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Some Notes on an Approximate Solution for the Free Oscillation Characteristics of Non-Linear Systems Typified by $\ddot{x} + F(x, \dot{x}) = 0$

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Summary.

A modified form of the approximate method developed by Kryloff and Bogoliuboff is derived and applied to particular cases with a large frequency dependence on amplitude. The results suggest that some relaxation of the constraints of the earlier method may be justified.

The method provides an approximate solution of the frequency and the amplitude damping envelope in terms of the amplitude, and suggests itself as a means of cataloguing, in matrix form, the characteristics of practical non-linear systems.

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1. *Introduction.*

Very many problems in dynamics involve second order systems, expressible generally as $\ddot{x}+F(x, \dot{x}) = 0$, characterised by an oscillatory behaviour. The linear system, in which $F = A\dot{x}+Bx$, is a familiar example; here the frequency is constant $\sqrt{B-A^2/4}$ and the amplitude varies exponentially, the exponent being $-A/2\,t$. It is striking how many systems, which, although non-linear in x and $\dot{x}$, nevertheless display free oscillation characteristics qualitatively similar to these, i.e. 'a periodic' motion, the amplitude envelope of which is time-dependent. At first sight it seems logical to express the solution to $\ddot{x}+F(x, \dot{x}) = 0$ as $x = x(t)$, which enables the motion—the displacement, velocity and acceleration—to be fully described at any instant. However, this has some drawbacks, particularly for non-linear equations, in that the solution is unnecessarily precise, and hence more difficult to obtain. Much more information on the behaviour may be gleaned by expressing the solution as a function of two variables, *viz.* one of the localised frequency, the other the amplitude envelope, and with both of them as functions of amplitude. To use the linear case again as an example, we have that the formal solution is $x = x_0\, e^{-A/2t} \cos(wt+\varepsilon)$ which is dependent upon the initial conditions (Fig. 1). We may, however, consider the solution as $x = \Theta(t) \cos \phi\, (t)$, so that $\lambda\left(\equiv \dfrac{\dot{\Theta}}{\Theta}\right)$ is one variable and $\omega\,(\equiv \dot{\phi})$ is the other; for the linear equation both are constants and do not involve the initial conditions. In essence they provide the frequency and the envelope within which the motion is contained, but without the detailed displacement history. These parameters, in addition to being more readily obtained, are usually more pertinent for system stability studies.

This approach is, of course, not new. Kryloff and Bogoliuboff produced a classical paper[1] on these lines in 1937, and introduced the concept of an 'equivalent linear system'. In some ways, however, their treatment was unnecessarily restrictive. For example, frequency variation was confined to be small compared with the basic frequency defined by the coefficient of x in $F(x, \dot{x})$. Thus, the variation of frequency during the whole motion, not just during any one cycle, was assumed to be small in order to justify the omission of certain inconvenient terms in the analysis.

It is the purpose of this Report to re-derive some of the elementary results of Ref. 1, to retain and recast the omitted terms, and thereby to remove some of the restrictions in application. It does not purport to be a rigorous treatise on non-linear systems, but rather a pointer to the way in which their physical properties may often by simply and accurately derived and described.

2. *Method.*

By analogy with the solution for the linear system we express the solution of

$$\ddot{x}+F(x,\dot{x}) = 0 \qquad \text{as} \qquad x = \Theta \cos \phi\,, \tag{1}$$

where Θ and ϕ are the amplitude and phase respectively, and are both functions of time.

Then

$$\dot{x} = \dot{\Theta} \cos \phi - \Theta\, \dot{\phi} \sin \phi$$

and

$$\ddot{x} = \ddot{\Theta} \cos \phi - 2\dot{\Theta}\, \dot{\phi} \sin \phi - \Theta\, \ddot{\phi} \sin \phi - (\dot{\phi})^2\, \Theta \cos \phi\,.$$

Rearranging, and writing ω ($\equiv \omega(t)$) for $\dot{\phi}$ as the instantaneous rate of change of phase angle,

$$[\omega^2\, \Theta - \ddot{\Theta}] \cos \phi + [2\omega\, \dot{\Theta} + \dot{\omega}\, \Theta] \sin \phi = -\ddot{x} = F[\Theta \cos \phi, \dot{\Theta} \cos \phi - \omega\, \Theta \sin \phi]\,. \tag{2}$$

From the assumption that the amplitude and frequency change during any one cycle is small enough, we may integrate over the cycle and interpret the bracketed terms not as instantaneous values but as values applicable at a particular Θ within the cycle. Herein this is taken at mid-cycle. Then,

$$2\omega\, \dot{\Theta} + \dot{\omega}\, \Theta = \frac{1}{\pi} \int_0^{2\pi} F \sin \phi \,.\, d\phi \tag{3}$$

$$\omega^2\, \Theta - \ddot{\Theta} = \frac{1}{\pi} \int_0^{2\pi} F \cos \phi \,.\, d\phi\,. \tag{4}$$

In Ref. 1 the terms $\dot{\omega}\, \Theta$ in equation (3) and $\ddot{\Theta}$ in equation (4) were disregarded, giving simply

$$\frac{\dot{\Theta}}{\Theta} \doteqdot \frac{1}{2\omega\, \pi\, \Theta} \int_0^{2\pi} F \sin \phi \,.\, d\phi \left(= \frac{I_1}{2\omega}, \text{say} \right) \tag{5}$$

and

$$\omega^2 \doteqdot \frac{1}{\pi\, \Theta} \int_0^{2\pi} F \cos \phi \,.\, d\phi \,(= I_2\,, \text{say})\,. \tag{6}$$

We may demonstrate the inadequacy of this approximation by comparison with the known exact solution for a linear system with damping, *viz*:

$$\left.\begin{aligned} \lambda \equiv \frac{\dot{\Theta}}{\Theta} &= -\frac{A}{2} \\ \omega^2 &= B - \frac{A^2}{4} \end{aligned}\right\} \tag{7}$$

The approximations of equations (5) and (6) give, on substituting for $x = \Theta \cos \phi$, and $\dot{x} = \dot{\Theta} \cos \phi - \omega \Theta \sin \phi$ in F.

$$I_1 = \frac{1}{\pi \Theta} \int_0^{2\pi} [A(\dot{\Theta} \cos \phi - \omega \Theta \sin \phi) + B \Theta \cos \phi] \sin \phi \, . \, d\phi = -A \omega$$

and

$$I_2 = B + A \lambda$$

from which

$$\left.\begin{aligned} \lambda &\doteqdot -\frac{A}{2} \\ \omega^2 &\doteqdot B - \frac{A^2}{2} \end{aligned}\right\} \quad (8)$$

The frequency is seen to be in error, the influence of the damping being overestimated by a factor of two.

Although not a fully satisfactory approximation, the above Kryloff and Bogoliuboff result nevertheless serves to bring out the importance of the parameters ω^2 and $\lambda \left(\equiv \frac{\dot{\Theta}}{\Theta} \right)$ as quantities which describe the characteristics of the system in terms of the envelope of the motion. These, rather than the instantaneous displacement and velocity within the envelope, will be regarded as the significant parameters throughout this Report. Hence in the special case of the linear system, both ω^2 and λ are constant. In the general case they will be shown to be dependent upon the amplitude, Θ.

Returning to equation (3) we may recast this in terms of λ, ω^2 and Θ. Thus, since $2\omega \dot{\Theta} + \dot{\omega} \Theta \equiv \left[2\omega^2 + \frac{\Theta}{2} \frac{d(\omega^2)}{d\Theta} \right] \frac{\lambda \Theta}{\omega}$, we have that

$$\lambda = \frac{I_1}{2\omega \left[1 + \frac{\Theta}{4\omega^2} \cdot \frac{d(\omega^2)}{d\Theta} \right]} \quad (9)$$

where $I_1 = \frac{1}{\pi \Theta} \int_0^{2\pi} F \sin \phi \, . \, d\phi$.

Similarly, since $\ddot{\Theta} \equiv \Theta \left[\lambda^2 + \lambda \Theta \frac{d\lambda}{d\Theta} \right]$,

$$\omega^2 = \lambda^2 + I_2 + \lambda \Theta \frac{d\lambda}{d\Theta} \quad (10)$$

where $I_2 = \frac{1}{\pi \Theta} \int_0^{2\pi} F \cos \phi \, . \, d\phi$

If now as our first approximation we disregard $\frac{d(\omega^2)}{d\Theta}$ and $\frac{d\lambda}{d\Theta}$, then

$$\lambda \doteqdot \frac{1}{2\omega} \cdot I_1 \tag{11}$$

$$\omega^2 \doteqdot \lambda^2 + I_2 \,. \tag{12}$$

Equation (11) is identical with equation (5), but equation (12) differs from that of equation (6) by the addition of the term λ^2. This latter removes the discrepancy when the approximate method is applied to the damped linear system, and equation (12) gives the frequency exactly.

