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Relative Accuracy of Deflections and Bending Moments (or Stresses) Derived by the Method of R.A.E. Report No. Structures 168

By

D. Williams, D.Sc., M.I.Mech.E., F.R.Ae.S.

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Relative accuracy of deflections and bending moments (or stresses) derived by the method of RAE Report No. Structures 168*

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D. Williams, D.Sc., M.I.Mech.E., F.R.Ae.S.

SUMMARY

The relative accuracies of moments and deflections, as derived by the method of ref.1, are discussed, and it is shown that moments are not expected in general to be less accurate than the deflections.

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* "A general method (depending on the aid of a digital computer) for deriving the structural influence coefficients of aeroplane wings" by D. Williams. ARC 17,438

1 Introduction

RAE Report No. Structures 168 describes a general method of deriving the bending moments and deflections of any type of aeroplane wing under any transverse load distribution. The method depends on representing the transverse deflection of the wing by the deflections of a number of stations distributed uniformly over the surface of the wing. Starting with a hypothetical set of deflections, one writes down in finite-difference form the bending and twisting moments at each station in terms of the deflections of the small group of stations surrounding it. The reaction at each station (necessary to hold the wing to its hypothetical contour) is, in turn, written down in terms of these moments and hence in terms of the deflections. Having in this way expressed by standard formulae the reactions in terms of the deflections, one finally solves (in matrix form by digital computer) the relevant set of linear algebraic equations to obtain the deflections in terms of the reactions. With the deflections for any type of loading thus known, it is a straightforward operation to derive the bending and twisting moments and hence the stresses.

An example[2] worked out for the case of a square plate mounted as a cantilever and loaded at an outer corner showed that the deflections derived by the use of this method agreed very satisfactorily with experimental values. As only the deflections of the plate were recorded in these experiments there was no means of judging how accurate were the stresses obtained by the method. It was therefore not easy to refute the natural argument that, since moments are obtained from deflections by two successive differentiations, any errors in the deflections tend to be magnified in the moments.

The writer was inclined to accept this criticism at its face value until it was found* that, in practice, exactly the opposite often happens; in other words, the moments are obtained with greater accuracy than the deflections. This seems somewhat paradoxical at first sight, but examination of the factors involved shows that the original criticism was based on a false analogy. It is perfectly true that, if only the <u>transverse deflections</u> of a beam-like structure at a number of points uniformly distributed over it are given, the second differences will not in general give the true curvature (i.e. bending moments). Still more inaccurate will be the loading as expressed by the fourth differences. The position is radically different, however, if what is given is the <u>loading</u> at a number of stations distributed over the structure. For the moments are then obtained by integrating the loading and the deflections by integrating the moments. The moments are thus in a more direct relation with the loading than the deflections, and on that account can, in many practical cases, be derived more accurately. Not always however, for much depends on the type of structure and the type of loading.

In many cases, as exemplified by a beam under uniform loading, the bending moments obtained at the various stations are exact. The deflections derived from these correct moments (or curvature) cannot however be exact because the curvature, unlike the loading is not constant.

Examples will now be given to demonstrate the truth of the above remarks and at the same time to throw some light on the factors governing the relative accuracies of moments and deflections as derived by the method of ref.1. This falls far short of a rigorous demonstration that, in general, moments derived by that method are closer to the true values than the corresponding deflections. What can however be claimed as having been demonstrated is that in most practical cases there is no basis for the criticism that moments are necessarily less accurate than the deflections.

---

* by Mr. Miller of Vickers-Armstrong (Supermarines).

