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by

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1976

PRICE £2-30 NET

UDC 533.6.013.42 : 533.6.013 : 532.511

*CP No.1336
July 1973

AN ASSESSMENT OF THE IMPORTANCE OF THE RESIDUAL FLEXIBILITY OF NEGLECTED MODES IN THE DYNAMICAL ANALYSIS OF DEFORMABLE AIRCRAFT

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SUMMARY

The mathematical framework for a unified approach to the dynamical problems of deformable aircraft, set up by Taylor in his contribution to R & M 3776, is used as the basis for a limited numerical investigation of the usefulness of the residual flexibility concept in truncated modal analyses. A finite-element model of a supersonic transport aircraft of slender-delta configuration is the subject of stability and response calculations, in which various representations of the structural deformability are used. These comprise up to four natural modes both with and without the residual flexibility of the remaining modes.

It is concluded that the addition of residual flexibility to a structural model which comprises only one or two modes significantly improves the accuracy of estimates of low-frequency characteristics. However, if a single model is to be used in an integrated approach to the aeroelastic problems of an aircraft, it must incorporate a fairly large number of modes in order to deal with the higher-frequency problems. In these circumstances the residual flexibility concept would seem to have little practical value.

* Replaces RAE Technical Report 73119, ARC 35085

CONTENTS

1 INTRODUCTION

A series of recent reports by Taylor[1,2], Woodcock[3] and Niblett[4] has dealt with the topic of establishing mathematical models suitable for studying dynamical problems of deformable aircraft. One of the central issues involved is the representation of the structural deformation of the aircraft. For the study of small perturbations from an initially steady flight condition, each author adopts a specification of the structural deformation primarily in terms of normal modes (which are defined as the natural modes of free vibration of the undamped structure *in vacuo*). In theory, a continuous structure has an infinite number of such modes, while for a discrete-element model of this structure only a finite number exist. In practice a further simplification is usually made by retaining only a limited number of these modes in the analyses. In addition, Taylor advocates the use of the residual-flexibility approximation, first suggested by Schwendler and MacNeal[5], to incorporate the effect of the neglected modes. However, Niblett[4] suggests that no real increase in accuracy of calculations is likely to be obtained because of the basic inaccuracies inherent in the aeroelastic data. Instead, both Niblett and Woodcock suggest the use of a few selected arbitrary modes to supplement the basic normal-mode approach when the structure is subjected to certain discrete loads. All of these reports are concerned with establishing mathematical models, the usefulness of which can be determined only by numerical studies of specific problems. A certain amount of numerical experimentation with the residual-flexibility concept was reported in Schwendler and MacNeal's paper[5] and related papers by Pearce *et al.*[6,7] but no firm conclusions were drawn as to its usefulness. It was with the object of providing further evidence on this score, as well as acquiring experience in the practical application of Taylor's method[2], that the numerical study described in the present Report was undertaken.

The study was of limited scope, it being restricted to a consideration of the stability and response characteristics of one particular aircraft layout (slender delta) under a single set of operating conditions. A fairly crude finite-element (or 'lumped-parameter') representation of the aircraft was adopted, with the inertia properties represented by n point-masses distributed along the centre-line of the aircraft. Structural flexibility was represented by a matrix of influence coefficients, relating to the n mass points and derived, by simple beam theory, from an assumed longitudinal distribution of bending rigidity (EI). Two matrices of aerodynamic influence coefficients, relating to the same set of points, and derived respectively from piston theory and slender-body theory, were used to provide alternative evaluations of the incremental aerodynamic loading due to structural deformation. These two theories were chosen because they are simple to apply in the present context. Neither was expected to give results of great absolute accuracy, but this was not of prime importance in an investigation of the influence of residual flexibility. It was necessary only to observe the general trends of calculated stability and response characteristics as the structural representation was varied, and it was expected that either aerodynamic theory would suffice for this purpose. The stability calculations were made using both sets of aerodynamic influence coefficients to check that conclusions as to trends are substantially independent of the choice of aerodynamic theory. A single set of calculations, based on piston-theory influence coefficients, was then considered sufficient for the response investigation.

In attempting to assess the validity and relative importance of the residual-flexibility concept, one may adopt the n-mass finite-element representation as a datum mathematical model whose longitudinal-symmetric stability and response characteristics may (in theory at least) be determined 'exactly' in terms of the characteristics of the (n - 2) natural modes of the structure and those of the two relevant 'rigid-body' modes. One may expect to obtain progressively better approximations to these 'exact' solutions by performing a series of truncated modal analyses in which the number of retained modes is progressively increased. In fact, one may perform two series of analyses, the first ignoring any effect of the excluded modes and the second taking account of them *via* a residual-flexibility approximation. Then, if the residual-flexibility concept is sound in principle and has been correctly implemented, the second set of results should converge rather more rapidly to the 'exact' values than the first set. In practice the 'exact' values for the datum model cannot be directly determined because the characteristics of its higher natural modes cannot themselves be accurately calculated. Thus, in the present application, only the first four natural modes have been determined and solutions of the stability and response problems have been obtained by analyses (with and without residual flexibility) in which 0, 1, 2, 3 and 4 modes were retained. Comparison of the accuracies of the two sets of solutions is thus dependent to some extent on intuition, guided by judgment of the degree of convergence already achieved when 4 modes are retained.

To make the present Report reasonably self-contained the development of the appropriate equations of motion is presented in some detail in section 2, which is based on Taylor's work[2]. The derivation of the structural and aerodynamic influence coefficients appropriate to slender-delta configurations of the type considered is discussed in section 3 and particulars of the mathematical model used in the numerical work are presented in section 4. Results are given in sections 5 and 6, which deal respectively with stability and response to sinusoidal gusts, and in section 7 an attempt is made to draw some conclusions from this limited numerical investigation.

2 EQUATIONS OF MOTION

The analysis in this section is based on the work of Taylor[2], which has antecedents in Bisplinghoff and Ashley[8] and Milne[9], while the residual flexibility theory is drawn from the work of Schwendler and MacNeal[5] and Pearce *et al*[6]. A finite-element representation of the aircraft is considered which is essentially two-dimensional in nature, no allowance being made for spanwise flexibility*. There are n discrete point-masses m_i, whose centres are at distances x_i from a chosen origin (measured positively in the forward direction). The main feature of the subsequent analysis is that it utilises two sets of influence coefficients through which the structural and aerodynamic properties of the aircraft are represented.

2.1 The basic equations

The undisturbed aircraft motion is considered to be level flight, at speed V_e, and mean axes through the centre of mass are chosen for the coordinate system, the x-axis being directed forward and the z-axis downward. In the mathematical model considered, the rigid-body displacement corresponding to the phugoid motion is ignored, although it could easily be incorporated if desired. Two rigid-body modes are required to define the aircraft short-period motion, the associated coordinates ζ_1, ζ_2 (which define the displacements of the nose in the positive z-direction) being related to the velocity-coordinates w, $q\left(=\frac{d\Theta}{dt}\right)$ of classical stability and control theory, by the transformations

$$\ddot{\zeta}_1 = \dot{w} - qV_e \ , \ \zeta_2 = -\Theta x_{ref} \tag{1}$$

* The aircraft configuration used as the basis of the numerical work described in section 4 is of the 'integrated' slender-delta type considered in the earliest project studies for a supersonic transport aircraft, for which longitudinal bending was considered to be the dominant elastic deformation.

where x_{ref} is the coordinate of the nose, here taken as reference point.

In the finite-element representation adopted here, the general motion of the aircraft structure may be assumed to be compounded of displacements in the two rigid-body modes and in $(n - 2)$ independent natural vibration modes of the elastic structure, whose shapes are defined relative to the mean axes. Thus, for small perturbations, the displacements, $\bar{\delta}_i$, $(i = 1,\ldots n)$ of the elements at points x_i, relative to their datum-path positions* are given by the matrix equation

$$\{\bar{\delta}(t)\} = [\Delta]\{\zeta(t)\} \quad . \tag{2}$$

The last $(n - 2)$ columns of the square matrix $[\Delta]$ describe the natural mode shapes of the flexible structure relative to mean axes through the centre of mass, while the first two columns, corresponding to the rigid-body displacement of the mean axes relative to the datum-path axes, are specified by

$$\Delta_{i1} = 1 \; ; \; \Delta_{i2} = \frac{x_i}{x_{ref}} \; ; \qquad (i = 1,\ldots n) \quad . \tag{3}$$

The $\zeta_j(t)$ represent the displacements, in the respective modes, of the reference point (nose) and are measured in the positive z-direction. The mode shapes are normalised with respect to the displacement at the nose.

For $j \geqslant 3$, the Δ_j's satisfy an eigenvalue equation of the form

$$[D]\{\Delta\} = \Lambda\{\Delta\} \tag{4}$$

where the matrix $[D]$ is defined by

$$[D] = [G][m_\lrcorner \quad . \tag{5}$$

Here $[G]$ is a matrix of influence coefficients describing the elastic properties of the unconstrained aircraft structure, which will be precisely defined in section 3.1. For the present we note only that it has the important

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* See Ref.2, section 3.6.2. Note that as we are concerned here only with incremental values of displacements and forces relative to datum values we will, for simplicity, use unprimed symbols in place of the primed symbols of Ref.2.

property of being symmetrical, by virtue of which it may be established from equations (4) and (5) that the mode shapes satisfy the orthogonality condition:

$$\{\Delta_i\}^T [m] \{\Delta_j\} = 0 \; ; \qquad i \neq j \; , \quad i,j \geqslant 3 \quad . \tag{6}$$

Furthermore the equation

$$\{\Delta_j\}^T [m] \{\Delta_j\} = M_j \; ; \qquad j \geqslant 3 \tag{7}$$

defines the generalised masses, M_j , of the structural modes, for which corresponding generalised stiffnesses, K_j , are defined by

$$K_j = M_j \omega_j^2 \; ; \qquad j \geqslant 3 \tag{8}$$

where the ω_j are the natural frequencies of the aircraft and are related to the eigenvalues Λ_j of equation (4) by the relationship $\Lambda_j = 1/\omega_j^2$. If the rigid-body modes are considered as zero-frequency structural modes (i.e. $\omega_1 = \omega_2 = 0$) with zero generalised stiffness $(K_1 = K_2 = 0)$ then equation (8) holds for $j \geqslant 1$. In virtue of the conditions defining mean axes it can be shown that the orthogonality condition (6) holds for $i,j \geqslant 1$, while equation (7) with $j = 1$ or 2 defines the generalised masses for the rigid-body modes. It is readily shown that

$$M_1 = m \text{ and } M_2 = \frac{I_y}{x_{ref}^2} \; ,$$

where m is the total mass of the aircraft and I_y is its pitching moment of inertia.

The equations of motion may now be written in terms of the $n \times n$ diagonal matrices $[M]$ and $[K]$ formed from the M_j and K_j :-

$$[Y]\{\zeta\} = \{F\} \tag{9}$$

where

$$[Y] = [MD^2 + K] \tag{10}$$

with $D \equiv \frac{d}{dt}$. The generalised forces $\{F\}$ consist of a contribution, $\{F_M\}$, due to the perturbations $\{\bar{\delta}\}$, and one, $\{F_D\}$, due to control deflections or atmospheric disturbances. The former may be expressed as

$$\{F_M\} = [\Delta]^T[R]\{\bar{\delta}\} \tag{11}$$

where $[R]$ is the aerodynamic influence-coefficient matrix, which is defined later, in section 3.2. We note that, in general, the elements of $[R]$ are quadratic expressions in the differential operator D . Thus the basic equation may be expressed as

$$[Y]\{\zeta\} = [\Delta]^T[R][\Delta]\{\zeta\} + [\Delta]^T\{F_D\} \tag{12}$$

where $\{F_D\}$ is the column of incremental forces due to control deflections or atmospheric disturbances, acting at the nodal points.

2.2 Approximate treatments of the equations of motion

2.2.1 Truncated modal analysis

The basic matrix equation (12) consists of n equations where n is the number of masses specified in the finite-element representation of the aircraft. As already mentioned it is not possible to determine accurately the characteristics of all (n - 2) natural modes of the structure, so that an accurate formulation of the complete set of equations is not generally feasible. However, one is usually only interested in the response of the aircraft over a (lower-frequency) part of the frequency spectrum covered by the natural modes and it is often argued, on a semi-intuitive basis, that a sufficiently accurate solution can be obtained by using a truncated system of equations, in which the effects of modes with frequencies outside the range of interest are completely ignored. It is then only necessary to evaluate the frequencies and mode shapes of (say) the lowest k structural modes, to retain only the first (k + 2) of the component equations in equation (12) and to reject the terms which coupled them to the other equations.

2.2.2 Residual flexibility of the neglected modes

The concept of residual flexibility was introduced by Schwendler and MacNeal as a means of making a partial allowance for the effects of the neglected higher modes in truncated modal analyses. Essentially it is assumed that, when the system is responding to excitation in the frequency range of the retained (lower-frequency) modes, the inertial and damping forces arising from any motion in the higher-frequency modes will be negligible in comparison with the corresponding forces due to the (static) displacements in those modes. Accordingly the equations of motion that are completely rejected in the truncated modal approach may be retained in an approximate form, in which they may be regarded as a set of algebraic equations for the coordinates of the higher modes in terms of those of the lower modes. As such they may be used to eliminate the coordinates of the higher modes from the other (retained) set of equations and, thereby, to obtain a set of equations identical in form with those used in the truncated modal analysis, albeit with modified derivatives.

If we denote the rigid-body and lower-frequency structural modes by the suffices 0 and 1 respectively, and the higher-frequency structural modes by the suffice 2 , then we may partition the matrices in equation (12) to obtain:

$$\left[\begin{array}{c|c} Y_{0+1} & 0 \\ \hline 0 & Y_2 \end{array}\right]\left\{\begin{array}{c} \zeta_{0+1} \\ \hline \zeta_2 \end{array}\right\} = \left[\left[\begin{array}{c} \Delta_{0+1}^T \\ \hline \Delta_2^T \end{array}\right][R]\left[\begin{array}{c|c} \Delta_{0+1} & \Delta_2 \end{array}\right]\right]\left\{\begin{array}{c} \zeta_{0+1} \\ \hline \zeta_2 \end{array}\right\} + \left[\begin{array}{c} \Delta_{0+1}^T \\ \hline \Delta_2^T \end{array}\right]\{F_D\} \quad . \qquad (13)$$

Equation (13) is essentially two simultaneous matrix equations:

$$[Y_{0+1}]\{\zeta_{0+1}\} = [\Delta_{0+1}]^T[R][\Delta_{0+1}]\{\zeta_{0+1}\} + [\Delta_{0+1}]^T[R][\Delta_2]\{\zeta_2\} + [\Delta_{0+1}]^T\{F_D\} \qquad (14)$$

$$[Y_2]\{\zeta_2\} = [\Delta_2]^T[R][\Delta_{0+1}]\{\zeta_{0+1}\} + [\Delta_2]^T[R][\Delta_2]\{\zeta_2\} + [\Delta_2]^T\{F_D\} \ . \qquad (15)$$

At this stage we introduce the crucial assumption, mentioned in the preamble to this section, that, when we are concerned with the response of the aircraft to disturbances which have frequencies much smaller than those of the higher structural modes, we may neglect the D and D^2 terms in the aerodynamic operator $[R]$ in those terms of equations (14) and (15) which involve $\{\zeta_2\}$. It is also assumed that we may neglect the D^2 term in $[Y_2]$, so that $[Y_2] \approx [K_2]$. With these assumptions it is a straightforward matter to obtain from equation (15) an approximate solution for $\{\zeta_2\}$ in terms of $\{\zeta_{0+1}\}$ and substitution of this into the modified equation (14) leads, after some simplification, to

$$[Y_{0+1}]\{\zeta_{0+1}\} = [\Delta_{0+1}]^T[A][R][\Delta_{0+1}]\{\zeta_{0+1}\} + [\Delta_{0+1}]^T[A]\{F_D\} \tag{16}$$

where

$$[A]^{-1} = [I] - [R'][X] \tag{17}$$

with $[R']$ denoting the steady influence-coefficient matrix, and

$$[X] = [\Delta_2][K_2]^{-1}[\Delta_2]^T \tag{18}$$

representing the residual flexibility of the eliminated modes. Equation (16) is of an essentially similar nature to the nth order matrix equation (12) but consists of a smaller number $(k + 2)$ equations, where $k(< n - 2)$ is the number of structural modes retained in the analysis. Comparing equation (16) with the equation which governs the truncated modal analysis of section 2.2.1, (i.e. equation (14) with the term in $\{\zeta_2\}$ deleted) we note that the matrix expressions operating on $\{\zeta_{0+1}\}$ and $\{F_D\}$ are each modified by the presence of the aeroelastic correction matrix $[A]$.

It is a simple matter to express $[X]$ in terms of $[G]$ and other known quantities, i.e. those with suffix 1. This is due to a basic result that $[G]$ may be expressed as

$$[G] = [\Delta_1][K_1]^{-1}[\Delta_1]^T + [\Delta_2][K_2]^{-1}[\Delta_2]^T \quad . \tag{19}$$

The proof of this result is as follows. From equation (4) we have

$$[D][\Delta_{1+2}] = [\Delta_{1+2}][\Lambda] \tag{20}$$

where $[\Lambda]$ is the $(n - 2) \times (n - 2)$ diagonal matrix whose jth diagonal element is $\Lambda_{j+2} = 1/\omega^2_{j+2}$, $(j = 1,\ldots \overline{n-2})$. From equations (5) and (20) we have

$$[G][m][\Delta_{1+2}] = [\Delta_{1+2}][\Lambda] \tag{21}$$

where, from equation (8),

$$[\Lambda] = [K_{1+2}]^{-1}[M_{1+2}] \tag{22}$$

and, from equation (7)

$$[M_{1+2}] = [\Delta_{1+2}]^T [m] [\Delta_{1+2}] \quad . \tag{23}$$

On substituting (22) and (23) in (21) we obtain

$$\Big[[G] - [\Delta_{1+2}][K_{1+2}]^{-1}[\Delta_{1+2}]^T\Big][m][\Delta_{1+2}] = 0$$

from which it follows* that

$$[G] = [\Delta_{1+2}][K_{1+2}]^{-1}[\Delta_{1+2}]^T \tag{24}$$

and, by partitioning of the matrices,

$$[G] = \left[\begin{array}{c|c} \Delta_1 & \Delta_2 \end{array}\right] \left[\begin{array}{c|c} K_1^{-1} & 0 \\ \hline 0 & K_2^{-1} \end{array}\right] \left[\begin{array}{c} \Delta_1^T \\ \hline \Delta_2^T \end{array}\right] \quad , \tag{25}$$

from which the result (19) follows immediately. Hence $[X]$, the residual-flexibility matrix, as defined by equation (18), is expressible in terms of known quantities by

$$[X] = [G] - [\Delta_1][K_1]^{-1}[\Delta_1]^T \quad . \tag{26}$$

2.2.3 Mathematical models for comparative numerical investigation

The preceding two sections have provided two alternative mathematical models with which to seek approximations to the 'exact' solutions for the dynamical behaviour of a finite-element idealisation of an aircraft**. In both models the set of n equations which describes the behaviour of the n-element idealisation 'exactly', is truncated to a set of $(k + 2)$ equations where k , the number of structural modes to be retained explicitly in the analysis, is less than $(n - 2)$. Ostensibly, the more refined of the two models is that described by equations (16), (17) and (26), which incorporates an allowance for the neglected modes in terms of the residual-flexibility matrix $[X]$. In the other model the effect of the higher-frequency structural modes is completely neglected at the outset by simply deleting the last $(n - k - 2)$ of the n equations comprised by equation (12) and also the corresponding coupling terms in the remaining equations. Mathematically the resulting set of equations can also be derived from the set which incorporates the residual-flexibility matrix $[X]$, by setting $[X] = [0]$, which leads to $[A] = [I]$. In the limiting case

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* For a rigorous proof of this statement see the Appendix to the R & M version of Ref.2.

** We are not directly concerned here with the accuracy of the finite-element idealisation itself.

of $k = (n - 2)$ it is, of course, exactly true that $[X] = [0]$ and the sets of equations for the two models are identical.

Thus, for various values of $k < (n - 2)$, the respective orders of magnitude of the elements of the corresponding matrices $[X]$ will give some indication of the relative importance of including residual flexibility in the different cases. Furthermore, if we consider two corresponding series of models, with and without residual flexibility, in which the number of retained modes is progressively increased, we may expect results for each series of models to converge towards the same limiting values, *viz.* the 'exact' results for the chosen finite-element idealisation. Moreover, if the residual flexibility concept is soundly based, the convergence should be initially more rapid in the case of the models which incorporate it.

3 THE INFLUENCE COEFFICIENTS

3.1 Structural influence coefficients

For the purposes of the present investigation, the aircraft is treated as a longitudinal beam with mass density and stiffness varying along its length; spanwise it is assumed to be infinitely stiff. The existence and nature of influence functions for beams is well known (see, for example, Milne[9]). At the outset, in defining the influence function for the beam, it is necessary to consider it to have sufficient kinematic constraint to prevent bodily motion. For our purposes it is convenient to restrain the beam at the point which coincides with the centre of mass when the beam is undeformed (i.e. at $x = 0$).

Then the transverse displacement $w(x)$ due to a loading distribution $p(x)$ is determined by the equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2w}{dx^2}\right) = p(x)\,, \qquad (x_T < x < x_N) \tag{27}$$

subject to the boundary conditions

$$w = \frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 \tag{28}$$

and

$$EI\frac{d^2w}{dx^2} = \frac{d}{dx}\left(EI\frac{d^2w}{dx^2}\right) = 0 \quad \text{at} \quad x = x_T, x_N \tag{29}$$

where $EI(x)$ is the bending stiffness of the beam. A formal solution of the differential equation is given by

$$w(x) = \int_{x_T}^{x_N} C(x,x')p(x')dx' \tag{30}$$

where the influence function C satisfies the differential equation

$$\frac{d^2}{dx^2}\left(EI\frac{d^2C}{dx^2}\right) = \delta(x - x') \quad , \tag{31}$$

δ being the Dirac function, and the boundary conditions

$$C = \frac{dC}{dx} = 0 \quad \text{at} \quad x = 0 \tag{32}$$

$$EI\frac{d^2C}{dx^2} = \frac{d}{dx}\left(EI\frac{d^2C}{dx^2}\right) = 0 \quad \text{at} \quad x = x_T, x_N \quad . \tag{33}$$

The differential operator in (31) is self-adjoint, so that the function $C(x,x')$ is symmetrical. For a freely flying aircraft we have to consider a beam subject to no kinematic constraint and having unloaded ends. For such a beam the boundary conditions (29) still apply and it is further necessary that

$$\int_{x_T}^{x_N} p(x)dx = \int_{x_T}^{x_N} xp(x)dx = 0 \quad . \tag{34}$$

In order to define an influence function for this case it is necessary to postulate a loading system to equilibrate the unit load $\delta(x - x')$. Any convenient load distribution which, in combination with the unit load, satisfies the equations (34) may be used. For since, in any real motion, the resultant load distribution $p(x)$, including inertia forces, must itself satisfy equation (34) (by virtue of the overall equations of motion) it will follow that the resultant of the arbitrary balancing systems will also satisfy equation (34); i.e. the balancing systems will constitute a null set and have no effect on the motion. If we think of the unit load being reacted by inertia loads, a convenient form for the balancing system is $[a(x') + b(x')x]m(x)$, where $m(x)$ is the mass per unit length. The influence function $G(x,x')$ for the unconstrained beam then satisfies the differential equation

$$\frac{d^2}{dx^2}\left(EI\,\frac{d^2G}{dx^2}\right) = \delta(x - x') - [a(x') + b(x')x]m(x) \tag{35}$$

where, from equation (34), a and b are defined by

$$\int_{x_T}^{x_N} \{\delta(x - x') - (a + bx)m(x)\}\,dx = \int_{x_T}^{x_N} x\{\delta(x - x') - (a + bx)m(x)\}\,dx = 0 \; . \quad \ldots.(36)$$

By using the condition that the origin is at the longitudinal location of the centre of mass, we have

$$a = \frac{1}{m} , \quad b = \frac{x'}{I_y} \tag{37}$$

where

$$m = \int_{x_T}^{x_N} m(x)\,dx \; ; \quad I_y = \int_{x_T}^{x_N} m(x)x^2\,dx \tag{38}$$

are respectively the total mass of the beam and its moment of inertia.