In many cases equations (11) and (12) will suffice; but where further refinement is justified the $\frac{d(\omega^2)}{d\Theta}$ and $\frac{d\lambda}{d\Theta}$ terms may be readily obtained (by differentiating the first approximation) to give a second approximation, and so on. The neglect of these terms, rather than $\ddot{\Theta}$ and $\dot{\omega}$, can be shown to be better by an order for systems with significant damping.

The method is best demonstrated by examples, as in the following sections.

3. *Direct Applications.*

The examples given in this section are such that the integrations from 0 to 2π may be performed directly.

3.1. *Damped, Linear System with an Additional Cubic Stiffness Term (Duffing's Equation).*

$$F = A\dot{x} + Bx + Cx^3 \,. \tag{13}$$

$$I_1 = \frac{1}{\pi\,\Theta} \int_0^{2\pi} F \sin\phi \,.\, d\phi \qquad \text{in which } x = \Theta\cos\phi$$

$$\dot{x} = \dot{\Theta}\cos\phi - \omega\,\Theta\sin\phi$$

$$= \frac{1}{\pi\,\Theta} \int_0^{2\pi} [A(\dot{\Theta}\cos\phi - \omega\,\Theta\sin\phi) + B\,\Theta\cos\phi + C\,\Theta^3\cos^3\phi]\sin\phi \,.\, d\phi$$

$$= -A\,\omega \tag{14}$$

$$I_2 = \frac{1}{\pi\,\Theta} \int_0^{2\pi} F\cos\phi \,.\, d\phi = B + A\,\lambda + \frac{3}{4}C\,\Theta^2 \,. \tag{15}$$

First approximation

$$\left.\begin{aligned} \lambda &\doteqdot -\frac{A}{2} \\ \omega^2 &\doteqdot B - \frac{A^2}{4} + \frac{3}{4}C\,\Theta^2 \end{aligned}\right\} . \tag{16}$$

Second approximation

$$\frac{d\lambda}{d\Theta} \doteqdot 0$$

$$\frac{d\omega^2}{d\Theta} \doteqdot \frac{3}{2} C\,\Theta$$

$$\left.\begin{aligned} \lambda &\doteqdot \frac{-A\,\omega^2}{2\omega^2 + \frac{3}{4} C\,\Theta^2} \\ \omega^2 &\doteqdot B - \frac{A^2}{4} + \frac{3}{4} C\,\Theta^2 . \end{aligned}\right\} \tag{17}$$

The presence of the $\frac{3}{4} C\,\Theta^2$ term due to the non-linearity is a well known result when the motion is undamped.

In Fig. 2 is shown a comparison between the approximations of equations (16) and (17) and the solution from digital computations by the Kutta-Merson method[2]. The upper half of the figure shows the rapid departure with amplitude of the damping parameter, λ, from the constant value predicted by Ref. 1 and by the first approximation of the present method. The inclusion of the $\frac{d\omega^2}{d\Theta}$ term, however, (equation (17)) produces very close agreement depending only slightly on the value of $\frac{A^2}{4B}$.

The correlation with the frequency parameter in the lower half of the figure is quite good. It is worthy of comment that it is only the presence of the $\frac{A^2}{4B}$ term, hitherto omitted, in the ordinates, $\left(\frac{\omega^2}{B} + \frac{A^2}{4B}\right)$, which has produced the correlation for the data near the origin, corresponding to the largest values of $\frac{A^2}{4B}$.

3.2. *Damped, Linear System with an Additional nth Order Stiffness Term.*

$$F = A\dot{x} + Bx + C_n x^n \tag{18}$$

$$I_1 \equiv \frac{1}{\pi\,\Theta} \int_0^{2\pi} F \sin\phi \,.\, d\phi = -A\,\omega \tag{19}$$

$$I_2 \equiv \frac{1}{\pi\,\Theta} \int_0^{2\pi} F \cos\phi \,.\, d\phi = B + A\lambda + \gamma_n$$

where

$$\left.\begin{aligned} \gamma_n &= 0 && \text{for } n \text{ even} \\ &= \frac{C_n (n+1)\,!\,\Theta^{n-1}}{\left(\frac{n+1}{2}\,!\right)^2 2^n} && \text{for } n \text{ odd}. \end{aligned}\right\} \tag{20}$$

First approximation

$$\left.\begin{aligned} \lambda &\doteqdot -\frac{A}{2} \\ \omega^2 &\doteqdot B-\frac{A^2}{4}+\frac{C_n (n+1)\,!\,\Theta^{n-1}}{\left(\frac{n+1}{2}\,!\right)^2 2^n} \end{aligned}\right\} \tag{21}$$

The above reduces to that in the previous example, 3·1, when $n = 3$. The approximate value of ω^2 is unaffected by $\frac{d\lambda}{d\Theta}$, so that, in common with linear differential equations, to the order of the second approximation, the solutions may be superposed. This is an interesting result since it enables the influence of the items in a polynomial stiffness representation to be separately assessed.

3.3. *Van der Pol's Equation.*

$$F = A\dot{x}(Cx^2-1)+Bx\,. \tag{22}$$

Hence

$$I_1 = \omega A\left(1-C\frac{\Theta^2}{4}\right) \tag{23}$$

$$I_2 = B-A\lambda\left(1-\frac{3}{4}C\,\Theta^2\right). \tag{24}$$

The first approximation gives

and

$$\left.\begin{aligned} \lambda &\doteqdot \frac{A}{2}\left(1-\frac{C\,\Theta^2}{4}\right) \\ \omega^2 &\doteqdot B-\frac{A^2}{4}\left(1-\frac{5}{4}C\,\Theta^2\right)\left(1-\frac{C\,\Theta^2}{4}\right). \end{aligned}\right\} \tag{25}$$

This shows the characteristic behaviour of a limit cycle, λ being positive, and hence the motion divergent, for $\Theta < 2/\sqrt{C}$, and the converse for $\Theta > 2/\sqrt{C}$. The frequency in the limit cycle (i.e. when $\Theta = 2/\sqrt{C}$) is $\sqrt{B}$. The behaviour when A is large is discussed further in Section 5.

3.4. *Large amplitude motion of a pendulum with damping.*

$$F = A\dot{x}+B\sin x\,. \tag{26}$$

This is a particularly interesting application. Without damping the solution is well known as a complete elliptic integral of the first kind, and this provides us with a yardstick by which to judge the effectiveness of the present method in conditions where F is not dominated by the linear term in x.

To evaluate I_1 and I_2 we first utilise the Sonine expansion form, whereby

$$\sin(\Theta\cos\phi) = 2\sum_{n=1}^{\infty}(-1)^{n-1}J_{2n-1}(\Theta)\cos(2n-1)\phi$$

where $J_n(\Theta)$ is the Bessel function of the first kind of integer order n.

Hence

$$I_1 = -A\,\omega \tag{27}$$

and

$$I_2 = \frac{2B\,J_1(\Theta)}{\Theta} + A\lambda \tag{28}$$

First approximation

$$\left.\begin{aligned} \lambda &\doteqdot -\frac{A}{2} \\ \omega^2 &\doteqdot \frac{2B\,J_1(\Theta)}{\Theta} - \frac{A^2}{4}. \end{aligned}\right\} \tag{29}$$

Second approximation

$$\frac{d\lambda}{d\Theta} \doteqdot 0$$

$$\frac{d(\omega^2)}{d\Theta} \doteqdot -\frac{2B\,J_2(\Theta)}{\Theta}.$$

Hence

$$\left.\begin{aligned} \lambda &\doteqdot \frac{-\dfrac{A}{2}}{1 - \dfrac{\dfrac{\Theta}{4}\,\dfrac{J_2(\Theta)}{J_1(\Theta)}}{1 - \dfrac{A^2}{8B}\,\dfrac{\Theta}{J_1(\Theta)}}} \\ &\doteqdot \frac{-\dfrac{A}{2}}{1 - \dfrac{\Theta}{4}\cdot\dfrac{J_2(\Theta)}{J_1(\Theta)}} \end{aligned}\right\} \tag{30}$$

to within 1 per cent for $\Theta = 2$, and $\dfrac{A^2}{B} \leqslant 0{\cdot}05$

$$\omega^2 \doteqdot \frac{2B\,J_1(\Theta)}{\Theta} - \frac{A^2}{4}.$$

When $A = 0$ the exact solution gives

$$\omega^2 = \frac{\pi^2}{4K^2\left(\sin\dfrac{\Theta}{2}\right)}.B \tag{31}$$

where $K\left(\sin\frac{\Theta}{2}\right) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\sin^2\frac{\Theta}{2}\sin^2\phi}}$.

The present approximation gives $\omega^2 \doteqdot \frac{2J_1(\Theta)}{\Theta} B$.

The exact and approximate solutions are compared in Fig. 3. It is noticeable that the agreement is remarkably close—even up to an amplitude of $\pi/2$, the error is only 0·5 per cent*. Since at this amplitude Θ is clearly not a valid approximation for $\sin\Theta$, the restriction to cases where F is predominantly linear in x is probably unnecessary.