2 Beam examples

2.1 1st beam example

Consider first a cantilever beam of constant cross-section under a uniform loading q per unit length (Fig.1)

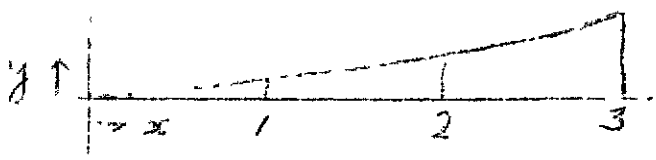

Fig.1

We proceed to derive the moments and deflections of the cantilever at the four stations shown. This we do by first writing down (as described in ref.1) the moments and reactions associated with a hypothetical set of displacements y_1, y_2 and y_3 at the stations 1, 2 and 3. Following this programme, we obtain the bending moments

$$\left.\begin{aligned} M_3 &= 0 \\ M_2 &= \left(\frac{EI}{a^2}\right)(y_1 + y_3 - 2y_2) \\ M_1 &= \left(\frac{EI}{a^2}\right)(y_2 - 2y_1) \\ M_o &= \left(\frac{EI}{a^2}\right)(2y_1) \end{aligned}\right\} \qquad (1)$$

and the loadings

$$\left.\begin{aligned} \frac{R_3}{a} &= \frac{(M_2 + M_4 - 2M_3)}{a^2}, \\ &\text{where } M_4 = 0 \text{ (because } I_4 \text{ is zero), or} \\ \frac{R_3}{a} &= \frac{M_2}{a^2} = \frac{EI}{a^4}(y_1 + y_3 - 2y_2) \\ \frac{R_2}{a} &= \frac{(M_1 - 2M_2)}{a^2} = \frac{EI}{a^4}(5y_2 - 4y_1 - 2y_3) \\ \frac{R_1}{a} &= \frac{(M_o + M_2 - 2M_1)}{a^2} = \frac{EI}{a^4}(7y_1 - 4y_2 + y_3). \end{aligned}\right\} \qquad (2)$$

From (2) and (1), which have been written down at sight, we solve for the moments and deflections in terms of the reactions R and obtain

$$\left.\begin{aligned} y_1 &= \tfrac{1}{2}(R_1 + 2R_2 + 3R_3)\left(\frac{a^3}{EI}\right) \\ y_2 &= (R_1 + 3R_2 + 5R_3)\left(\frac{a^3}{EI}\right) \\ y_3 &= \tfrac{1}{2}(3R_1 + 10R_2 + 19R_3)\left(\frac{a^3}{EI}\right) \end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned} M_2 &= a\,R_3 \\ M_1 &= a\,(R_2 + 2R_3) \\ M_o &= a\,(R_1 + 2R_2 + 3R_3)\,. \end{aligned}\right\} \quad (4)$$

It is important to note that we can derive the bending moments appropriate to any given set of forces R either by first obtaining the deflections and taking their second differences or by direct solution of the equations giving the reactions in terms of the moments. Thus any errors involved in deriving deflections from moments or moments from deflections have no effect on the relation connecting moments and reactions. It is clear, for instance, that the moments given by (4) which are here exact, would be just as exact if the constant-section beam were replaced by a strongly varying section. The accuracy of the deflections obtained would, however, be much reduced. The fact of the matter is that the relation between moments and loading is one of equilibrium and is independent of the beam characteristics in this particular example.

Reverting to the present example, we see that the moments given by (4) are exact. In terms of an applied loading q, since

$$R_1 = R_2 = 2R_3 = qa \quad (5)$$

$$\left.\begin{aligned} M_2 &= \frac{qa^2}{2}\,, \\ M_1 &= 2qa^2\,, \\ M_o &= 4.5\,qa^2\,. \end{aligned}\right\} \quad (6)$$

The deflections given by (3), compared with the true deflections are as follows:-

Table I

Station	Deflection	True Deflection	% Inaccuracy
1	$\frac{2.25\ a^4 q}{EI}$	$\frac{1.79\ a^4 q}{EI}$	12.5
2	$\frac{6.5\ a^4 q}{EI}$	$\frac{5.65\ a^4 q}{EI}$	11.5
3	$\frac{11.25\ a^4 q}{EI}$	$\frac{10.125\ a^4 q}{EI}$	11.1

The deflections are all too large, as would be expected from the fact that they are obtained on the assumption of constant curvature over each set of three successive stations. For this neglects the interference effect introduced by the fact that the curvatures defined for example by the set of three displacements y_0, y_1, y_2 and the succeeding set y_1, y_2, y_3, are not compatible over the overlapping region between y_1 and y_2.

