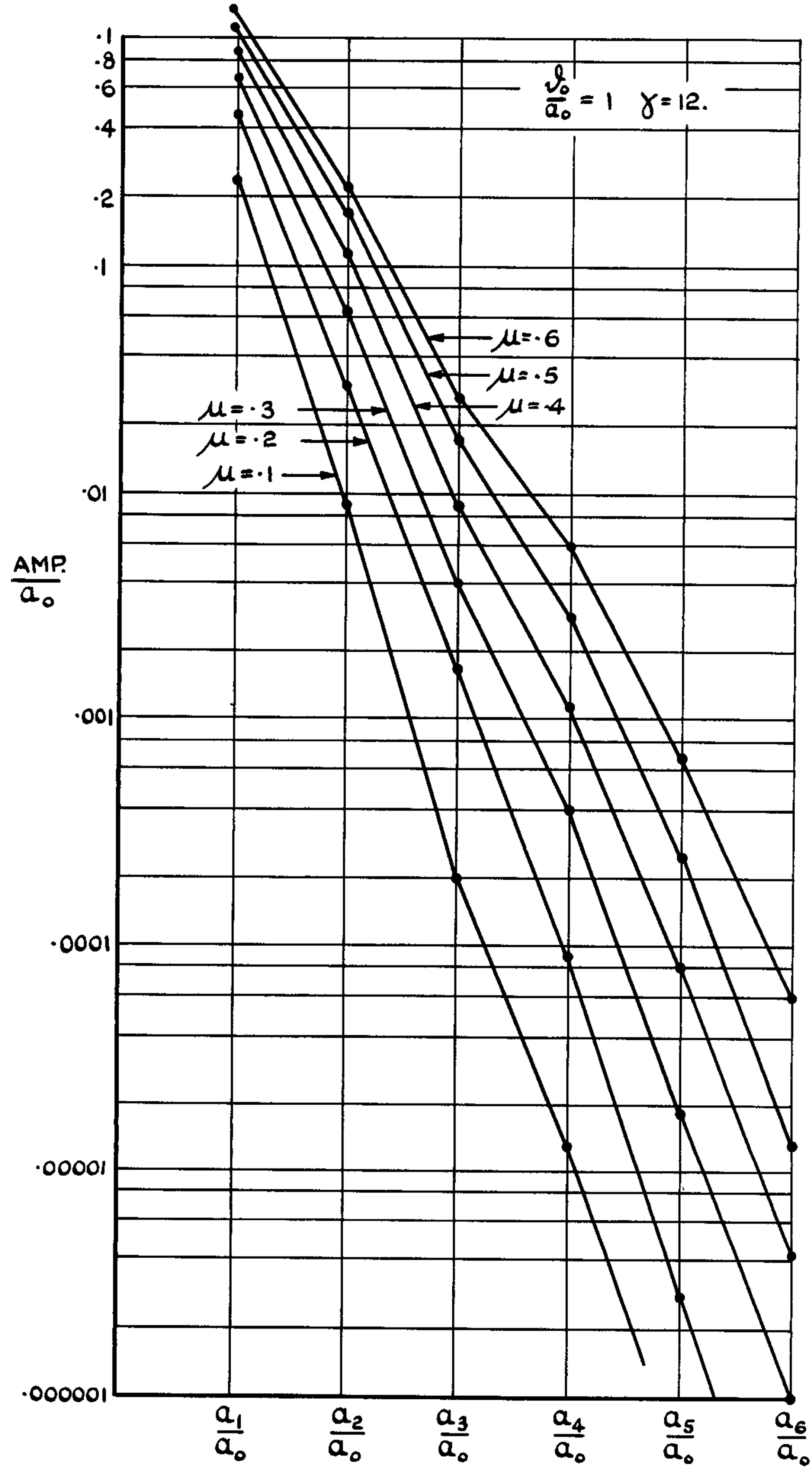

FIG. 13. a-HARMONIC AMPLITUDES

FIG. 14.

