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by

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THE FLUTTER OF A TWO-DIMENSIONAL WING WITH SIMPLE AERODYNAMICS

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Ll. T. Niblett

SUMMARY

The flutter stability of a rigid wing with two degrees-of-freedom and subject to the simplest aerodynamic forces including damping is considered. The limits of combinations of nodal axis positions which can lead to flutter are found and a fairly simple expression from which the flutter speed can be found is given. The results are compared with those from simple frequency-coalescence theory. The comparison shows that the present theory indicates that flutter will occur more extensively than indicated by frequency-coalescence theory both in terms of nodal axis combinations and range of airspeed.

* Replaces RAE Technical Report 75008 - ARC 36164

CONTENTS

I INTRODUCTION

Large flexibly-mounted stores can have a considerable effect on the flutter stability of an aircraft wing. The flutter speed of the wing, store combination will vary not only with the inertial properties of the store but also with the flexibilities of the connections. Since these latter are difficult to estimate it is generally thought prudent, in the design of aircraft, to calculate flutter speeds for enough values of the mounting stiffnesses to cover all those probable as well as covering the range of inertial properties of the stores. Whilst there is a case for treating specific aircraft problems in this way the results obtained add little to the general knowledge of the effect of wing stores on flutter. From this point of view it might be more profitable to split an investigation in two; one part being the determination of the effect of the stores on the deflections of the wing in the normal modes of the aircraft and the other the determination of the effect of the modal shapes of the wing on the flutter stability - for in general the wing is effectively the sole source of aerodynamic force. What follows is concerned with the second part of the problem.

It was thought that the investigation of the heave, pitch flutter of a rigid wing under aerodynamic forces as given by Minhinnick derivatives (see section 6.2) would be a useful preliminary study of the effect of wing deflection shapes on flutter, combining the simplest structural assumptions with the simplest credible aerodynamic assumptions to comprehend the damping forces even though the concept of a typical section would be needed to apply the results to a deforming wing.

In what follows the theory is first developed for the flutter of a rigid wing which can pitch about two spanwise axes, in modes which are orthogonal with respect to the structural mass and stiffness, under the general type of aerodynamic forces to which Minhinnick and piston-theory derivatives[1] belong. The ranges of combinations of pitching axes positions over which flutter is possible are found and it is shown that the stiffness at which flutter occurs at a nominated critical equivalent speed is the sum of two terms, one independent of the relative density of the body and the fluid and the other, which always reduces the stiffness, linear in the relative density. A comparison is then made with the results of applying frequency-coalescence theory, in which the damping terms are omitted, to the same system. The stability of the system is also examined with the help of a graphical representation[2] of the flutter equations.

2 EQUATIONS OF PARTICULAR SYSTEM

2.1 Basic flutter equation

The system considered is a two-dimensional rigid wing which can pitch about two axes. The wing's positions at any particular time is described in terms of two generalised coordinates which are orthogonal with respect to inertia and structural stiffness. The unit amplitudes in the generalised coordinates are such that each generalised inertia coefficient is unity when the equations have been made non-dimensional.

The flutter equation, for unit span, can be written

$$\left| -\mu\ell^4\omega^2 I + i\rho\ell^3\omega\upsilon B + \rho\ell^2\upsilon^2 C + \mu\ell^4\omega_0^2 \left\lceil \hat{\omega}^2 \right\rfloor \right| = 0 \qquad (1)$$

where μ is a nominal wing mass density, (mass per unit span)/ℓ^2

ℓ is the chord of the wing

ω is the flutter frequency

I is the unit matrix

ρ is the air density

υ is the flutter speed

B and C are real square matrices of non-dimensional aerodynamic coefficients

ω_0 is a nominal frequency, and

$\omega_0^2 \left\lceil \hat{\omega}^2 \right\rfloor = \left\lceil \omega_1^2, \omega_2^2 \right\rfloor$ where ω_1 and ω_2 are the still-air frequencies pertinent to the normal coordinates, ω_2 being the higher.

Dividing each element by $\rho\ell^2\upsilon^2$ and substituting σ for ρ/μ , λ for $i\mu^{\frac{1}{2}}\ell\omega(\rho^{\frac{1}{2}}\upsilon)^{-1}$ and χ for $\mu\ell^2\omega_0^2(\rho\upsilon^2)^{-1}$ the flutter equation can be written

$$\left| I\lambda^2 + \sigma^{\frac{1}{2}}B\lambda + C + \chi\left\lceil \hat{\omega}^2 \right\rfloor \right| = 0 \qquad (2)$$

χ is hereinafter called the stiffness number and χ_f its value at a critical flutter speed, i.e. a value that satisfies equation (2), is called the flutter stiffness number. The value of this number can be taken as a measure of the propensity of the system to flutter in that the higher it is the lower will be the critical flutter speed.

2.2 Generalised coordinates

The inertia matrix in equation (1) is diagonal as a consequence of the assumption that the generalised coordinates are orthogonal with respect to inertia. That the inertia matrix is unit implies that the generalised coordinates are normalised in some way.

Each of the generalised coordinates that will be used is a rotation about a nodal axis at some chordwise position. It will be convenient to describe the distance of the nodal axis in front of the aerodynamic axis as $\ell \tan \phi$ and unit amplitude in the unnormalised coordinate as $\cos \phi$.

An inertia coefficient is derived by summing the products of masses and the squares of their deflections when there is unit amplitude in the coordinate. Consider a wing, of mass per unit span M , with an inertia axis a distance x_g behind the aerodynamic axis and radius of gyration about the inertia axis of k . The dimensioned unnormalised inertia coefficient will be

$$M\left\{(\ell \tan \phi + x_g)^2 + k^2\right\} \cos^2 \phi \quad .$$

The factor used to non-dimensionalise the inertia coefficients is $(\mu\ell^4)^{-1}$ (see equation (1)) so the dimensionless unnormalised coefficient will be

$$(M/\mu\ell^2)\left\{(\sin \phi + x_g \ell^{-1} \cos \phi)^2 + k^2 \ell^{-2} \cos^2 \phi\right\} \quad .$$

The normalising factor, κ , is the inverse of the square root of this coefficient and unit deflection in the normalised coordinate is a rotation of $\kappa \cos \phi$.

2.3 Aerodynamic coefficients

Let the aerodynamic matrices B and C be given by

$$\left.\begin{aligned} B &= \begin{bmatrix} \zeta_1 & \alpha_1 \\ \zeta_2 & \alpha_2 \end{bmatrix} \begin{bmatrix} \ell_{\dot{z}} & \ell_{\dot{\alpha}} \\ -m_{\dot{z}} & -m_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \zeta_1 & \zeta_2 \\ \alpha_1 & \alpha_2 \end{bmatrix} \\ C &= \begin{bmatrix} \zeta_1 & \alpha_1 \\ \zeta_2 & \alpha_2 \end{bmatrix} \begin{bmatrix} \ell_z & \ell_\alpha \\ -m_z & -m_\alpha \end{bmatrix} \begin{bmatrix} \zeta_1 & \zeta_2 \\ \alpha_1 & \alpha_2 \end{bmatrix} \end{aligned}\right\} \qquad (3)$$

where $\ell\zeta_r$ is the vertical amplitude of a reference axis and a_r is the incidence for unit amplitude in the rth coordinate. The particular aerodynamic forces used here are those given by aerodynamic derivatives which are related to each other by the equations

$$\begin{bmatrix} \ell_z & \ell_\alpha \\ -m_z & -m_\alpha \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ell_\alpha$$

and (4)

$$\begin{bmatrix} \ell_{\dot z} & \ell_{\dot\alpha} \\ -m_{\dot z} & -m_{\dot\alpha} \end{bmatrix} = \begin{bmatrix} 1 & \beta \\ 0 & \gamma \end{bmatrix} \ell_\alpha \quad .$$

The derivatives are thus referred to an axis which is both the aerodynamic axis (m_a = 0) and the axis of independence ($m_{\dot z}$ = 0).

Let the non-dimensional vertical deflection of this reference axis and the pitch of the wing be given by

$$\begin{bmatrix} \zeta \\ \alpha \end{bmatrix} = \begin{bmatrix} \kappa_1 \sin\phi_1 & \kappa_2 \sin\phi_2 \\ \kappa_1 \cos\phi_1 & \kappa_2 \cos\phi_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \kappa_1 s_1 & \kappa_2 s_2 \\ \kappa_1 c_1 & \kappa_2 c_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{5}$$

say, where the factors κ_1, κ_2 have been included so that account may be taken of the normalisation of the coordinates to give unit generalised inertias. The ϕ_r lie in the range $-\pi/2 \leqslant \phi_r \leqslant \pi/2$ and the coordinates are rotations about axes $\ell \tan \phi_r$ in front of the reference axes.

Substituting from equations (4) and (5) in equations (3) gives the aerodynamic coefficient matrices as

$$C = \begin{bmatrix} \kappa_1^2 s_1 c_1 & \kappa_1 \kappa_2 s_1 c_2 \\ \kappa_1 \kappa_2 s_2 c_1 & \kappa_2^2 s_2 c_2 \end{bmatrix} \ell_\alpha \tag{6}$$

and

$$|C| = 0$$

and

$$B = \begin{vmatrix} \kappa_1^2(s_1^2 + \beta s_1 c_1 + \gamma c_1^2) & \kappa_1\kappa_2(s_1 s_2 + \beta s_1 c_2 + \gamma c_1 c_2) \\ \kappa_1\kappa_2(s_1 s_2 + \beta s_2 c_1 + \gamma c_1 c_2) & \kappa_2^2(s_2^2 + \beta s_2 c_2 + \gamma c_2^2) \end{vmatrix} \ell_\alpha \tag{7}$$

and

$$|B| = \begin{vmatrix} \kappa_1 s_1 & \kappa_2 s_2 \\ \kappa_1 c_1 & \kappa_2 c_2 \end{vmatrix}^2 \begin{vmatrix} 1 & \beta \\ 0 & \gamma \end{vmatrix} \ell_\alpha^2 = \kappa_1^2 \kappa_2^2 (s_1 c_2 - s_2 c_1)^2 \gamma \ell_\alpha^2 \ .$$

Further,

$$|B + C| = \kappa_1^2 \kappa_2^2 (s_1 c_2 - s_2 c_1)^2 \begin{vmatrix} 1 & 1 + \beta \\ 0 & \gamma \end{vmatrix} \ell_\alpha^2 = |B| \ . \qquad (8)$$

Hence,

$$\begin{vmatrix} b_{11} & b_{12} \\ c_{21} & c_{22} \end{vmatrix} + \begin{vmatrix} c_{11} & c_{12} \\ b_{21} & b_{22} \end{vmatrix} = |B + C| - |B| - |C| = 0 \ . \qquad (9)$$

2.4 Conditions for stability

If the expansion of equation (2) is written

$$\lambda^4 + p_1 \lambda^3 + p_2 \lambda^2 + p_3 \lambda + p_4 = 0 \ , \qquad (10)$$

the full conditions for stability are that all the p coefficients and

$$T_3 (= p_1 p_2 p_3 - p_1^2 p_4 - p_3^2)$$

are positive. From equations (6) to (9)

$$\left.\begin{aligned} p_1 &= p_{10} = \sigma^{\frac{1}{2}} (b_{11} + b_{22}) \\ p_2 &= p_{20} + \chi p_{21} = c_{11} + c_{22} + \sigma |B| + \chi (\hat{\omega}_1^2 + \hat{\omega}_2^2) \\ p_3 &= \chi p_{31} = \sigma^{\frac{1}{2}} \chi (b_{11} \hat{\omega}_2^2 + b_{22} \hat{\omega}_1^2) \\ p_4 &= \chi (p_{41} + \chi p_{42}) = \chi (c_{11} \hat{\omega}_2^2 + c_{22} \hat{\omega}_1^2 + \chi \hat{\omega}_1^2 \hat{\omega}_2^2) \end{aligned}\right\} \qquad (11)$$

$$T_3 = \chi\left\{p_{10}p_{20}p_{31} - p_{10}^2 p_{41} + \chi(p_{10}p_{21}p_{31} - p_{10}^2 p_{42} - p_{31}^2)\right\} =$$

$$= \sigma\chi\Big[(b_{11} + b_{22})\left\{(c_{11} + c_{22} + b)(b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2) - (b_{11} + b_{22})(c_{11}\hat{\omega}_2^2 + c_{22}\hat{\omega}_1^2)\right\}$$

$$+ \chi\Big\{(b_{11} + b_{22})\left[(\hat{\omega}_1^2 + \hat{\omega}_2^2)(b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2) - (b_{11} + b_{22})\hat{\omega}_1^2\hat{\omega}_2^2\right]$$

$$- (b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)^2\Big\}\Big] =$$

$$= \sigma\chi\Big[(b_{11} + b_{22})\left\{(c_{11}b_{22} - c_{22}b_{11})(\hat{\omega}_1^2 - \hat{\omega}_2^2) + b(b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)\right\}$$

$$+ \chi b_{11}b_{22}(\hat{\omega}_2^2 - \hat{\omega}_1^2)^2\Big] \qquad (12)$$

where $b \equiv \sigma|B|$.