Before leaving this example it is interesting to note that, whereas the deflections obtained from the given set of reactions of equation (7) are only 10-12% inaccurate, the reactions derived from the correct deflections appropriate to these reactions (by equation (4) and Table I) are respectively:

$$0.615qa, \quad 0.59qa \quad \text{and} \quad 0.055qa \quad \text{for} \quad R_3, R_2 \text{ and } R_1 \tag{7}$$

instead of

$$0.5qa, \quad 1.0qa \quad \text{and} \quad 1.0qa, \tag{8}$$

the true values. The errors are +23%, -16% and -95.5%. The corresponding errors in the moments M_2, M_1 and M_0 are respectively +23%, +3.5% and -20.5%. Yet, starting with a hypothetical set of deflections to derive the corresponding set of reactions, we find that on giving these reactions the numerical values appropriate to a particular system of loading, we obtain the moments with exactness and the deflections with only 10-12% error.

2.2 2nd beam example

Take next a beam example in which the loading is not constant, and where therefore the moments derived from it will not be exact.

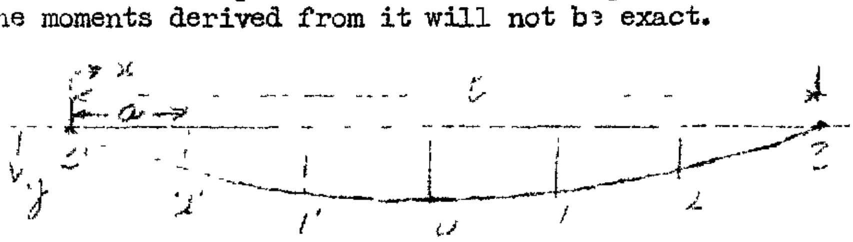

Fig. 2

The beam – of constant section – is shown in Fig. 2 under a loading

$$q = q_o \sin \frac{\pi x}{\ell} . \tag{9}$$

Taking account of symmetry, we write down the moments (hogging moments taken as positive)

$$\left.\begin{aligned} -\frac{M_o}{EI} &= \frac{2(y_1 - y_o)}{a^2} \\ -\frac{M_1}{EI} &= \frac{(y_2 + y_o - 2y_1)}{a^2} \\ -\frac{M_2}{EI} &= \frac{(y_1 - 2y_2)}{a^2} . \end{aligned}\right\} \tag{10}$$

Then

$$\left.\begin{aligned} \frac{R_o}{a} &= \frac{2(M_1 - M_o)}{a^2} \\ \frac{R_1}{a} &= \frac{(M_2 + M_o - 2M_1)}{a^2} \\ \frac{R_2}{a} &= \frac{(M_1 - 2M_2)}{a^2} . \end{aligned}\right\} \tag{11}$$

Solving for moments and deflections, we have

$$\left.\begin{aligned} M_o &= -a\left(\tfrac{3}{2} R_o + 2R_1 + R_2\right) \\ M_1 &= -a\left(R_o + 2R_1 + R_2\right) \\ M_2 &= -a\left(\frac{R_o}{2} + R_1 + R_2\right) \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} y_o &= \left(\frac{a^3}{EI}\right)(4.75\, R_o + 8R_1 + 4.5\, R_2) \\ y_1 &= \left(\frac{a^3}{EI}\right)(4R_o + 7R_1 + 4R_2) \\ y_2 &= \left(\frac{a^3}{EI}\right)(2.25\, R_o + 4R_1 + 2.5\, R_2) . \end{aligned}\right\} \tag{13}$$

Introducing now the loading given by (9), which means that,

$$\left.\begin{aligned} q \text{ for station } 0 &= q_o \\ q \text{ for station } 1 &= \frac{q_o \sin 2\pi}{3} = 0.866\, q_o \\ q \text{ for station } 2 &= \frac{q_o \sin 5\pi}{6} = 0.5\, q_o \end{aligned}\right\} \quad (14)$$

and putting

$$\left.\begin{aligned} R_o &= q_o a \\ R_1 &= 0.866\, q_o a \\ R_2 &= 0.5\, qa \end{aligned}\right\} \quad (15)$$

we obtain the moments and deflections from (12) and (13). The moments and deflections compared with their true values are as follows:-