The influence function $G(x,x')$ is then given by

$$G(x,x') = C(x,x') - \int_{x_T}^{x_N} C(x,\xi)[am(\xi) + b(x')\xi m(\xi)]\,d\xi \tag{39}$$

which is not a symmetrical function. It still satisfies the conditions $G = dG/dx = 0$ at $x = 0$ but, in its most general form, G is not required to satisfy these conditions since, for a fixed x', it should define displacements which are arbitrary to the extent of a small rigid-body displacement. Thus the most general form of the unconstrained influence function may be taken as

$$G(x,x') = C(x,x') - \int_{x_T}^{x_N} C(x,\xi)\left[am(\xi) + b(x')\xi m(\xi)\right]d\xi + c(x') + d(x')x \qquad (40)$$

where the functions c and d are determined by the particular choice of body axes. For mean axes through the centre of mass

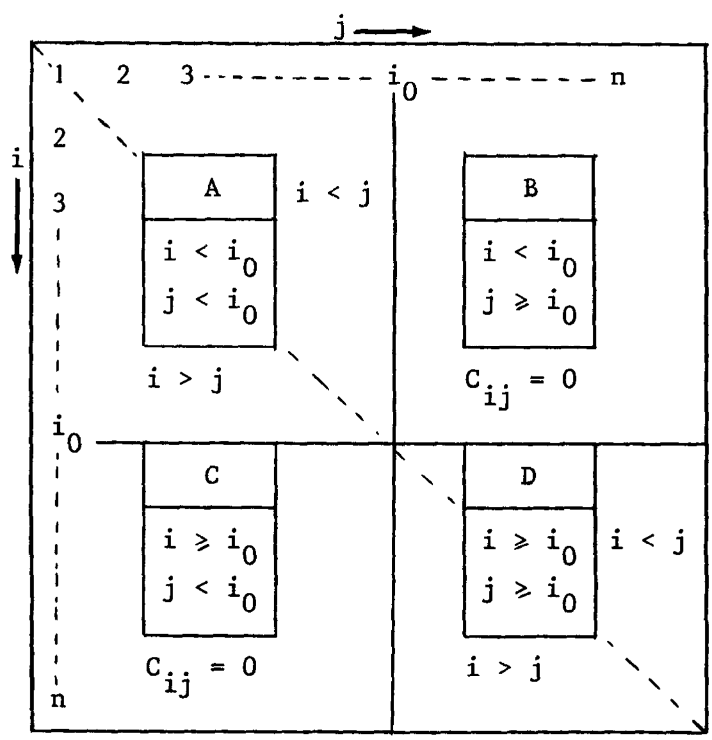
$$\int_{x_T}^{x_N} m(\xi')G(\xi',x')d\xi' = \int_{x_T}^{x_N} m(\xi')\xi'G(\xi',x')d\xi' = 0 \qquad (41)$$

which yields two simultaneous equations for c and d. Upon solving and substituting into equation (40), together with the expressions for a and b given by equation (37) we obtain the following expression for the influence function of the unconstrained beam:-

$$\begin{aligned} G(x,x') = {} & C(x,x') - \frac{1}{m}\int_{x_T}^{x_N} C(x,\xi)m(\xi)d\xi - \frac{1}{m}\int_{x_T}^{x_N} C(\xi,x')m(\xi)d\xi \\ & - \frac{x}{I_y}\int_{x_T}^{x_N} C(\xi,x')m(\xi)\xi d\xi - \frac{x'}{I_y}\int_{x_T}^{x_N} C(x,\xi)m(\xi)\xi d\xi \\ & + \frac{1}{m^2}\int_{x_T}^{x_N}\int_{x_T}^{x_N} C(\xi,\xi')m(\xi)m(\xi')d\xi d\xi' + \frac{x'}{mI_y}\int_{x_T}^{x_N}\int_{x_T}^{x_N} C(\xi,\xi')m(\xi)m(\xi')\xi' d\xi d\xi' \\ & + \frac{x}{mI_y}\int_{x_T}^{x_N}\int_{x_T}^{x_N} C(\xi,\xi')m(\xi)m(\xi')\xi d\xi d\xi' \\ & + \frac{xx'}{I_y^2}\int_{x_T}^{x_N}\int_{x_T}^{x_N} C(\xi,\xi')m(\xi)m(\xi')\xi\xi' d\xi d\xi' \quad . \end{aligned} \qquad (42)$$

It is readily shown that for this particular choice of axes, the influence function $G(x,x')$ is symmetrical.

It now remains to construct a finite-element representation of the above integrals. The first step is to replace the continuous influence function $C(x,x')$, defined for values of x and x' in the range $[x_T, x_N]$ by an $n \times n$ symmetric matrix of influence coefficients $[C_{ij}]$, where i and j range from 1 to n, C_{ij} denoting the deflection at the ith mass point due to unit load applied at the jth mass point. Let i_0 refer to the origin $\left(x_{i_0} = 0\right)$ if it is a mass point or otherwise to the nearest mass point ahead of the origin. Physically we may regard the beam as two beams separately encastré at the origin, so that a load applied at a point on one beam will produce no deflection at points on the other beam. Hence, in constructing $[C_{ij}]$, if we consider the matrix to be partitioned along $i = i_0$ and $j = i_0$, we may note at once that all elements in the top right-hand and bottom left-hand corners, B and C, are zero, i.e. $C_{ij} = 0$ for $i < i_0$, $j \geqslant i_0$ and for $i \geqslant i_0$, $j < i_0$ (see sketch, which shows the combinations of values of i and j which characterise the remaining parts of the matrix)



The elements in the bottom right-hand corner D $(i \geqslant i_0, j \geqslant i_0)$ may be determined by consideration of the bending of the part of the beam ahead of the origin $(x \geqslant 0)$. From equations (31) and (33) we have, for x and $x' \geqslant 0$,

$$EI(x) \frac{d^2C}{dx^2} = 0 , \qquad (x > x') \tag{43}$$

and

$$EI(x) \frac{d^2C}{dx^2} = x' - x , \qquad (x < x') \quad . \tag{44}$$

Then the values of C_{ij} in the upper triangular part of region D $(i < j)$ may be determined from a double integration of equation (44), subject to the boundary conditions, equation (32), and the values in the lower triangular part $(i > j)$ may then be written down by invoking the condition that $[C]$ is symmetrical.

The elements in the top left-hand corner A $(i < i_0 , j < i_0)$ may be evaluated similarly by considering the bending of the part of the beam which is aft of the origin. Finally, the elements of the matrix $[G]$ for the unconstrained aircraft can be determined by transforming the integrals in equation (42) into summations, i.e.

$$G_{ij} = C_{ij} - \sum_{\ell=1}^{n} C_{i\ell} \left[\frac{1}{m} + \frac{x_j x_\ell}{I_y}\right] m_\ell - \sum_{\ell=1}^{n} C_{\ell j} \left[\frac{1}{m} + \frac{x_i x_\ell}{I_y}\right] m_\ell$$

$$+ \sum_{\ell=1}^{n} \sum_{k=1}^{n} C_{\ell k} \left[\frac{1}{m^2} + \frac{x_j x_k + x_i x_\ell}{mI_y} + \frac{x_i x_j x_\ell x_k}{I_y^2}\right] m_\ell m_k \quad . \tag{45}$$

3.2 Aerodynamic influence coefficients

The essence of the aerodynamic lifting problem is the relationship between the distribution of lifting pressure and the surface distribution of downwash velocity. The mathematical formulation of the problem leads to an integral equation for the lifting pressure. The most general numerical formulation that can be obtained from the aerodynamic integral equation leads to a matrix of aerodynamic influence coefficients that relate the lifting forces (local surface integrals of pressure) at a set of points to the downwash velocities at those points. Accordingly, if L_i is the lift force and h_i the surface deflection at the point x_i , there exists a relationship of the form

$$\{L\} = -[R]\{h\} \tag{46}$$

where, in general, R_{ij} is second-order in time, so that we may write

$$R_{ij} = R_{ij}^{(2)} \frac{d^2}{dt^2} + R_{ij}^{(1)} \frac{d}{dt} + R_{ij}^{(0)} \quad . \tag{47}$$

The aerodynamic influence coefficients which are considered in the present application are derived from two approximate aerodynamic theories: (1) slender-body theory and (2) piston theory, which were used (in their quasi-steady forms) by Taylor and Urich[10] in their investigation of static aero-elastic characteristics of the configuration to be considered here. Although these theories are not expected to give particularly accurate representations of the incremental aerodynamics due to deformation, they are considered to be adequate for an investigation which is directed primarily towards an assessment of the relative importance of the residual-flexibility concept.

A convenient basis for deriving the influence coefficients from slender-body theory is provided by Rodden and Revell[11], who cite Bisplinghoff, Ashley and Halfman (Ref.12, p.418) as the source of the relevant equations. We adapt the presentation of Ref.11 to our system of axes and nomenclature. Thus we relate the transverse displacement $h(x,t)$ of the body's longitudinal centre line to datum path axes; i.e. x is measured positively forward and h positively in the (downward) direction of Oz .

The lift force per unit length of the body, $\frac{dL}{dx}$, at station x is the reaction to the substantial rate of change of the downward z-component of momentum of the virtual mass per unit length of the body at x . The downwash velocity at x is

$$w(x,t) = \frac{\partial h(x,t)}{\partial t} - V_e \frac{\partial h(x,t)}{\partial x} \tag{48}$$

and the corresponding component of momentum of fluid per unit length is

$$\frac{dI}{dx} = \rho_e Sw = \rho_e S\left(\frac{\partial h}{\partial t} - V_e \frac{\partial h}{\partial x}\right) \quad , \tag{49}$$

where the effective cross-sectional area S is taken to be

$$S \equiv S(x) = \pi s^2(x) \quad , \tag{50}$$

$s(x)$ being the semi-span of the planform. Hence

$$\frac{dL}{dx} = \frac{D}{Dt}\left\{\frac{dI}{dx}\right\}$$

$$= \left(\frac{\partial}{\partial t} - V_e \frac{\partial}{\partial x}\right)\left\{\rho_e S\left(\frac{\partial h}{\partial t} - V_e \frac{\partial h}{\partial x}\right)\right\} \tag{51a}$$

i.e.

$$\frac{dL}{dx} = -\rho_e V_e \frac{\partial}{\partial x}\left\{S\left(-V_e \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\right)\right\} + \rho_e S \frac{\partial}{\partial t}\left(-V_e \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\right)$$

$$= -\pi\rho_e V_e \frac{\partial}{\partial x}\left\{s^2(x)\left(-V_e \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\right)\right\} + \pi\rho_e s^2(x) \frac{\partial}{\partial t}\left(-V_e \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\right) \tag{51b}$$

To obtain the lift on a specified length of body it is necessary to integrate equation (51b) over that length. The total length of the body is divided into n sections with mid-points coinciding with the mass points except in the case of the end sections. The ith section $(1 < i < n)$ thus has its mid-point at x_i and its length Δ_i is defined to be $(x_{i+1} - x_{i-1})/2$. Then, if the locations of its aft and forward extremities are denoted by $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ respectively,

$$\left.\begin{aligned} x_{i-\frac{1}{2}} &= x_i - \frac{\Delta_i}{2} = \frac{x_{i-1}}{4} + x_i - \frac{x_{i+1}}{4} \\ x_{i+\frac{1}{2}} &= x_i + \frac{\Delta_i}{2} = -\frac{x_{i-1}}{4} + x_i + \frac{x_{i+1}}{4} \end{aligned}\right\} \tag{52}$$

The semi-spans at these locations are denoted by $s_{i-\frac{1}{2}}$ and $s_{i+\frac{1}{2}}$ respectively. For $i = 1$ and $i = n$, only half-sections, extending from x_1 to $x_{1\frac{1}{2}}$ and from $x_{n-\frac{1}{2}}$ to x_n respectively are considered.

The lift, L_i , acting on the ith section at the point x_i is given by

$$L_i = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{dL}{dx}\, dx \tag{53a}$$

$$= \pi\rho_e V\left[V\left(s^2_{i+\frac{1}{2}}h'_{i+\frac{1}{2}} - s^2_{i-\frac{1}{2}}h'_{i-\frac{1}{2}}\right) - \left(s^2_{i+\frac{1}{2}}\dot{h}_{i+\frac{1}{2}} - s^2_{i-\frac{1}{2}}\dot{h}_{i-\frac{1}{2}}\right)\right]$$

$$- \pi\rho_e \frac{\partial}{\partial t} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} s^2(x)\,(V_e h' - \dot{h})\,dx \tag{53b}$$

where dashes and dots denote differentiation with respect to x and t respectively.

In the numerical application of equation (53b), evaluation of h and h' at $x_{i+\frac{1}{2}}$ and $x_{i-\frac{1}{2}}$ is effected by parabolic interpolation:

$$h = \frac{(x - x_i)(x - x_{i+1})}{(x_{i-1} - x_i)(x_{i-1} - x_{i+1})} h_{i-1}$$

$$+ \frac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})} h_i + \frac{(x - x_{i-1})(x - x_i)}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)} h_{i+1} \quad . \tag{54}$$

For evaluation of the integrals in equation (53b) only linear interpolation is used. The evaluation of the expression for L_i in terms of h_{i-1} , h_i and h_{i+1} , provides the values of the $(i-1)$th , ith and $(i + 1)$th elements in the ith row of each of the influence-coefficient matrices $\left[R^{(0)}\right]$, $\left[R^{(1)}\right]$ and $\left[R^{(2)}\right]$ when $1 < i < n$; the remaining elements in these rows are zero. The first and last rows in each case contain only two non-zero elements. Thus each of $\left[R^{(0)}\right]$, $\left[R^{(1)}\right]$, and $\left[R^{(2)}\right]$ is a band matrix with entries occurring in the leading diagonal and the two off-diagonals.

In piston theory it is postulated that under certain conditions the instantaneous pressure Δp at a point on the aerofoil is proportional to the instantaneous downwash w at the point. In the present context we may write

$$\Delta p(x,t) = \frac{2\rho_e V_e}{M} w(x,t) \tag{55}$$

where M is the Mach number; then the lift per unit length of the slender delta configuration is given by

$$\frac{dL}{dx} = \Delta p \times 2s(x)$$

$$= \frac{4\rho_e V_e}{M} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left(-V_e \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t}\right) \tag{56}$$

and the concentrated lift L_i at the ith point by

$$L_i = \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{dL}{dx}\, dx$$

$$= \frac{4\rho_e V_e}{M} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} s(x)(-V_e h' + \dot{h})dx \quad . \tag{57}$$

In this case the matrix of influence coefficients is such that $R_{ij}^{(2)} = 0$ for all i,j . A numerical procedure similar to that used before leads to band matrices for $\left[R^{(0)}\right]$ and $\left[R^{(1)}\right]$.

4 THE NUMERICAL MODEL

In a series of studies[13,14] around 1960, various calculations were performed for an integrated slender-delta shaped aircraft. In these studies the aircraft's mass was about 317 500 kg and its length 69.19 m. Subsequently Taylor and Urich[10] used a model which was basically similar but which had smaller dimensions appropriate to the size and shape of the integrated supersonic transport aircraft configurations which were being considered prior to the inception of the Concorde project. Both of these models differed from the Concorde inasmuch as they postulated an integrated all-wing configuration which could be considered infinitely stiff in the spanwise direction, while the

Concorde design featured a discrete fuselage, associated with a thin wing, whose spanwise flexibility could certainly not be ignored. However, the primary aim of the present investigation is to assess the relative importance of the residual-flexibility concept in aeroelastic investigations and, for this purpose, the integrated configuration provides a suitable model, free from too much structural complexity. Accordingly the model considered by Taylor and Urich is employed in the present analysis.

4.1 Presentation of results in non-dimensional form

For the presentation of various derived aircraft data and of the results of stability and response calculations, it will be convenient to introduce the non-dimensional systems of units advocated by Hopkin[15], which are summarised in Items 67001-67003 of Ref.16. The structural and aerodynamic influence coefficients will be expressed in the aero-normalised system while inertial data and stability and response quantities will be expressed in the dynamic-normalised system. In each of these systems the units of velocity and force are taken respectively equal to V_e and $\frac{1}{2}\rho_e V_e^2 S_w$, where V_e, ρ_e are the datum values of aircraft speed and air density respectively and S_w is a representative area, here taken to be the planform area. A basic triad of units is completed in the aero-normalised system by specifying a representative length ℓ_0 (here taken to be the overall length of the planform) as the unit of length, and in the dynamic-normalised system by taking the datum mass of the aircraft, m_e, as unit of mass. A parameter which is useful in the consequential definition of other units in the two systems is the relative density given by $\mu = m_e \big/ (\frac{1}{2}\rho_e S_w \ell_0)$.

We now introduce the auxiliary coordinate $\xi = x_N - x$, denoting distance aft of the nose and define the aero-normalised quantities $\check{\xi}$, $\check{h}$ and $\check{t}$ by

$$\left.\begin{aligned} \check{\xi} &= \frac{\xi}{\ell_0} = \frac{(x_N - x)}{\ell_0} \\ \check{h}(\check{\xi},\check{t}) &= \frac{h(x,t)}{\ell_0} \\ \check{t} &= \frac{tV_e}{\ell_0} \quad . \end{aligned}\right\} \qquad (58)$$

It will be noted that, in accordance with Ref.15, the aero-normalised coordinates are denoted by the overscript dressing $\breve{}$ (dip). Similarly, we shall denote dynamic-normalised quantities by the overscript dressing $\hat{}$ (cap). Further details of the normalised notation used in subsequent sections are given in Appendix A.

4.2 The basic characteristics of the aircraft

The selected aircraft is essentially a delta-like wing of overall length $\ell_0 = 51.206$ m and semi-span at the trailing-edge $s_T = 12.802$ m. In terms of the non-dimensional coordinate $\breve{\xi}$ defined by equation (58), the planform, which is illustrated in Fig.1, is defined by

$$\left|\frac{s(\breve{\xi})}{s_T}\right| = 0.68\breve{\xi} - 1.7\breve{\xi}^2 + 0.476\breve{\xi}^4 \; ; \qquad 0 \leqslant \breve{\xi} \leqslant 0.328 \tag{59}$$

$$\left|\frac{s(\breve{\xi})}{s_T}\right| = 1.1(1.26\breve{\xi} - 0.26) - 0.1(1.26\breve{\xi} - 0.26)^{11} \; ; \qquad 0.328 \leqslant \breve{\xi} \leqslant 1.0 \tag{60}$$

The planform area is $S_W = 592.35\ \mathrm{m}^2$. For the weight distribution it was considered appropriate to adopt the 'middle-of-the-cruise' condition defined in Ref.10. The relevant weight distribution is shown in Fig.2, the total weight being 1205.89 kN with a corresponding CG position of $\breve{\xi}_G = 0.6471$*. As no parametric study is attempted, only the one condition of flight is considered.

For the purposes of this investigation the aircraft is idealised as a beam of varying cross-section, which is infinitely stiff in the spanwise direction. Thus the only structural data which needs to be specified is the distribution of (longitudinal) bending rigidity $B(\breve{\xi}) = EI(\breve{\xi})$ along the representative beam. This is taken to be the 'basic' distribution defined in Ref.10 and is shown here in Fig.3. This curve is treated as a stepwise continuous function with the following values of the various parameters: $B_1/B_M = \frac{1}{3}$, $B_2/B_M = \frac{4}{15}$, $\breve{\xi}_1 = 0.4045$, $\breve{\xi}_M = 0.6824$ and $\breve{\xi}_2 = 0.8015$. The value of B_M is $7.748 \times 10^9\ \mathrm{N\ m}^2$.

In order to construct the aerodynamic influence-coefficient matrix the only remaining data which need to be specified are the design cruising conditions. The selected conditions are:-

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* These figures are consistent with the discrete-mass representation subsequently adopted (see Table 1).

Cruising altitude	:	19202 m
Corresponding air density ρ_e	:	0.0969 kg/m^3
Cruising speed V_e	:	649.10 m/s
Mach number M	:	2.2

4.3 Derived structural and aerodynamic quantities

4.3.1 The natural modes and the associated generalised masses and stiffnesses

Given an n-point discrete-element representation of the aircraft it is theoretically possible to calculate its lowest (n - 2) structural frequencies and mode shapes from the (n - 2) distinct eigenvalues and related eigenvectors of the eigenvalue equation (4). In practice it is difficult to achieve numerical accuracy of solutions corresponding to the higher-frequency modes so that usually only a limited number k are calculated. The value of n is arbitrary and the upper limit is determined almost certainly by the available store-size in a computer. In turn this limits the size of k . At the other extreme Huntley[14] considered a model in which n = 6 and calculated only the lowest (fundamental) structural mode. This model was perfectly acceptable for the problems which were studied. For our purposes there is no advantage to be gained by choosing n as large as possible, since we do not wish to model accurately a particular aircraft. However, we do need n sufficiently large to permit the accurate determination of an adequate number, k , of modes so that the models with and without residual flexibility may be seen to be converging to the same solution as k is increased. Two possible models were investigated initially, with n = 14 and n = 26 respectively. The features of residual flexibility appeared to be demonstrated with sufficient accuracy by the 14-point model, and therefore this is the one we choose to describe in this Report. In this model it is considered that only the first four natural frequencies and mode shapes are of sufficient accuracy.

The selected finite-element scheme is related to 14 points whose coordinates (aero-normalised) and associated masses (dynamic-normalised) are shown in Table 1. The continuous functions $s(\check{\xi})$ and $EI(\check{\xi})$ may be evaluated at the discrete points $\check{\xi}_i$ and it is then possible to construct the matrix of structural influence coefficients G , the elements of which may be calculated according to the procedure specified in section 3. Aero-normalised values are shown in Table 2. In turn, equation (5) may be used to obtain the matrix $[D]$ which is relevant to the eigenvalue problem.