If χ , which must obviously be positive, is varied continuously from an initial stable condition flutter occurs at the first value which satisfies $T_3 = 0$. At low airspeeds χ is large and the sign of T_3 is the same as that of the coefficient of χ^2 in equation (12) and will be positive if single degree-of-freedom instability is absent. There are two values of χ for which $T_3 = 0$. One is zero which indicates that all systems tend to neutral stability as the airspeed tends to infinity. The other is given by

$$\chi_f = \frac{(b_{11} + b_{22})\left\{(c_{11}b_{22} - c_{22}b_{11}) - b(b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)(\hat{\omega}_2^2 - \hat{\omega}_1^2)^{-1}\right\}}{b_{11}b_{22}(\hat{\omega}_2^2 - \hat{\omega}_1^2)}$$

$$= \frac{(\kappa_1^2 b_1 + \kappa_2^2 b_2)\ell_\alpha\left\{(s_1c_2 - s_2c_1)(\gamma c_1c_2 - s_1s_2) - \sigma\gamma\ell_\alpha(s_1c_2 - s_2c_1)^2(\kappa_1^2 b_1\hat{\omega}_2^2 + \kappa_2^2 b_2\hat{\omega}_1^2)(\hat{\omega}_2^2 - \hat{\omega}_1^2)^{-1}\right\}}{b_1b_2(\hat{\omega}_2^2 - \hat{\omega}_1^2)}$$

$$= \frac{(\kappa_1^2 b_1 + \kappa_2^2 b_2)\ell_\alpha F_0}{b_1b_2(\hat{\omega}_2^2 - \hat{\omega}_1^2)}\left\{1 - \sigma\gamma\ell_\alpha f_\sigma\,\frac{(\kappa_1^2 b_1\hat{\omega}_2^2 + \kappa_2^2 b_2\hat{\omega}_1^2)}{(\hat{\omega}_2^2 - \hat{\omega}_1^2)}\right\}, \qquad (13)$$

after substitution from equations (6) and (7) and with

$$b_r = s_r^2 + \beta s_r c_r + \gamma c_r^2$$

$$FO = (s_1c_2 - s_2c_1)(\gamma c_1c_2 - s_1s_2)$$

and

$$f_\sigma = (s_1c_2 - s_2c_1)(\gamma c_1c_2 - s_1s_2)^{-1} .$$

The flutter frequency, ω_f , is given by the solution of the imaginary part of equation (10) when λ is purely imaginary, i.e.

$$(b_{11} + b_{22})\lambda_f^2 + (b_{11}\hat\omega_2^2 + b_{22}\hat\omega_1^2)\chi_f = 0$$

$$\left(\frac{\omega_f}{\omega_\alpha}\right)^2 = \left(-\frac{\lambda_f^2}{\chi_f}\right) = \frac{b_{11}\hat\omega_2^2 + b_{22}\hat\omega_1^2}{b_{11} + b_{22}} . \qquad (14)$$

There is a possibility that the system diverges steadily. This first happens at an airspeed which corresponds to a value of χ which satisfies $p_4 = o$. Again the system is stable at low airspeeds since the coefficient of χ^2 is positive (last of equations (11)). The divergence stiffness number is given by

$$\chi_d = -\frac{c_{11}\hat\omega_2^{-2} + c_{22}\hat\omega_1^{-2}}{\hat\omega_1^2\hat\omega_2^2}$$

$$= -\frac{(\kappa_1^2 s_1 c_1\hat\omega_2^2 + \kappa_2^2 s_2 c_2\hat\omega_1^2)}{\hat\omega_1^2\hat\omega_2^2} \qquad (15)$$

3 UPPER BOUND OF FLUTTER STIFFNESS NUMBER

The flutter stiffness number, χ_f , has the form of a typical structural stiffness divided by the dynamic head at a critical flutter speed and hence, if the structural stiffness and air density are constant, the flutter speed varies inversely as its square root. Equation (13) gives the flutter stiffness number as the difference between two terms, the second of which has the density of the fluid relative to the body, a , as a factor. It can be seen from the penultimate form of the equation that each of the factors in this second term is positive since $4\gamma > \beta^2$ (equation (4)) for freedom from instability in a single degree-of-freedom and that the flutter stiffness number decreases with increase in the relative density σ . Thus the flutter speed, as an equivalent airspeed, increases with increase in relative density and indeed a flutter might be eliminated by such an increase. Therefore if the second term is taken to be zero an upper bound of the flutter stiffness number, corresponding to a lower bound of the equivalent flutter speed is obtained.

The abbreviation of equation (13) which gives an upper bound of the flutter stiffness number can be written

$$X_f = \frac{\mu\omega_0^2\ell^2}{\rho v_f^2} = \frac{\ell_\alpha}{(\omega_2^2 - \omega_1^2)}\left(\frac{\kappa_1^2}{b_2} + \frac{\kappa_2^2}{b_1}\right) F_0 \qquad (16)$$

The following comments are of interest in the context of the evaluation of structural stiffnesses from measurements of natural frequencies and generalised inertias in still-air resonance tests. If the mass density of the wing is changed whilst its stiffnesses and nodal axes remain the same the flutter speed v_f will be constant for ω_0^2 will vary inversely as the mass density. But if the frequencies rather than the stiffnesses remain the same the flutter speed will vary as the square root of the mass density. Thus uniform inaccuracies in the generalised inertias assumed for the modes have no effect on the estimate of the lower bound of the flutter speed unless they are used in conjunction with the natural frequencies to obtain the structural stiffnesses. Non-uniform inaccuracies affect the κ and through these the estimate of the lower bound.

The κ were introduced in section 2.2 to allow for the normalisation of the unit amplitudes in the coordinates with a consequent simplification of the flutter equation. From the expressions given in that section it can be seen that unit amplitude before normalisation tends to a pure heave deflection of a chord's length as the distance between the nodal and aerodynamic axes tends to infinity and is a pitch deflection of one radian when the nodal and aerodynamic axes are coincident. For a wing of mass per unit span of $\mu\ell^2$, κ is unity in the first case and the inverse of the radius of gyration about the aerodynamic axis (in chords) in the second. If the mass per unit span is $\mu\ell^2\bar{M}$, these values are factored by $\bar{M}^{-\frac{1}{2}}$.

If $\hat{\omega}_2$ is always the larger of the $\hat{\omega}$ the terms on the right-hand side of equation (16) are all necessarily positive apart from F_0 ; ℓ_α , $(\hat{\omega}_2^2 - \hat{\omega}_1^2)$, κ_1^2 and κ_2^2 by definition and b_1 and b_2 because only systems free from single degree-of-freedom instabilities are considered. Thus the system will be assuredly stable* if F_0 is negative.

F_0 , it will be remembered (equation (13)) is $(s_1c_2 - s_2c_1)(\gamma c_1c_2 - s_1s_2)$ and will have the same sign as $(t_1 - t_2)(\gamma - t_1t_2)$. Hence the stability boundaries in the $\phi_1 - \phi_2$ plane are given by $\phi_1 = \phi_2$ and $\tan\phi_1 \tan\phi_2 = \gamma$. When γ is zero the second of these equations reduces to $\phi_1\phi_2 = 0$, so the axes themselves are boundaries. $(t_1 - t_2)$ is negative in the second to fifth octants inclusive and $(-t_1t_2)$ is negative in the first, second, fifth and sixth octants. Thus stability is assured for systems represented by points in the first, third, fourth and sixth octant: (see Fig.2); i.e. when the nodal line of the graver mode is downwind of the reference axis and the other nodal line is upwind of the axis or when both nodal lines are on the same side of the axis and that of the graver mode is the further upwind.

When γ is positive the axes are replaced as stability boundaries by two curves, one of which lies wholly in the first quadrant and the other completely in the third (see Fig.3). This change in the boundaries changes the stability position only when both nodal lines are on one side of the reference axis and one of them is sufficiently close to the axis.

4 COMPARISON WITH FREQUENCY-COALESCENCE THEORY

The values of χ at flutter which result from the application of frequency-coalescence theory are those for which $p_2^2 = 4p_4$, with b taken as zero (equations (10) and (11)), i.e. the values given by

$$\left\{c_{11} + c_{22} + \chi(\hat{\omega}_1^2 + \hat{\omega}_2^2)\right\}^2 = 4\chi\left\{c_{11}\hat{\omega}_2^2 + c_{22}\hat{\omega}_1^2 + \chi\hat{\omega}_1^2\hat{\omega}_2^2\right\} \quad (17)$$

which can be reduced to

$$(\hat{\omega}_2^2 - \hat{\omega}_1^2)^2\chi^2 - 2(c_{11} - c_{22})(\hat{\omega}_2^2 - \hat{\omega}_1^2)\chi + (c_{11} + c_{22})^2 = 0 \quad (18)$$

$$(\omega_2^2 - \omega_1^2)\chi_{fc} = c_{11} - c_{22} \pm \sqrt{(c_{11} - c_{22})^2 - (c_{11} + c_{22})^2}$$

$$= c_{11} - c_{22} \pm 2\sqrt{-c_{11}c_{22}} \quad (19)$$

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* 'Assuredly stable' means here 'stable at all airspeeds'.

$$(\hat{\omega}_2^2 - \hat{\omega}_1^2)\ell_\alpha^{-1}\chi_{fc} = \kappa_1^2 s_1 c_1 - \kappa_2^2 s_2 c_2 \pm 2\kappa_1\kappa_2\sqrt{-s_1 c_1 s_2 c_2} \quad . \qquad (20)$$

The condition for χ_{fc} to be real is that $c_{11}c_{22}$ is negative which means that ϕ_1 and ϕ_2 have to be of opposite sign. Now $(c_{11} - c_{22})^2$ is not less than $(-4c_{11}c_{22})$ and χ_{fc} will have the sign of $(c_{11} - c_{22})$. Hence the conditions for a positive real value of χ_{fc} are that c_{11} must be positive and c_{22} negative, i.e. the point in the $\phi_1 - \phi_2$ plane representing the system must lie in the fourth quadrant. For χ_{fc} to correspond to true flutter there is a further condition which is that the pertinent λ^2 must be negative, i.e. the frequency must be real.