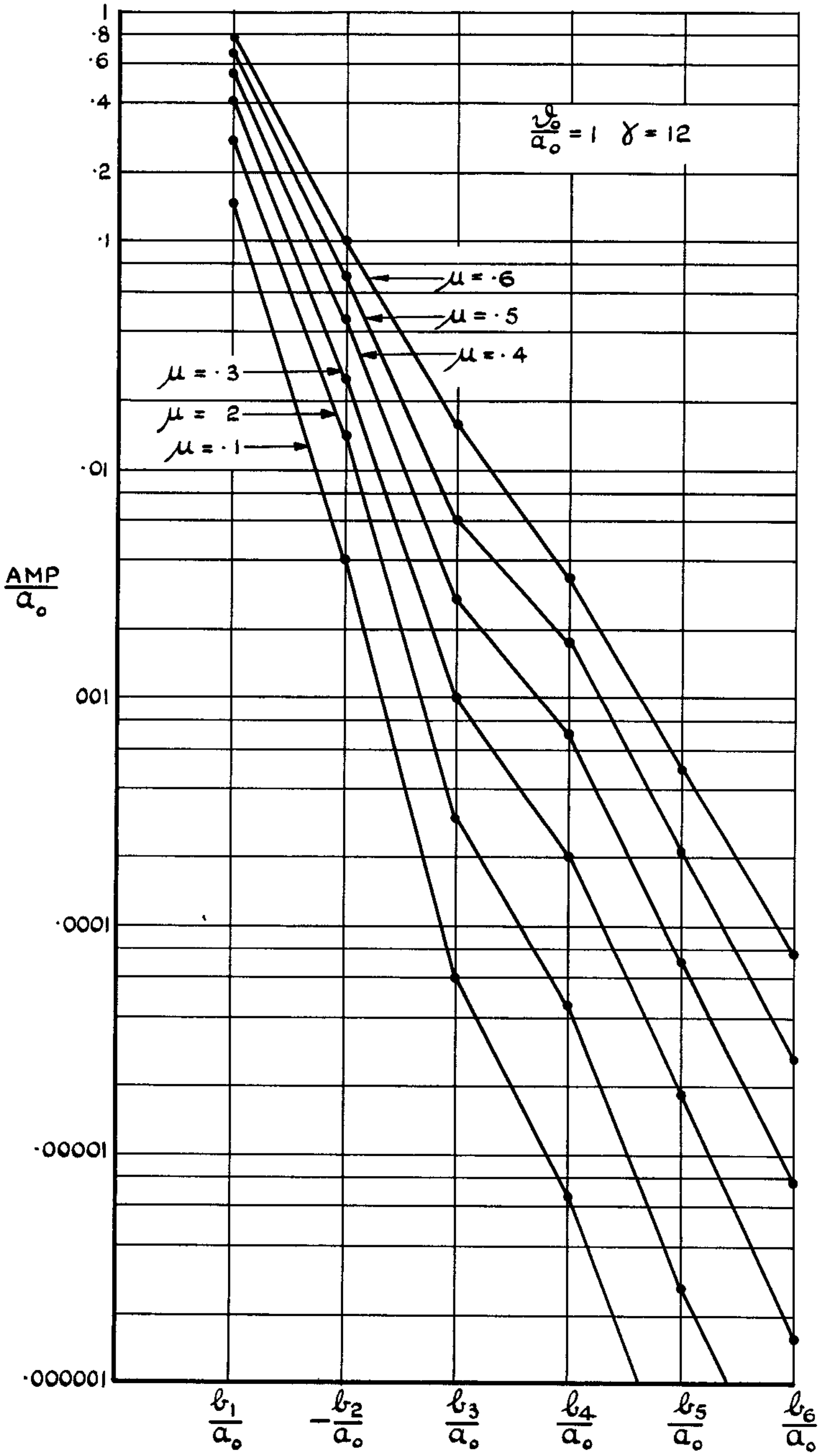

FIG. 14. b – HARMONIC AMPLITUDES

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