In order to construct a numerical procedure for solving the eigenvalue equation, it is noted that only the first few largest eigenvalues (corresponding to the lowest natural frequencies) and the associated eigenvectors are required. As the eigenvalues are also real and widely spaced, the Power Method[17] provides the simplest analytical technique for the solution of the equation, which is performed with the aid of a standard scientific subroutine. The first four natural frequencies (dynamic-normalised) and mode shapes are specified in Table 3 and the mode shapes are also plotted in Fig.4.

From the calculated frequencies and mode shapes the diagonal matrices of generalised masses and stiffnesses, $\left[M_{0+1}\right],\left[K_{0+1}\right]$, have been calculated by applying equations (7) and (8) for $i,j = 1, 2, \ldots (k + 2)$, and remembering that the rigid-body mode shapes are given by equation (3) and that $\omega_1 = \omega_2 = 0$. Dynamic-normalised values of the diagonal elements of the two matrices are presented in Table 4.

4.3.2 The matrices of aerodynamic influence coefficients

A description of the method of calculating the matrices of aerodynamic influence coefficients is provided in section 3.2. For the planform defined by equations (59) and (60) the aero-normalised elements of the matrices (see Appendix A) have been evaluated and, for one of the two theories, namely slender-body theory, the elements are specified in Table 5. A side-investigation, described in Appendix B, indicates that the numerical scheme based on influence coefficients should provide tolerably accurate estimates of the aerodynamic loading due to deformation.

4.4 Magnitude of the residual-flexibility effects

The effect of residual flexibility is included in the equations of motion (16) through the non-dimensional matrix A which, by equation (17), is given by

$$[A]^{-1} = [I] - [E] \tag{61}$$

where

$$[E] = [R'][X] \quad . \tag{62}$$

If residual flexibility is neglected we have $[E] = [0]$. If all the elements of the matrix $[E]$ are small compared with unity, then, to a first approximation, we have

$$[A] = [I] + [E] \tag{63}$$

so that the values of the elements of the matrix $[E]$ determine the magnitude of the residual-flexibility effects. Two factors are involved in determining $[E]$, namely the steady aerodynamic influence-coefficient matrix, $[R']$, and the residual-flexibility matrix $[X]$. However, each acts in a different manner. The first factor, $[R']$, is independent of the number of structural modes that are retained in the model and, as far as determining the magnitude of $[E]$ is concerned, it acts as a constant for a given aerodynamic theory. The second factor, $[X]$, determines the variation in magnitude of $[E]$ as the number of structural modes retained in the model is varied. From equations (26) and (24), it is seen that, as the number of structural modes retained is varied from zero to $(n - 2)$, so $[X]$ varies from $[G]$ to $[0]$. Thus, an upper bound to the magnitude of $[E]$ is given by $[\bar{E}] = [R'][G]$.

With the results obtained from the numerical model so far, it is a simple matter to estimate the magnitudes of the elements of $[E]$. Firstly, the residual-flexibility matrix $[X]$ is calculated from equation (26), using the already calculated matrices $[G]$, (Table 2), $[K_1]$ (Table 4) and $[\Delta_1]$ (Table 3). This has been done for various situations, in which different numbers of structural modes have been retained in the analysis, and the aero-normalised results for two cases are presented in Table 7*. Finally, an upper bound to the magnitude of $[E]$ is obtained by calculating the elements of the matrix $[\bar{E}]$, selected elements of which are shown in Table 8. It is to be noted that these results are for the situation in which the aerodynamic influence coefficients are derived from slender-body theory. However, the use of influence coefficients based on piston theory would not alter the general order of magnitude of the elements of $[E]$. Table 8 also gives values of the corresponding elements of the aeroelastic correction matrix $[\bar{A}]$ derived from the approximate relationship $[\bar{A}] = [I] + [\bar{E}]$ (see equation (79)). $[\bar{A}]$ is the correction matrix appropriate to the quasi-static (no modes retained) solution.

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* Since one is interested primarily in the order of magnitude of $[E]$, results are given to only one significant figure.

5 STABILITY

In this section we examine the simplest of the problems which are governed mathematically by equation (16). This is the problem of stability of the aircraft when subjected to an infinitesimal disturbance, the relevant form of equation (16) being obtained by setting $\{F_D\} = \{0\}$. This gives a set of (k + 2) ordinary differential equations which are linear and second-order in the differential operator $D \equiv \frac{d}{dt}$. In the dynamic-normalised system they take the concise form

$$\left([\hat{a}]\hat{D}^2 + [\hat{b}]\hat{D} + [\hat{c}]\right)\{\hat{\xi}_{0+1}\} = 0 \tag{64}$$

where, from equation (A-18) of Appendix A, the $(k + 2) \times (k + 2)$ matrices $[\hat{a}]$, $[\hat{b}]$ and $[\hat{c}]$ are defined by

$$[\hat{a}] = [I] + \left[\hat{e}_M^{(2)}\right]' \tag{65}$$

$$[\hat{b}] = \left[\hat{e}_M^{(1)}\right]' \tag{66}$$

$$[\hat{c}] = [\hat{\omega}^2] + \left[\hat{e}_M^{(0)}\right]' \quad . \tag{67}$$

Equation (64) may be reduced to a set of first-order differential equations by setting

$$\hat{\xi}_{j+k+2} = \frac{d\hat{\xi}}{d\hat{t}}j , \qquad (j = 1,\ldots \overline{k+2}) \tag{68}$$

and then solutions are sought in the form

$$\{\hat{\xi}\} = \{\underline{\hat{\xi}}\}e^{\hat{\mu}\hat{t}} \tag{69}$$

where, in general, $\hat{\mu}$ may be complex. The presence of a positive real value for $\hat{\mu}$ will indicate a divergence, while the presence of a pair of conjugate complex values with positive real part will indicate an oscillatory instability. By substituting equations (68) and (69) into equation (64), we obtain the eigenvalue equation

$$\left([\hat{d}] - \hat{\mu}[I]\right)\{\hat{\underline{\zeta}}\} = 0 \qquad (70)$$

where the $2(k+2) \times 2(k+2)$ partitioned matrix $[\hat{d}]$ is defined by

$$[\hat{d}] = \begin{bmatrix} 0 & I \\ -[\hat{a}]^{-1}[\hat{c}] & -[\hat{a}]^{-1}[\hat{b}] \end{bmatrix} \qquad (71)$$

The matrix $[\hat{d}]$ possesses no special properties so that we are faced with the solution of the general algebraic eigenvalue problem, for which the QR algorithm has proved to be the most effective of known methods of solution (see Wilkinson's[17] 'The algebraic eigenvalue problem'). The matrix is first transformed into an upper Heisenberg form, i.e. a matrix of an upper triangular form with one extra sloping line below the leading diagonal. In the present application the calculation of the eigenvalues has been effected by use of two standard scientific subroutines.

The eigenvalues have been calculated for the various cases in which $k = 0,1,2,3$ and 4, and for the two situations in which residual flexibility is respectively included and neglected. Dynamic-normalised results obtained by using the two different aerodynamic theories in turn are shown in Table 9. In general the order of the matrix $[\hat{d}]$ is $2(k+2)$, which leads to $(k+2)$ pairs of complex conjugate eigenvalues. In fact there are only $(k+1)$ non-zero pairs, corresponding to the rigid-body short-period mode and the lowest k structural modes. The remaining two eigenvalues are zero in virtue of the fact that, in the whole-body sense, the aircraft is neutrally stable with respect to displacements in the coordinates ζ_1 and ζ_2. In the numerical-model evaluation these eigenvalues actually occur as very small non-zero values.

If the results for the two aerodynamic theories, given in Table 9, are compared, little agreement is seen to exist. This is not unexpected since both theories represent a very rough approximation to the true aerodynamics. However, there is a measure of agreement with the results obtained by Huntley[14] who used piston theory in his calculation of the stability of an aircraft having a delta planform roughly similar to, but larger than that of the aircraft considered here. An indication of the effects of residual flexibility is obtained if we examine the results for each aerodynamic theory separately. These show clearly certain main features. Firstly, the overall effects of aeroelasticity in this example are quite small. Secondly, the two sets of solutions for a given aerodynamic theory converge towards the same set of values, as the number of structural modes retained in the basic model is increased. (This common set of limiting values may effectively be considered as the 'exact' solution.) The convergence is achieved very rapidly in the case where residual flexibility is included. For example, if we consider the piston-theory results, then as far as the rigid-body short-period mode is concerned, the quasi-static solution $(k = 0)$ already gives a reasonable approximation $(\hat{\mu}_0 = -1.7309 \pm 17.361i)$ to the 'exact' solution $(\hat{\mu}_0 = -1.7133 \pm 17.303i)$ and convergence to the 'exact' values is, for practical purposes, complete when $k = 1$ $(\hat{\mu}_0 = -1.7146 \pm 17.304i)$. Similarly, a fair approximation to the pair of eigenvalues corresponding to the first structural mode is given by the $k = 1$ solution, $(\hat{\mu}_1 = -1.2137 \pm 107.61i)$ with convergence to the 'exact' solution $(\hat{\mu}_1 = -1.1956 \pm 107.50i)$ being virtually complete at $k = 2$ $(\hat{\mu}_1 = -1.1962 \pm 107.50i)$. When residual flexibility is not included, it is necessary to increase k by 1 or 2 to achieve the same degree of convergence as in the case where it is included. However, in an example like the present, where overall effects of aeroelasticity are fairly small, it appears likely that a generally acceptable approximation to the rth pair of eigenvalues (which corresponds to the $(r - 1)$th structural mode*) will be provided by a model which incorporates the first r structural modes, with or without the residual flexibility of the neglected modes.

6 RESPONSE TO A SINUSOIDAL GUST DISTURBANCE

In this section we study another simple problem, namely the response of the aircraft to a sinusoidal gust disturbance, and examine the effect of residual flexibility on the calculated accelerations at various stations along the length of the aircraft. If we consider the gust to be represented by a simple standing wave, of spatial frequency k rad/unit length, then the gust velocity, w_g, experienced by the aircraft travelling at a speed V_e, is given by

$$w_g = \underline{w}_g \sin (k\xi - \omega t) \tag{72}$$

where $\omega = kV_e$ is the temporal frequency in rad/unit time. The local incidence due to the gust is

* The first pair of eigenvalues corresponds to the rigid-body short-period mode.

$$\alpha_g = \frac{w_g}{V_e} \quad . \tag{73}$$

In this instance we consider only piston-theory aerodynamics, according to which the lift per unit length along the aircraft is

$$\begin{aligned}\frac{dL(\xi,t)}{d\xi} &= \tfrac{1}{2}\rho_e V_e^2 \frac{4}{M} \alpha_g 2s(\xi) \\ &= \frac{4\rho_e V_e s(\xi)}{M} \underline{w}_g \sin(k\xi - \omega t) \quad .\end{aligned} \tag{74}$$

Thus the lift on the ith element (extending from $\xi_{i+\frac{1}{2}}$ to $\xi_{i-\frac{1}{2}}$) is

$$L_i = \frac{4\rho_e V_e \underline{w}_g}{M} \int_{\xi_{i+\frac{1}{2}}}^{\xi_{i-\frac{1}{2}}} s(\xi) \sin(k\xi - \omega t) d\xi \quad , \tag{75}$$

This may be expressed in aero-normalised form as

$$\breve{L}_i = \frac{L_i}{\frac{1}{2}\rho_e V_e^2 S_w} = \frac{2A}{M}\left(\frac{\ell_0}{s_T}\right) \breve{\underline{w}}_g \left[p_i(\breve{\omega}) \cos \breve{\omega}\breve{t} + q_i(\breve{\omega}) \sin \breve{\omega}\breve{t}\right] \tag{76}$$

where

$$p_i(\breve{\omega}) = \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \frac{s(\breve{\xi})}{s_T} \sin \breve{\omega}\breve{\xi} d\breve{\xi} \tag{77}$$

$$q_i(\breve{\omega}) = - \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \frac{s(\breve{\xi})}{s_T} \cos \breve{\omega}\breve{\xi} d\breve{\xi} \tag{78}$$

and A is the aspect ratio of the planform, while

$$\breve{\underline{w}}_g = \frac{\underline{w}_g}{V_e} ; \quad \breve{\omega} = \frac{\omega \ell_0}{V_e} . \tag{79}$$

To simplify the integration procedure in equations (77) and (78) for the numerical work, the actual planform given by equations (59) and (60) was approximated by a triangular shape, defined by

$$\left| \frac{s(\breve{\xi})}{\ell_0} \right| = 0.25\breve{\xi} \quad , \tag{80}$$

which has an area $S_w' = 655.52 \text{ m}^2$ and an aspect ratio $A' = 1.0$. With the substitution of equation (80), the integrals in equations (77) and (78) are readily performed and, by use of equations in $\breve{\xi}$ corresponding to equation (52) in x , $p_i(\breve{\omega})$ and $q_i(\breve{\omega})$ can be expressed in terms of $\breve{\xi}_{i-1}$, $\breve{\xi}_i$ and $\breve{\xi}_{i+1}$.

The response problem is governed by a set of $(k + 2)$ ordinary differential equations which, from equation (A-18) of Appendix A, take the concise dynamic-normalised form

$$\left([\hat{a}]\hat{D}^2 + [\hat{b}]\hat{D} + [\hat{c}] \right) \{ \hat{\zeta}_{0+1} \} + [\hat{e}_D]' = 0 \tag{81}$$

where $[\hat{a}]$, $[\hat{b}]$ and $[\hat{c}]$ are defined by equations (65)-(67) and

$$[\hat{e}_D]' = \left(\frac{2A}{M} \left(\frac{\ell_0}{s_T} \right) \breve{\underline{w}}_g \right) \left[\frac{1}{\hat{M}} \right] \left[\Delta_{0+1} \right]^T [A] \left\{ \{p\} \cos \hat{\omega}\hat{t} + \{\hat{q}\} \sin \hat{\omega}\hat{t} \right\} \tag{82}$$

by use of equations (A-17), (A-15) and (76), it being noted that $\{ \breve{F}_D \} = - \{ \breve{L} \}$, $\breve{\omega}\breve{t} = \hat{\omega}\hat{t}$, and $\breve{\underline{w}}_g = \hat{\underline{w}}_g$.

The solution of equation (81) may be expressed in the form

$$\hat{\zeta}_i = \left(\frac{2A}{M} \left(\frac{\ell_0}{s_T} \right) \breve{\underline{w}}_g \right) \left(B_i \cos \hat{\omega}\hat{t} + C_i \sin \hat{\omega}\hat{t} \right) \tag{83}$$

where B_i and C_i may be determined from the set of $2(k + 2)$ equations described by

$$\begin{bmatrix} [\hat{c}] - \hat{\omega}^2[\hat{a}] & \hat{\omega}[\hat{b}] \\ - \hat{\omega}[\hat{b}] & [\hat{c}] - \hat{\omega}^2[\hat{a}] \end{bmatrix} \begin{Bmatrix} B \\ C \end{Bmatrix} = \begin{Bmatrix} - \left[\frac{1}{M}\right]\left[\Delta_{0+1}\right]^T [A]\{p\} \\ - \left[\frac{1}{M}\right]\left[\Delta_{0+1}\right]^T [A]\{q\} \end{Bmatrix} . \tag{84}$$

For a particular value of $\hat{\omega}$, the solution for $\begin{Bmatrix} B \\ C \end{Bmatrix}$ can be obtained directly by using a standard scientific subroutine, which is based on a Gaussian elimination method.

The displacements $\{\bar{\delta}\}$ and accelerations $\{f\}$ of the mass-points normal to the datum flight path are related by $\{f\} = \{\ddot{\bar{\delta}}\}$, and the displacements may be expressed as

$$\{\bar{\delta}\} = [\Delta_{0+1}]\{\zeta_{0+1}\} + [\Delta_2]\{\zeta_2\} . \tag{85}$$

The column vector $\{\zeta_2\}$ is determined from equation (15) which, with the approximations used in developing the residual-flexibility concept, may be written as

$$[K_2]\{\zeta_2\} = \left[[\Delta_2]^T[R'][\Delta_{0+1}]\right]\{\zeta_{0+1}\} + \left[[\Delta_2]^T[R'][\Delta_2]\right]\{\zeta_2\} + [\Delta_2]^T\{F_D\} . \tag{86}$$

Premultiplication of equation (86) by $[\Delta_2][K_2]^{-1}$ and use of equation (18) leads to

$$\left[[I] - [X][R']\right][\Delta_2]\{\zeta_2\} = [X][R'][\Delta_{0+1}]\{\zeta_{0+1}\} + [X]\{F_D\} . \tag{87}$$

On substituting for $[\Delta_2]\{\zeta_2\}$ in equation (85) we obtain, in terms of normalised quantities previously introduced,

$$\{\hat{\bar{\delta}}\} = \frac{\{\bar{\delta}\}}{\mu \ell_0} = [A']\left[[\Delta_{0+1}]\hat{\zeta}_{0+1} - \frac{1}{\mu}[\ddot{X}]\{\bar{L}\}\right] \tag{88}$$

where

$$[A']^{-1} = [I] - [X][R'] . \tag{89}$$

Finally, the displacement and acceleration at the ith point may be expressed in the form

$$\frac{\hat{f}_i}{\hat{\underline{w}}_g} = -\hat{\omega}^2 \frac{\hat{\bar{\delta}}_i}{\hat{\underline{w}}_g} = -\hat{\omega}^2\hat{\bar{\delta}}_{i_0}(\hat{\omega}) \sin\left\{\hat{\omega}\hat{t} + \varepsilon_i(\hat{\omega})\right\} \tag{90}$$

where

$$\hat{\bar{\delta}}_{i_0}(\hat{\omega}) = -\left(\frac{24}{M}\frac{\ell_0}{s_T}\right)\left(P_i'^2 + Q_i'^2\right)^{\frac{1}{2}} \tag{91}$$

and

$$\varepsilon_i(\hat{\omega}) = \tan^{-1}\left(\frac{P_i'}{Q_i'}\right) \tag{92}$$

while, through the use of (76) and (83), $\{P'\}$ and $\{Q'\}$ are defined by

$$\left.\begin{aligned} \{P'\} &= [A']\left[[\Delta_{0+1}]\{B\} - \frac{1}{\mu}[\check{X}]\{p\}\right] \\ \{Q'\} &= [A']\left[[\Delta_{0+1}]\{C\} - \frac{1}{\mu}[\check{X}]\{q\}\right] \end{aligned}\right\} \tag{93}$$

If residual flexibility is neglected, $[A'] = [I]$ and $[\check{X}] = [0]$. Then, in equations (91) and (92), P_i' and Q_i' are replaced by P_i and Q_i where

$$\left.\begin{aligned} \{P\} &= [\Delta_{0+1}]\{B\} \\ \{Q\} &= [\Delta_{0+1}]\{C\} \end{aligned}\right\} \tag{94}$$

The vectors $\{\hat{\bar{\delta}}_0\}$ and $\{\varepsilon\}$ have been evaluated for ranges of values of $\hat{\omega}$ centred about the frequencies of the aircraft short-period mode and the first four structural modes. The calculations were performed for a number of cases both with and without allowance for residual flexibility. A selection of these results has been used in preparing Figs.5-14, which illustrate the relative accuracies of the computed responses derived by use of the various structural models.

Pairs of figures: 5 and 6, 7 and 8, and 9 and 10 show the amplitudes and phase angles of the acceleration response at Stations 14, 7 and 1 for gust excitation frequencies in the vicinity of the aircraft's short-period frequency and of the first structural-mode frequency. These stations are at the nose, near the CG and at the tail-end of the aircraft. The amplitude of the response per unit amplitude of gust velocity is expressed in dynamic-normalised form $\hat{\underline{f}}_G / \hat{\underline{w}}_g$ where $\hat{\underline{f}}_G$ denotes the dynamic-normalised amplitude of acceleration expressed in 'g'-units, rather than in the basic unit of acceleration in the dynamic-normalised system; i.e. $\hat{\underline{f}}_G = \hat{\underline{f}}/\hat{g}$, where $\hat{g} = \left(m_e / \tfrac{1}{2}\rho_e V_e^2 S\right) g = 0.09974$, (in UK or SI system). Now

$$\frac{\hat{\underline{f}}_G}{\hat{\underline{w}}_g} = \frac{\hat{\underline{f}}/\hat{g}}{\hat{\underline{w}}_g} = \frac{\underline{f}/g}{\underline{w}_g/V_e} = V_e \frac{\underline{f}/g}{\underline{w}_g} .$$

Thus the amplitude of acceleration in 'g'-units, per unit amplitude of gust velocity may be obtained, in a system of 'ordinary' units, from the plotted results, by use of the relationship

$$\frac{(\underline{f}/g)}{\underline{w}_g} = \frac{1}{V_e}\left(\frac{\hat{\underline{f}}_G}{\hat{\underline{w}}_g}\right) .$$

In examining the figures it may be considered that the curves for '4 modes retained, with residual flexibility' represent the 'exact' solution for the 14-element model, to engineering accuracy. The '0 modes retained, without residual flexibility' curves correspond to the 'rigid-aircraft' solution, while the '0 modes retained, with residual flexibility' curves represent the 'quasi-static' solution for the flexible aircraft.

From the left-hand halves of Figs.5-10 it is clear that the response of the actual (flexible) aircraft to sinusoidal gusts in the frequency-range of the aircraft short-period mode is very inaccuractely predicted by a 'rigid-aircraft' calculation. The quasi-static solutions and the solutions based on the retention of a single mode, without residual flexibility, give tolerably good approximations to the 'exact' solution, the errors in the two cases being of roughly the same magnitude, albeit of opposite sign. The addition of residual flexibility to the '1 mode retained' model provides a solution which approximates the 'exact' solution with engineering accuracy. The computer results show a progressive increase in the accuracy of the calculated response as more modes are incorporated in the structural model (without or with residual flexibility) but for practical purposes the (1 mode + residual flexibility) model or the (2 modes without residual flexibility) model provides an acceptable solution.

From the right-hand halves of Figs.5-10 it is seen that, in order to predict response at frequencies near that of the first structural mode, it is, of course, essential to retain that mode in the structural model. Moreover, if residual flexibility is included, the retention of this one mode is sufficient to ensure a solution that approximates the 'exact' solution with engineering accuracy. The computer results show that a solution of comparable accuracy is obtained from the (2 modes without residual flexibility) model and that the addition of residual flexibility and/or further modes then results in refinements of accuracy that are of academic interest only.

The computer results for frequencies near those of the second and higher structural modes suggest that acceptable results are provided by models which incorporate all the modes with frequencies up to and including that at which the response is required, with or without residual flexibility. Typical results, supporting this conclusion, are illustrated in Fig.11, which shows the amplitude of the acceleration response at Stations 1 and 14 to sinusoidal gusts of frequencies near that of the second structural mode. (Station 7 lies nearly at a node for this mode and response there is consequently negligible.) It will be noted that solutions for the (2 modes without residual flexibility) and (4 modes with residual flexibility) models are practically identical.