$$\lambda_{fc}^2 = -\frac{P_2}{2}$$

and, using equation (19) to substitute for χ_f, the flutter frequency is given by

$$\left(\frac{\omega_{fc}}{\omega_0}\right)^2 = \left(-\frac{\lambda_{fc}^2}{\chi_{fc}}\right) = \frac{P_{20} + P_{21}\chi_{fc}}{2\chi_{fc}} =$$

$$= \frac{(c_{11} + c_{22})(\hat{\omega}_2^2 - \hat{\omega}_1^2) + (\hat{\omega}_2^2 + \hat{\omega}_1^2)\left\{c_{11} - c_{22} \pm 2\sqrt{-c_{11}c_{22}}\right\}}{2\left\{c_{11} - c_{22} \pm 2\sqrt{-c_{11}c_{22}}\right\}} =$$

$$= \frac{c_{11}\hat{\omega}_2^2 - c_{22}\hat{\omega}_1^2 \pm (\hat{\omega}_2^2 + \hat{\omega}_1^2)\sqrt{-c_{11}c_{22}}}{c_{11} - c_{22} \pm 2\sqrt{-c_{11}c_{22}}} \qquad (21)$$

The sign of this depends solely on the sign of the numerator since the denominator is necessarily positive in the cases considered. The sign of the numerator will be positive for the higher χ_{fc} since $(c_{11}\hat{\omega}_2^2 - c_{22}\hat{\omega}_1^2)$ is positive. The lower χ_{fc} will not be meaningful if $-c_{11}c_{22}(\hat{\omega}_2^2 + \hat{\omega}_1^2)^2$ is greater than $(c_{11}\hat{\omega}_2^2 - c_{22}\hat{\omega}_1^2)^2$, a condition that can be reduced to

$$-c_{22}\hat{\omega}_1^4 < c_{11}\hat{\omega}_2^4 < -c_{22}\hat{\omega}_2^4 \quad . \qquad (22)$$

The expressions for χ_f given by frequency-coalescence and the present theory with zero mass density ratio σ seem to have few points in common which is perhaps not surprising since frequency-coalescence theory takes no account of the damping terms which are prominent in equation (16). Frequency-coalescence theory says that only systems represented by points in the fourth quadrant of the $\phi_1 - \phi_2$ plane are flutter-prone whilst the present theory includes parts of the first and third quadrants as well. Coalescence theory says that there will be an upper critical speed under certain circumstances whilst the present theory says that all systems will tend to oscillations of constant amplitude when the airspeed approaches infinity. Comparing (16) and (20) it will be seen that χ_{fc} is given by a more complicated expression than χ_f in that it is not possible to extract an expression as simple as that for F_0.

The result's for systems in which the graver mode is pure heave $(c_{11} = 0)$ typify the kind of discrepancy that exists between the two theories. Substitution of zero for c_{11} in equation (19) immediately gives the square of the frequency-coalescence flutter speed as proportional to $(-c_{22})^{-1}$. Substitution for c_{11} in the first form of equation (13) (with b zero) gives the square of the flutter speed as proportional to $b_{22}(b_{11} + b_{22})^{-1}(-c_{22})^{-1}$. Further in the case of coalescence theory the upper and lower speeds are identical and there is no speed at which the amplitude of the oscillation grows.

The relationship between frequency-coalescence and more comprehensive theories of flutter is clarified by a graphical representation of the flutter equations and this is examined next.

5 GRAPHICAL REPRESENTATION

A graphical representation[2] of the flutter equations obtained by tracing the curves whose equations are the real and imaginary parts of equation (10) when λ is purely imaginary has been found to be an aid in distinguishing between types of flutter. In the present case the equations of the curves can be written.

$$\left.\begin{aligned} \lambda^4 + p_2\lambda^2 + p_4 &= 0 \\ p_1\lambda^2 + p_3 &= 0 \end{aligned}\right\} \qquad (23)$$

If $|c_{11} - c_{22}| \gg \sigma|B|$, p_2 can be approximated to be $(c_{11} + c_{22}) + (\hat{\omega}_1^2 + \hat{\omega}_2^2)\chi$ and the two equations can be written

$$\left.\begin{aligned} \lambda^4 + \left\{(c_{11} + c_{22}) + (\bar{\omega}_1^2 + \hat{\omega}_2^2)\chi\right\}\lambda^2 + \left\{(c_{11}\bar{\omega}_2^2 + c_{22}\bar{\omega}_1^2) + \bar{\omega}_1^2\bar{\omega}_2^2\chi\right\}\chi &= 0 \\ (b_{11} + b_{22})\lambda^2 + (b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)\chi &= 0 . \end{aligned}\right\} \quad (24)$$

Then, remembering that $\lambda\chi^{-\frac{1}{2}} = i\hat{\omega}$, replacing χ^{-1} by y and choosing $\omega_0^2 = \omega_1^2 + \omega_2^2$, equivalents of the equations can be written

$$\left.\begin{aligned} \hat{\omega}^4 - (c_{11} + c_{22})\hat{\omega}^2 y - \hat{\omega}^2 + (c_{11}\hat{\omega}_2^2 + c_{22}\hat{\omega}_1^2)y + \hat{\omega}_1^2\hat{\omega}_2^2 &= 0 \\ (b_{11} + b_{22})\hat{\omega}^2 &= b_{11}\hat{\omega}_2^2 + b_{22}\bar{\omega}_1^2 \end{aligned}\right\} \quad (25)$$

which can be recognised as the equations of a conic and a straight line in the $y, \hat{\omega}^2$ plane. It is shown in Ref.2 that critical flutter speeds are given by intersections of the straight line and the conic in the first quadrant and the possibilities are analysed. Employing the findings of this analysis in the present case it can be said that

(a) the conic will intersect the $\hat{\omega}^2$ axis at $\bar{\omega}_1^2$ and $\hat{\omega}_2^2$ with slopes at the intersections of c_{11} and c_{22} respectively;

(b) the conic will be an NS hyperbola for systems represented by points in the first and third ϕ_1, ϕ_2 quadrants and an EW hyperbola for systems in the second and fourth quadrants;

(c) the centre of the hyperbola will be at negative y for systems represented by points in the second ϕ_1, ϕ_2 quadrant and at positive y for systems in the fourth quadrant, i.e. $(c_{11} - c_{22})$ is positive;

(d) the slopes of the asymptotes of the hyperbola will be $(c_{11} + c_{22})$ and zero and their intersections with the $\hat{\omega}^2$ axis will be at $(c_{11}\hat{\omega}_1^2 + c_{22}\hat{\omega}_2^2)(c_{11} + c_{22})^{-1}$ and $(c_{11}\hat{\omega}_2^2 + c_{22}\hat{\omega}_1^2)(c_{11} + c_{22})^{-1}$ respectively;

(e) the damping line will be a line of zero slope at an $\bar{\omega}^2$ of $(b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)(b_{11} + b_{22})^{-1}$.

The stabilities of systems which are represented by EW hyperbolas with centres at positive or negative ys (fourth or second ϕ_1, ϕ_2 quadrants respectively) are easily seen to be consistent with these properties. In the case of the system considered in the last section in which the graver mode is pure heave the conic degenerates into two straight lines, the flutter speed from the present theory is given by the intersection of the damping line with the conic line of non-zero slope and the identical upper and lower frequency coalescence speeds by the centre of the conic where the straight lines intersect each other.

In the first and third $\phi_1 - \phi_2$ quadrants signum y at the centre of the (NS) hyperbolas is given by signum $(c_{11} - c_{22})$ and since c_{11} and c_{22} depend on the κ as well as the ϕ, it is not immediately obvious how the stability boundaries given by the present theory are independent of κ. Fig.4 shows the two types of hyperbola possible when the system can be represented by a point in the first $\phi_1 - \phi_2$ quadrant. For both types of hyperbola flutter at some air-speed will occur if the damping line is closer to the origin than the zero-slope asymptote. From (d) and (e) above the inequality to be satisfied is

$$(b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)(b_{11} + b_{22})^{-1} < (c_{11}\hat{\omega}_2^2 + c_{22}\hat{\omega}_1^2)(c_{11} + c_{22})^{-1}$$

or

$$(c_{11}\hat{\omega}_2^2 + c_{22}\hat{\omega}_1^2)(b_{11} + b_{22}) - (b_{11}\hat{\omega}_2^2 + b_{22}\hat{\omega}_1^2)(c_{11} + c_{22}) > 0$$

which reduces to

$$(c_{11}b_{22} - c_{22}b_{11})(\hat{\omega}_2^2 - \hat{\omega}_1^2) \equiv \kappa_1^2\kappa_2^2F_0(\hat{\omega}_2^2 - \hat{\omega}_1^2) > 0 \ . \qquad (26)$$

A similar inequality holds for points in the third ϕ_1, ϕ_2 quadrant. This confirms that the flutter boundary is independent of κ but the critical air-speeds of systems that are unstable will depend on the values of the κ.

6 F_0 CONTOURS

6.1 Piston theory

Piston theory, without thickness effects, gives aerodynamic forces of the type considered. The aerodynamic axis is at mid chord and the only non-zero aerodynamic derivatives with this as reference axis are ℓ_α (and hence $\ell_{\dot{z}}$) and $m_{\dot{\alpha}}$. ℓ_α has the value $2M^{-1}$, where M is the Mach number, and $(-m_{\dot{\alpha}})$ is $\ell_\alpha/12$, i.e. γ is 1/12. Since $\ell_{\dot{\alpha}}$ is zero, β is zero and the aerodynamic axis is also the axis of minimum damping coefficient (b_r).

Contours of positive F_0 for the $\phi_1 - \phi_2$ plane are given in Fig.5. The maximum value of F_0 occurs when the nodal line in the graver mode is about

5/3 chords in front of the aerodynamic axis and the other nodal line is the same distance behind. The contours can be plotted as continuous curves if the appropriate ranges of ϕ_1 and ϕ_2 are chosen, i.e. $(-\arc\tan\gamma^{\frac{1}{2}}) < \phi < (2\pi - \arc\tan\gamma^{\frac{1}{2}})$ and $(-\arc\tan\gamma^{\frac{1}{2}}) > \phi_2 > (-2\pi + \arc\tan\gamma^{\frac{1}{2}})$ but it is thought that the presentation given here allows easier appreciation of the results.

FO gives the dependence of the flutter stiffness number on the nodal line positions only in part. The more comprehensive expression is $(\kappa_1^2/b_2 + \kappa_2^2/b_1)F_0$. Contours of positive $F_0b_1^{-1}$ are given in Fig.6. The maximum value of $F_0b_1^{-1}$ occurs when the nodal line of the graver mode is just aft of 20% chord and the other nodal line is about three chords behind the trailing edge. Since b_r is symmetric about the reference axis, contours of $F_0b_2^{-1}$ are the reflections of those of $F_0b_1^{-1}$ in the line $\phi_1 + \phi_2 = 0$, i.e. Fig.6 gives contours of $F_0b_2^{-1}$ if the positive ϕ_1 axis is taken to be the negative ϕ_2 axis and vice versa. The maximum value of $F_0b_2^{-1}$ occurs when the nodal line of the graver mode is about three chords in front of the leading edge and the other nodal line is just forward of 80% chord.

With the aid of Fig.6 one can obtain an upper limit of the flutter stiffness number from the mass, stiffness, natural frequency ratio, nodal line positions and Mach number.

6.2 Minhinnick derivatives

Minhinnick suggested that the oscillatory aerodynamic derivatives for wings of fairly-low aspect ratio in incompressible flow were approximated to by the steady state values where there were relatives. In this way values can be obtained for ℓ_z, m_z, $\ell_{\dot z}$, $m_{\dot z}$, ℓ_α and m_α from the fact that steady lift is independent of vertical position and from the lift coefficient and the position of the aerodynamic axis. Minhinnick further suggested that the ratios of $\ell_{\dot\alpha}$ and $m_{\dot\alpha}$ to ℓ_α should be taken to have the values given by the two-dimensional derivatives when the frequency parameter tends to infinity. These asymptotic values are 10/11 for $(\ell_{\dot\alpha}/\ell_\alpha)$ and 9/22 for $(-m_{\dot\alpha}/\ell_\alpha)$ when the leading edge of the wing is the reference axis. The aerodynamic axis for this type of flow is at the quarter chord and this is the axis for which equation (4) is applicable. β and γ are 29/44 and 2/11 respectively, b_r has its minimum value, 0.0696, when the nodal line is at about 60% chord and its maximum value, 1.116, when the nodal line is about $2\frac{1}{2}$ chords in front of the leading edge. This contrasts with piston theory which gives β zero and only one turning point.