Table II

Station	Moment $\div q_o \ell^2$	True Moment	error	Deflection $\div \frac{q_o \ell^4}{EI}$	True Deflection	error
0	0.1037	0.1013	2.2%	0.01076	0.0103	4.4%
1	0.0898	0.088	2.2%	0.00932	0.00893	4.4%
2	0.0518	0.0506	2.2%	0.00538	0.00515	4.4%

It is seen from this table that the error in the deflections is approximately twice that in the moments (or stresses).

3 Plate examples

3.1 1st plate example

We shall next consider two plate examples for which the theoretical solutions - obtained by a series method - are known. Consider first the problem of the square plate clamped along all four edges. By taking a square plate we take maximum advantage of symmetry.

/Fig.4

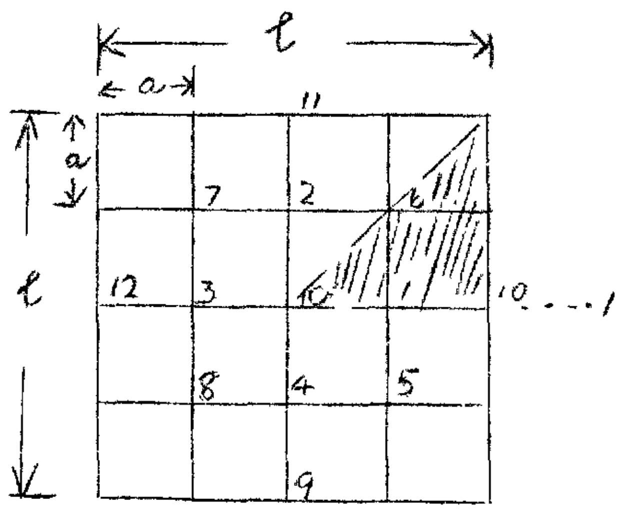

Fig.3

With eight-fold symmetry as shown in Fig.3 it is necessary only to consider the three stations 0, 1 and 6 to obtain the solution for the whole plate for a constant loading.

From the standard difference formula that gives the reaction at any station in terms of the deflections of the group of surrounding stations we have, for the central station 0,

$$\frac{R_o}{a^2} = \frac{D}{a^4}\left[20\, w_o - 8\sum_{r=1}^{4} w_r + 2\sum_{r=5}^{8} w_r + \sum_{r=9}^{12} w_r\right] \qquad (16)$$

where w_r = deflection at station r

and D = plate stiffness.

By using the symmetrical properties we reduce this to the formula

$$\left.\begin{aligned} R_o &= \frac{D}{a^2}(20\, w_o - 32\, w_1 + 8\, w_6)\,. \\ \text{Similarly} \quad R_1 &= \frac{D}{a^2}(-8\, w_o + 26\, w_1 - 16\, w_6) \\ \text{and} \quad R_6 &= \frac{D}{a^2}(2\, w_o - 16\, w_1 + 24\, w_6)\,. \end{aligned}\right\} \qquad (17)$$

Deflections in terms of loads are found by solving (17) to give:

$$
\left.\begin{aligned}
w_o &= \frac{a^2}{D}(0.1295\,R_o + 0.225\,R_1 + 0.107\,R_6) \\
w_1 &= \frac{a^2}{D}(0.0564\,R_o + 0.163\,R_1 + 0.0901\,R_6) \\
w_6 &= \frac{a^2}{D}(0.0269\,R_o + 0.0895\,R_1 + 0.0929\,R_6)\,.
\end{aligned}\right\} \quad (18)
$$

For a constant loading q per unit area

$$R_o = R_1 = R_6 = qa^2 \quad (19)$$

so that

$$
\left.\begin{aligned}
w_o &= \left(\frac{qa^4}{D}\right)(0.4615) \\
w_1 &= \left(\frac{qa^4}{D}\right)(0.3095) \\
w_6 &= \left(\frac{qa^4}{D}\right)(0.2093)\,.
\end{aligned}\right\} \quad (20)
$$

The bending moment per unit width M_o at the centre of the plate is (taking $\nu = 0.3$)

$$M_o = \frac{D}{a^2}(1 + \nu)\,2(w_1 - w_o)$$

$$= 0.02445\,q\ell^2 \quad (21)$$

(where $\ell = 4a$).