Similar conclusions as to the adequacy of the various structural models are suggested by consideration of Figs.12-14, which compare displacements of the aircraft centre-line from the datum flight path, as calculated for selected models, responding at frequencies close to the resonance frequencies indicated by Figs.5-11. As indicated on the figures, the displacements shown are those occurring at the instant when the displacement at Station 14 (nose) is at its maximum. Because the phase-angle of the response varies along the aircraft, the displacements at stations other than the nose are somewhat below their maxima.

7 CONCLUDING SUMMARY AND DISCUSSION

A simple fourteen-element idealisation of an aircraft of slender-delta configuration has been used as the basis of a numerical investigation of the validity and usefulness of the residual-flexibility concept in the analysis of the dynamical behaviour of deformable aircraft. The only form of elastic deformation admitted by this model is longitudinal bending and, in assessing the incremental aerodynamic effects of such bending, two simple but very approximate theories - slender-body theory and linearised piston theory have been employed. Thus no great reliance can be placed on the *absolute* values of any of the calculated quantities. However, the main object of the investigation has been to assess the *relative* accuracies of various truncated modal analyses in which only a limited number (up to four) of the natural modes of the basic model have been used, in conjunction with 'rigid-body' modes, to specify the displacement of the aircraft from its datum-flight-path configuration. For this purpose, the accuracy of the aerodynamic theory employed is relatively unimportant.

The effect of residual flexibility has been examined in the context of stability and response calculations which were performed for structural models incorporating 0,1,2,3 or 4 structural modes, with or without an allowance for the residual flexibility of the neglected modes. The 'zero modes-retained' cases, without and with residual flexibility, respectively provide the 'rigid-aircraft' and 'quasi-static' aeroelastic solutions. For the models incorporating the rigid-body mode and k elastic modes $(k = 0,1,...4)$ the stability calculations provided $(k + 1)$ pairs of complex eigenvalues, representing approximations to the lowest $(k + 1)$ eigenvalue-pairs of the basic (14-element) model. The corresponding response calculations yielded displacement- and acceleration-responses to harmonic gusts of frequencies up to the vicinity of the kth natural structural frequency. In general, results for the two series of models (i.e. without and with residual flexibility) exhibit a pattern of convergence which justifies the following conclusions:-

(1) For practical purposes the model incorporating four structural modes, with residual flexibility, may be considered to give 'exact' solutions for the stability and response characteristics of the basic (14-element) model in a frequency range extending up to the fourth natural frequency.

(2) Results for both series of models converge quite rapidly to the 'exact' values. However, for a given value of k, the model with residual flexibility provides the better approximation. It is roughly true that the addition of residual flexibility to a model without residual flexibility effects an improvement of the same magnitude as would the addition of another mode.

(3) At the lower end of the frequency range the residual-flexibility effect is quite significant, and its inclusion in the '0-modes retained' model leads to the well-known 'quasi-static' solution for the aircraft's short-period characteristics, whereby the considerable errors involved in the 'rigid-aircraft' solution are largely eliminated. Addition of residual flexibility to the '1-mode retained' model converts solutions of tolerable accuracy into ones which, for engineering purposes, are virtually exact.

(4) As one progresses up the frequency range and necessarily incorporates more modes in the model, the effect of including residual flexibility, though seen to be consistently favourable, diminishes quantitatively. Thus, in the present example, there is little to be gained practically by adding residual flexibility to a model which incorporates more than two modes.

(5) Summing up the results of this limited numerical study we may conclude that the concept of residual flexibility is sound in principle and that, as applied here, it leads to results that are consistently meaningful in a mathematical sense. Moreover, when the low-frequency characteristics of an aircraft are being investigated by means of a structural model which incorporates very few modes dynamically, the inclusion of residual flexibility may result in an increase in accuracy which is of engineering significance. However, in the case of the models incorporating more modes, which are necessary to evaluate the higher-frequency characteristics, the small increase in accuracy resulting from the inclusion of residual flexibility is of academic interest only. In the context of the so-called integrated approach to aeroelastic problems, wherein a structural model incorporating a fairly large number of modes must be used in order to deal with the higher-frequency problems, the residual-flexibility concept would thus seem to have little practical value.

Appendix A

THE AERO-NORMALISED AND DYNAMIC-NORMALISED SYSTEMS OF UNITS

A.1 Units for fundamental quantities

As mentioned in section 4.1 of the main text, it was considered convenient to present numerical results of this investigation in terms of the aero-normalised and dynamic-normalised systems of units introduced by Hopkin[15]. The following table gives expressions for the units of five fundamental quantities - force, speed, mass, length and time - in the two systems. Evaluation of these expressions for values of V_e, ρ_e, S_w, ℓ_0, m_e in a given system of ordinary units provides the normalising divisors for converting quantities in that system to corresponding quantities in the appropriate non-dimensional system. Table A1 below shows the values of these divisors in UK ordinary units (lb,ft,second) and in SI units (kg, metre, second).

Table A1

Quantity	Aero-normalised system			Dynamic-normalised system		
	Unit	Divisor		Unit	Divisor	
		UK units	SI units		UK units	SI units
Force	$\frac{1}{2}\rho_e V_e^2 S_w$	0.27181×10^7 lbf	0.12091×10^8 newtons	$\frac{1}{2}\rho_e V_e^2 S_w$	0.27181×10^7 lbf	0.12091×10^8 newtons
Speed	V_e	0.21296×10^4 ft/s	0.64910×10^3 m/s	V_e	0.21296×10^4 ft/s	0.64910×10^3 m/s
Mass	m_e/μ $(= \frac{1}{2}\rho_e S_w \ell_0)$	0.10069×10^3 slugs	0.14695×10^4 kg	m_e	0.84259×10^4 slugs	0.12297×10^6 kg
Length	ℓ_0	0.168×10^3 ft	0.51206×10^2 m	$\mu\ell_0$	0.14059×10^5 ft	0.42850×10^4 m
Time	$\tau_A = \ell_0/V_e$	0.78888×10^{-1} s	0.78888×10^{-1} s	$\tau_D = \mu\ell_0/V_e$	0.66015×10 s	0.66015×10 s

$\mu = m_e/(\frac{1}{2}\rho_e S_w \ell_0) = 0.836818 \times 10^2$; $g = 0.32174 \times 10^2$ ft/s^2 $= 0.98067 \times 10$ m/s^2

A.2 Normalised forms of some other quantities

The aerodynamic influence coefficient R_{ij} and the related coefficients $R_{ij}^{(2)}$, $R_{ij}^{(1)}$ and $R_{ij}^{(0)}$ are defined by equations (46) and (47) of the main text which, with the substitution of equation (B-1) of Appendix B, may be written in the aero-normalised forms

$$\{\breve{L}\} = -[\breve{R}]\{\breve{\underline{h}}\}e^{\breve{\lambda}\breve{t}} \tag{A-1}$$

and

$$\breve{R}_{ij} = \breve{R}_{ij}^{(2)}\breve{\lambda}^2 + \breve{R}_{ij}^{(1)}\breve{\lambda} + \breve{R}_{ij}^{(0)} \quad . \tag{A-2}$$

Basically, R_{ij} is a force per unit deflection (length) and is thus aero-normalised by means of the divisor $\left(\frac{1}{2}\rho_e V_e^2 S_w/\ell_0\right)$. Then it is evident from equation (A-2) and equation (B-3) that the aero-normalising divisors for $R_{ij}^{(0)}$, $R_{ij}^{(1)}$ and $R_{ij}^{(2)}$ are $\left(\frac{1}{2}\rho_e V_e^2 S_w/\ell_0\right)$, $\left(\frac{1}{2}\rho_e V_e S_w\right)$ and $\frac{1}{2}\rho_e S_w \ell_0$ respectively. The following table gives values of these divisors appropriate to the UK and SI systems of ordinary units

Table A2

Aero-normalised quantity	Relationship with ordinary quantity	Value of normalising divisor in:	
		UK units	SI units
$\breve{R}_{ij}^{(0)}$	$R_{ij}^{(0)}\Big/\left(\frac{1}{2}\rho_e V_e^2 S_w/\ell_0\right)$	0.16179×10^5 lbf/ft	0.23612×10^6 newton/m
$\breve{R}_{ij}^{(1)}$	$R_{ij}^{(1)}\Big/\left(\frac{1}{2}\rho_e V_e S_w\right)$	0.12763×10^4 lbf s/ft	0.18627×10^5 newton s/m
$\breve{R}_{ij}^{(2)}$	$R_{ij}^{(2)}\Big/\left(\frac{1}{2}\rho_e S_w \ell_0\right)$	0.10068×10^3 lbf s^2/ft	0.14694×10^4 newton s^2/m

By use of equation (B-2), equation (53b), which relates to slender-body theory, may be expressed in the aero-normalised form

$$\breve{L}_i = \frac{L_i}{\frac{1}{2}\rho V_e^2 S_w} = e^{\breve{\lambda}\breve{t}}\,\frac{\pi A}{2}\left[\left\{\left(\frac{s_{i-\frac{1}{2}}}{s_T}\right)^2 \breve{\underline{h}}'_{i-\frac{1}{2}} - \left(\frac{s_{i+\frac{1}{2}}}{s_T}\right)^2 \breve{\underline{h}}'_{i+\frac{1}{2}}\right\} + \right.$$
$$+ \breve{\lambda}\left\{\left(\frac{s_{i-\frac{1}{2}}}{s_T}\right)^2 \breve{\underline{h}}_{i-\frac{1}{2}} - \left(\frac{s_{i+\frac{1}{2}}}{s_T}\right)^2 \breve{\underline{h}}_{i+\frac{1}{2}} + \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \left(\frac{s(\breve{\xi})}{s_T}\right)^2 \breve{\underline{h}}'d\breve{\xi}\right\}$$
$$\left. + \breve{\lambda}^2 \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \left(\frac{s(\breve{\xi})}{s_T}\right)^2 \breve{\underline{h}}d\breve{\xi}\right] \tag{A-3}$$

where S_w , A are the planform area and aspect ratio, which are related to the trailing-edge semi-span, s_T , by

$$S_w = \frac{4s_T^2}{A} \ . \tag{A-4}$$

Then the elements of $\left[\breve{R}^{(0)}\right]$, $\left[\breve{R}^{(1)}\right]$ and $\left[\breve{R}^{(2)}\right]$ may be determined by considering, in turn, the expressions

$$-\breve{L}_i^{(0)} e^{-\breve{\lambda}\breve{t}} = -\frac{\pi A}{2}\left\{\left(\frac{s_{i-\frac{1}{2}}}{s_T}\right)^2 \underline{\breve{h}}'_{i-\frac{1}{2}} - \left(\frac{s_{i+\frac{1}{2}}}{s_T}\right)^2 \underline{\breve{h}}'_{i+\frac{1}{2}}\right\} \tag{A-5}$$

$$-\breve{L}_i^{(1)} e^{-\breve{\lambda}\breve{t}} = -\frac{\pi A}{2}\left\{\left(\frac{s_{i-\frac{1}{2}}}{s_T}\right)^2 \underline{\breve{h}}_{i-\frac{1}{2}} - \left(\frac{s_{i+\frac{1}{2}}}{s_T}\right)^2 \underline{\breve{h}}_{i+\frac{1}{2}} + \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \left(\frac{s(\breve{\xi})}{s_T}\right)^2 \underline{\breve{h}}' d\breve{\xi}\right\} \tag{A-6}$$

$$-\breve{L}_i^{(2)} e^{-\breve{\lambda}\breve{t}} = -\frac{\pi A}{2} \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \left(\frac{s(\breve{\xi})}{s_T}\right) \underline{\breve{h}} d\breve{\xi} \tag{A-7}$$

and using the aero-normalised form of the parabolic interpolation formula, equation (54), (i.e. with $\underline{\breve{h}}$ in place of h and $\breve{\xi}$ in place of x , etc.).

In similar fashion, equation (57), for the piston-theory case may be expressed in aero-normalised form:

$$\breve{L}_i = \frac{L_i}{\frac{1}{2}\rho V_e^2 S_w} = e^{\breve{\lambda}\breve{t}} \frac{2A}{M} \frac{\ell_0}{s_T} \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \frac{s(\breve{\xi})}{s_T} (\underline{\breve{h}}' + \breve{\lambda}\underline{\breve{h}}) d\breve{\xi} \tag{A-8}$$

and the matrices $\left[\breve{R}^{(0)}\right]$ and $\left[\breve{R}^{(1)}\right]$ may be derived from consideration of the expressions

$$-\breve{L}_i^{(0)} e^{-\breve{\lambda}\breve{t}} = -\frac{2A}{M} \frac{\ell_0}{s_T} \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \frac{s(\breve{\xi})}{s_T} \underline{\breve{h}}' d\breve{\xi} \quad , \tag{A-9}$$

and

$$- \breve{L}_i^{(1)} e^{-\breve{\lambda}\breve{t}} = - \frac{2A}{M} \frac{\ell_0}{s_T} \int_{\breve{\xi}_{i+\frac{1}{2}}}^{\breve{\xi}_{i-\frac{1}{2}}} \frac{s(\breve{\xi})}{s_T} \underline{\breve{h}} d\breve{\xi} \quad , \tag{A-10}$$

respectively. (The matrix $\left[\breve{R}^{(2)}\right]$ is null in this case.) The structural flexibility influence coefficient G_{ij} is a deflection (length) per unit force and is thus aero-normalised by means of the divisor $\left(\ell_0 \big/ \frac{1}{2}\rho_e V_e^2 S_w\right)$, i.e.

$$\breve{G}_{ij} = G_{ij} \times \left(\tfrac{1}{2}\rho_e V_e^2 S_w \big/ \ell_0\right)$$

where the values of the multiplier $\left(\frac{1}{2}\rho_e V_e^2 S_w \big/ \ell_0\right)$ in UK and SI units are equal to the respective values of the corresponding normalising divisors for $\breve{R}_{ij}^{(0)}$ in Table A2. Elements of the residual-flexibility matrix $[X]$ have the same dimensions as G_{ij}. Hence

$$\breve{X} = X \times \left(\tfrac{1}{2}\rho_e V_e^2 S_w \big/ \ell_0\right) \quad .$$

As the generalised masses and stiffnesses of the aircraft modes (M_j, K_j) are defined for use in the dynamical equations they are best normalised with respect to the dynamic-normalised system of units. From equation (7) of the main text, since the mode shape $\{\Delta\}$ is non-dimensional, it is clear that M_j has the dimensions of mass and hence should be normalised by dividing by m_e, i.e.

$$\hat{M}_j = \frac{M_j}{m_e}$$

where the values of m_e appropriate to the UK and SI systems are given in Table A1. From equation (8) it is seen that K_j has the dimensions Mass/Time^2 and hence the appropriate normalising divisor is $m_e \big/ \tau_D^2$, i.e.

$$\hat{K}_j = \frac{K_j}{m_e \tau_D^{-2}}$$

where the divisor $m_e\tau_D^{-2}$ assumes the values:-

$$0.19334 \times 10^3 \text{ slugs/s}^2 \quad \text{in UK units}$$

$$0.28217 \times 10^4 \text{ kg/s}^2 \quad \text{in SI units.}$$

The frequency of a structural mode, ω_j , has the dimensions of $(\text{time})^{-1}$ so that the appropriate normalising divisor is τ_D^{-1} , i.e.

$$\hat{\omega} = \frac{\omega}{\tau_D^{-1}}$$

where the divisor τ_D^{-1} has the value

$$0.15148 \text{ s}^{-1} \quad \text{in both UK and SI units.}$$

A.3 Normalised form of the equations of motion

The non-dimensional form of the equations of motion recommended in Refs.15 and 16 is the concise dynamic-normalised form. To obtain equations in this form we start from the expanded aero-normalised form of equation (16)

$$\left[\breve{M}(\breve{D}^2 + \breve{\omega}^2)\right]\{\breve{\zeta}_{0+1}\} = [\Delta_{0+1}]^T[A]\left[\breve{R}^{(2)}\breve{D}^2 + \breve{R}^{(1)}\breve{D} + \breve{R}^{(0)}\right][\Delta_{0+1}]\{\breve{\zeta}_{0+1}\}$$
$$+ [\Delta_{0+1}]^T[A]\{\breve{F}_D\} \quad , \tag{A-11}$$

and write

$$\left[\breve{E}_M^{(0)}\right] = [\Delta_{0+1}]^T[\breve{R}^{(0)}][\Delta_{0+1}] \tag{A-12}$$

and

$$\left[\breve{E}_M^{(0)}\right]' = [\Delta_{0+1}]^T[A][\breve{R}^{(0)}][\Delta_{0+1}] \quad . \tag{A-13}$$

Similarly, write $\left[\breve{E}_M^{(1)}\right]$ and $\left[\breve{E}_M^{(1)}\right]'$ for the corresponding expressions with $\breve{R}_{ij}^{(0)}$ replaced by $\breve{R}_{ij}^{(1)}$, and $\left[\breve{E}_M^{(2)}\right]$, $\left[\breve{E}_M^{(2)}\right]'$ for those with $\breve{R}_{ij}^{(0)}$ replaced by $\breve{R}_{ij}^{(2)}$. Also, write

$$[\breve{E}_D] = [\Delta_{0+1}]^T\{\breve{F}_D\} \tag{A-14}$$

and

$$[\breve{E}_D]' = [\Delta_{0+1}]^T[A]\{\breve{F}_D\} \quad . \tag{A-15}$$

(The primes here denote 'modified' values of the quantities concerned, which incorporate the effect of residual flexibility, through the matrix $[A]$.) Then equation (A-11) becomes

$$\left[\breve{M}(\breve{D}^2 + \breve{\omega}^2)\right]\{\breve{\zeta}_{0+1}\} = \left[\left[\breve{E}_M^{(2)}\right]'\breve{D}^2 + \left[\breve{E}_M^{(1)}\right]'\breve{D} + \left[\breve{E}_M^{(0)}\right]'\right]\{\breve{\zeta}_{0+1}\} + [\breve{E}_D]' \tag{A-16}$$

and, if both sides are pre-multiplied by $\left[\frac{1}{\breve{M}}\right] = \left[\frac{\mu}{\breve{M}}\right]$ and we write

$$\left.\begin{aligned} \left[\hat{e}_M^{(2)}\right]' &= -\left[\frac{1}{\mu\hat{M}}\right]\left[\breve{E}_M^{(2)}\right]' \;;\; \left[\hat{e}_M^{(1)}\right]' = -\left[\frac{1}{\hat{M}}\right]\left[\breve{E}_M^{(1)}\right]' \\ \left[\hat{e}_M^{(0)}\right]' &= -\left[\frac{\mu}{\hat{M}}\right]\left[\breve{E}_M^{(0)}\right]' \;;\; [\hat{e}_D]' = -\left[\frac{1}{\hat{M}}\right][\breve{E}_D]' \end{aligned}\right\} \tag{A-17}$$

the required concise form of the equations of motion is obtained as

$$\left[(\hat{D}^2 + \hat{\omega}^2)\right]\{\hat{\zeta}_{0+1}\} + \left[\left[\hat{e}_M^{(2)}\right]'\hat{D}^2 + \left[\hat{e}_M^{(1)}\right]'\hat{D} + \left[\hat{e}_M^{(0)}\right]'\right]\{\hat{\zeta}_{0+1}\} + [\hat{e}_D]' = 0 \quad . \tag{A-18}$$

Appendix B

A CHECK ON THE ACCURACY OF AERODYNAMIC LOADING DERIVED BY USE OF INFLUENCE COEFFICIENTS

For two simple situations it is possible to compare the approximate results for total lift and pitching moment obtained from the numerical scheme based on influence coefficients with 'exact' results obtained analytically.

Firstly, we set

$$h(\xi,t) = e^{\lambda t}\underline{h}(\xi) \tag{B-1}$$

or, in aero-normalised form:

$$\breve{h}(\breve{\xi},\breve{t}) = e^{\breve{\lambda}\breve{t}}\underline{\breve{h}}(\breve{\xi}) \tag{B-2}$$

where $\breve{h}(\breve{\xi},\breve{t})$ and $\breve{t}$ are defined by equation (58) of the main text, and

$$\breve{\lambda} = \frac{\lambda \ell_0}{V_e} \quad . \tag{B-3}$$

Then we consider in turn the two displacement mode shapes

$$\begin{aligned} \underline{\breve{h}}(\breve{\xi}) &= \breve{\xi} \tan \underline{\alpha} \\ &\approx \breve{\xi}\underline{\alpha} , \qquad \text{for small } \underline{\alpha} \end{aligned} \tag{B-4}$$

and

$$\underline{\breve{h}}(\breve{\xi}) = \breve{\xi}(1 - \breve{\xi})\underline{\alpha} \quad . \tag{B-5}$$

The corresponding profiles are sketched in Fig.5; for the first, the amplitude of the angle of attack, $\underline{\alpha}$, is constant along the aircraft; for the second it varies linearly from $\underline{\alpha}$ at the nose to $-\underline{\alpha}$ at the tail-end.