Contours of positive FO are given in Fig.7. The maximum value occurs when the nodal line in the graver mode is about 8/5 chords in front of the reference axis and the other nodal line is the same distance behind. Contours of positive $F_0b_1^{-1}$ are given in Fig.8. The maximum value occurs when the nodal line in the graver mode is almost 1/5 of the chord in front of the leading edge and the other nodal line is just over $1\frac{1}{2}$ chords aft of the trailing edge. Since the value of the damping coefficient is not symmetric about the aerodynamic axis there is no simple relationship between $F_0b_1^{-1}$ and $F_0b_2^{-1}$. Contours of positive $F_0b_2^{-1}$ are given in Fig.9. The maximum value occurs when the nodal line in graver mode is two chords in front of the leading edge and the other nodal line is at almost 70% chord, The maximum of $F_0b_2^{-1}$ is almost ten times that of $F_0b_1^{-1}$.

7 EFFECT OF DENSITY RATIO

The fractional reduction in flutter stiffness number when the density ratio is non-zero is

$$\sigma\gamma\ell_\alpha f_\sigma(\kappa_1^2 b_1\hat{\omega}_2^2 + \kappa_2^2 b_2\hat{\omega}_1^2)(\hat{\omega}_2^2 - \hat{\omega}_1^2)^{-1}$$

from equation (13). All the terms in this expression are necessarily positive except for f_σ. f_σ is $(S_1C_2 - S_2C_1)(\gamma C_1C_2 - S_1S_2)^{-1}$ and hence has the same sign as F_0 but whereas F_0 tends to zero at the boundaries of the region in which it is positive, f_σ tends to zero at the ($\phi_1 = \phi_2$) limit and infinity at the ($\tan\phi_1 \tan\phi_2 = \gamma$) limit. Full account of the effect of nodal line position requires consideration of $f_\sigma b_r$ rather than f_σ alone (cf. $F_0b_r^{-1}$ and F_0). Contours of positive f_σ and $f_\sigma b_2$ for piston theory are given in Figs.10 and 11. The $f_\sigma b_1$ contours are the reflection of the $f_\sigma b_2$ contours in the line $\phi_1 + \phi_2 = 0$. Contours of positive f_σ, fob,, $f_\sigma b_2$ for Minhinnick derivatives are given in Figs.12-14.

Some of the other factors in the expression are identical to factors in the expression for the upper bound of flutter stiffness number (equation (16)) and the effects tend to cancel each other. Thus if the upper bound is high due to large ℓ_a or small $(\hat{\omega}_2^2 - \hat{\omega}_1^2)$, the fractional decrease will also tend to be large for the same reason.

8 CONCLUDING REMARKS

The flutter of a two-dimensional wing under simple aerodynamic forces has been analysed. The forces differ from those assumed in frequency-coalescence theory in that forces in phase with the velocity of displacement are included. The inclusion of these damping forces results in increased possibilities of flutter in terms of combinations of nodal-line positions which can lead to instability over those given by frequency-coalescence theory. Which extra combinations of nodal-line positions make flutter a possibility is dependent to some extent on the aerodynamic damping moment about the aerodynamic axis (Cf. Figs.2 and 3). This damping moment in pitch is also a significant factor in determining which combinations of nodal lines lead to minimum flutter speeds as well as the actual value of these minimum speeds. It also determines the only nodal-line positions, one for the lower- and one for the higher-frequency mode, which eliminate the possibility of flutter altogether.

The expression for the upper bound of the flutter stiffness number given in equation (16) is disappointing in that it contains terms depending on generalised masses in the modes. However, contour plots are given from which it is possible to obtain the value of the upper bound for any combination of nodal line positions once the generalised masses are known. This dependence on generalised masses is also present in the case of frequency-coalescence theory but complicates the equations to an extent such that comparable contours cannot be drawn.

The effect of the density ratio is complicated and involves the relative frequencies of the modes as well as the generalised masses. Contour plots, however, are again given to aid the evaluation of the effect in specific cases. Two points are of general applicability. One is that the drop in flutter stiffness number is proportional to the damping in pitch. The other is that if the value of the upper bound of flutter stiffness number is large due to the proximity of the frequencies, the drop in stiffness number due to density-ratio effects will also be large.

Appendix

MAXIMA OF F_0 AND $F_0 b_r^{-1}$ (EQUATION (16))

F_0 will have maxima with respect to ϕ_1 when 92 is constant and it can be shown that these will occur when

$$\tan \phi_1 = -T_2 + (T_2^2 + 1)^{\frac{1}{2}} \tag{A-1}$$

where $T_r = (1 - \gamma)t_r(t_r^2 + \gamma)^{-1}$. It can also be shown that the maxima with respect to ϕ_2, ϕ_1 constant, occur when

$$\tan \phi_2 = -T_1 - (T_1^2 + 1)^{\frac{1}{2}} \quad . \tag{A-2}$$

F_0 will also have an absolute maximum with respect to ϕ_1 and 92 . This lies on the line $\phi_1 = -\phi_2$ and is at

$$2 \tan^2 \phi_1 = 2 \tan^2 \phi_2 = 3(1 - \gamma) + \Gamma \tag{A-3}$$

where $\Gamma \equiv \sqrt{9(1 - \gamma)^2 + 4\gamma}$ and this maximum value will be

$$F_{0_{max}} = \frac{\sqrt{2}\{3(1 - \gamma) + \Gamma\}^{\frac{1}{2}}\{3(1 + \gamma) + \Gamma\}\{5 - 3\gamma + \Gamma\}}{64(1 - \gamma)} \quad . \tag{A-4}$$

$F_0 b_1^{-1}$ and $F_0 b_2^{-1}$ will also have maxima and absolute maxima. The maxima of $F_0 b_1^{-1}$, ϕ_1 constant, and those of $F_0 b_2^{-1}$, ϕ_2 constant, will have the sam locations as those of F_0 under the same conditions but the maxima of $F_0 b_1^{-1}$ with 92 constant are all at $\tan \phi_1 = \gamma^{\frac{1}{2}}$ and those of $F_0 b_2^{-1}$ with ϕ_1 constant are all at $\tan \phi_2 = -\gamma^{\frac{1}{2}}$. The values of the maxima are

$$F_0 b_1^{-1} = (\gamma^{\frac{1}{2}} - t_2)^2 (2\gamma^{\frac{1}{2}} + \beta)^{-1} (1 + t_2^2)^{-1} \tag{A-5a}$$

and

$$F_0 b_2^{-1} = (\gamma^{\frac{1}{2}} - t_1)^2 (2\gamma^{\frac{1}{2}} - \beta)^{-1} (1 + t_1^2)^{-1} \quad . \tag{A-5b}$$

The absolute maximum of $F_0 b_1^{-1}$ is located at (arc tan $\gamma^{\frac{1}{2}}$, arc tan $-\gamma^{-\frac{1}{2}}$) and

$$(F_0 b_1^{-1})_{max} = (\gamma + 1)(2\gamma^{\frac{1}{2}} + \beta)^{-1} \quad . \tag{A-6a}$$

That of $F_0 b_2^{-1}$ is located at (arc tan $\gamma^{-\frac{1}{2}}$, arc tan $-\gamma^{\frac{1}{2}}$) and

$$(F_0 b_2^{-1})_{max} = (\gamma + 1)(2\gamma^{\frac{1}{2}} - \beta)^{-1} \quad . \tag{A-6b}$$

REFERENCES

No.	Author	Title, etc.
1	Bisplinghoff Ashley Halfman	Aeroelasticity. First edition, p.362, Cambridge, Mass., Addison Wesley Publishing Co. (1955)
2	Ll. T. Niblett	A graphical representation of the binary flutter equations in normal coordinates. ARC R&M 3496 (1966)

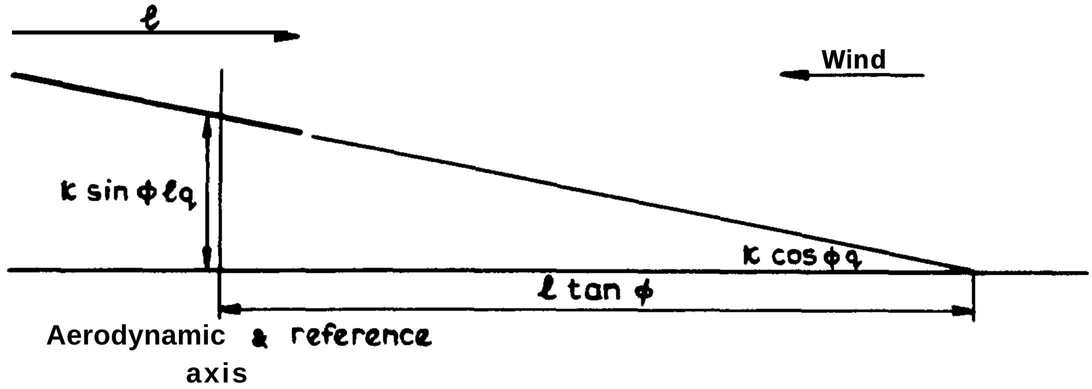
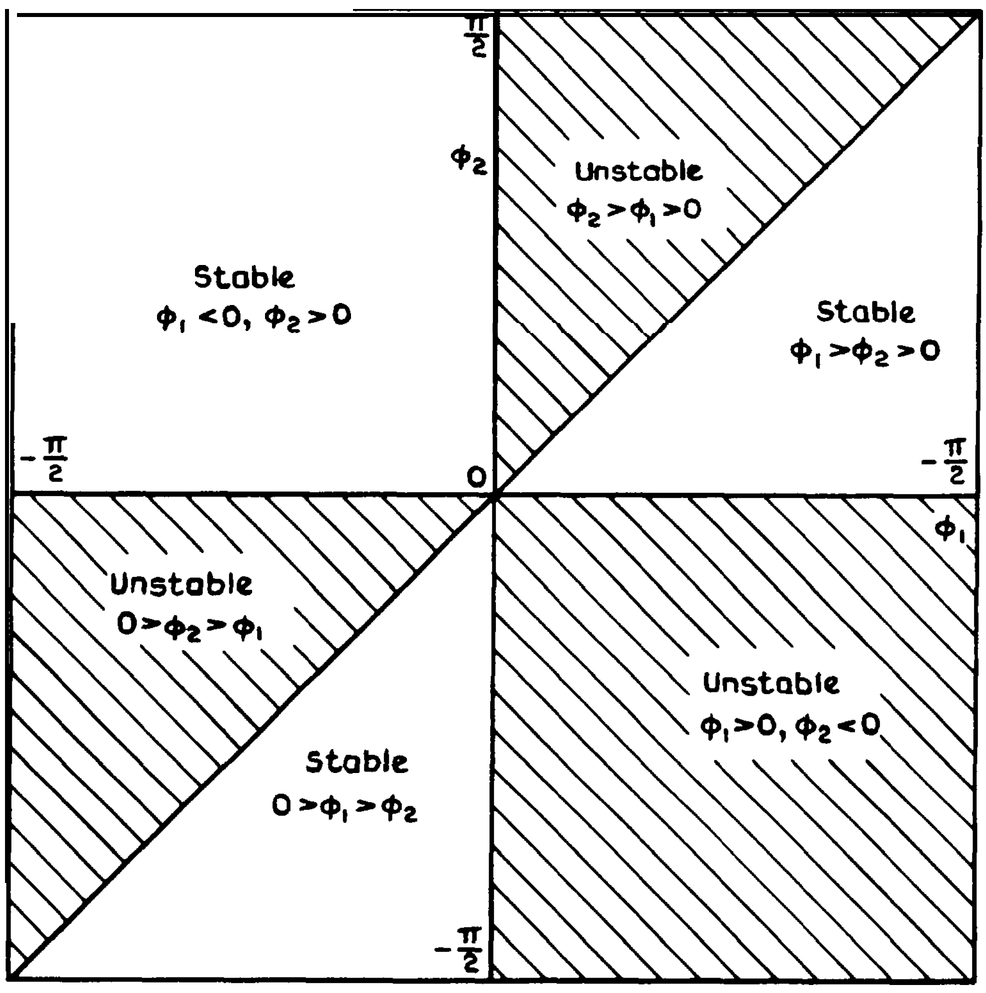