An exact solution with damping present is not known to the authors. In this case digital computations using the Kutta-Merson variable step programme[2], and the method given in Appendix A, were made, and the results are compared with those from the present approximations in Figs. 3a and 3b. These confirm the influence of A, the coefficient of $\dot{x}$, on the natural frequency (equation (30)), *viz.* that, as in the linear case, $\frac{\omega^2+A^2/4}{B}$ collapses the results to one curve over a wide range of $A^2/4B$ (0 to 0·6).

Regarding the damping, the computed data show a high order dependence on amplitude (Fig. 3b). The first approximation predicts a constant value of $\frac{\lambda}{A/2}$. The second approximation (equation (30)) predicts the correct trend and agrees quantitatively quite well up to amplitudes of 75° or so. At really large amplitudes the approximation noticeably overestimates the damping, probably because of the increasing sensitivity to the diminishing denominator. For example, although not justifiable analytically, replacement by $1-\frac{J_2(\Theta)}{5J_1(\Theta)}$ secures much better agreement. A brief examination of the effect of a third approximation suggests that, as in Duffing's equation of Section 3.1, the modifications are mainly on the influence of $k\,(\equiv A^2/4B)$.

4. *Other Applications.*

Consider a general term, $x^m \dot{x}^n$,

$$= \Theta^m \cos^m\phi\,(\dot{\Theta}\cos\phi - \Theta\,\omega\sin\phi)^n$$

$$= \sum_{r=0}^{n} A_r(\Theta,\dot{\Theta})\cos^{(m+n-r)}\phi\,\sin^r\phi\,. \tag{32}$$

The contribution to

$$I_1 = \frac{1}{\pi\Theta}\int_0^{2\pi}\sum_{r=0}^{n} A_r\cos^{(m+n-r)}\phi\,\sin^{(r+1)}\phi\,d\phi$$

---

*The power series expansion for $\frac{\pi^2}{4K^2(\Theta)}$ is $1-\frac{\Theta^2}{8}+\frac{7}{1536}\Theta^4\ldots\ldots$

and for $\frac{2J_1(\Theta)}{\Theta}$ is $1-\frac{\Theta^2}{8}+\frac{8}{1536}\Theta^4\ldots\ldots$

and to (33)

$$I_2 = \frac{1}{\pi\,\Theta}\int_0^{2\pi}\sum_{r=0}^{n} A_r \cos^{(m+n-r+1)}\phi \sin^r \phi \, d\phi \, .$$

$$\text{Now} \int_0^{2\pi} \cos^M \phi \sin^N \phi \equiv 0 \text{ unless both } M \text{ and } N \text{ are even} \, . \tag{34}$$

Thus terms contributing to I_1 and I_2 are those for which r is odd and even respectively. Both cases, however, require that $m+n$ shall be odd. The important corollary is that any terms in the equation of motion which consist of an even-powered product of x and $\dot{x}$ may, within the limits of the present method, be removed without affecting the external characteristics, *viz.* the amplitude envelope and the frequency. For example, even-powered stiffness terms do not contribute.

4.1. *Aerodynamic Systems with Cross-Flow Forces.*

There is an important type of non-linear stiffness, however, which has the form of an even-power term, but is nevertheless an odd function of x. This may be represented by Sgn x. $\sin^2 x$ and is readily accommodated by the present method inasmuch as

$$\begin{aligned} \operatorname{Sgn} x &= \operatorname{Sgn} \cos\phi \\ &= +ve \text{ for } -\frac{\pi}{2} < \phi < \frac{\pi}{2} \\ &= -ve \text{ for } \ \frac{\pi}{2} < \phi < \frac{3\pi}{2} \, . \end{aligned}$$

Thus

$$\int_0^{2\pi} \operatorname{Sgn} x . F \mathop{\sin\phi}_{(\cos\phi)} d\phi = 2 \int_{-\pi/2}^{\pi/2} F \mathop{\sin\phi}_{(\cos\phi)} d\phi \, . \tag{35}$$

since F is an even function of ϕ.

As an illustration let us take the case of a body at high incidence. Here the stiffness term is often regarded as coming from two sources; (i) the potential flow aerodynamics giving a contribution proportional to sin $2x$, and (ii) a term giving a contribution proportional to Sgn $x.\sin^2 x$. If we include damping as before, then

$$F = A\dot{x} + B \sin 2x + C \operatorname{Sgn} x . \sin^2 x \, . \tag{36}$$

The contributions of the first two terms to I_1 and I_2 may be derived directly from previous examples (3.4 etc.).

The third term may be written as

$$\frac{C}{2}[1-\cos(2\Theta\cos\phi)] \operatorname{Sgn} x = \frac{C}{2}\left[1 - J_0(2\Theta) - 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(2\Theta)\cos 2n\phi\right] \operatorname{Sgn} x \, . \tag{37}$$

Using equation (35), we have that the contribution of this term to I_1 is

$$\frac{C}{\pi\Theta}\int_{-\pi/2}^{\pi/2}\left[1-J_0(2\Theta)-2\sum_{n=1}^{\infty}(-1)^n J_{2n}(2\Theta)\cos 2n\phi\right]\sin\phi\,.\,d\phi\,. \tag{38}$$

The general term

$$\int_{-\pi/2}^{\pi/2}\cos 2n\phi\,.\sin\phi\,d\phi=\frac{1}{2}\left[\frac{\cos(2n-1)\phi}{2n-1}-\frac{\cos(2n+1)\phi}{2n+1}\right]=0 \tag{39}$$

whereas

$$\int_{-\pi/2}^{\pi/2}\cos 2n\phi\,.\cos\phi\,.\,d\phi=\frac{1}{2}\left[\frac{\sin(2n-1)\phi}{2n-1}+\frac{\sin(2n-1)\phi}{2n+1}\right]=-\frac{2(-1)^n}{4n^2-1}. \tag{40}$$

Thus the contribution to

$$I_1=0$$

$$I_2=\frac{2C}{\pi\Theta}\left[1-J_0(2\Theta)+2\sum_{n=1}^{\infty}\frac{J_{2n}(2\Theta)}{4n^2-1}\right]. \tag{41}$$

Thus, complete

$$I_1=-A\omega$$

and

$$I_2=A\lambda+\frac{2B\,J_1(2\Theta)}{\Theta}+\frac{2C}{\pi\Theta}\left[1-J_0(2\Theta)+2\sum_{n=1}^{\infty}\frac{J_{2n}(2\Theta)}{4n^2-1}\right]. \tag{42}$$

First approximation

$$\lambda\doteqdot-\frac{A}{2}$$

$$\omega^2\doteqdot-\frac{A^2}{4}+\frac{2B\,J_1(2\Theta)}{\Theta}+\frac{2C}{\pi\Theta}\left[1-J_0(2\Theta)+2\sum_{n=1}^{\infty}\frac{J_{2n}(2\Theta)}{4n^2-1}\right] \tag{43}$$

$$\doteqdot-\frac{A^2}{4}+\frac{2}{\Theta}\left[B\,J_1(2\Theta)+\frac{C}{\pi}\left\{1-J_0(2\Theta)+\frac{2}{3}J_2(2\Theta)\right\}\right] \tag{44}$$

to within 1 per cent up to $\Theta=\dfrac{\pi}{2}$.

4.2. *Systems with Stiffness Discontinuities Dependent upon the Direction of Motion.* These are of the type

$$\ddot{x}+A\dot{x}+Bx+f(x)\,.\,\mathrm{Sgn}\,\dot{x} = 0\,. \tag{45}$$

Now

$$\dot{x} = \dot{\Theta}\cos\phi - \omega\,\Theta\sin\phi \equiv \Theta\,[\lambda\cos\phi - \omega\sin\phi]$$

so that

$$\mathrm{Sgn}\,\dot{x} = -ve \text{ for } \phi_c < \phi < \pi+\phi_c$$

$$= +ve \text{ for } \pi+\phi_c < \phi < 2\pi+\phi_c\,,$$

where $\tan\phi = \dfrac{\lambda}{\omega}$.

If $f(x)$ may be represented by the polynomial $\sum_0^N a_n x^n$, then

$$\int_0^{2\pi} f(x)\,\mathrm{Sgn}\,\dot{x}\,\begin{matrix}\cos\phi\\(\sin\phi)\end{matrix}\,d\phi = -2\int_{\phi_c}^{\pi+\phi_c}\sum a_n\,\Theta^n\cos^n\phi\,\begin{matrix}\cos\phi\\(\sin\phi)\end{matrix}\,d\phi \text{ for } n \text{ even} \tag{46}$$

and $\qquad = 0$ for n odd

Then, for the first approximation,

$$\lambda \doteqdot \frac{I_1}{2\omega} = -\frac{A}{2}-\frac{2}{\pi}\sum_0^N \frac{a_n\,\Theta^{n-1}}{n+1}\cdot\frac{\omega^n}{[\omega^2+\lambda^2]^{\frac{n+1}{2}}} \text{ for } n \text{ even} \tag{47}$$

and

$$\omega^2 \doteqdot \lambda^2+I_2$$

$$= \lambda^2+B+A\lambda+\frac{4}{\pi}\sum_0^N a_{2n}\,\Theta^{2n-1}\sum_0^n(-1)^\vartheta\binom{n}{\vartheta}\frac{\lambda^{2\vartheta+1}}{[\omega^2+\lambda^2]^{\frac{2\vartheta+1}{2}}} \text{ for } n \text{ even} \tag{48}$$

where $\dbinom{n}{\vartheta} = \dfrac{n\,!}{(n-\vartheta)\,!\,\vartheta\,!}$.