The total aero-normalised lift is given by

$$\breve{L} = \frac{L}{\frac{1}{2}\rho_e V_e^2 S_w} = \frac{1}{\frac{1}{2}\rho_e V_e^2 S_w} \int_0^1 \frac{dL}{d\breve{\xi}} \, d\breve{\xi} = \int_0^1 \frac{d\breve{L}}{d\breve{\xi}} \, d\breve{\xi} \tag{B-6}$$

while the aero-normalised nose-up pitching-moment about the nose, $\breve{M}$, is given by

$$\breve{M} = \frac{M}{\frac{1}{2}\rho_e V_e^2 S_w \ell_0} = -\frac{1}{\frac{1}{2}\rho_e V_e^2 S_w \ell_0} \int_0^1 \ell_0 \breve{\xi} \frac{dL}{d\breve{\xi}} d\xi = -\int_0^1 \xi \frac{d\breve{L}}{d\breve{\xi}} d\xi . \qquad \text{(B-7)}$$

The longitudinal distribution of lift $\frac{dL}{dx}$, according to slender-body theory, is given in the main text by equation (51b) which, by use of equations (58), (61) and (62), can be written in the aero-normalised form

$$\frac{d\breve{L}}{d\breve{\xi}} = \frac{\pi A}{2} e^{\breve{\lambda}\breve{t}} \left[\frac{\partial}{\partial\breve{\xi}} \left\{\left(\frac{s(\breve{\xi})}{s_T}\right)^2 \underline{h}'\right\} + \breve{\lambda} \left[\frac{\partial}{\partial\breve{\xi}} \left\{\left(\frac{s(\breve{\xi})}{s_T}\right)^2 \underline{\breve{h}}\right\} + \left\{\left(\frac{s(\xi)}{s_T}\right)^2 \underline{\breve{h}'}\right\}\right] + \breve{\lambda}^2 \left(\frac{s(\breve{\xi})}{s_T}\right)^2 \underline{\breve{h}}\right] \qquad \text{... (B-8)}$$

where dashes denote differentiation with respect to $\breve{\xi}$, and A is the aspect ratio of the planform $\left(A = 4s_T^2/S_w\right)$. Hence, if we set

$$\breve{L} = \underline{L}e^{\breve{\lambda}\breve{t}} = \left(\underline{\breve{L}}^{(0)} + \breve{\lambda}\underline{\breve{L}}^{(1)} + \breve{\lambda}^2\underline{\breve{L}}^{(2)}\right) e^{\breve{\lambda}\breve{t}} \qquad \text{(B-9)}$$

we have

$$\left.\begin{aligned} \underline{\breve{L}}^{(0)} &= \frac{\pi A}{2} \underline{\breve{h}}'\Big|_{\breve{\xi}=1} \\ \underline{\breve{L}}^{(1)} &= \frac{\pi A}{2} \left\{\underline{\breve{h}}\Big|_{\breve{\xi}=1} + \int_0^1 \left[\left(\frac{s(\breve{\xi})}{s_T}\right)^2 \breve{h}'\right] d\breve{\xi}\right\} \\ \underline{\breve{L}}^{(2)} &= \frac{\pi A}{2} \int_0^1 \left(\frac{s(\breve{\xi})}{s_T}\right)^2 \breve{h} d\breve{\xi} \end{aligned}\right\} \qquad \text{(B-10)}$$

Similarly, if we set

$$\breve{M} = \underline{\breve{M}}e^{\breve{\lambda}\breve{t}} = \left(\underline{\breve{M}}^{(0)} + \breve{\lambda}\underline{\breve{M}}^{(1)} + \breve{\lambda}^2\underline{\breve{M}}^{(2)}\right) e^{\breve{\lambda}\breve{t}} \qquad \text{(B-11)}$$

we have

$$\left.\begin{aligned}
\underline{\check{M}}^{(0)} &= -\frac{\pi A}{2}\left[\underline{\check{h}}'\Big|_{\check{\xi}=1} - \int_0^1 \left(\frac{s(\check{\xi})}{s_T}\right)^2 \underline{\check{h}}' d\check{\xi}\right] \\
\underline{\check{M}}^{(1)} &= -\frac{\pi A}{2}\left[\underline{\check{h}}\Big|_{\check{\xi}=1} - \int_0^1 \left(\frac{s(\check{\xi})}{s_T}\right)^2 \underline{\check{h}} d\check{\xi} + \int_0^1 \check{\xi}\left(\frac{s(\check{\xi})}{s_T}\right)^2 \underline{\check{h}}' d\check{\xi}\right] \\
\underline{\check{M}}^{(2)} &= -\frac{\pi A}{2}\int_0^1 \check{\xi}\left(\frac{s(\xi)}{s_T}\right)^2 \underline{\check{h}} d\check{\xi}
\end{aligned}\right\} \qquad \text{(B-12)}$$

The expressions for $\underline{\check{L}}^{(r)}$, $\underline{\check{M}}^{(r)}$, $r = 0,1,2$, in equations (B-10) and (B-12) may be evaluated analytically for the distributions of $\underline{\check{h}}(\check{\xi})$ defined by equations (B-4) and (B-5). Results obtained in this way are shown in Table 6, together with values of these quantities calculated from the numerical model. Reasonably good agreement is seen to exist and, therefore, the exercise has not been repeated for the case where the aerodynamic influence coefficients are derived from piston theory.

It may be noted that, since in the cases considered, $\underline{\check{L}}^{(r)}$ and $\underline{\check{M}}^{(r)}$ will be proportional to $\underline{\alpha}$, then, if we write $\alpha = \underline{\alpha}e^{\check{\lambda}\check{t}}$, we can express equations (B-9) and (B-11) in the forms

$$\check{L} = \frac{\partial \check{L}}{\partial \alpha}\alpha + \frac{\partial \check{L}}{\partial \dot{\alpha}_\gamma}\dot{\alpha}_\gamma + \frac{\partial \check{L}}{\partial \ddot{\alpha}_\gamma}\ddot{\alpha}_\gamma \qquad \text{(B-13)}$$

$$\check{M} = \frac{\partial \check{M}}{\partial \alpha}\alpha + \frac{\partial \check{M}}{\partial \dot{\alpha}_\gamma}\dot{\alpha}_\gamma + \frac{\partial \check{M}}{\partial \ddot{\alpha}_\gamma}\ddot{\alpha}_\gamma \qquad \text{(B-14)}$$

where

$$\dot{\alpha}_\gamma = \frac{\partial \alpha}{\partial \check{t}} = \frac{\ell}{V}\frac{\partial \alpha}{\partial t} \quad ; \quad \ddot{\alpha}_\gamma = \frac{\partial^2 \alpha}{\partial \check{t}^2} = \frac{\ell^2}{V^2}\frac{\partial^2 \alpha}{\partial \check{t}^2} \qquad \text{(B-15)}$$

are the normalised angular velocity and angular acceleration. Then

$$\frac{\partial \check{L}}{\partial \alpha} = \frac{\underline{\check{L}}^{(0)}}{\underline{\alpha}}, \quad \frac{\partial \check{L}}{\partial \dot{\alpha}_\gamma} = \frac{\underline{\check{L}}^{(1)}}{\underline{\alpha}}, \quad \frac{\partial \check{L}}{\partial \ddot{\alpha}_\gamma} = \frac{\underline{\check{L}}^{(2)}}{\underline{\alpha}} \qquad \text{(B-16)}$$

$$\frac{\partial \check{M}}{\partial \alpha} = \frac{\underline{\check{M}}^{(0)}}{\underline{\alpha}}, \quad \frac{\partial \check{M}}{\partial \dot{\alpha}_\gamma} = \frac{\underline{\check{M}}^{(1)}}{\underline{\alpha}}, \quad \frac{\partial \check{M}}{\partial \ddot{\alpha}_\gamma} = \frac{\underline{\check{M}}^{(2)}}{\underline{\alpha}} \quad . \qquad \text{(B-17)}$$

Table 1

DISTRIBUTION OF DISCRETE MASSES

Element number i	Location $\breve{x}_i = x_i/\ell_0$	$\breve{\xi}_i = \left(x_{14} - x_i\right)/\ell_0$	Normalised mass $\hat{m}_i = m_i/m_e$
1	-0.3529	1.0000	0
2	-0.3384	0.9855	0.037736
3	-0.2809	0.9280	0.047954
4	-0.2064	0.8535	0.173740
5	-0.1224	0.7695	0.165993
6	-0.0363	0.6834	0.146812
7	0.0438	0.6033	0.126893
8	0.1248	0.5223	0.089932
9	0.2045	0.4426	0.059536
10	0.3036	0.3435	0.059665
11	0.4111	0.2360	0.060200
12	0.5011	0.1460	0.030314
13	0.5941	0.0530	0.001225
14	0.6471	0	0

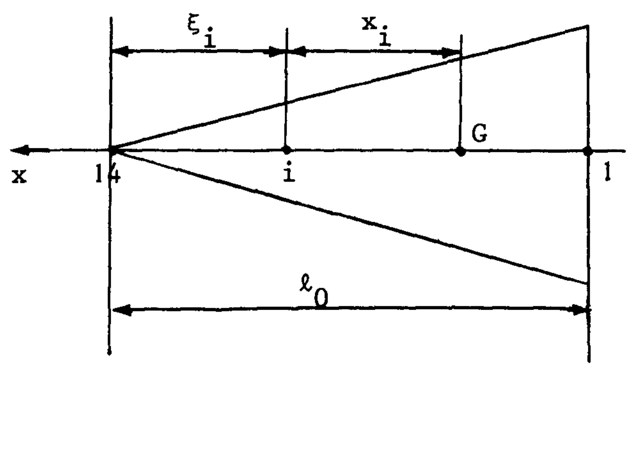

Table 2

THE AERO-NORMALISED STRUCTURAL FLEXIBILITY INFLUENCE COEFFICIENTS $\breve{G}_{ij}$

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.0613	0.0545	0.0299	0.0060	-0.0093	-0.0157	-0.0167	-0.0140	-0.0076	0.0053	0.0243	0.0424	0.0621	0.0735
2	0.0545	0.0482	0.0272	0.0059	-0.0082	-0.0143	-0.0154	-0.0131	-0.0072	0.0047	0.0222	0.0392	0.0574	0.0680
3	0.0299	0.0272	0.0160	0.0050	-0.0043	-0.0090	-0.0104	-0.0093	-0.0056	0.0024	0.0146	0.0265	0.0395	0.0469
4	0.0060	0.0059	0.0050	0.0025	-0.0001	-0.0029	-0.0043	-0.0045	-0.0034	0.0000	0.0057	0.0115	0.0178	0.0215
5	-0.0093	-0.0082	-0.0043	-0.0001	0.0019	0.0023	0.0015	0.0004	-0.0005	-0.0015	-0.0022	-0.0026	-0.0029	-0.0030
6	-0.0157	-0.0143	-0.0090	-0.0029	0.0023	0.0052	0.0059	0.0049	0.0026	-0.0018	-0.0080	-0.0137	-0.0197	-0.0233
7	-0.0167	-0.0154	-0.0104	-0.0043	0.0015	0.0059	0.0084	0.0083	0.0055	-0.0011	-0.0112	-0.0210	-0.0315	-0.0375
8	-0.0140	-0.0131	-0.0093	-0.0045	0.0004	0.0049	0.0083	0.0095	0.0077	0.0006	-0.0116	-0.0239	-0.0374	-0.0451
9	-0.0076	-0.0072	-0.0056	-0.0034	-0.0005	0.0026	0.0055	0.0077	0.0076	0.0028	-0.0084	-0.0207	-0.0345	-0.0422
10	0.0053	0.0047	0.0024	0.0000	-0.0015	-0.0018	-0.0011	0.0006	0.0028	0.0037	0.0012	-0.0046	-0.0120	-0.0163
11	0.0243	0.0222	0.0146	0.0057	-0.0022	-0.0080	-0.0112	-0.0116	-0.0084	0.0012	0.0155	0.0299	0.0429	0.0503
12	0.0424	0.0392	0.0265	0.0115	-0.0026	-0.0137	-0.0210	-0.0239	-0.0207	-0.0046	0.0299	0.0684	0.1158	0.1427
13	0.0621	0.0574	0.0395	0.0178	-0.0029	-0.0197	-0.0315	-0.0374	-0.0345	-0.0120	0.0429	0.1158	0.2103	0.2767
14	0.0735	0.0680	0.0469	0.0215	-0.0030	-0.0233	-0.0375	-0.0451	-0.0422	-0.0163	0.0503	0.1427	0.2767	0.3689

Note: (1) $\breve{G}_{ij} = G_{ij} \times \left(\frac{1}{2}\rho_e V_e^2 S_w / \ell_0\right)$ where values of the normalising factor $\left(\frac{1}{2}\rho_e V_e^2 S_w / \ell_0\right)$ in UK and SI units may be obtained from Table A2.

(2) For ease of presentation, values of $\breve{G}_{ij}$ are shown only to 4 places of decimals; this is sufficient to indicate relative orders of magnitude.

Table 3

PROPERTIES OF THE FIRST FOUR NATURAL MODES OF THE STRUCTURE

Mode number	1	2	3	4
Dynamic-normalised frequency $\bar{\omega}$ rad/dysecond	102.54	234.45	452.19	848.63
Mode shape: Displacements normalised with respect to displacement at station 14 (nose)				
Station				
1	0.44963	-0.51998	0.25530	-0.18956
2	0.41027	-0.43955	0.18919	-0.10896
3	0.26340	-0.17236	0.00986	0.04255
4	0.09597	0.06315	-0.08319	0.05640
5	-0.04593	0.15835	-0.05114	-0.01132
6	-0.14222	0.10630	0.05581	-0.08695
7	-0.19131	-0.01420	0.11441	-0.00965
8	-0.19427	-0.14364	0.06560	0.09362
9	-0.14135	-0.23077	-0.06274	0.08809
10	+0.01032	-0.20516	-0.18254	-0.05507
11	0.26757	0.02238	-0.08584	-0.09143
12	0.53266	0.34402	0.20283	0.08275
13	0.82926	0.75666	0.69288	0.61186
14	1.00000	1.00000	1.00000	1.00000

Table 4

ELEMENTS OF THE DIAGONAL MATRICES OF GENERALISED MASSES AND STIFFNESSES $\lceil M_{0+1} \rfloor$ AND $\lceil K_{0+1} \rfloor$

Dynamic-normalised values

i	$\hat{M}_{ii} = M_{ii}/m_e$	$\hat{K}_{ii} = K_{ii}/m_e \tau_D^{-2}$
1	1.00	0
2	0.10990	0
3	0.03762	395.26
4	0.02706	1489.6
5	0.01005	2053.4
6	0.00483	3480.9

Values of the normalising factors

Normalising factor	Value in UK units	Value in SI units
m_e	0.84259×10^4 slugs	0.12297×10^6 kg
$m_e \tau_D^{-2}$	0.19334×10^6 slugs/s^2	0.28217×10^4 kg/s^2

Table 5

NORMALISED AERODYNAMIC INFLUENCE COEFFICIENTS DERIVED FROM SLENDER-BODY THEORY

$$\check{R}_{ij} = \check{R}^{(2)}_{ij}\check{\lambda}^2 + \check{R}^{(1)}_{ij}\check{\lambda} + \check{R}^{(0)}_{ij}\check{\lambda}$$

The tables below give values of the elements $\check{R}_{ii}$ in the leading diagonal and of elements $\check{R}_{i,i-1}$ and $\check{R}_{i,i+1}$ in the off-diagonals of the matrices $[\check{R}^{(2)}]$, $[\check{R}^{(1)}]$ and $[\check{R}^{(0)}]$. All other elements are zero.

Matrix $[\check{R}^{(2)}]$

i	$\check{R}^{(2)}_{i,i-1}$	$\check{R}^{(2)}_{ii}$	$\check{R}^{(2)}_{i,i+1}$
1	-	0.00000	0.00000
2	-0.00017	-0.06206	0.00001
3	-0.00065	-0.10635	0.00039
4	-0.00108	-0.10775	0.00085
5	-0.00111	-0.08928	0.00106
6	-0.00084	-0.06307	0.00098
7	-0.00072	-0.04226	0.00071
8	-0.00056	-0.02674	0.00058
9	-0.00065	-0.01639	0.00042
10	-0.00035	-0.00642	0.00030
11	-0.00007	-0.00337	0.00010
12	-0.00007	-0.00140	0.00006
13	-0.00001	-0.00018	0.00003
14	0.00000	0.00000	-

Matrix $[\check{R}^{(1)}]$

i	$\check{R}^{(1)}_{i,i-1}$	$\check{R}^{(1)}_{ii}$	$\check{R}^{(1)}_{i,i+1}$
1	-	0.0000	0.0000
2	-6.8455	6.3850	0.4326
3	-2.1111	0.7061	1.2216
4	-1.5731	0.1098	1.1728
5	-1.1173	-0.1781	0.9886
6	-0.7386	-0.3064	0.7829
7	-0.5602	-0.1478	0.4952
8	-0.3508	-0.1377	0.3194
9	-0.2551	-0.0220	0.1364
10	-0.0879	-0.0441	0.0575
11	-0.0304	-0.0267	0.0374
12	-0.0184	-0.0116	0.0132
13	-0.0021	-0.0076	0.0037
14	0.0000	0.0000	-

Matrix $[\check{R}^{(0)}]$

i	$\check{R}^{(0)}_{i,i-1}$	$\check{R}^{(0)}_{ii}$	$\check{R}^{(0)}_{i,i+1}$
1	-	0.00	0.00
2	-120.15	149.96	-29.81
3	-29.68	50.12	-20.44
4	-20.32	34.87	-14.56
5	-14.40	24.88	-10.48
6	-10.32	18.15	-7.83
7	-7.95	13.19	-5.24
8	-5.21	8.38	-3.11
9	-3.40	4.71	-1.31
10	-1.21	1.64	-0.43
11	-0.39	0.64	-0.25
12	-0.27	0.36	-0.08
13	-0.06	0.05	0.01
14	0.00	0.00	-

Note: For values of the aero-normalising constants in UK and SI units, see Table A2 of the Appendix.

Table 6

DERIVATIVES OF AERODYNAMIC LIFT AND PITCHING MOMENT COEFFICIENTS AS DETERMINED ANALYTICALLY AND NUMERICALLY

(Slender body theory)

Test Case 1

Derivative	Analytical	Numerical
$\frac{\underline{\underline{\check{L}}}^{(0)}}{\underline{\alpha}} = \frac{\partial \check{L}}{\partial \alpha}$	1.7343	1.7104
$\frac{\underline{\underline{\check{L}}}^{(1)}}{\underline{\alpha}} = \frac{\partial \check{L}}{\partial \dot{\alpha}_\gamma}$	2.2719	2.2281
$\frac{\underline{\underline{\check{L}}}^{(2)}}{\underline{\alpha}} = \frac{\partial \check{L}}{\partial \ddot{\alpha}_\gamma}$	0.4246	0.4151
$\frac{\underline{\underline{\check{M}}}^{(0)}}{\underline{\alpha}} = \frac{\partial \check{M}}{\partial \alpha}$	-1.1970	-1.1727
$\frac{\underline{\underline{\check{M}}}^{(1)}}{\underline{\alpha}} = \frac{\partial \check{M}}{\partial \dot{\alpha}_\gamma}$	-1.7344	-1.6913
$\frac{\underline{\underline{\check{M}}}^{(2)}}{\underline{\alpha}} = \frac{\partial \check{M}}{\partial \ddot{\alpha}_\gamma}$	-0.3498	-0.3407

Test Case 2

Derivative	Analytical	Numerical
$\frac{\underline{\underline{\check{L}}}^{(0)}}{\underline{\alpha}} = \frac{\partial \check{L}}{\partial \alpha}$	-1.7343	-1.6931
$\frac{\underline{\underline{\check{L}}}^{(1)}}{\underline{\alpha}} = \frac{\partial \check{L}}{\partial \dot{\alpha}_\gamma}$	-0.3120	-0.2954
$\frac{\underline{\underline{\check{L}}}^{(2)}}{\underline{\alpha}} = \frac{\partial \check{L}}{\partial \ddot{\alpha}_\gamma}$	0.07478	0.07378
$\frac{\underline{\underline{\check{M}}}^{(0)}}{\underline{\alpha}} = \frac{\partial \check{M}}{\partial \alpha}$	1.4223	1.3812
$\frac{\underline{\underline{\check{M}}}^{(1)}}{\underline{\alpha}} = \frac{\partial \check{M}}{\partial \dot{\alpha}_\gamma}$	0.3498	0.3345
$\frac{\underline{\underline{\check{M}}}^{(2)}}{\underline{\alpha}} = \frac{\partial \check{M}}{\partial \ddot{\alpha}_\gamma}$	-0.05268	-0.05232

Table 7

ELEMENTS OF THE NORMALISED RESIDUAL FLEXIBILITY MATRIX $[\check{X}]$

(a) One mode retained

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	0.02	0.02	0.005	-0.003	-0.005	-0.002	0.001	0.005	0.006	0.005	-0.001	-0.008	-0.02	-0.02
2	0.02	0.01	0.05	-0.002	-0.005	-0.002	0.001	0.003	0.005	0.003	-0.001	-0.006	-0.01	-0.02
3	0.005	0.005	0.001	-0.0003	-0.002	-0.001	0.0002	0.001	0.002	0.002	-0.0003	-0.003	-0.006	-0.008
4	-0.003	-0.002	-0.0003	0.0005	0.0008	0.0000	0.0005	-0.0006	-0.0005	-0.0002	0.0003	0.0006	0.001	0.001
5	-0.005	-0.005	-0.002	0.0008	0.001	0.0008	-0.0005	-0.001	-0.002	-0.001	0.0003	0.003	0.005	0.006
6	-0.002	-0.002	-0.001	0.0000	0.0008	0.001	0.0002	-0.001	-0.002	-0.001	0.0001	0.002	0.005	0.006
7	0.001	0.001	0.0002	-0.0005	-0.0005	0.0002	0.0006	0.0005	-0.0002	-0.0006	-0.0003	0.0006	0.002	0.003
8	0.005	0.003	0.001	-0.0006	-0.001	-0.001	0.0005	0.002	0.002	0.001	-0.0006	-0.002	-0.003	-0.005
9	0.006	0.005	0.002	-0.0005	-0.002	-0.002	-0.0002	0.002	0.003	0.003	-0.0005	-0.005	-0.01	-0.01
10	0.005	0.003	0.002	-0.0002	-0.001	-0.001	-0.0006	0.001	0.003	0.003	0.0006	-0.006	-0.01	-0.02
11	-0.001	-0.001	-0.0003	0.0003	0.0003	0.0001	-0.0003	-0.0006	-0.0005	0.0006	0.0003	-0.0003	-0.003	-0.006
12	-0.008	-0.006	-0.003	0.0006	0.003	0.002	0.0006	-0.002	-0.005	-0.006	-0.0003	0.008	0.02	0.03
13	-0.02	-0.01	-0.006	0.001	0.005	0.005	0.002	-0.003	-0.01	-0.01	-0.003	0.02	0.06	0.1
14	-0.02	-0.02	-0.008	0.001	0.006	0.006	0.003	-0.005	-0.01	-0.02	-0.006	0.03	0.1	0.2

Notes: (1) $\check{X}_{ij} = X_{ij} \times \left(\frac{1}{2}\rho_e V_e^2 S_w / \ell_0\right)$ where values of the normalising factor $\left(\frac{1}{2}\rho_e V_e^2 S_w / \ell_0\right)$ in UK and SI units may be obtained from Table A2.

(2) When no modes are retained, $\check{X}_{ij} = \check{G}_{ij}$. The elements of $[\check{G}]$ are given in Table 2.

(3) Values of $\check{X}_{ij}$, quoted above to only one significant figure and a maximum of 4 decimal places, are intended only to indicate order of magnitude, for comparison with corresponding values of $\check{G}_{ij}$.