Fig. I Typical generalised coordinate



Fig, 2 Stability boundaries for no damping in pitch

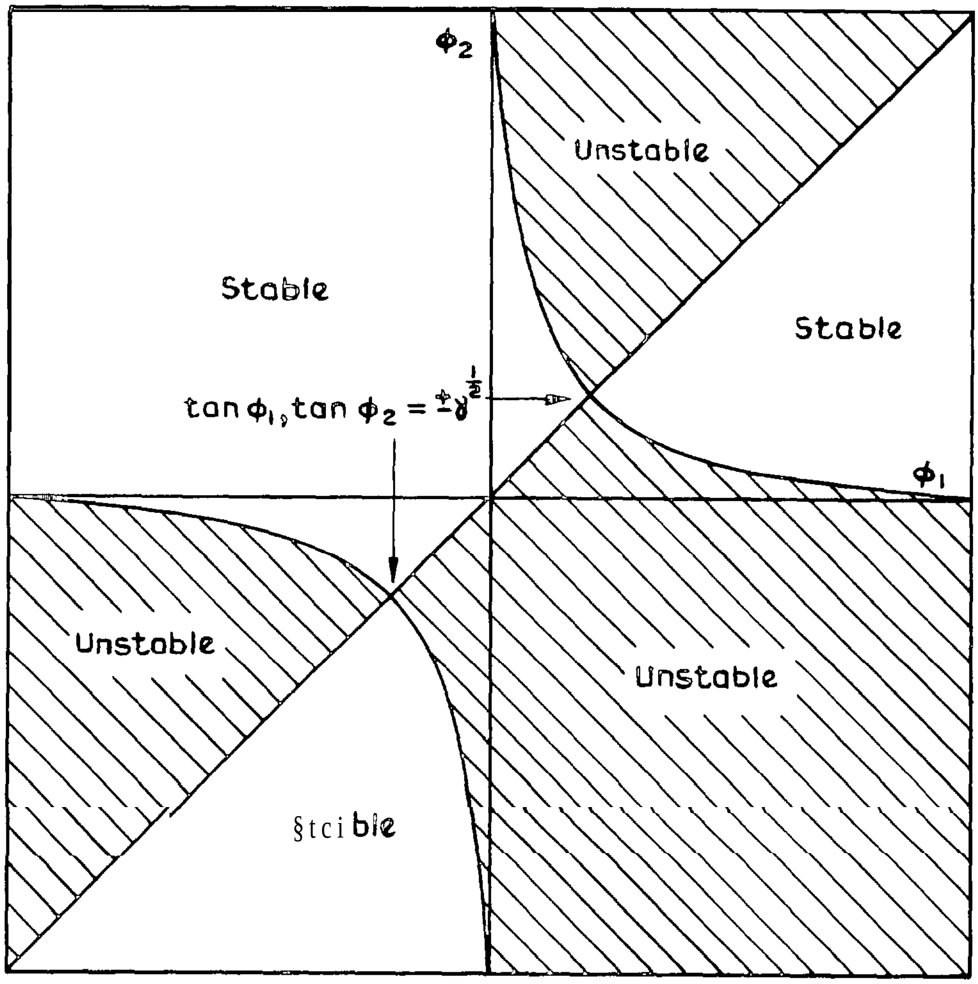
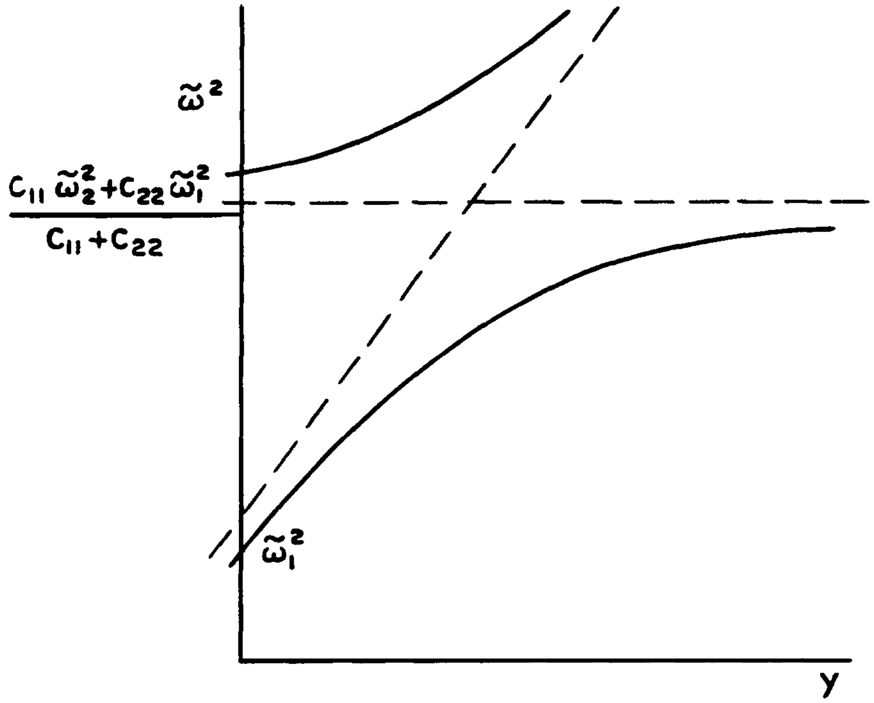