As a particular example, consider the effect of Coulomb friction, *viz.* of constant magnitude but always opposing the motion. For this case $f(x) = a_0 =$ constant. Then from the first approximation

$$\lambda \doteqdot -\frac{A}{2}-\frac{2}{\pi}\,\frac{a_0}{\sqrt{\omega^2+\lambda^2}}\cdot\frac{1}{\Theta} \tag{49}$$

$$\omega^2 \doteqdot \lambda^2+B+A\lambda+\frac{4}{\pi}\,\frac{a_0\,.\,\lambda}{\sqrt{\omega^2+\lambda^2}}\cdot\frac{1}{\Theta} \tag{50}$$

$$= B-\lambda^2 \text{ from equation (49)},$$

i.e. $\omega^2+\lambda^2 = B$, so that

and

$$\left.\begin{aligned} -\frac{\lambda}{A/2} &\doteqdot 1+\frac{2}{\pi}\left(\frac{a_0}{B}\right)\cdot\left(\frac{2B^{\frac{1}{2}}}{A}\right)\cdot\frac{1}{\Theta} \\ \frac{\omega^2+\dfrac{A^2}{4}}{B} &\doteqdot 1-\frac{4}{\pi}\left(\frac{a_0}{B}\right)\cdot\left(\frac{A}{2B^{\frac{1}{2}}}\right)\cdot\frac{1}{\Theta}+0\left(\frac{1}{\Theta^2}\right). \end{aligned}\right\} \tag{51}$$

From equation (45) it follows that below an amplitude of $\dfrac{a_0}{B}$, there will be no motion.

5. *Influence of Distortion in the Response Waveform.*

In the preceding sections it has been assumed that the displacement is nearly enough sinusoidal with time, and is capable of representation adequately by $\Theta\,(t)\,.\cos\phi\,(t)$. The present method may be generalised to include, and to obtain the amplitude and phase of, harmonics of the fundamental which produce the distortion. Thus

$$x = \sum_{n=1}^{N} \Theta_n \cos\,(n\,\phi+\varepsilon_n)$$

$$\dot{x} = \sum_{n=1}^{N} [\dot{\Theta}_n \cos\,(n\,\phi+\varepsilon_n)-\Theta_n\, n\,\omega \sin\,(n\,\phi+\varepsilon_n)]\,.$$

From these, and the method used for deriving equations (9) and (10) for the fundamental, it follows that for the $(m-1)th$ harmonic,

$$\lambda_m = \frac{I_{1m}}{2m\,\omega\left[1+\dfrac{\Theta_m}{4\omega^2}\cdot\dfrac{d(\omega^2)}{d\Theta_m}\right]} \tag{52}$$

and

$$(m\,\omega)^2 = \lambda_m^2+I_{2m}+\lambda_m\,.\,\Theta_m\,.\,\frac{d\lambda_m}{d\Theta_m} \tag{53}$$

where

$$I_{1m} = \frac{1}{\pi\,\Theta_m}\int_0^{2\pi} F\,.\sin\,(m\,\phi+\varepsilon_m)\,d\phi$$

and

$$I_{2m} = \frac{1}{\pi\,\Theta_m}\int_0^{2\pi} F\,.\cos\left(m\,\phi+\varepsilon_m\right)d\phi\,.$$

These give a set of $2N$ equations for the unknowns.

This could be of particular interest in the study of systems producing limit cycles. In the steady case, $\frac{d(\omega^2)}{d\Theta_m} = 0$, and since by definition the steady state amplitude is not zero, whereas $\dot{\Theta}_m$ is, λ_m must be zero. Then

$$\left.\begin{aligned} \frac{I_{1m}}{m} &\equiv 0 \quad \text{for all } m \\ \text{and} \qquad \omega^2 &= \text{constant} \equiv \frac{I_{2m}}{m^2} \quad \text{for all } m\,. \end{aligned}\right\} \qquad (54)$$

A good example of a system in which the distortion is important is the van der Pol equation (equation (22)) in which A is very large compared with B. Fig. 4 shows the displacement waveforms for a range of $\frac{A}{B}$ from 0·1 to 10. Accompanying the distortion in the latter case is a large change in frequency which is not predicted from a consideration of the fundamental alone (equation (25)) inasmuch as in the limit cycle, ($\lambda = 0$), it gives $\omega^2 \doteqdot B$, and thus independent of A.

The displacement limit cycle waveforms of Fig. 4 have been harmonically analysed utilising 96 ordinates per cycle. Because of symmetry the amplitudes of even multiples of the fundamental are zero. The amplitudes of the first few odd-multiple components, together with the phase angles referred to the fundamental are given below.

$$\ddot{x}+A\left(x^2-1\right)\dot{x}+x = 0\,, \qquad \text{for which} \qquad x = \sum_1^{\infty} \Theta_n \cos\left(n\,\phi-\varepsilon_n\right)$$

	$A = 0{\cdot}1$		$A = 1{\cdot}0$		$A = 10{\cdot}0$	
Fundamental frequency	$0{\cdot}999_4$		$0{\cdot}943_2$		$0{\cdot}329_7$	
n	Θ_n	$\varepsilon_n(°)$	Θ_n	$\varepsilon_n(°)$	Θ_n	$\varepsilon_n(°)$
1	1·997	0	2·020	0	2·128	0
3	0·022	90°	0·231	54	0·639	16
5	$< 1\%$ of Θ_1 (5–11)		0·048	85	0·358	13
7			$< 1\%$ of Θ_1 (7–11)		0·241	12
9					0·178	16
11					0·139	19

In principle at least, it should be possible to predict these coefficients from equation (54). The inter-relationships which produce zero damping (i.e. $I_{1m} = 0$) should then give ω^2 from I_{2m}.

5.1. *Duffing's Equation.*

As an example, we consider the extent of distortion present in the motion characterised by the example in Section 3.1 but without damping, *viz.* $\ddot{x}+Bx+Cx^3 = 0$. In this case, $\lambda = 0$, and the amplitude remains as the initial value. Consider the effect of a second harmonic content such that

$$x = \Theta_1 \cos\phi+\Theta_3 \cos(3\phi+\varepsilon).$$

Thus

$$F = B[\Theta_1 \cos\phi+\Theta_3 \cos(3\phi+\varepsilon)]+C\sum_0^3 k_n \Theta_1^{3-n}\Theta_3^n \cos^n(3\phi+\varepsilon) . \cos^{3-n}\phi \tag{55}$$

where $k_n = 1$ for $n = 0$ and 3

$k_n = 3$ for $n = 1$ and 2.

After some arithmetic, the relevant integrals give:

(i) $$\frac{1}{\pi}\int_0^{2\pi} F . \cos\phi \, d\phi = B\Theta_1+C\left[\frac{3}{4}\Theta_1^3+\frac{3}{4}\Theta_1^2\Theta_3 \cos\varepsilon+\frac{3}{2}\Theta_1\Theta_3^2\right]. \tag{56}$$

(ii) $$\frac{1}{\pi}\int_0^{2\pi} F . \cos(3\phi+\varepsilon)\, d\phi = B\Theta_3+C\left[\frac{\Theta_1^3}{4}\cos\varepsilon+\frac{3}{2}\Theta_1^2\Theta_3+\frac{3}{4}\Theta_3^3\right]. \tag{57}$$

(iii) $$\frac{1}{\pi}\int_0^{2\pi} F . \sin\phi \, d\phi = -\frac{3}{4}C\Theta_1^2\Theta_3 \sin\varepsilon. \tag{58}$$

(iv) $$\frac{1}{\pi}\int_0^{2\pi} F . \sin(3\phi+\varepsilon)\, d\phi = C\frac{\Theta_1^3}{4}\sin\varepsilon. \tag{59}$$

From equations (52), (58) and (59), since the system is conservative

$$\sin\varepsilon = 0.$$

Hence, from equations (53), (56) and (57)

$$\omega^2 = B+\frac{3}{4}C[\Theta_1^2+\Theta_1\Theta_3+2\Theta_3^2]. \tag{60}$$

$$9\Theta_3\omega^2 = B\Theta_3+C\left[\frac{3}{4}\Theta_3^3+\frac{3}{2}\Theta_1^2\Theta_3+\frac{\Theta_1^3}{4}\right]. \tag{61}$$

Substituting for ω^2 in equation (61), and omitting higher order terms in Θ_3 in comparison with Θ_1, we have:

$$\frac{C}{4}[\Theta_1^3 - 21\Theta_1^2\Theta_3] \doteqdot 8B\Theta_3 .$$

Putting

$$\frac{B}{C\Theta_1^2} = R \quad \text{and} \quad \frac{\Theta_3}{\Theta_1} = r ,$$

this becomes:

$$r \doteqdot \frac{1}{32R+21} \tag{62}$$

i.e. $\Theta_3 \to 0$ for the linear case, and to about 2 per cent the amplitude of Θ_1 where the maximum non-linear contribution to the stiffness is equal to the linear.