Table 7 (continued)

(b) Four modes retained

i \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	-0.0003	0.0002	-0.0001	-0.0001	0.0002	0.0000	-0.0002	0.0000	0.0002	0.0000	-0.0001	0.0001	0.001	0.002
2	0.0002	0.0000	0.0001	-0.0001	0.0001	0.0000	0.0001	0.0000	0.0001	0.0000	-0.0001	0.0000	0.001	0.001
3	-0.0001	0.0001	-0.0003	0.0002	-0.0002	0.0000	0.0001	0.0000	0.0001	0.0000	0.0000	0.0000	0.0003	0.0005
4	-0.0001	-0.0001	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0002	-0.0003
5	0.0002	0.0001	-0.0002	0.0001	-0.0001	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0002	0.0003
6	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0003	0.0006
7	-0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0001	0.0000	0.0001	0.0000	-0.0005	-0.0006
8	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0005	-0.001
9	0.0002	0.0001	0.0001	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	0.0001	-0.0001	0.0001	0.0006	0.001
10	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	-0.0002	0.0001	-0.0002	0.0008	0.002
11	-0.0001	-0.0001	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000	-0.0001	0.0001	-0.0001	0.0002	-0.0001	-0.002
12	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	-0.0002	0.0002	-0.0002	0.0008	0.0005
13	0.001	0.001	0.0003	-0.0002	0.0002	0.0003	-0.0005	-0.0005	0.0006	0.0008	-0.0001	0.0008	0.003	0.02
14	0.002	0.001	0.0005	-0.0003	0.0003	0.0006	-0.0006	-0.001	0.001	0.002	-0.002	0.0005	0.02	0.03

Table 8

VALUES OF SELECTED ELEMENTS OF THE MATRICES $[\bar{E}]$ AND $[\bar{A}]$

(a) Values of $\bar{E}_{ij}$

i \ j	2	4	7	10	13
2	-0.13	0.015	-0.0091	-0.0054	0.017
4	-0.23	-0.013	0.041	-0.027	-0.14
7	-0.022	-0.010	-0.020	-0.0031	-0.063
10	0.0070	0.0017	-0.0037	0.0021	0.0039
13	0.0012	-0.0004	-0.0007	-0.0005	0.0063

(b) Values of $\bar{A}_{ij}$

i \ j	2	4	7	10	13
2	0.87	0.015	-0.0091	-0.0054	0.017
4	-0.23	0.987	0.041	-0.027	-0.14
7	-0.022	-0.010	0.980	-0.0031	-0.063
10	0.0070	0.0017	-0.0037	1.0021	0.0039
13	0.0012	-0.0004	-0.0007	-0.0005	1.0063

Table 9

NORMALISED EIGENVALUES CORRESPONDING TO VARIOUS STRUCTURAL REPRESENTATIONS IN ASSOCIATION WITH TWO DIFFERENT AERODYNAMIC THEORIES

No. of flexible modes included	Slender-body theory aerodynamics		Piston-theory aerodynamics	
	No residual flexibility	Residual flexibility included	No residual flexibility	Residual flexibility included
	(a) Eigenvalues $\hat{\mu}_0$ corresponding to 'rigid-body' short-period mode			
0	-3.0707 ± 10.639i	-2.8584 ± 11.184i	-1.8613 ± 19.958i	-1.7309 ± 17.361i
1	-2.9577 ± 11.200i	-2.8924 ± 11.192i	-1.7254 ± 17.607i	-1.7146 ± 17.304i
2	-2.8925 ± 11.084i	-2.8891 ± 11.194i	-1.7132 ± 17.383i	-1.7133 ± 17.303i
3	-2.8851 ± 11.087i	-2.8890 ± 11.194i	-1.7125 ± 17.305i	-1.7133 ± 17.303i
4	-2.8847 ± 11.168i	-2.8899 ± 11.194i	-1.7126 ± 17.308i	-1.7133 ± 17.303i
	(b) Eigenvalues $\hat{\mu}_1$ corresponding to first structural mode ($\hat{\omega}_1$ = 102.54)			
1	-4.4431 ± 104.23i	-4.2314 ± 104.30i	-1.2586 ± 108.09i	-1.2137 ± 107.61i
2	-4.1852 ± 104.31i	-4.1736 ± 104.40i	-1.1982 ± 107.53i	-1.1962 ± 107.50i
3	-4.1494 ± 104.33i	-4.1737 ± 104.40i	-1.1947 ± 107.51i	-1.1955 ± 107.50i
4	-4.1510 ± 104.35i	-4.1754 ± 104.40i	-1.1953 ± 107.49i	-1.1956 ± 107.50i
	(c) Eigenvalues $\hat{\mu}_2$ corresponding to second structural mode ($\hat{\omega}_2$ = 234.45)			
2	-8.2162 ± 236.36i	-8.2208 ± 236.51i	-1.3263 ± 240.10i	-1.3194 ± 240.04i
3	-8.1271 ± 236.47i	-8.2241 ± 236.59i	-1.3173 ± 240.02i	-1.3162 ± 240.02i
4	-8.1555 ± 236.51i	-8.2321 ± 236.60i	-1.3157 ± 240.02i	-1.3161 ± 240.02i
	(d) Eigenvalues $\hat{\mu}_3$ corresponding to third structural mode ($\hat{\omega}_3$ = 452.19)			
3	-5.3441 ± 451.40i	-5.3841 ± 451.52i	-1.0599 ± 453.18i	-1.0603 ± 453.17i
4	-5.3545 ± 451.43i	-5.4006 ± 451.53i	-1.0607 ± 453.17i	-1.0605 ± 453.17i
	(e) Eigenvalues $\hat{\mu}_4$ corresponding to fourth structural mode ($\hat{\omega}_4$ = 848.63)			
4	-6.1560 ± 847.04i	-6.1573 ± 846.97i	-1.1811 ± 848.75i	-1.1821 ± 848.75i

Note: (1) $\hat{\omega}$ (rad/dysecond) = ω (rad/s) × τ_D where τ_D = 6.6015 s

(2) $\hat{\mu}$ (rad/dysecond) = μ (rad/s) × τ_D

SYMBOLS

$[A]$	aeroelastic correction matrix defined by equation (17)
$[\bar{A}]$	approximate aeroelastic correction matrix appropriate to quasi-static solution
A	aspect ratio of planform
A'	aspect ratio of approximate (triangular) planform
$B(\xi) = EI(\xi)$	bending rigidity of representative beam
B_1, B_2, B_3	parameters defining longitudinal stiffness distribution (Fig.3)
B_i	see equation (83)
$C(x,x')$	structural influence function for constrained beam
C_{ij}	structural influence coefficient for constrained beam
C_i	see equation (83)
$[D]$	matrix defined by equation (5)
$D \equiv \frac{d}{dt}$, $\hat{D} \equiv \frac{d}{d\hat{t}}$	differential operators
E	Young's modulus
$[E]$	matrix defined by equation (62)
$[\bar{E}]$	upper bound to matrix $[E]$, appropriate to quasi-static solution
$\left[E_M^{(r)}\right]$, r = 0,1,2	matrices of generalised aerodynamic forces due to perturbation motion
$\left[E_M^{(r)}\right]'$	matrices of generalised aerodynamic forces due to perturbation motion, modified to allow for residual flexibility
$[E_D],[E_D]'$	matrices of generalised aerodynamic forces due to control deflections or atmospheric disturbances, respectively unmodified and modified to allow for residual flexibility
$\{F\}$	column of generalised forces
$\{F_M\},\{F_D\}$	contributions to $\{F\}$ due to perturbation motion and external disturbances respectively
$\{\mathcal{F}_D\}$	column of incremental forces, due to external disturbances, acting at nodal points
$[G]$	matrix of structural influence coefficients, G_{ij} , for the unconstrained structure
$G(x,x')$	structural influence function for the unconstrained structure
$I(\xi)$ (section 3.1)	second moment of area of beam section at station ξ (about neutral axis)
(section 3.2)	momentum of cross-flow virtual mass at station ξ
I_y	pitching moment of inertia of aircraft

SYMBOLS (continued)

$[I]$	unit matrix
K_j	generalised stiffness of jth mode
$[K_1],[K_2]$	diagonal matrices of generalised stiffnesses relating respectively to modes retained and modes not retained in analysis
L_i	lift concentrated at ith station
L, $\underline{L}$	total unsteady lift and its amplitude
$\underline{L}^{(0)}$, $\underline{L}^{(1)}$, $\underline{L}^{(2)}$	coefficients in quadratic expression for $\underline{L}$ (equation (B-9))
M_j	generalised mass of jth mode
$[M_1],[M_2]$	diagonal matrices of generalised masses relating respectively to modes retained and modes not retained in analysis
M (sections 3.2 and 6)	Mach number
M, $\underline{M}$ (Appendix B)	total unsteady pitching moment and its amplitude
$\underline{M}^{(0)}$, $\underline{M}^{(1)}$, $\underline{M}^{(2)}$	coefficients in quadratic expression for $\underline{M}$ (equation (B-11))
$\{P'\},\{Q'\}$	column matrices defined by equation (93)
$[R],[R']$	matrices of unsteady and steady aerodynamic influence coefficients
$R_{ij}^{(2)}$, $R_{ij}^{(1)}$, $R_{ij}^{(0)}$	coefficients in quadratic expression for the unsteady aerodynamic influence coefficient R_{ij} (equation (47))
S	effective cross-sectional area in slender-body theory
S_w	representative area used in defining systems of units (taken as planform area)
S'_w	area of approximate (triangular) planform
V_e	equilibrium flight speed
$[X]$	residual-flexibility matrix
$Y_i \equiv$	$M_i D^2 + K_i$
$[Y_{0+1}],[Y_2]$	diagonal matrices of Y_i relating respectively to the modes (rigid body and structural) retained in the analysis and to the structural modes not retained
$[\hat{a}]$	defined by equation (65)
a(x'), b(x')	coefficients in expression for balancing load system used in defining structural influence functions (section 3.1)
$[\hat{b}]$	defined by equation (66)
$[\hat{c}]$	defined by equation (67)
c(x'), d(x')	arbitrary functions introduced in equation (40)
$[\hat{d}]$	defined by equation (71)

SYMBOLS (continued)

$\left[\hat{e}_M^{(r)}\right]'$, r = 0,1,2	matrices of concise generalised aerodynamic forces due to perturbation motion, modified to allow for residual flexibility $\left(\text{related to } \left[\check{E}_M^{(r)}\right]' \text{ by equation (A-17)}\right)$
$[\hat{e}_D]'$	matrix of concise generalised aerodynamic forces due to external disturbances, modified to allow for residual flexibility (related to $[\check{E}_D]'$ by equation (A-17))
f_i	acceleration of ith point-mass
g	acceleration due to gravity
$h_i = h(x_i,t)$	displacement of ith point-mass normal to surface
$\underline{h}_i$	amplitude of h_i
k	(i) number of structural modes retained in analysis; (ii) spatial frequency of standing wave
ℓ_0	reference length used in definition of systems of units (= overall length of aircraft)
m_i	mass of ith element
m(x)	mass per unit length at station x
n	number of elements in finite-element model
p(x)	longitudinal loading distribution
$p_i(\hat{\omega})$, $q_i(\hat{\omega})$	defined by equations (77) and (78)
q	incremental pitching velocity
s(x) , s_i	semi-spans of planform at station x , and at ith station
s_T	semi-span at trailing-edge
t	time
w(x)	transverse displacement of beam at station x
w	z-component of incremental aircraft velocity relative to air
w_g ; $\underline{w}_g$	velocity of sinusoidal gust; amplitude of gust velocity
x_i	x-coordinate of ith concentrated mass
$x_{ref} = x_N$	x-coordinate of reference point (nose)
x_T	x-coordinate of trailing-edge
Δ_{ij}	ith element of vector $\{\Delta_j\}$ defining jth mode shape
$[\Delta_{0+1}]$	modal matrix corresponding to rigid-body modes and structural modes retained in analysis
$[\Delta_2]$	modal matrix corresponding to structural modes not retained in analysis
Θ	increment in aircraft's angle of inclination to horizontal
Λ_j	jth eigenvalue of equation (4)

SYMBOLS (continued)

α, $\underline{\alpha}$	angle of attack and its amplitude
α_g	local incidence due to gust
$\dot{\alpha}_\gamma$, $\ddot{\alpha}_\gamma$	normalised angular velocity and angular acceleration
$\delta(x - x')$	Dirac function
$\bar{\delta} \equiv \bar{\delta}(x,t)$	displacement at station x relative to datum-flight-path
$\bar{\delta}_i(t)$	displacement of ith mass relative to datum-flight-path position
$\hat{\bar{\delta}}_{i_0}$	defined by equation (91)
ε_i	defined by equation (92)
ζ_1, ζ_2	generalised coordinates associated with rigid-body modes
ζ_j , $j \geq 3$	generalised coordinate associated with $(j - 2)$th structural mode
$\{\zeta_{0+1}\}$	vector of generalised coordinates associated with rigid-body modes and structural modes retained in analysis
$\{\zeta_2\}$	vector of generalised coordinates associated with structural modes not retained in analysis
λ	see equation (B-1)
μ	aircraft relative density $\left(= m_e \Big/ \left(\frac{1}{2}\rho_e S_w \ell_0\right)\right)$
$\hat{\mu}$	see equation (69)
$\xi = (x_N - x)$	auxiliary longitudinal coordinate
ξ_1, ξ_2, ξ_M	values of ξ at stations used in definition of stiffness distribution (Fig.3)
ρ	air density
τ_A , τ_D	units of time in aero-normalised and dynamic-normalised systems, respectively
ω_j, $j \geq 3$	frequency of $(j - 2)$th natural mode

Suffices

D	due to external disturbance
M	due to perturbation motion
N	related to nose station
T	related to trailing-edge station
e	related to equilibrium flight condition
(0+1)	associated with rigid-body modes and structural modes retained in analysis
2	associated with structural modes not retained in analysis

SYMBOLS (concluded)

Raised suffices

T	indicating transposed matrix
' (dash or prime)	(1) denoting differentiation with respect to x (2) denoting modification to allow for residual flexibility of neglected modes

Overscripts

˙ (dot)	denoting differentiation with respect to time
˘ (dip)	aero-normalised form or value
ˆ (cap)	dynamic-normalised form or value

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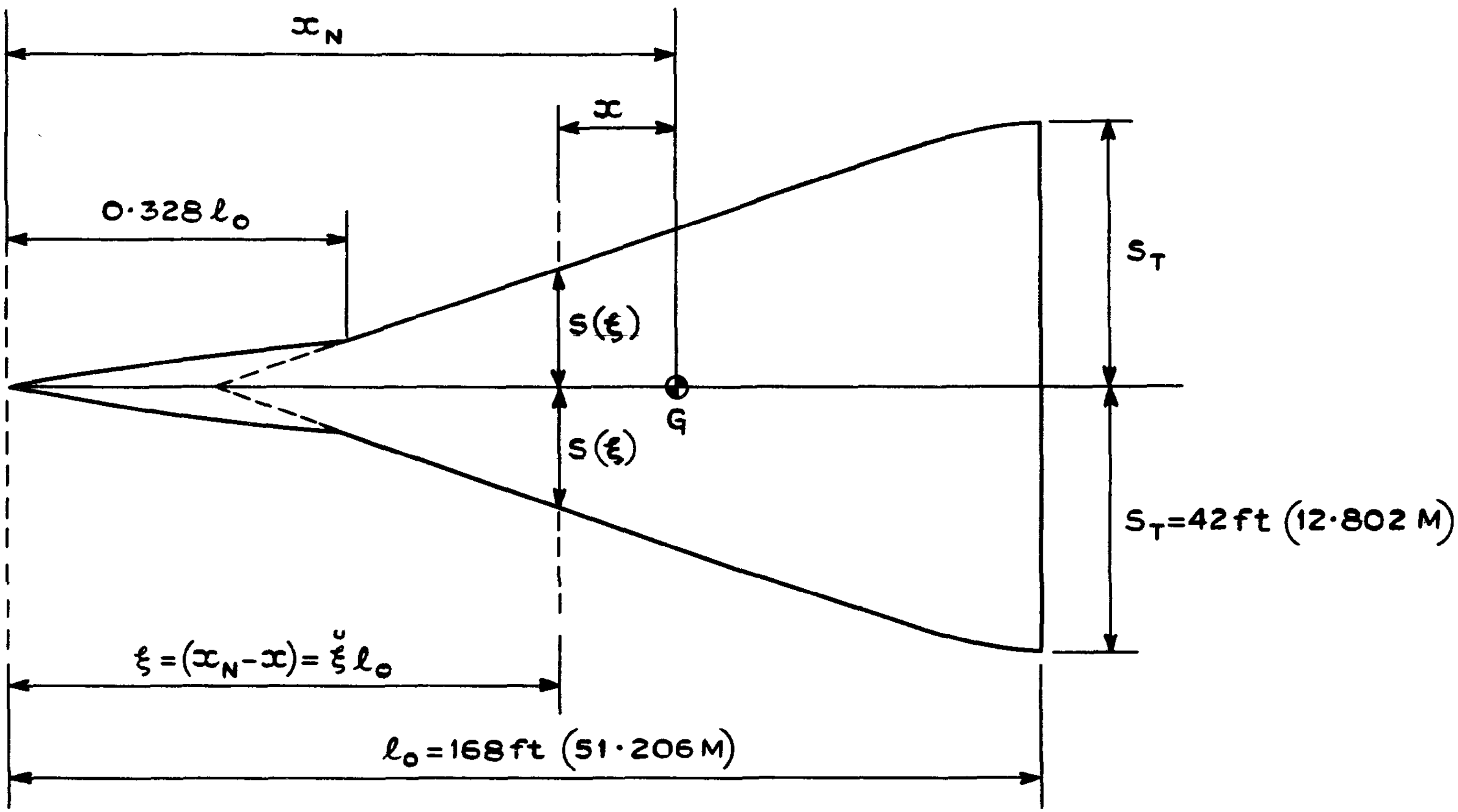