Fig. 3 Stability boundaries when there is aerodynamic damping in pitch



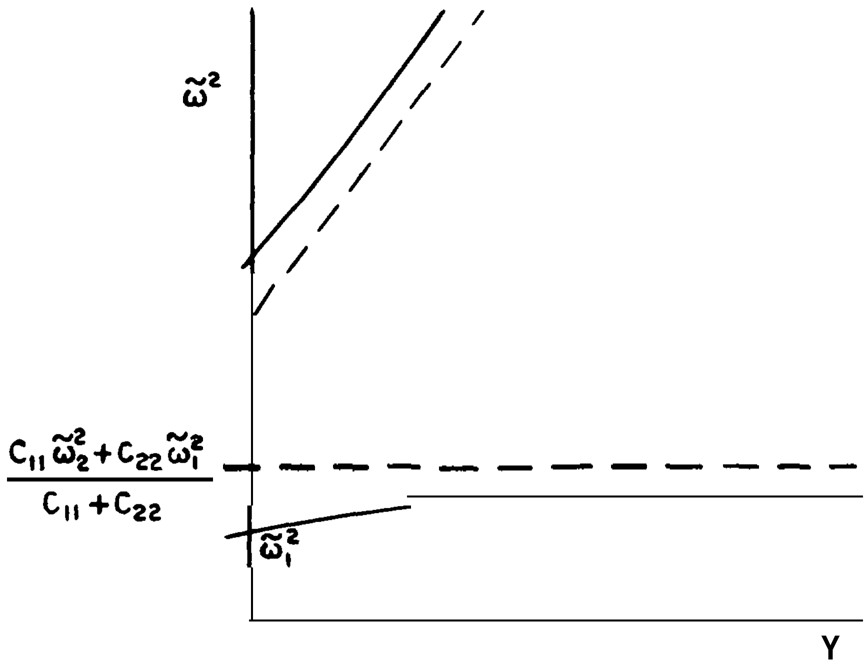


Fig.4 Graphical representation

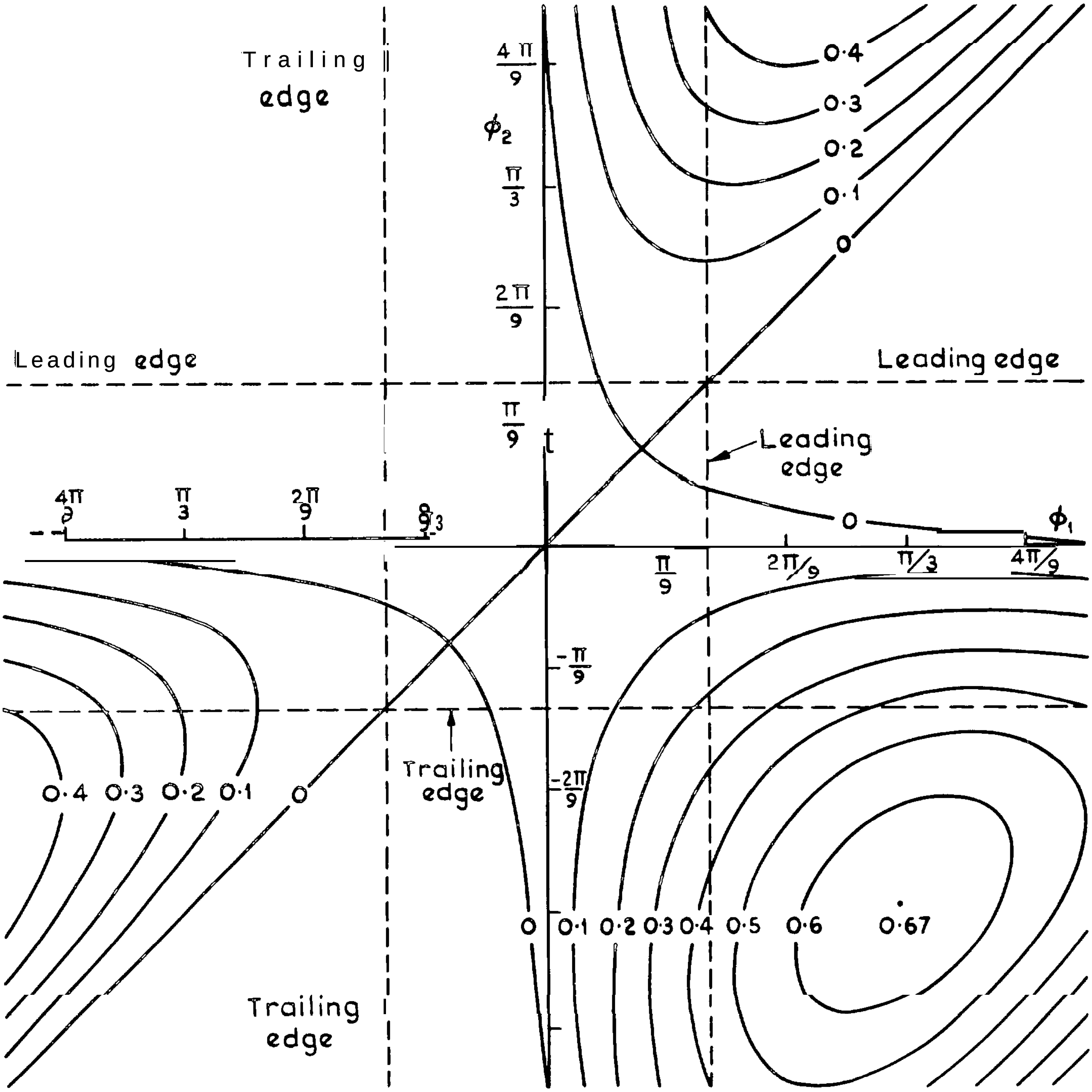


Fig.5 Contours of F_o – piston theory

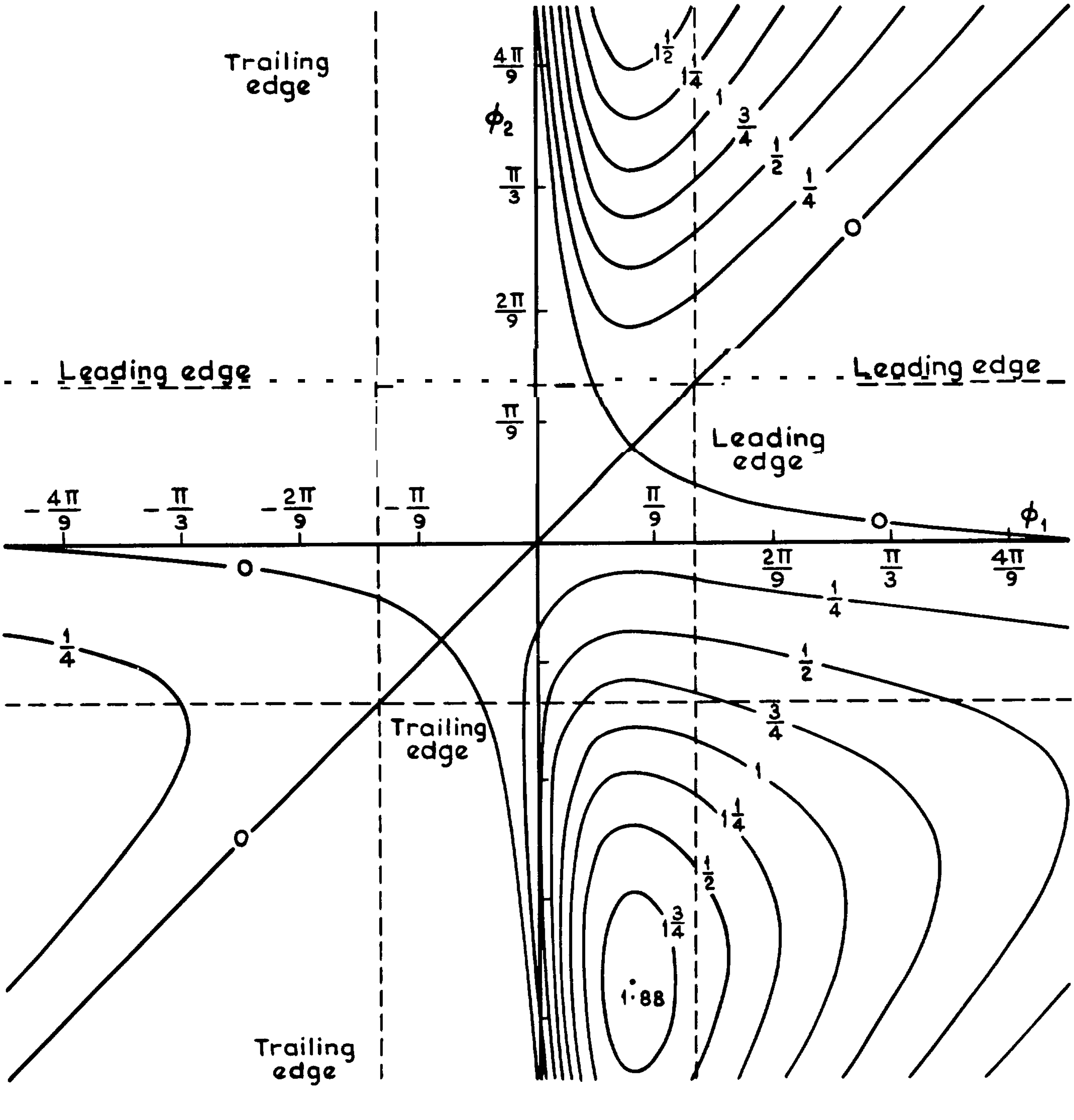


Fig.6 Contours of $F_o\ b_1^{-1}$ – piston theory

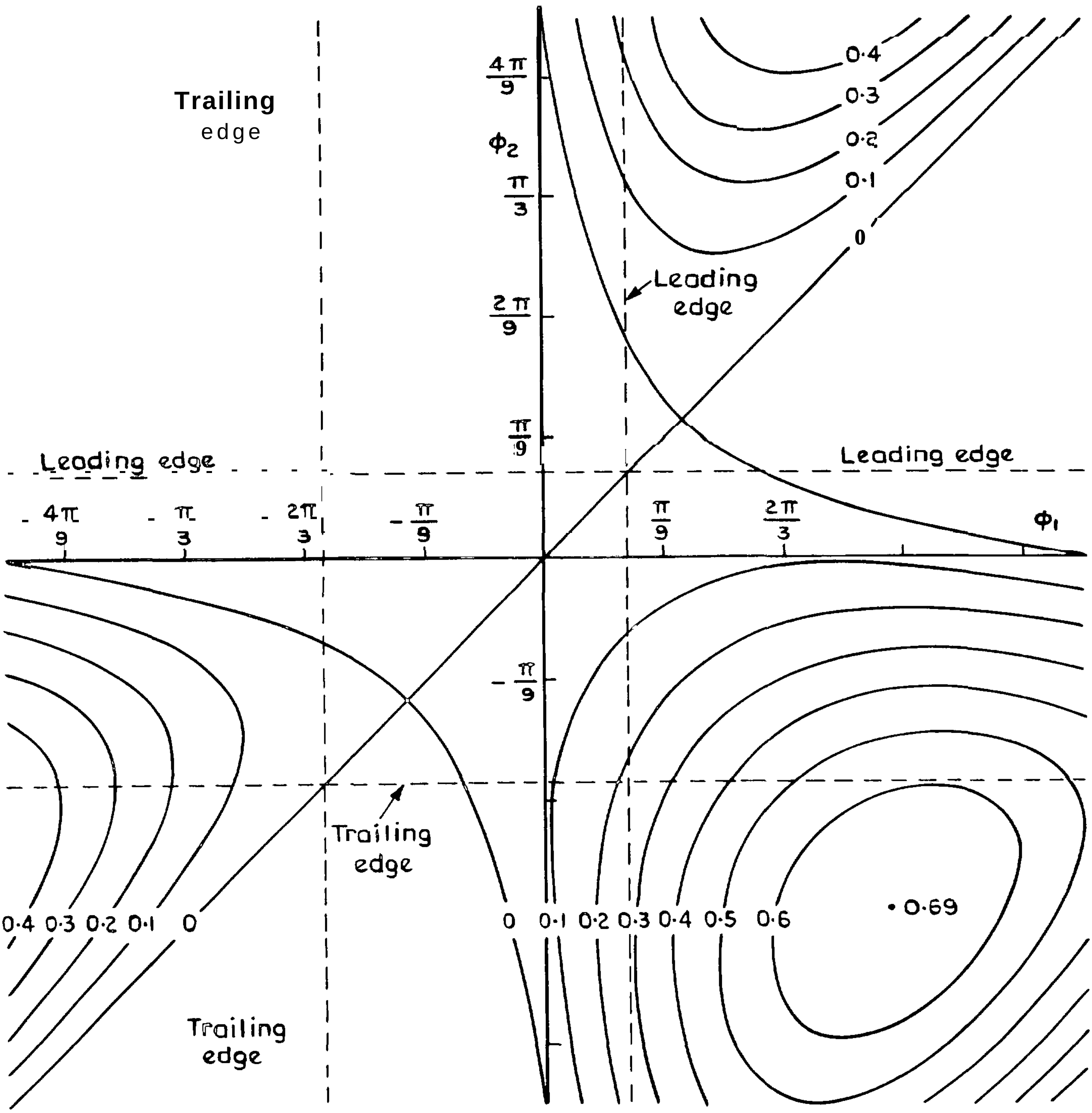