The influence on the frequency is no greater, and gives confidence in the application of the method using only the fundamental. Distortion of the waveform is, however, an inevitable consequence of non-linearity, and in extreme cases, such as the van der Pol Equation considered earlier, it will decide the degree of applicability of the fundamental mode method.

6. *Equivalent Linear Systems.*

In equations (9) and (10) we formulated expressions for the amplitude damping λ, and the (frequency)^2, ω^2, *viz.*

$$\left.\begin{aligned} \lambda &\doteqdot \frac{1}{2\omega} \cdot \frac{I_1}{1 + \dfrac{\Theta}{4\omega^2} \cdot \dfrac{d(\omega^2)}{d\Theta}} \\ \omega^2 &\doteqdot \lambda^2 + I_2 + \lambda\,\Theta \frac{d\lambda}{d\Theta} . \end{aligned}\right\} \tag{63}$$

Because I_1 and I_2 contain only Θ, $\dot{\Theta}$ and ω, i.e. Θ, λ and ω, equation (63) gives two relationships for these three variables, so that we may in principle express any two in terms of the third, i.e.

$$\left.\begin{aligned} \lambda &= f_1(\Theta) \\ \omega^2 &= f_2(\Theta) . \end{aligned}\right\} \tag{64}$$

We can now define the equivalent linear system as that which at a particular amplitude, Θ, has the same frequency and amplitude damping as the actual system. The equivalence will in general vary with amplitude (equation (64)).

Consider as a simple illustration the example of Section 3.1,

$$F = A\dot{x} + Bx + Cx^3 .$$

Let the equivalent linear system be

$$F = a\dot{x} + bx .$$

Then

$$\left.\begin{aligned} a &\equiv -2\lambda \doteqdot A \\ b &\equiv \omega^2+\lambda^2 \doteqdot B+\frac{3}{4}C\,\Theta^2 \end{aligned}\right\} \text{ from equation (16)}.$$

Note that the equivalent stiffness, b, is $\frac{1}{\pi}\int\limits_0^{2\pi} \frac{\partial F}{\partial x}\sin^2\phi\,.\,d\phi$, rather than $\frac{1}{2\pi}\int\limits_0^{2\pi} \frac{\partial F}{\partial x}\,.\,d\phi$, or the displacement average, $\int\limits_0^{\pi/2} \frac{\partial F}{\partial x}\,.\sin\phi\, d\phi$. In Ref. 1 the term $F(x, \dot{x})$ was predominantly linear in x, *viz.* $v^2 x+\varepsilon(x, \dot{x})$ and it was shown that, to the order of ε^2, the 'equivalent linear system' was a justifiable replacement for the original system. In our present attempt to extend the scope of the method, we have shown that the linear equivalent will vary with amplitude, and it suggests that the transfer function might be regarded as having different values according to the amplitude range, or even continuously varying coefficients. The ramifications of this have yet to be investigated, particularly as regards response to forcing.

Nevertheless it suggests that non-linear systems of this type might well be analysed and catalogued in terms of the dependence of the two parameters ω^2 and $\frac{\dot{\Theta}}{\Theta}(\equiv\lambda)$ on amplitude, and this can be done whether or not the basic differential equation of any particular system is known.

Table 1 gives a summary of the results obtained herein in the form:

$$\begin{bmatrix} \dfrac{\omega^2}{K_1}+K_2 \\[2ex] \dfrac{\lambda}{K_3} \end{bmatrix} = [A_{mn}]\,\Theta^{2n} \tag{65}$$

where A_{mn} defines the dynamic characteristics of the system

$n = 0, 1, 2$ etc.

$m = 1$ and 2 for top and bottom rows respectively,

and K_1, K_2 and K_3 are constants.

The resemblance in form to that of the linear system is striking. The constants on the l.h.s. are identical in every case. On the r.h.s. the constant term (i.e. for $n = 0$) is unity in every case, the essential difference being the non-zero coefficients of Θ^2, Θ^4, etc.* It is worth noting that because of its form very few terms in the expansion are needed to define this dependence over quite large amplitudes. For example, only the term in Θ^2 is necessary to specify the response for the damped pendulum system to within 1 per cent for 1 radian amplitude.

Also included in Table 1 is the suggested fit to the 'experimental' (i.e. computed) data. In all cases the adjustment required to the solution from the present approximate method is very small indeed for the representation of $\frac{\omega^2}{K_1}+K_2$. Rather more is required to fit the $\frac{\lambda}{K_3}$ data at large amplitudes (of the order of 90° or more).

---

*The even order powers of Θ arise naturally from the 'direct' applications considered in Section 3. Odd powers will arise if $F(x, \dot{x})$ includes Sgn x or Sgn $\dot{x}$ terms (Section 4).

The method offers possibilities for the classification of practical dynamic systems, and is being adopted for the presentation of the response data from the RAE wind tunnel/flight dynamics simulator facility. It is hoped that the 'response signatures' for aircraft and weapon flight dynamics, with or without auto-pilot and guidance representation, will prove to be amenable to presentation in the form, and that it will be possible eventually to extend the technique to the higher order differential equations necessary to specify the full 6 degrees-of-freedom motion behaviour.

7. *Conclusions.*

7.1. By a slight recasting of the results from Ref. 1 the external dynamic properties of the system, $\ddot{x}+F(x,\dot{x}) = 0$, *viz.* the frequency ω and the amplitude damping $\frac{\dot{\Theta}}{\Theta}$, may be expressed as:

$$\lambda\left(\equiv\frac{\dot{\Theta}}{\Theta}\right) \doteqdot \frac{I_1}{2\omega\left[1+\frac{\Theta}{4\omega^2}\frac{d(\omega^2)}{d\Theta}\right]}$$

$$\omega^2 \doteqdot \lambda^2+I_2+\lambda\,\Theta\frac{d\lambda}{d\Theta}$$

where $x = \Theta(t)\cos\phi(t)$

Θ = amplitude

$$I_1 = \frac{1}{\pi\Theta}\int_0^{2\pi} F\sin\phi\,.\,d\phi$$

$$I_2 = \frac{1}{\pi\Theta}\int_0^{2\pi} F\cos\phi\,.\,d\phi$$

7.2. Instead of neglecting $\ddot{\Theta}$ and $\dot{\omega}\,\Theta$ as in Ref. 1, a first approximation is obtained by omitting the $\frac{d}{d\Theta}$ terms; this, for example, gives the exact solution for the linear case. Where required, second order approximations are obtained by including the derivatives of the first approximation with respect to Θ.

7.3. In this form the present method has been found to give good agreement in particular cases for which $F(x,\dot{x})$ departs markedly from the constraints implicit in Ref. 1.

7.4. Since ω^2 and λ are generally functions of amplitude it may be convenient for analysis and application to catalogue the characteristics of this type of non-linear system in terms of this dependence.

LIST OF SYMBOLS

a	(i) coefficients of 'equivalent linear system' (Section 6)
b	(ii) coefficients of exponents (Appendix)
A	
B	Constants
C	
A_{mn}	2 row matrix giving coefficients of Θ^n for $\frac{\omega^2}{K_1}+K_2$ $(m=1)$ and $\frac{\lambda}{A/2}$ $(m=2)$
I_1	$\frac{1}{\pi\,\Theta}\int_0^{2\pi} F \sin\phi \,.\, d\phi$
I_2	$\frac{1}{\pi\,\Theta}\int_0^{2\pi} F \,.\, \cos\phi \,.\, d\phi$
I_{1m}	$\frac{1}{\pi\,\Theta_m}\int_0^{2\pi} F \,.\, \sin(m\,\phi+\varepsilon_m)\, d\phi$
I_{2m}	$\frac{1}{\pi\,\Theta_m}\int_0^{2\pi} F \,.\, \cos(m\,\phi+\varepsilon_m)\, d\phi$
$J_n(\Theta)$	Bessel function of the first kind of integer order n
$K\left(\frac{\Theta}{2}\right)$	Complete elliptic integral of first kind $= \int_0^{\pi/2} \frac{d\phi}{\left[1-\sin^2\left(\frac{\Theta}{2}\right).\sin^2\phi\right]^{\frac{1}{2}}}$
k	$\frac{A^2}{4B}$
$K_{1,2,3}$	Constants (Section 6)
Sgn	'Sign of'
t_n	Elapsed time to the point on the nth half cycle to which the amplitude envelope is tangential
t_{ref}	Time reference
Δn	$t_n - t_{ref}$
ε	Phase displacement
Θ	Amplitude
λ	$\frac{\dot{\Theta}}{\Theta}$

ν	Frequency, considering only linear stiffness term in $F(x, \dot{x})$ (Ref. 1)
ϕ	Phase angle
ω	Fundamental frequency ($\equiv \dot{\phi}$)

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APPENDIX

Numerical Evaluation of $\dot{\Theta}/\Theta$.