The true value of the bending moment as given in Table 30 of Timoshenko's "Theory of Plates and Shells"[3] is $0.0233\,q\ell^2$ and therefore the value above obtained is some 5% too large.

Comparing the central deflection w_o with the true deflection we find a much greater discrepancy - $0.0197\,q\ell^4/Eh^3$ as against the true value of $0.0138\,q\ell^4/Eh^3$ - and therefore an excess error of 42.5%.

The central bending moment is thus obtained with much greater accuracy than the central deflection. The bending moment across a side at its mid-point is, however, not nearly so accurate as that at the plate centre. It is given by

$$M_{mid\text{-}side} = \frac{D}{a^2}\,2w_1 = 0.0387\,q\ell^2 \quad (22)$$

compared with the correct value quoted by Timoshenko of $0.0513\,q\ell^2$. The error is therefore some 25% on the small side.

It is easy to explain the relative magnitudes of these errors. The reason why the central deflection is too large is that too little account is taken by the difference method of the clamping effect at the sides of the plate. It tried, so to speak, to satisfy the clamping condition by the only means available which is, in Fig.4 for example, to make the deflection at station 1' outside the plate equal to that at station 1 inside the plate. But this still leaves the plate simply supported at station 1' and no account is taken of the fact that the slope at 1' is equal and opposite to that at 1. The result is that the full clamping effect is not obtained with the natural consequence of an excessive deflection at the plate centre.

That the central bending moment is so near correct is due to the mutually cancelling effects of the central deflection being too large and the clamping moments being too small. As the number of stations used is increased, this error rapidly decreases since the closer stations 1 and 1' are to each other the nearer is a true clamping effect achieved.

3.2 2nd plate example - four sides simply supported

In the second plate example the four sides of the square plate are assumed simply-supported. The procedure is the same as for the first example except that deflections at stations 1 and 1' (Fig.4) are now equal but opposite in sign. It is found that

$$\left.\begin{aligned} w_o &= 0.0441\ q\ell^4/Eh^3 \\ \text{as against} \qquad w_o\ (\text{true}) &= 0.0443\ q\ell^4/Eh^3, \end{aligned}\right\} \tag{23}$$

the error being less than 1% on the small side.

The bending moment this time is not quite so accurate as the deflection, having a value

$$\left.\begin{aligned} M_o &= 0.0457\ qa^2 \\ \text{as against} \qquad M_o\ (\text{true}) &= 0.0479\ qa^2, \end{aligned}\right\} \tag{24}$$

an error of 4½% on the small side.

That the error is not due to loss of accuracy caused by deriving the moments from the deflections by a process of differentiation, is shown by the fact that it is possible (in this symmetrical example) to derive the bending moment at the plate centre directly from the loading and without reference to the deflections. This in fact is the method used by Timoshenko in working out this very example* by finite differences. His numerical solution agrees exactly with that given above. The procedure depends on writing the equation connecting the deflections with the loading in terms of a quantity M defined by the relation

$$\begin{aligned} M &= \frac{(M_x + M_y)}{(1+\nu)} \\ &= D\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right). \end{aligned} \tag{25}$$

* Ref.3 Chap.V Section 36.

It is then possible to write the standard equation

$$\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} = \frac{q}{D} \tag{26}$$

in the form of two separate equations

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} = q \tag{27a}$$

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{M}{D}. \tag{27b}$$

Thus M can be derived in terms of q without proceeding via the deflections. Errors in expressing moments in terms of deflections do not therefore arise.

4 Conclusions

Enough has been said, one would suppose, to prove that bending moments (and hence stresses) obtained by the method of ref.1 are not likely to be in general less accurate than the deflections. Indeed in many practical cases the reverse is true.

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