Fig.7 Contours of F_0 Minhinnick rules

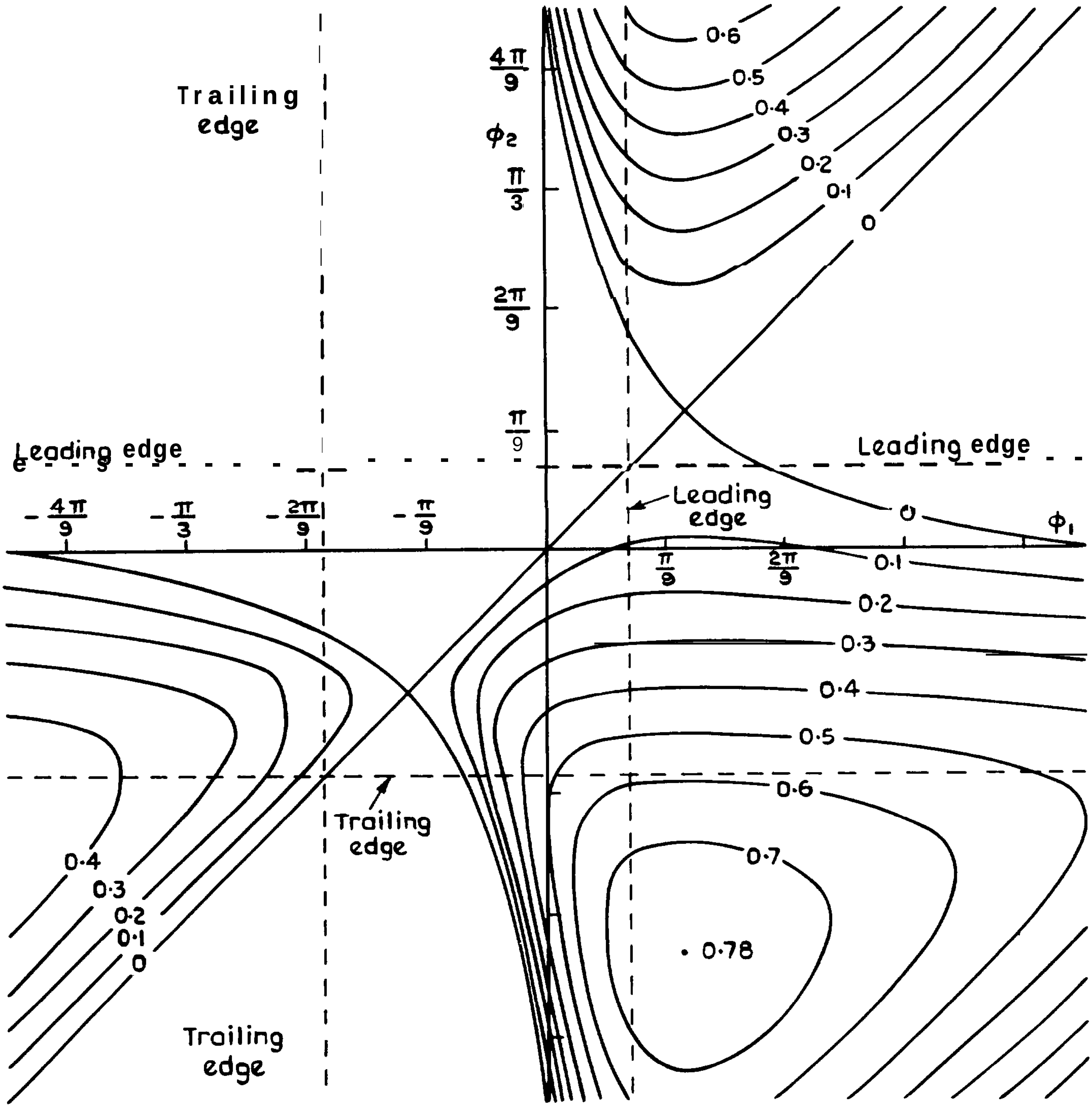


Fig.8 **Contours of $F_o\ b_1^{-1}$ Minhinnick rules**

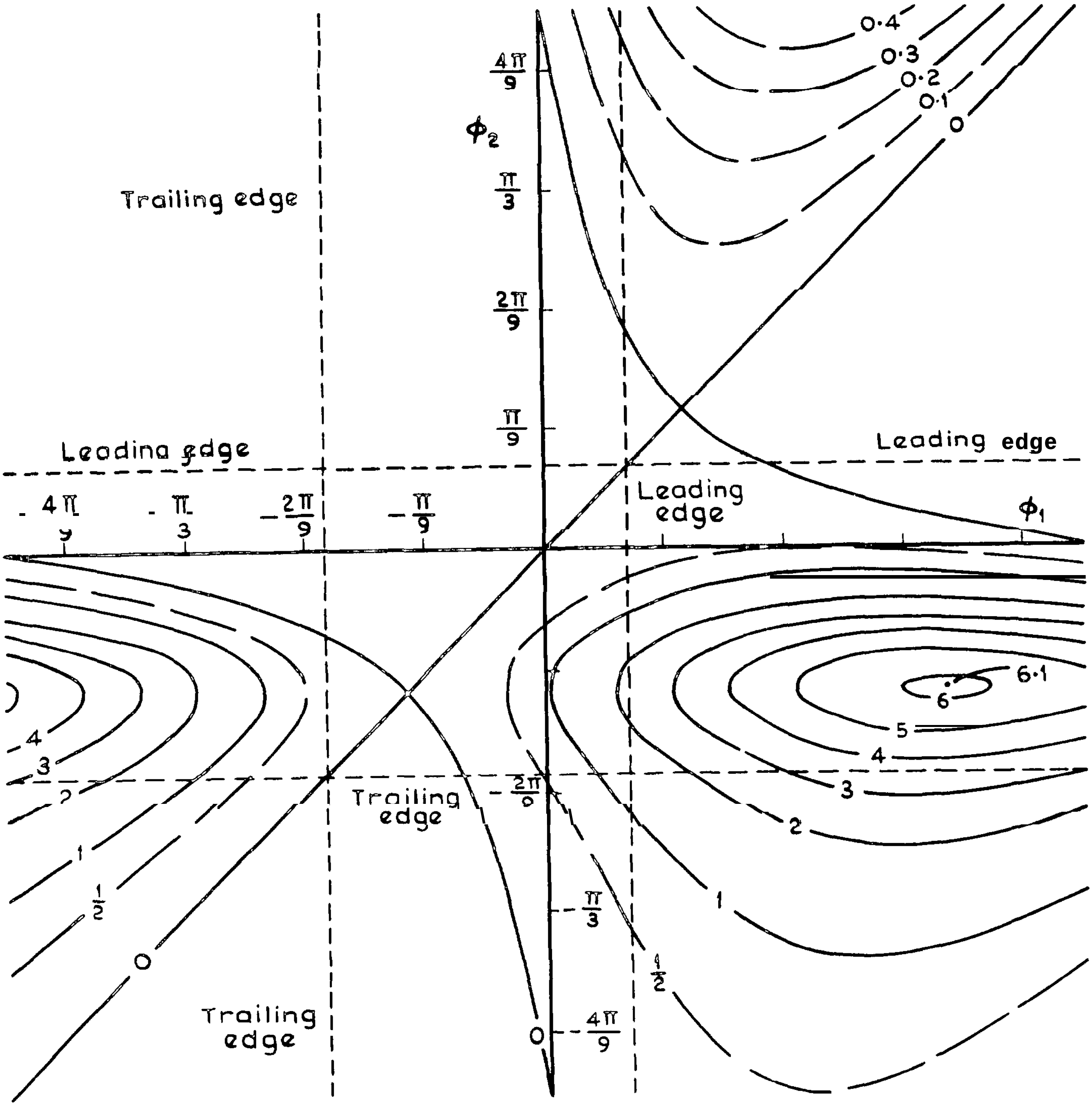


Fig.9 Contours of $F_0 b_2^{-1}$ Min hinnick rules

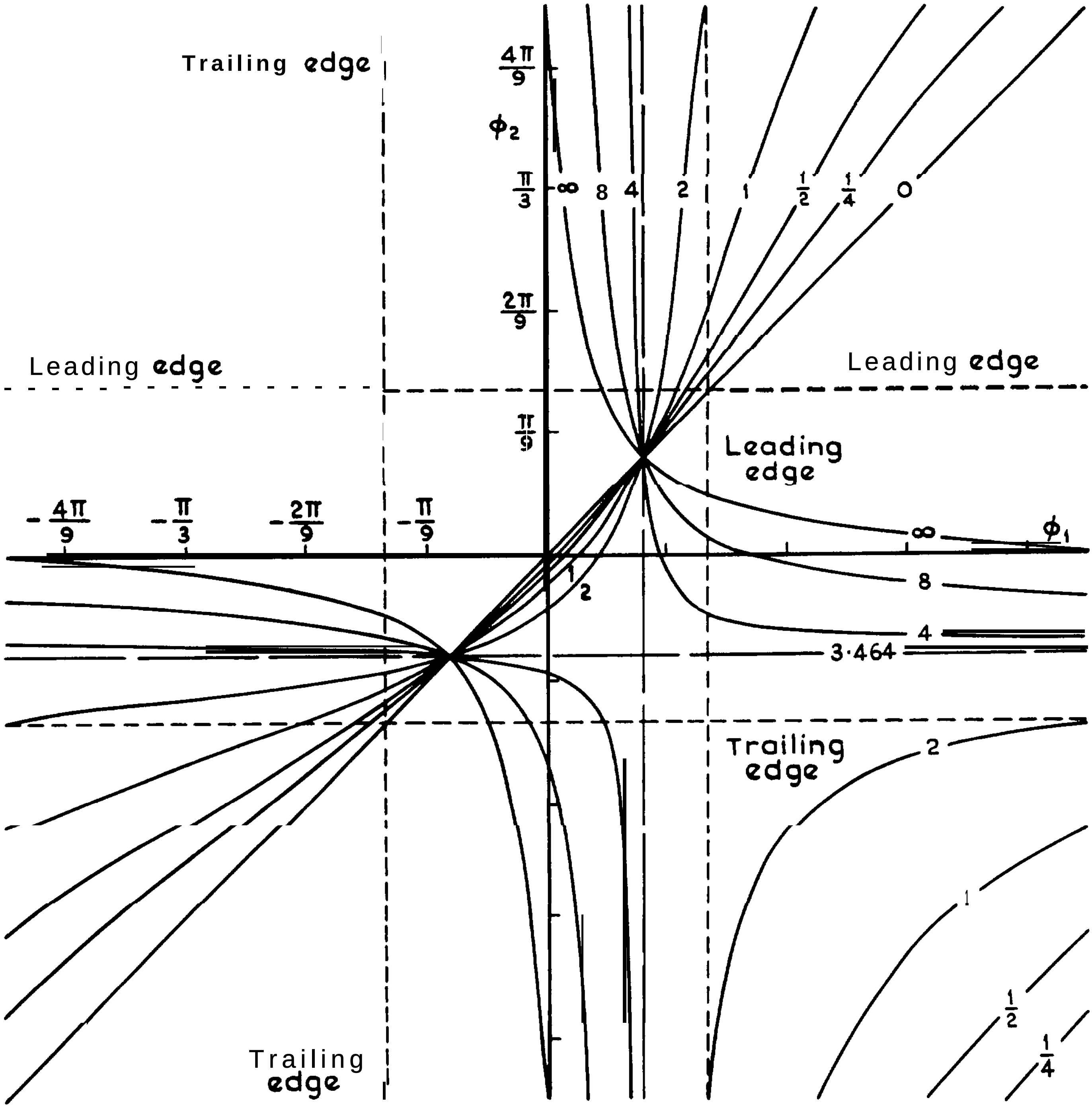


Fig.10 Contours of f_σ - piston theory

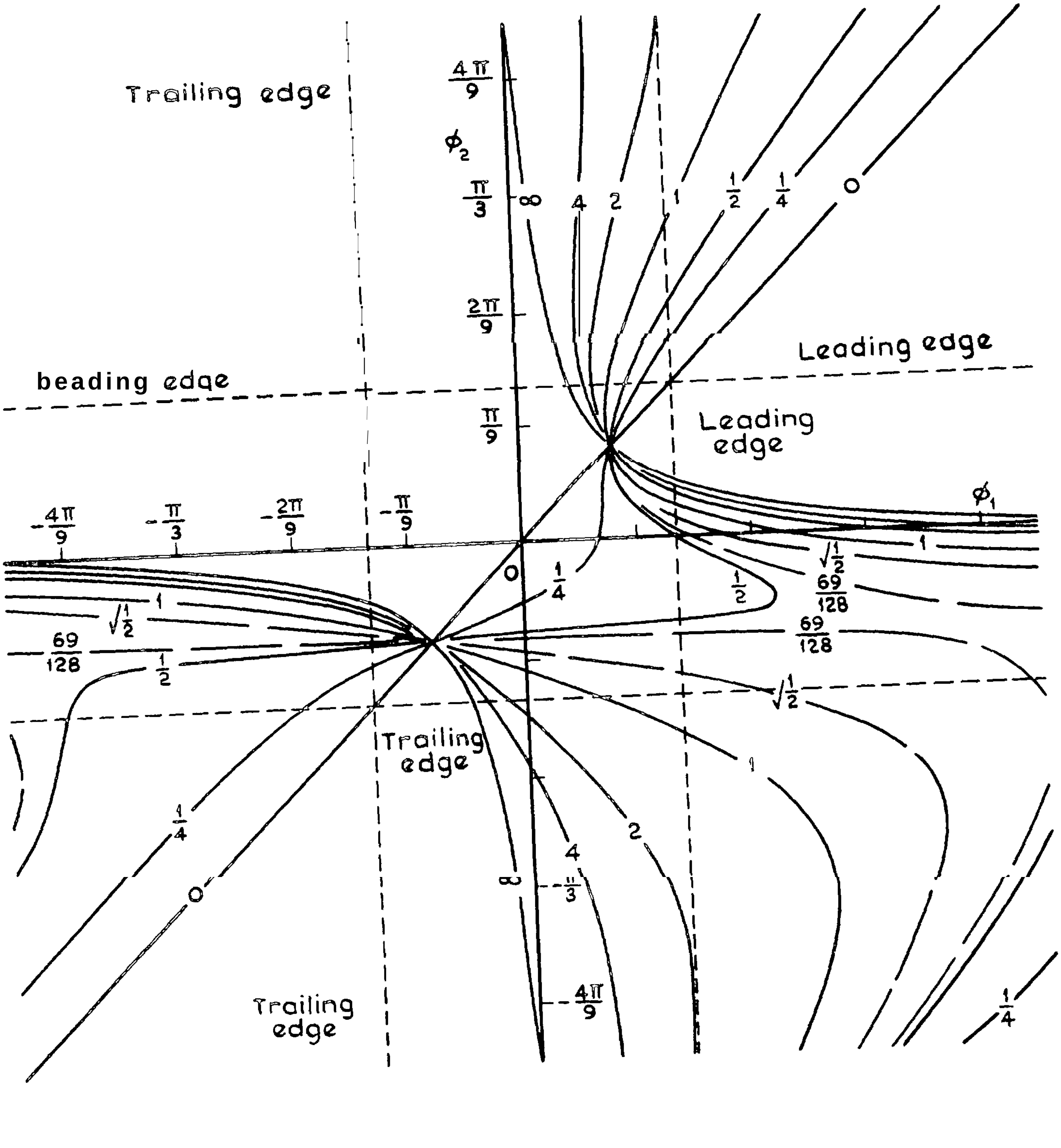