Let us assume that the envelope may be defined by

$$\Theta_n = \Theta_{n=\mathrm{ref}} \exp\left(\sum_{m=1} a_m \Delta_n^m\right) \tag{A.1}$$

where Θ_n is the amplitude at the point of tangency of the envelope on the nth half cycle peak

$\Delta_n = t_n - t_{\mathrm{ref}}$

$t_n =$ elapsed time to Θ_n.

If we take 3 points to define this dependence, then

$$\left.\begin{aligned} \Theta_3 &= \Theta_1\, e^{(a\Delta_3 + b\Delta_3^2)} \\ \Theta_2 &= \Theta_1\, e^{(a\Delta_2 + b\Delta_2^2)} \end{aligned}\right\} \tag{A.2}$$

Then, at Θ_2,

$$[\dot{\Theta}] = \Theta_1\, (a + 2b\, \Delta_2)\, e^{(a\Delta_2 + b\Delta_2^2)} \tag{A.3}$$

therefore

$$\left[\frac{\dot{\Theta}}{\Theta}\right]_2 = a + 2b\, \Delta_2\,. \tag{A.4}$$

From (A.2),

$$\left.\begin{aligned}\log_e\left[\frac{\Theta_3}{\Theta_1}\right] &= a\,\Delta_3 + b\,\Delta_3^2 \\ \log_e\left[\frac{\Theta_2}{\Theta_1}\right] &= a\,\Delta_2 + b\,\Delta_2^2\,.\end{aligned}\right\} \tag{A.5}$$

Eliminating a and b from (A.5) gives

$$\left.\begin{aligned} a &= \frac{\Delta_3^2 \log\left[\frac{\Theta_2}{\Theta_1}\right] - \Delta_2^2 \log\left[\frac{\Theta_3}{\Theta_1}\right]}{\Delta_3\,\Delta_2\,(\Delta_3 - \Delta_2)} \\ \text{and} \qquad & \\ b &= \frac{\Delta_2 \log\left[\frac{\Theta_3}{\Theta_1}\right] - \Delta_3 \log\left[\frac{\Theta_2}{\Theta_1}\right]}{\Delta_3\,\Delta_2\,(\Delta_3 - \Delta_2)}\,.\end{aligned}\right\} \tag{A.6}$$

Substituting for a and b in (A.4) gives

$$\left[\frac{\dot{\Theta}}{\Theta}\right]_2 = \frac{\Delta_2^2 \log_e\left[\frac{\Theta_3}{\Theta_2}\right] + [\Delta_3 - \Delta_2]^2 \log\left[\frac{\Theta_2}{\Theta_1}\right]}{\Delta_3\,\Delta_2\,(\Delta_3 - \Delta_2)}\,. \tag{A.7}$$

There is a phase difference between the peak of any half cycle and the point to which the amplitude envelope is tangential. If the latter is taken to occur at ϕ, the former is defined by:

$$\dot{x} = 0 \equiv \dot{\Theta}_n \cos(N\,\pi + \varepsilon) - \Theta_n\,\omega \sin(N\pi + \varepsilon)$$

or

$$\tan\varepsilon = \frac{1}{\omega}\left[\frac{\dot{\Theta}}{\Theta}\right]_n.$$

If the waveform is quasi-sinusoidal then the tangency points, $\Theta_{1,2,3}$ required for the foregoing are the peak values multiplied by $\cos\varepsilon$, i.e.

$$\Theta_{\text{peak}} = \Theta_n\sqrt{1 + \left(\frac{\lambda}{\omega}\right)^2}\,.$$

Since, by stipulation, λ and ω do not vary significantly during one cycle, we may cancel the $\sqrt{1 + \left(\frac{\lambda}{\omega}\right)^2}$ terms in deriving $\log\left[\frac{\Theta_3}{\Theta_2}\right]$, etc., and hence utilise *peak* values of Θ. By the same token, Δ_n is unaffected. However, it should be remembered that the Θ at which $\frac{\dot{\Theta}}{\Theta}(= \lambda)$ is evaluated from (A.7) is strictly the tangency value, *viz.* the peak value $\div \sqrt{1+\left(\frac{\lambda}{\omega}\right)^2}$; however, both λ and ω are now known, so this adjustment presents no problem.

Care is clearly necessary in using this numerical method where the waveform is significantly distorted (Section 5).

TABLE 1

External Dynamic Characteristics (Frequency and Amplitude Damping) of Various Systems for which $\ddot{x}+F(x,\dot{x})=0$.

$$x = \Theta(t)\cos\phi(t)$$

$$\omega = \dot{\phi}$$

$$\lambda = \frac{\dot{\Theta}}{\Theta}$$

$K_{1,\ 2\ \mathrm{and}\ 3}$ are constants

$$\begin{bmatrix} \frac{\omega^2}{K_1}+K_2 \\ \frac{\lambda}{K_3} \end{bmatrix} = [A_{mn}]\,\Theta^{2n}$$

$n = 0, 1, 2 \ldots$

$m = 1$ or 2 for upper and lower rows respectively

N.B. In the following table, $k = \frac{A^2}{4B}$

$F(x,\dot{x})$	K_1 K_2 K_3	A_{mn}														
		Column A - first approximation					Column B - second approximation					Column C - empirical fit				
		Eq.	n=0	1	2	... p	Eq.	n=0	1	2		n=0	1	2		
$A\dot{x}+Bx$	B k $\frac{A}{2}$		1	0	0	....						1	0 ...		- exact	
			-1	0	0	....						1	0 ...			
$A\dot{x}+Bx+Cx^3$	B k $\frac{A}{2}$	(16)	1	$\frac{3}{4}\frac{C}{B}$	0	....	(17)	1	$\frac{3}{4}\frac{C}{B}$	0	....					
			-1	0		....		-1	$+\frac{3}{4}\frac{C}{B(1-k)}$	$-\frac{27}{64}\frac{C^2}{B^2(1-k)^2}$	....					
$A\dot{x}+Bx+Cx^{2p+1}$	B k $\frac{A}{2}$	(21)	1	0	0 ...	$\frac{C(2p+2)!}{B(p+1)!^2\,2^{(2p+1)}}$ ...										
			-1	0	0	....										
$A(Cx^2-1)\dot{x}+Bx$	B k $\frac{A}{2}$	(25)	1	$\frac{3}{2}kC$	$-\frac{5}{16}(kC)^2$	0										
			1	$-\frac{C}{4}$	0	....										
$A\dot{x}+B\sin x$	B k $\frac{A}{2}$	(29)	1	$-\frac{1}{3}$	$\frac{1}{192}$	....	(30)	1	$-\frac{1}{8}$	$\frac{1}{192}$	....	1	- 0.125	0.004		
			-1	0		....		-1	$-\frac{1}{16(1-k)}$	$-\frac{(11+4k)}{768(1-k)^2}$	....	1	0.035	0.012		