Fig.1 Geometry of planform

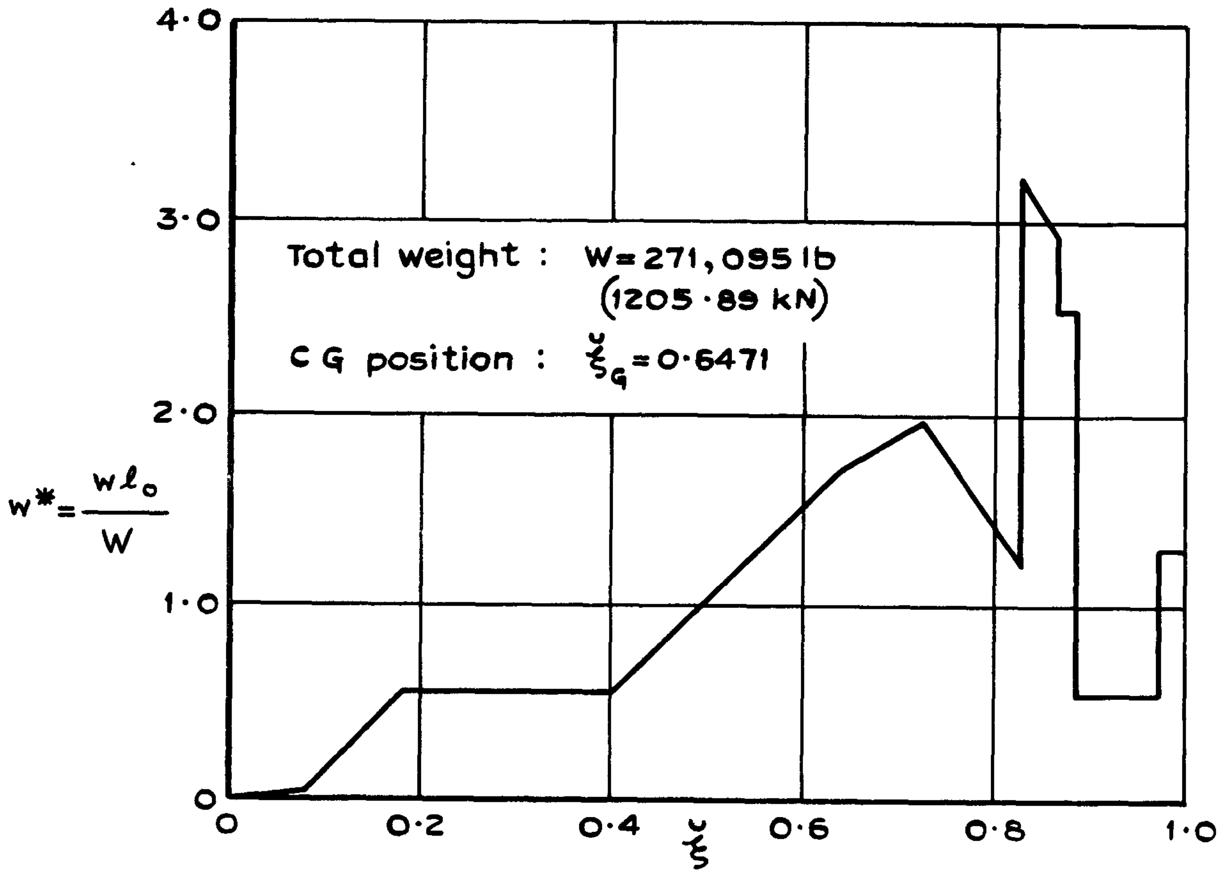


Fig.2 Assumed weight distribution

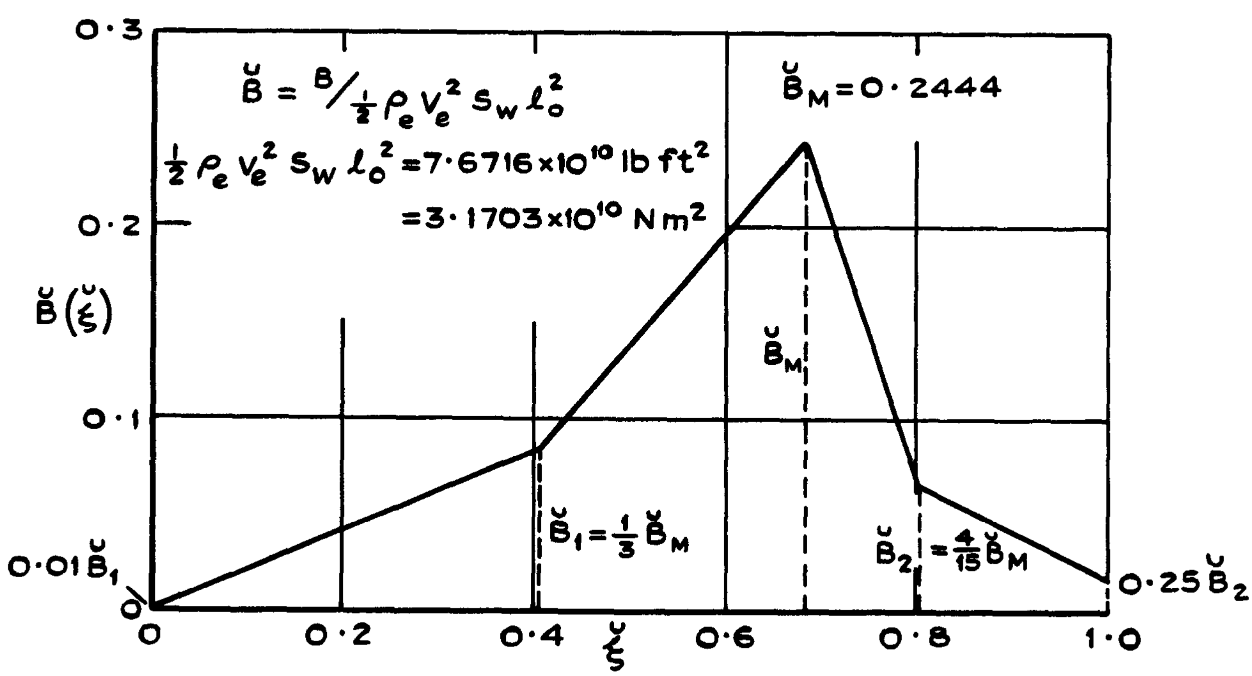


Fig 3 Assumed stiffness distribution

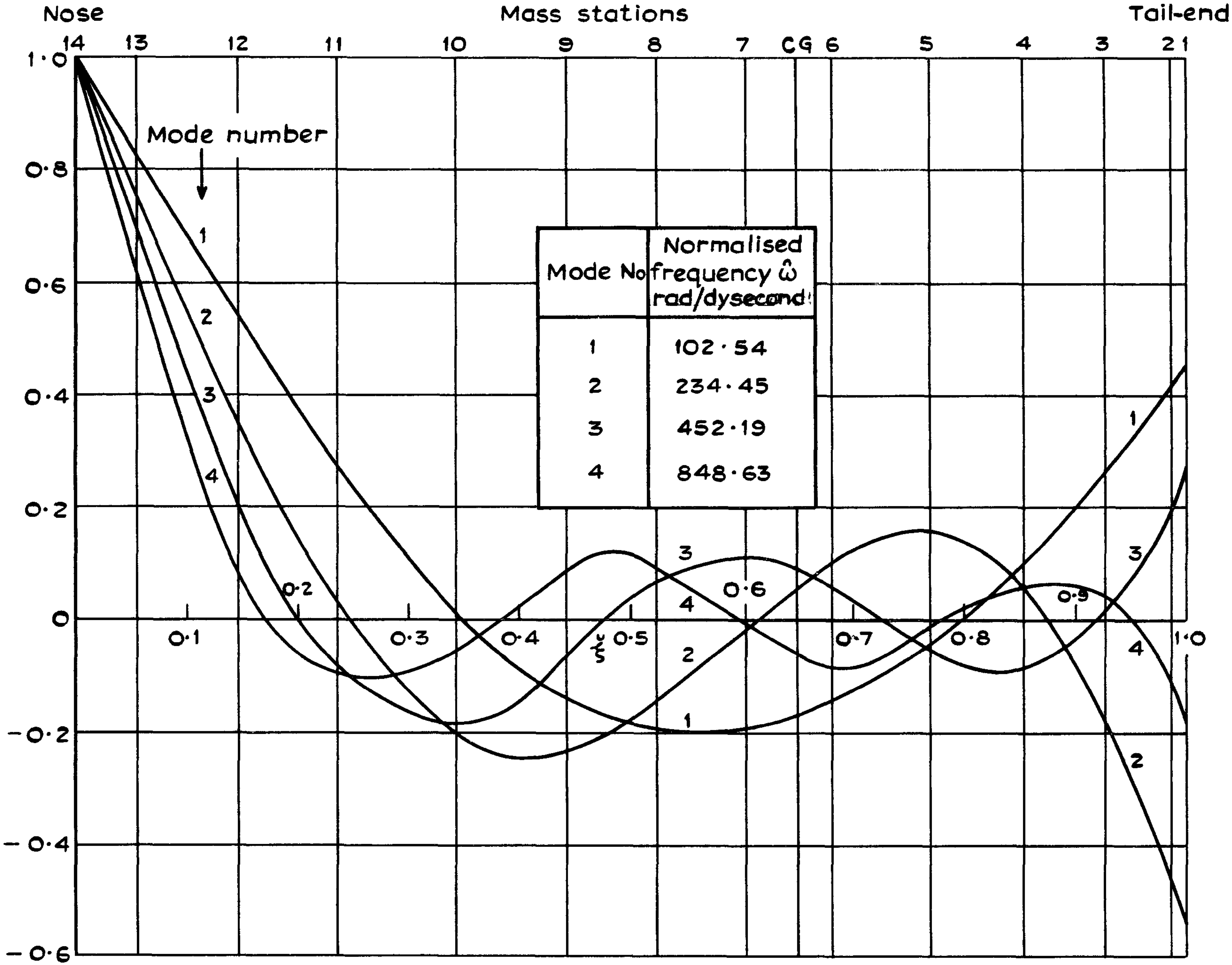


Fig. 4 Shapes and frequencies of the first four natural modes

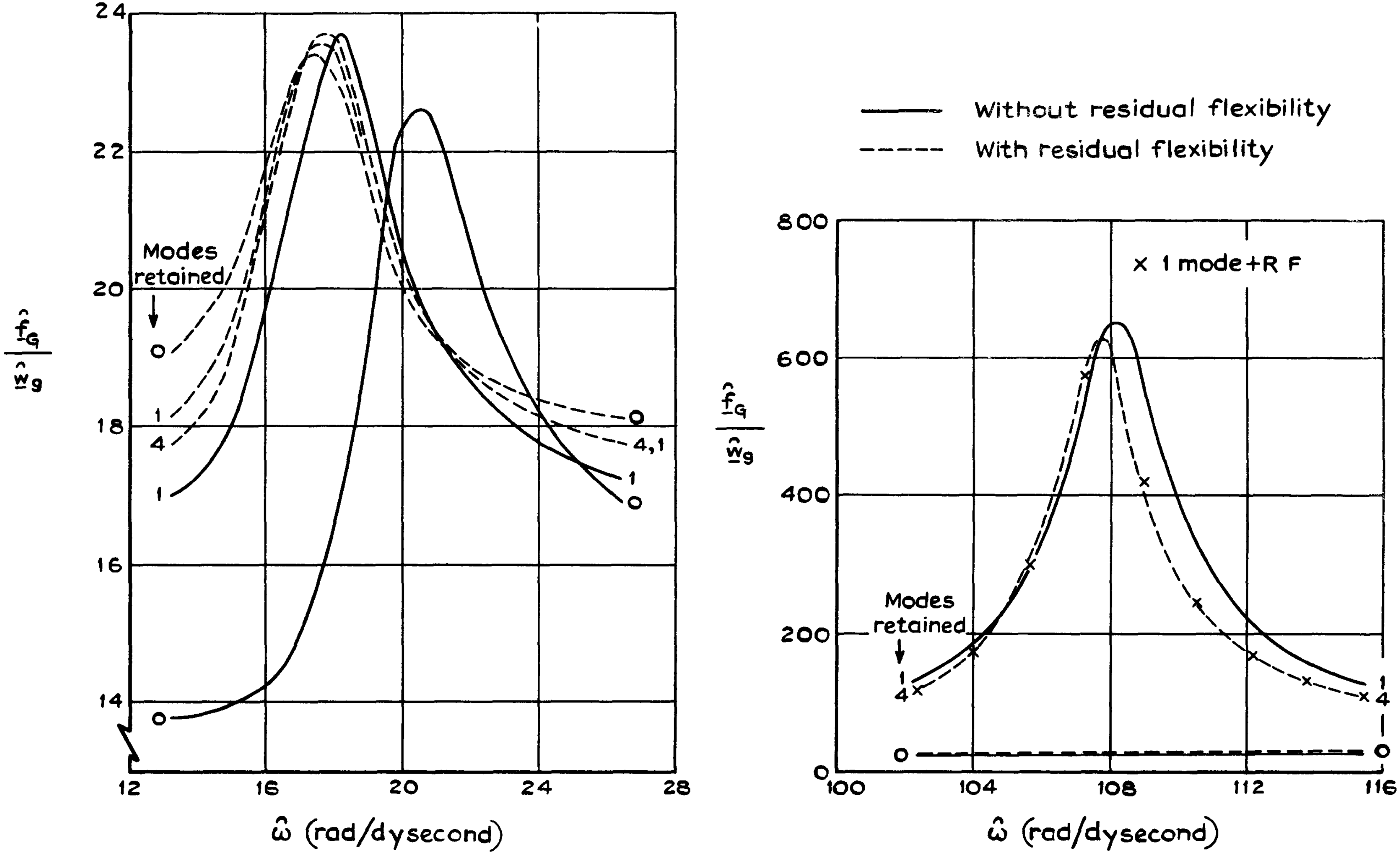


Fig. 5 Acceleration response at St. 14 (nose) to sinusoidal gusts of frequencies near those of the SP and 1st structural modes : amplitude

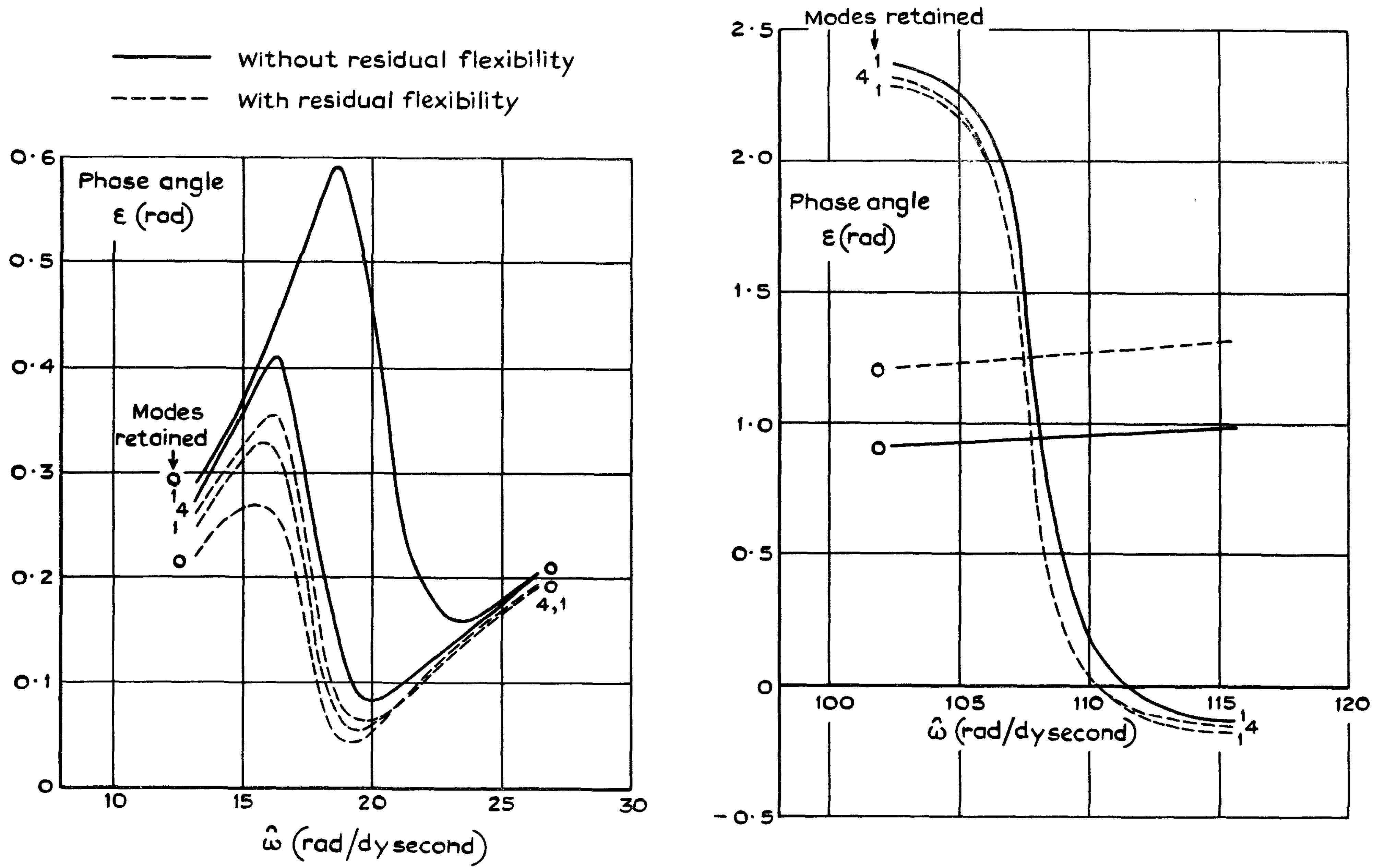


Fig. 6 Acceleration response at St. 14 (nose) to sinusoidal gusts of frequencies near those of the SP and 1st structural modes : phase

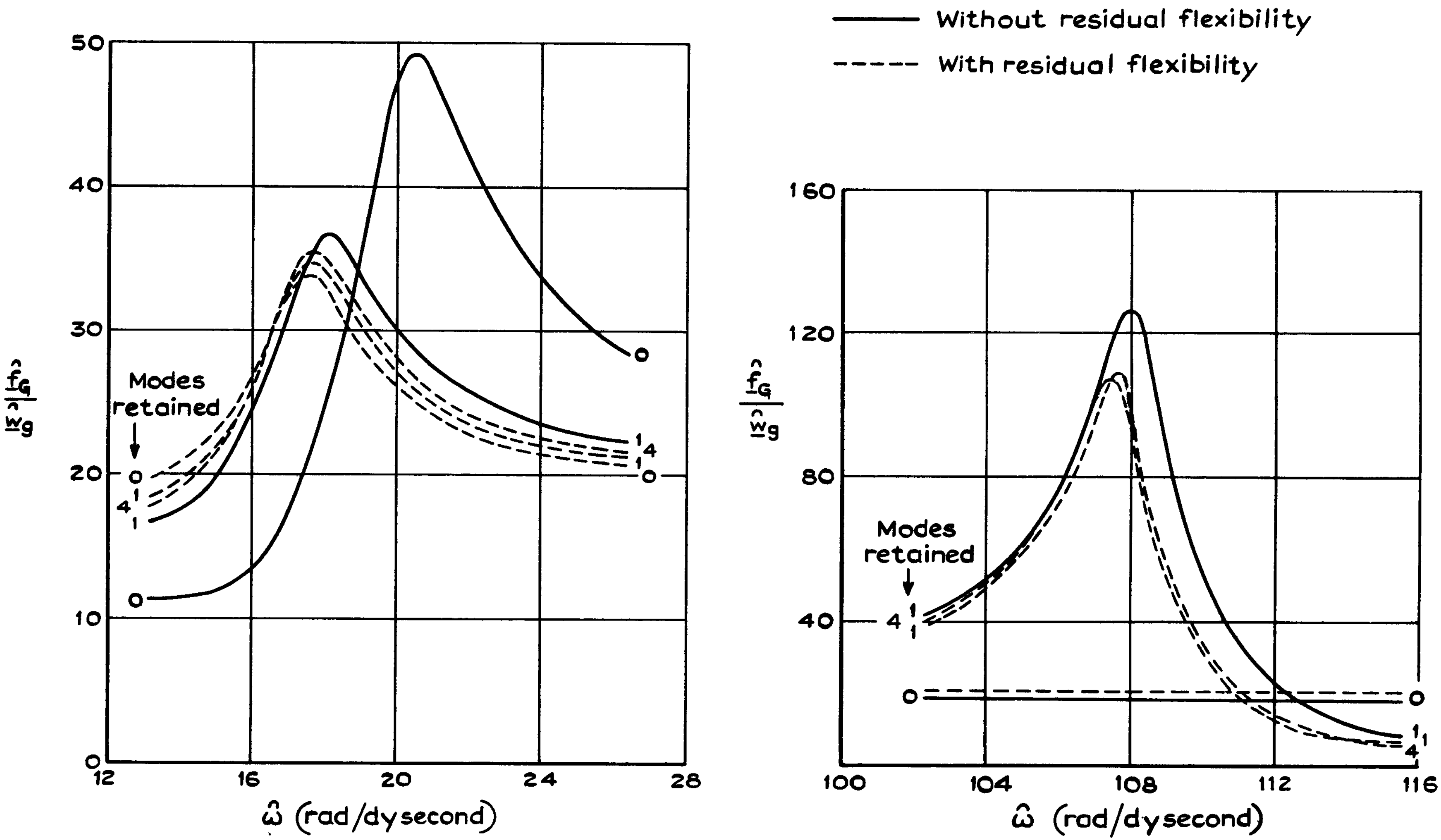


Fig.7 Acceleration response at St.7 (near CG) to sinusoidal gusts of frequencies near those of the SP and 1st structural modes : amplitude

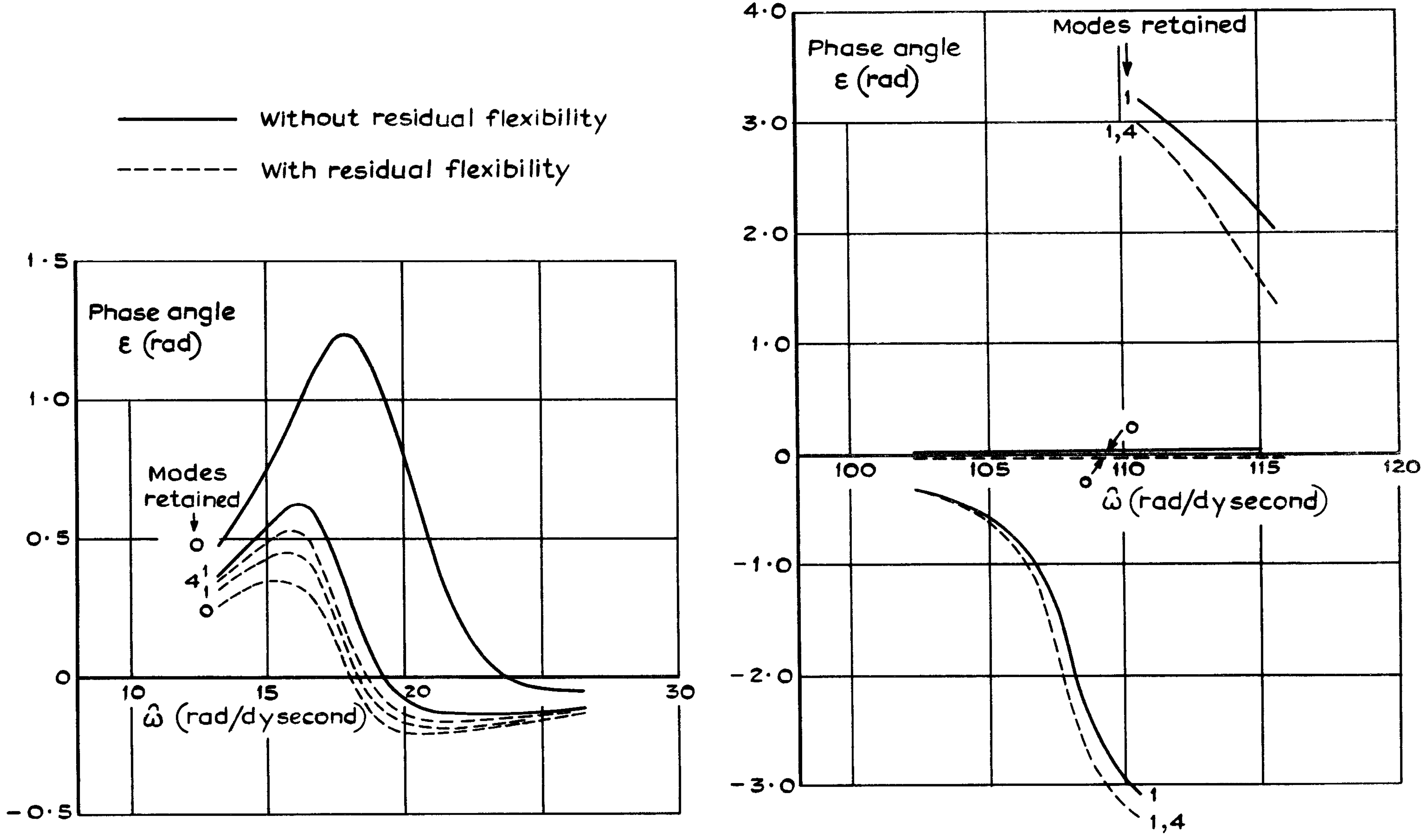


Fig. 8 Acceleration response at St. 7 (near CG) to sinusoidal gusts of frequencies near those of the SP and 1st structural modes: phase

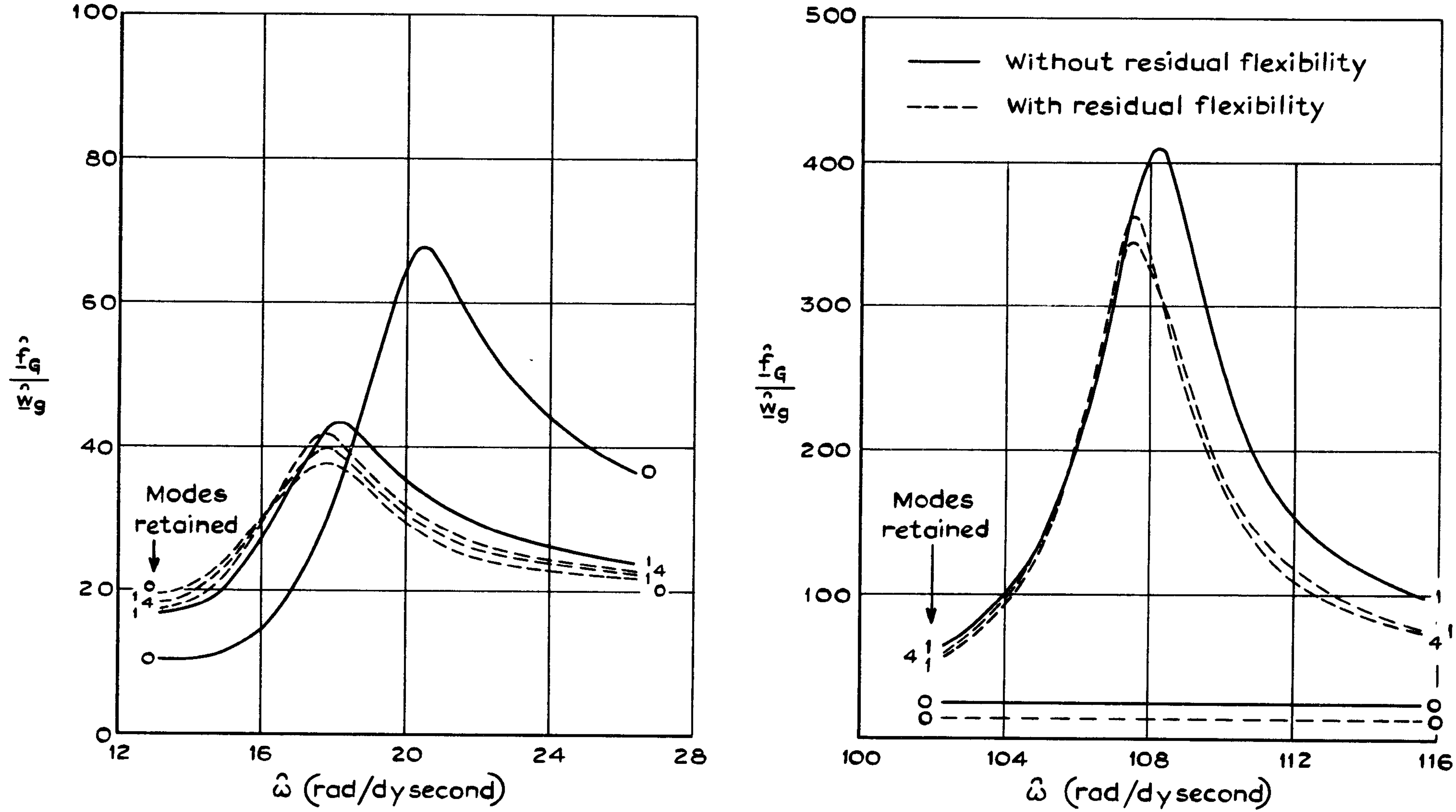


Fig.9 Acceleration response at St.1 (tail-end) to sinusoidal gusts of frequencies near those of the SP and 1st structural modes: amplitude