Fig.11 Contours of $f_\sigma b_2$ piston theory

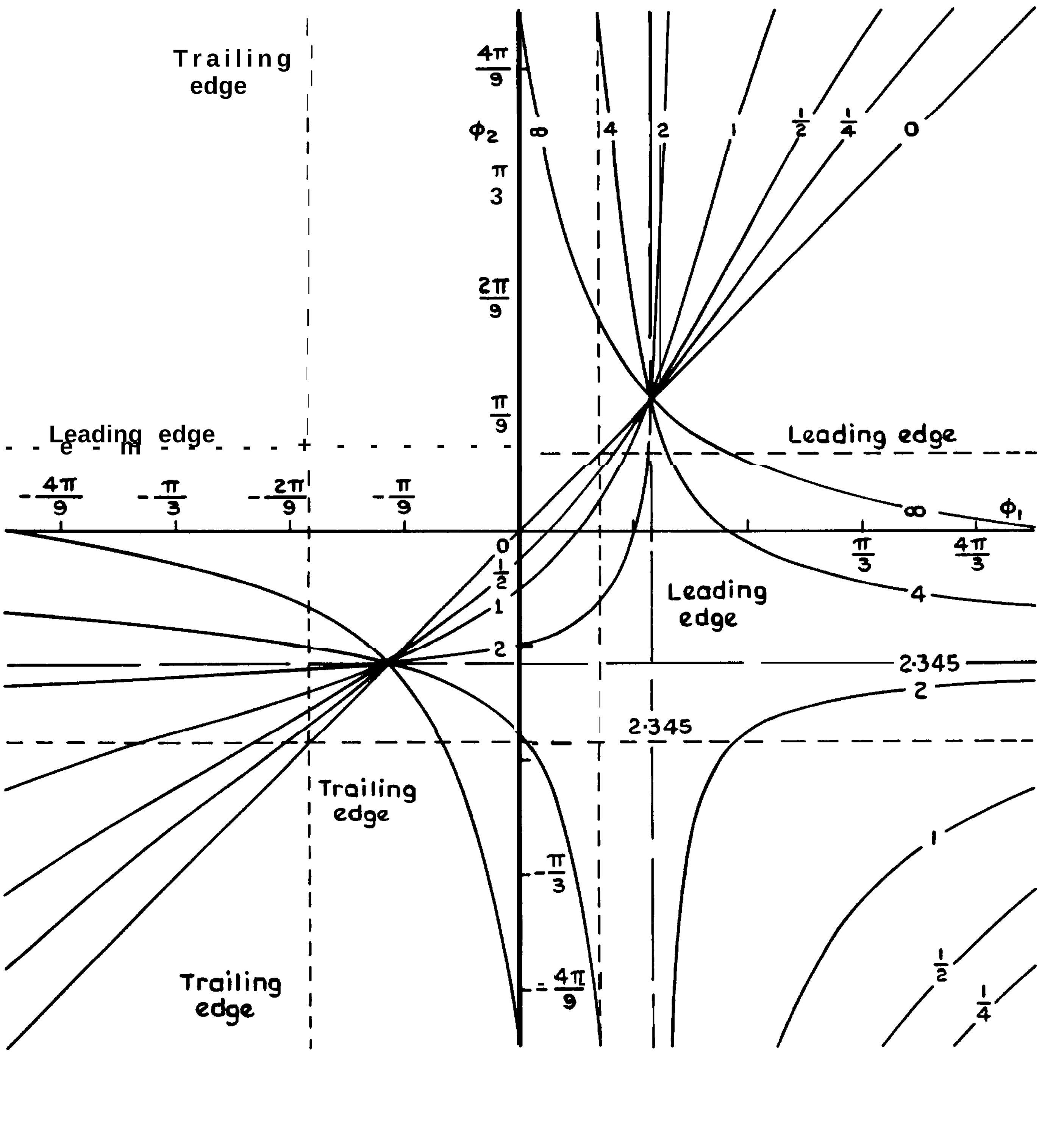


Fig.12 Contours of f_σ Minhinnick rules

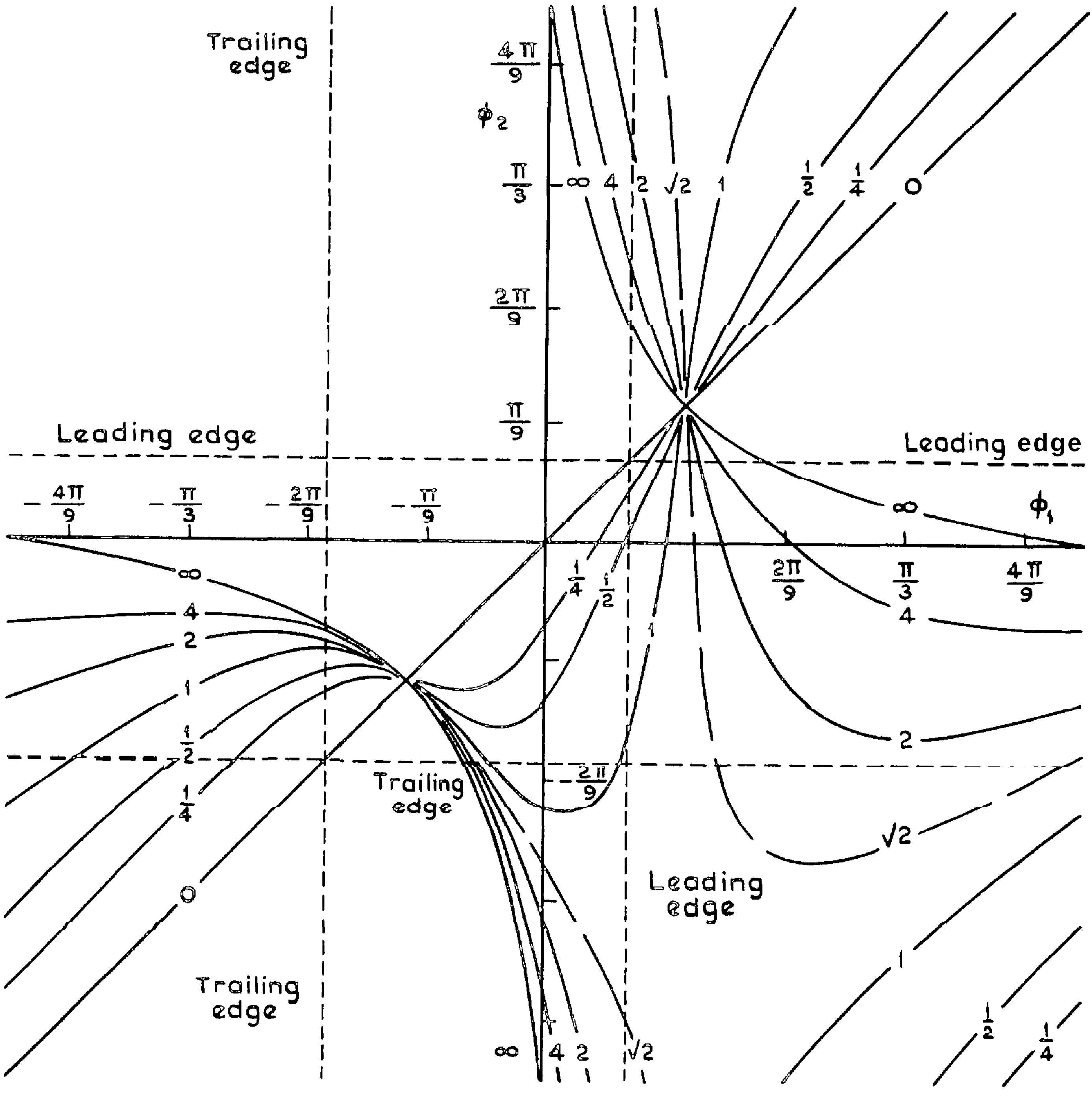


Fig.13 Contours of $f_\sigma b_1$ Minhinnick rules

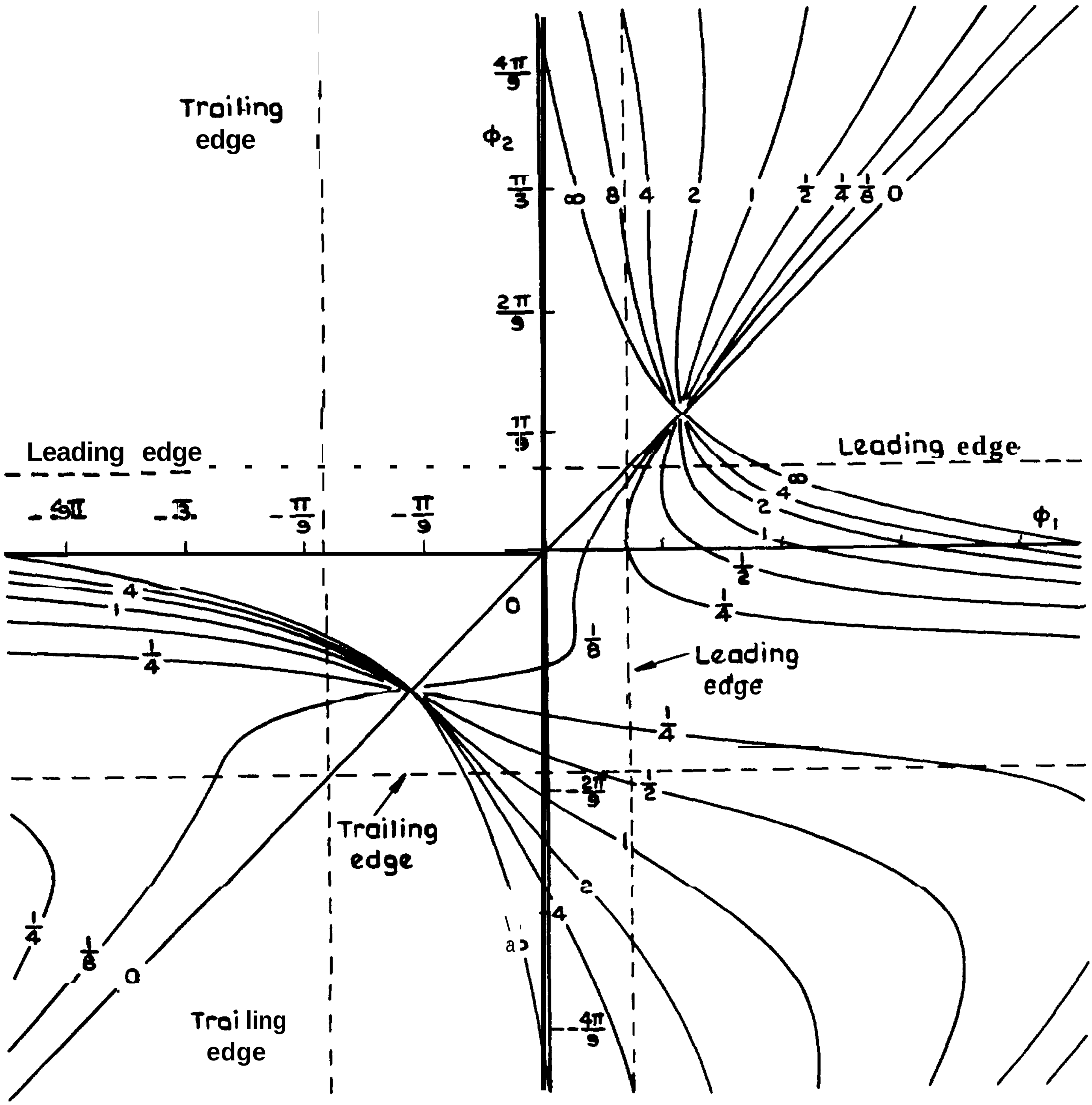


Fig.14 Contours of $f_\sigma b_2$ Minhinnick rules

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