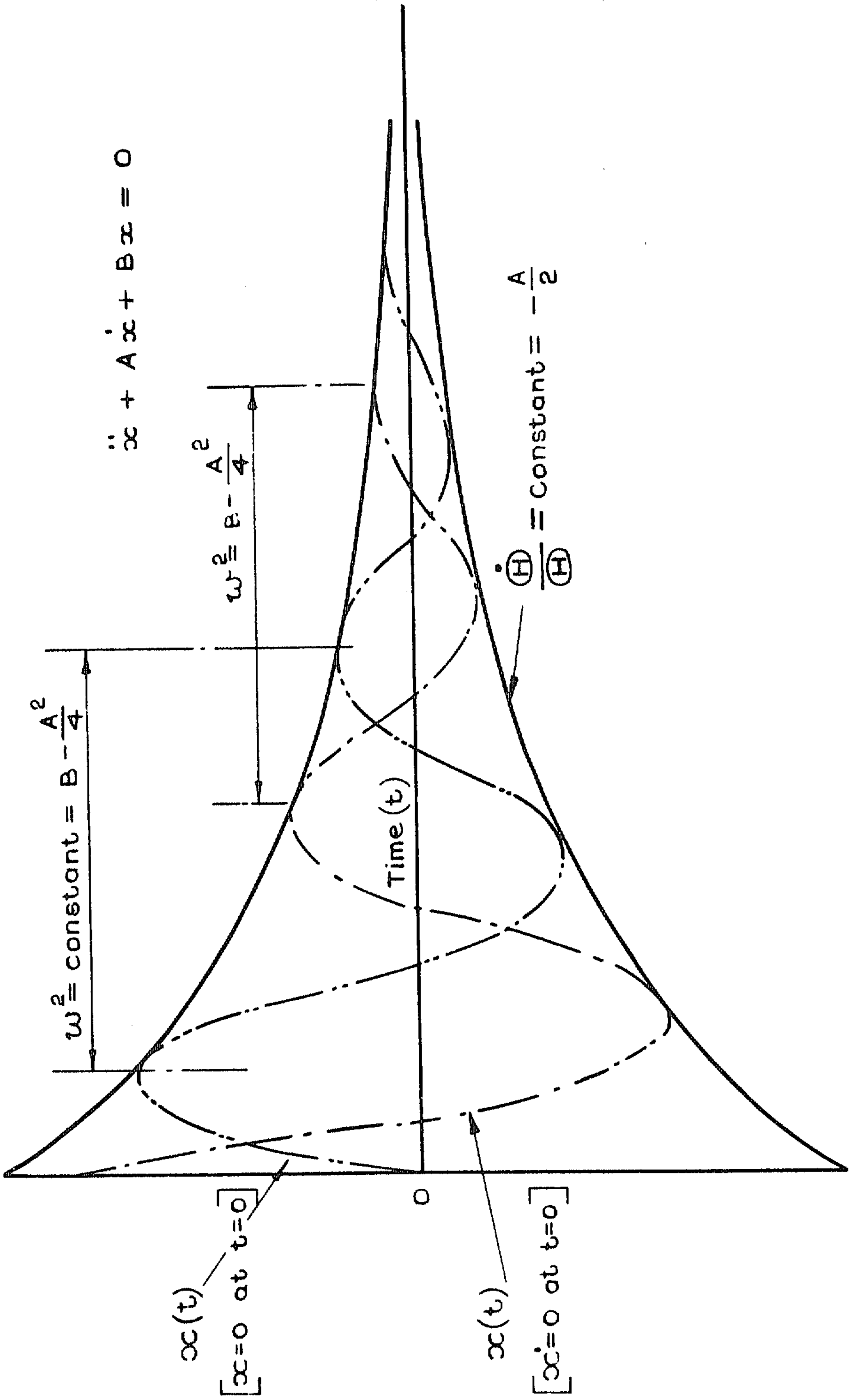

FIG. 1. Motion parameters of simple linear system.

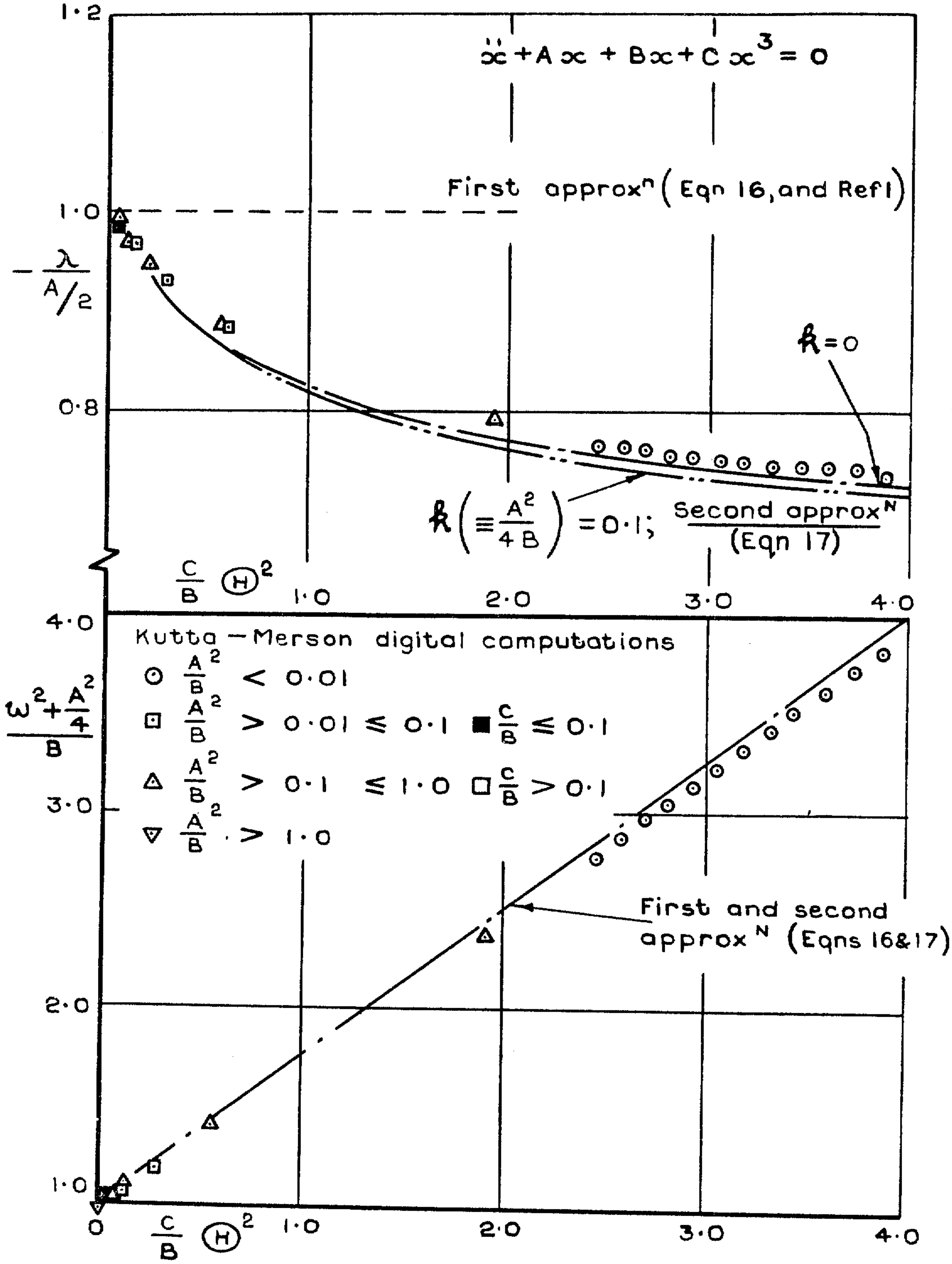

FIG. 2. Application to solution of Duffing's equation.

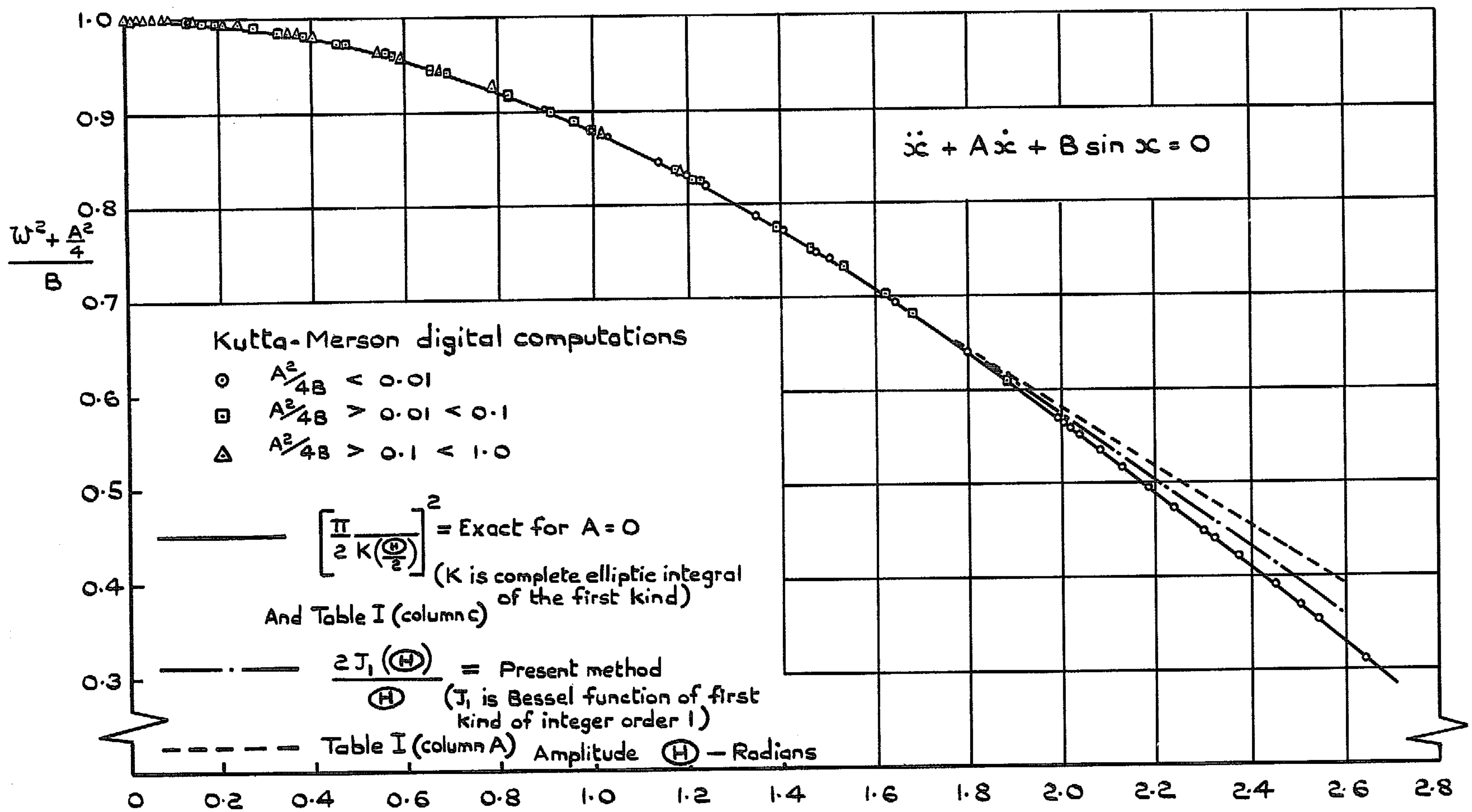

FIG. 3a. Large amplitude pendulum—frequency dependence on amplitude.