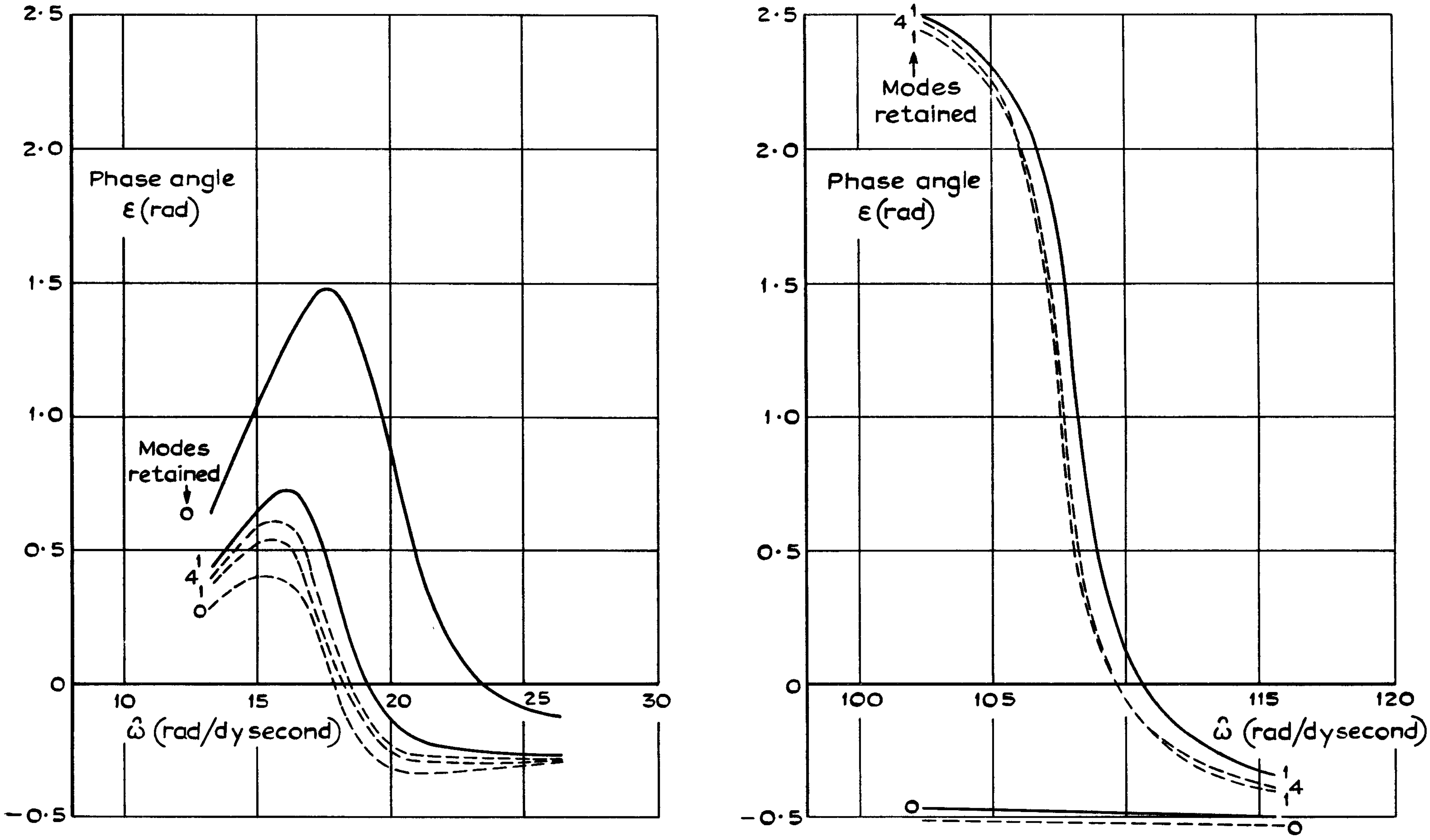
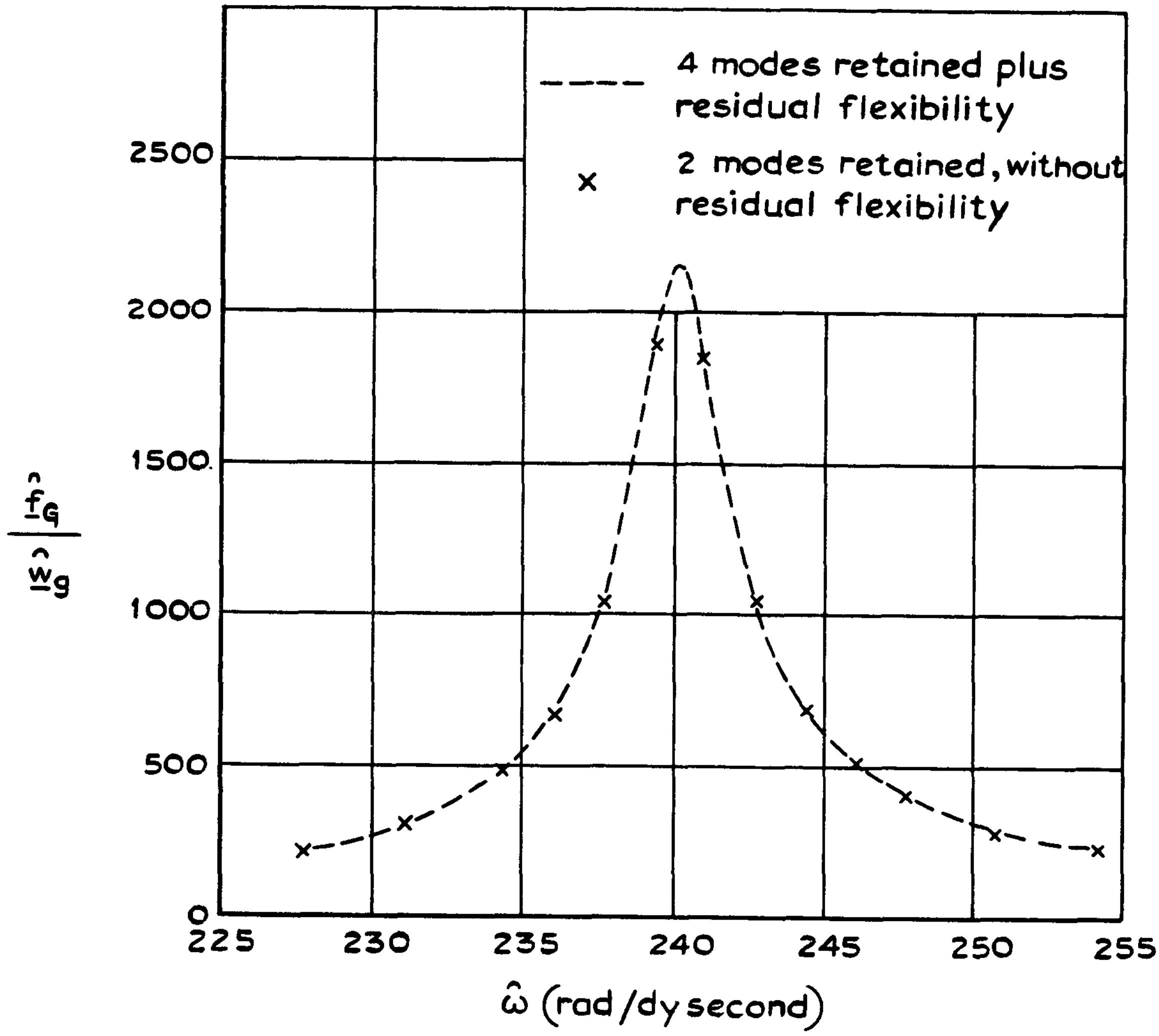
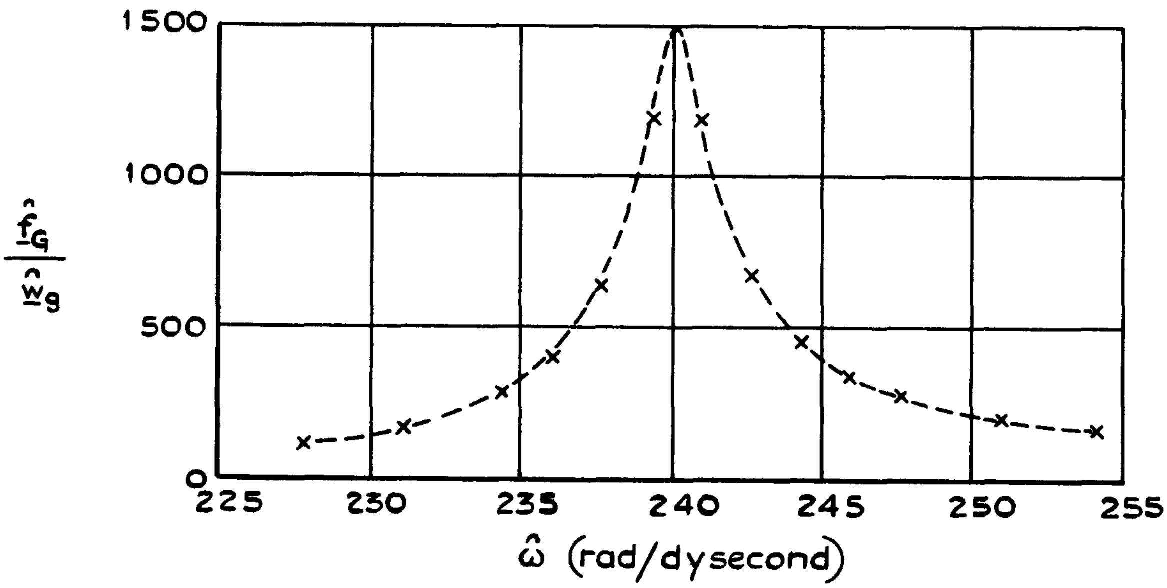


Fig. 10 Acceleration response at St. 1 (tail-end) to sinusoidal gusts of frequencies near those of the SP and 1st structural modes: phase



a Station 14 (nose)



b Station 1 (tail-end)

Fig.11 a&b Amplitude of the acceleration response at stations 1 & 14 to sinusoidal gusts of frequencies near that of the 2nd structural mode

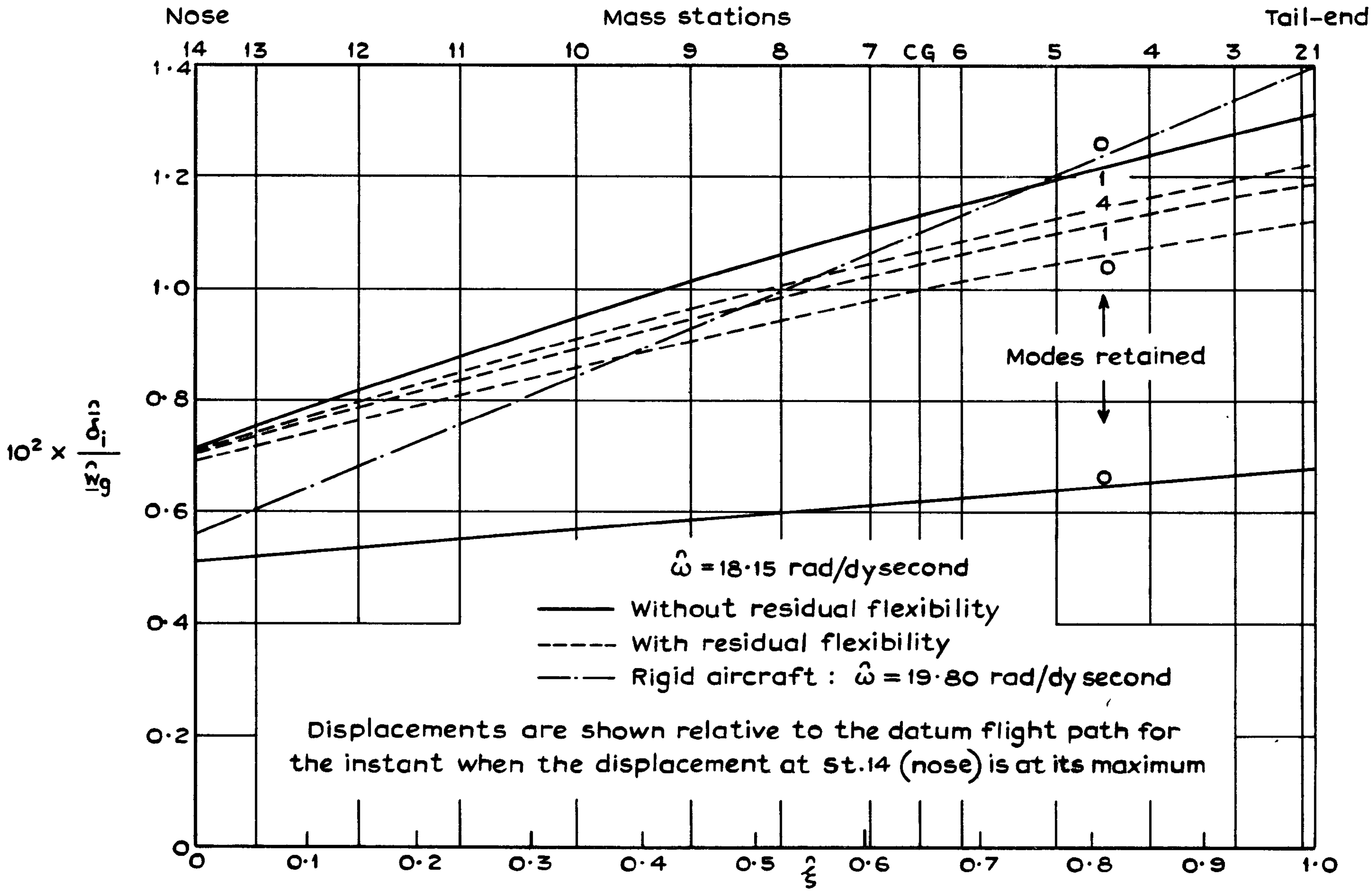
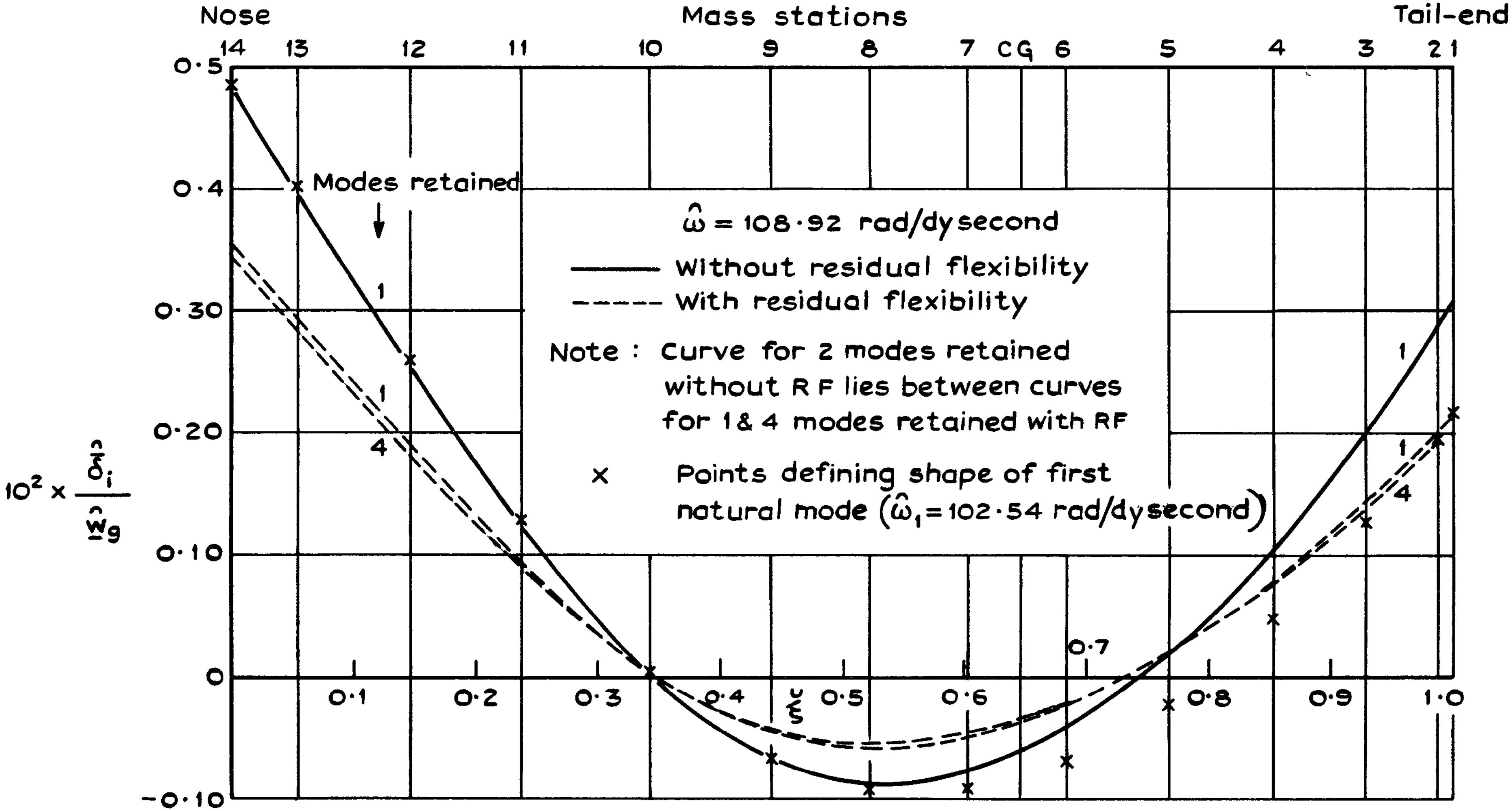


Fig. 12 Response to sinusoidal gust of frequency near to SP resonant frequency : displacements from datum flight path



Displacements are shown relative to the datum flight path for the instant when the displacement at St.14 (nose) is at its maximum

Fig. 13 Response to sinusoidal gust of frequency near to first structural reasonant frequency: displacements from datum flight path

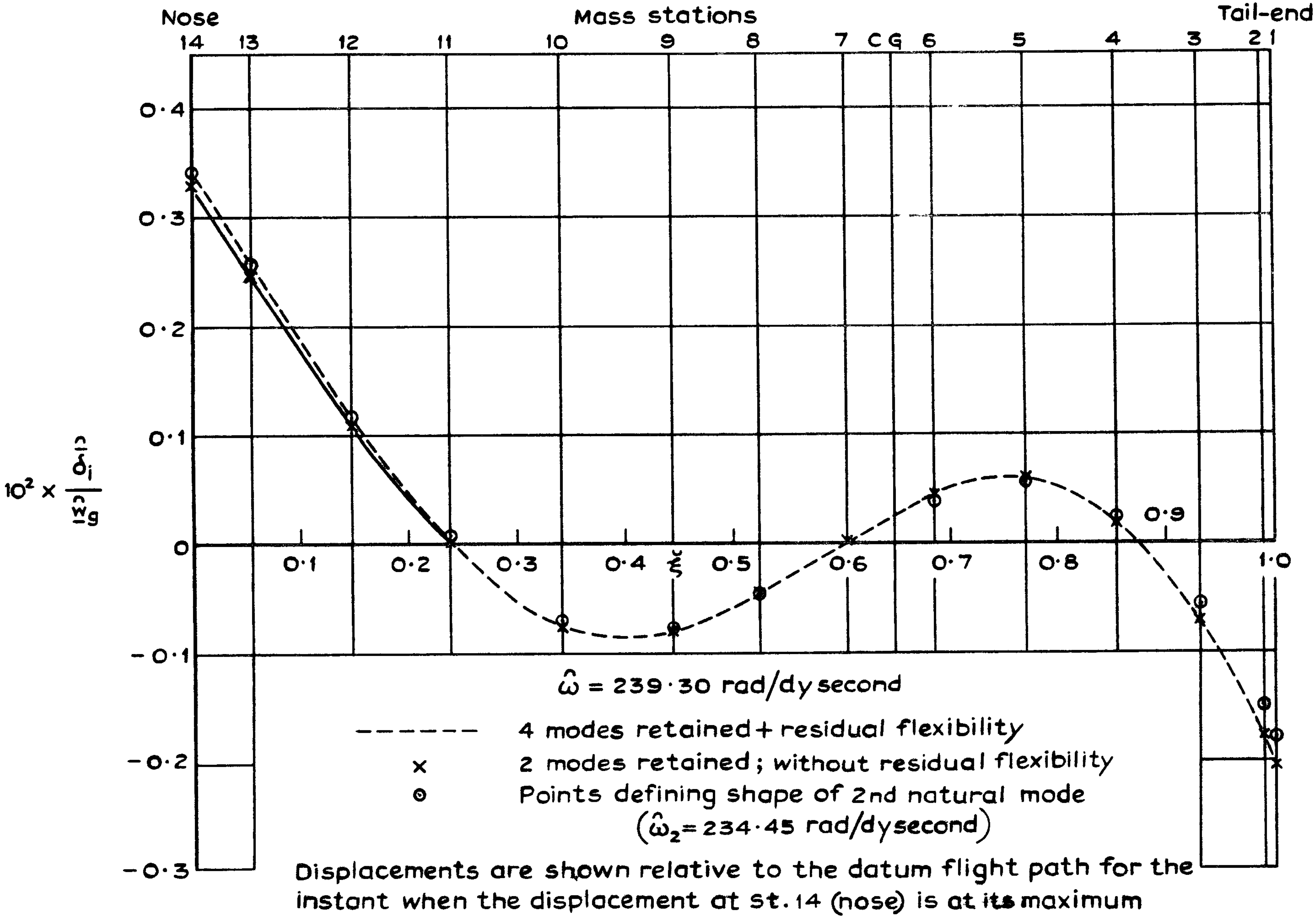


Fig. 14 Response to sinusoidal gust of frequency near to second structural resonant frequency: displacements from datum flight path

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ARC CP No.1336 533.6.013.42 :
July 1973 533.6.013 :
532.511

Taylor, A. S.
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AN ASSESSMENT OF THE IMPORTANCE OF THE RESIDUAL FLEXIBILITY OF NEGLECTED MODES IN THE DYNAMICAL ANALYSIS OF DEFORMABLE AIRCRAFT

The mathematical framework for a unified approach to the dynamical problems of deformable aircraft, set up by Taylor in his contribution to R & M 3776, is used as the basis for a limited numerical investigation of the usefulness of the residual flexibility concept in truncated modal analyses. A finite-element model of a supersonic transport aircraft of slender-delta configuration is the subject of stability and response calculations, in which various representations of the structural deformability are used. These comprise up to four natural modes both with and without the residual flexibility of the remaining modes.

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C.P. No. 1336

1976

Published by
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Southey House, Wine Street, Bristol BS1 2BQ
258 Broad Street, Birmingham B1 2HE
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C.P. No. 1336

ISBN 011 470946 7