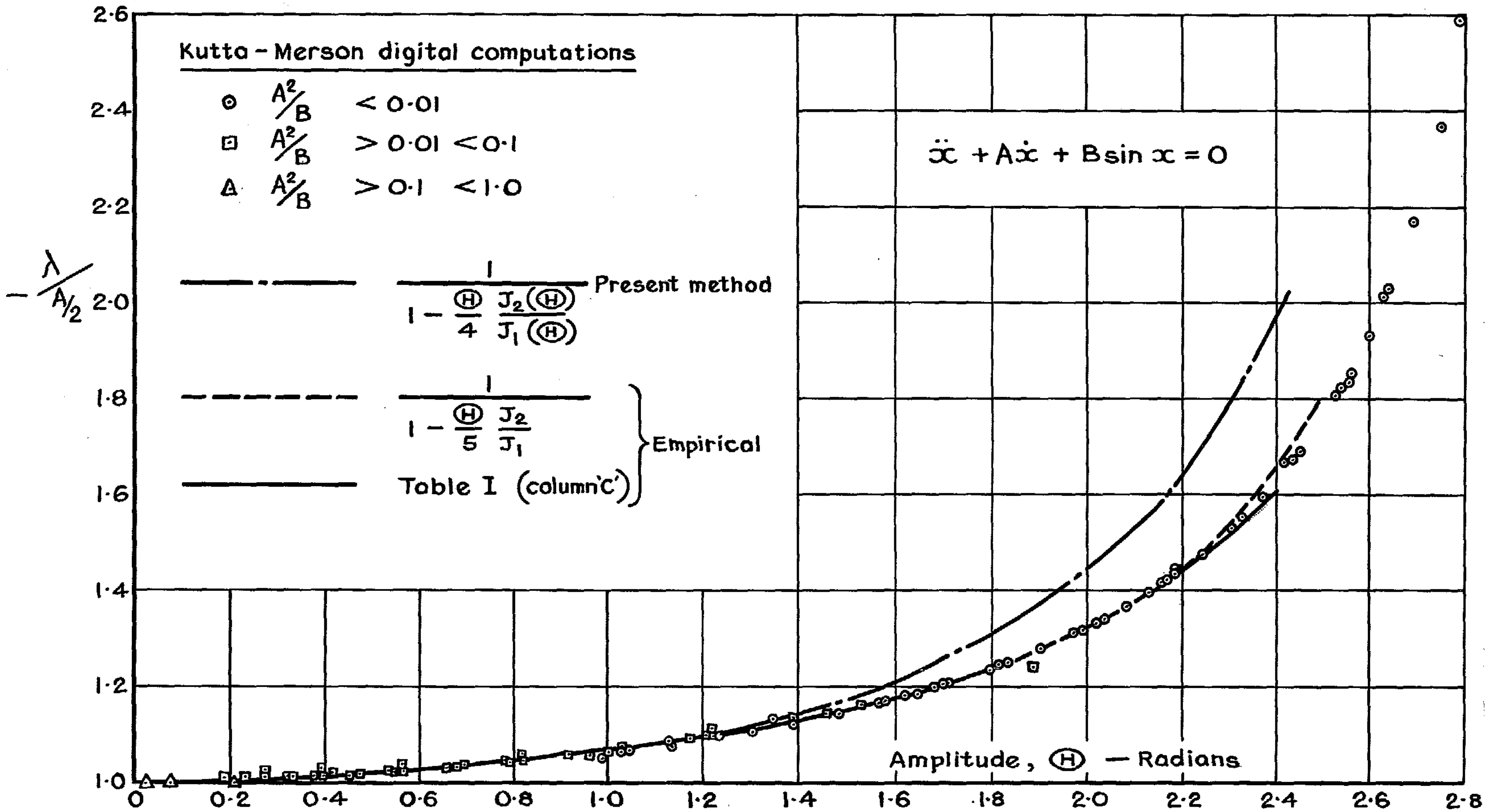

FIG. 3b. Large amplitude pendulum—damping dependence on amplitude.

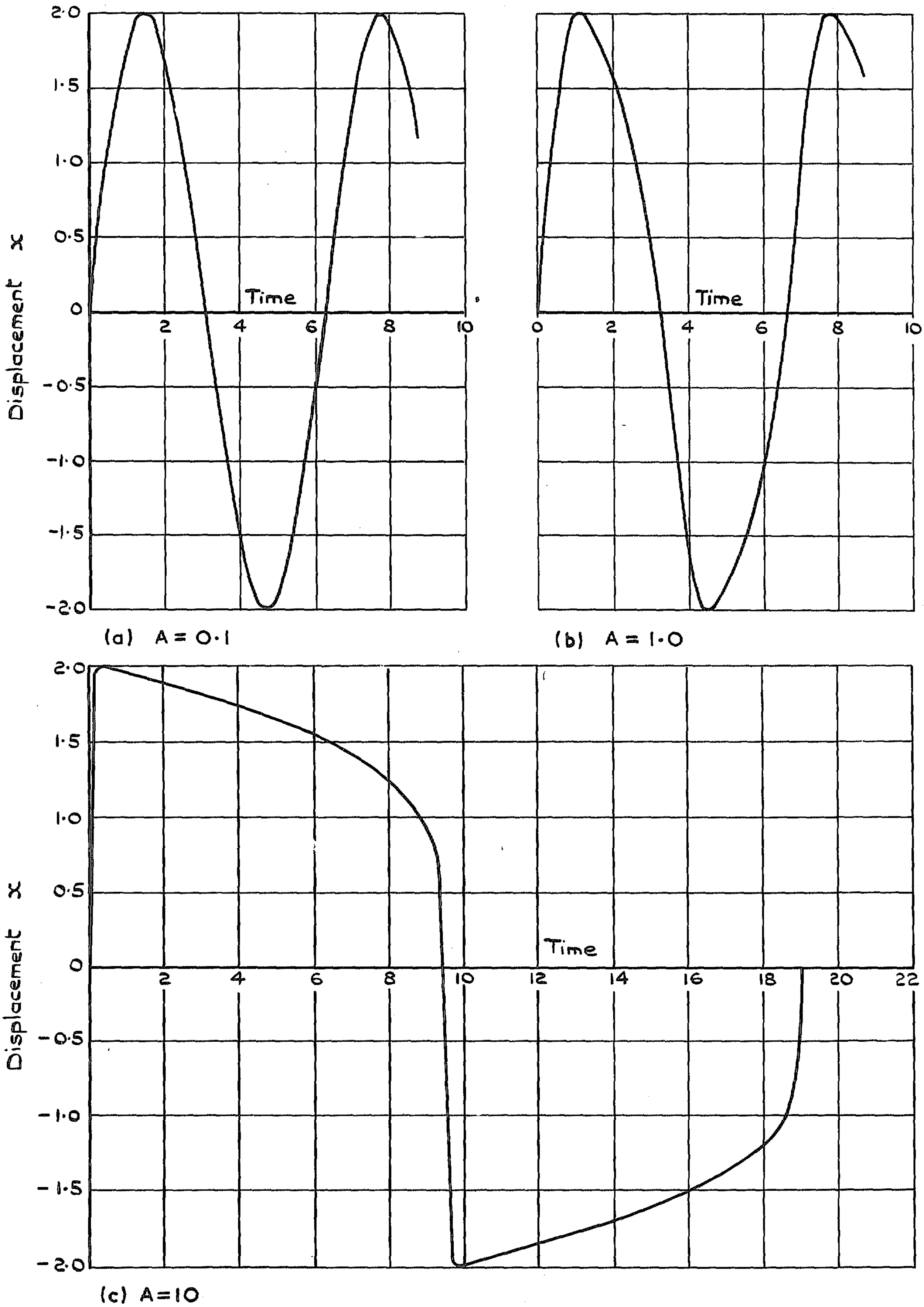

FIG. 4 a to c. Limit cycle displacement/time histories for van der Pol's Equations $\ddot{x} + A(x^2 - 1)\dot{x} + x = 0$.

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