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The Velocities Induced by Distributions of Infinite Kinked Source and Vortex Lines Representing Wings with Sweep and Dihedral in Incompressible Flow

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Summary.

Equations have been derived for the velocities induced in an incompressible flow by distributions of infinite source and vortex lines representing wings of infinite span and constant chord having both sweep and dihedral. Particular attention is paid to the centre section where the dihedral effects are large.

The equations showed that such a source distribution does not represent a wing with symmetrical sections, and that such a vortex distribution does not represent a thin wing. It is, therefore, not possible to separate the effects of wing thickness and wing load distribution, even when linear-theory assumptions are retained.

This work was done before the widespread use of electronic computers to calculate wing characteristics. It is published now to illustrate the complex nature of the equations for the velocities induced by non-planar singularity distributions.

*Replaces RAE Tech Report 70077—A.R.C. 32 389.

LIST OF CONTENTS

LIST OF APPENDICES

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1. *Introduction.*

Some twenty years ago, simple approximate methods were developed for calculating the pressure distribution and loading on plane swept wings in incompressible flow. In these methods the velocity distribution on such a wing was based on the velocities induced by distributions of parallel source and vortex lines, of infinite span and constant strength spanwise, kinked at the centre section to represent the sweep. Approximate expressions for these induced velocities are contained in Refs. 1 and 2, and the methods are summarised in Ref. 3.

About the same time, corresponding equations were derived* for the velocities induced by distributions of infinite singularities of constant spanwise strength which were kinked to represent not only sweep but dihedral. The distributions are therefore not planar, though each semi-infinite distribution on either side of the centreline is itself planar. These equations provide the solution to the design problem, in which the source and vorticity distributions are specified and the wing shape corresponding to these distributions is obtained by equating the induced velocities to the boundary conditions. As will be seen, the effect of the dihedral angle makes the application to the inverse problem—the velocity distribution on a given wing shape—extremely complicated.

Since this work was done, the advent of electronic computers has allowed some relaxation of the simplifying approximations in the above methods, which have thus to some extent been superseded by other methods. Nevertheless the results for swept, dihedralled sources and vortices are published now to illustrate the complex nature of the equations for the velocities induced by non-planar singularity distributions.

In Section 2, the geometry of the system is defined. In Section 3 the induced velocities at the kink section due to sources are derived, and in Section 4 those due to vortices. Section 5 contains some brief comments on certain features of the results.

2. *Geometric Definitions.*

Fig. 1 shows the axis system and geometric relations used in the derivation of the velocities induced by single, infinite kinked vortex and source lines having both sweep and dihedral with respect to a reference plane which contains the free-stream velocity vector V_0. The origin 0 is at the vertex of the kinked source or vortex line A0B, and the x-axis is in the free-stream direction, positive downstream. The y-axis is positive to starboard and the z-axis positive upwards, as shown. The x, z co-ordinates of points on the source or vortex lines are thus always positive. Two sweep angles, φ and Φ are defined: φ is the angle between the y-axis and the perpendicular projection of A0B on the xy plane; $\frac{\pi}{2}-\Phi$ is defined as the angle between A0B and the x-axis. φ and Φ are positive for sweepback. The dihedral angle, ψ, is defined as the angle between the xy plane and the perpendicular projection of the source or vortex line on a plane x = constant. ψ is positive for a dihedral angle, negative for anhedral. The angles Φ, φ and ψ are connected by the relation $\cos\psi = \dfrac{\tan\Phi}{\tan\varphi}$.

The distance along the source or vortex line is denoted by s, measured from the origin. The analysis deals primarily with the velocities induced in the xz plane.

In considering induced velocities far away from the plane of symmetry $y = 0$, it is convenient to work in terms of z', perpendicular to the plane which contains the source or vortex line and the x-axis, rather than z; $z' = 0$ on the plane. Clearly z' is in general, in a different direction on either side of the x-axis, because of the angle ψ.

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*Initially by D. E. Hartley, later by the present authors.

3. *Velocities Induced by a Distribution of Parallel Infinite Kinked Source Lines.*

3.1. *Single source line.*

Consider the velocity induced at a general point $R(x, 0, z)$ in the xy plane by a small element ds at a point $P(|y| \tan \varphi, y, |y| \tan \psi)$ on one half of a source line kinked at the origin as in Fig. 1.

In vector notation,

$$\mathbf{s} = \mathbf{0P} = (|y| \tan \varphi)\,\mathbf{i} + (y)\,\mathbf{j} + (|y| \tan \psi)\,\mathbf{k}$$

$$\mathbf{0R} = (x)\,\mathbf{i} + (0)\,\mathbf{j} + (z)\,\mathbf{k}\,.$$

Hence,

$$\mathbf{ds} = (\tan \varphi\, \mathbf{i} + \mathbf{j} + \tan \psi\, \mathbf{k})\, dy \tag{1}$$

and

$$\mathbf{r} = \mathbf{0R} - \mathbf{0P} = \mathbf{PR}$$

therefore

$$\mathbf{r} = (x - |y| \tan \varphi)\,\mathbf{i} + (-y)\,\mathbf{j} + (z - |y| \tan \psi)\,\mathbf{k}\,. \tag{2}$$

If the source line has strength E per unit length, the velocity at R due to the element ds at P is

$$\Delta \mathbf{v} = \frac{E\, ds\, \mathbf{r}}{4\pi r^3} \tag{3}$$

which has components along the axes

$$\left.\begin{aligned} \Delta v_x &= \frac{E\, ds\,(x - y \tan \varphi)}{4\pi r^3} \\ \Delta v_y &= -\frac{E\, ds\, y}{4\pi r^3} \\ \Delta v_z &= \frac{E\, ds\,(z - y \tan \psi)}{4\pi r^3} \end{aligned}\right\} \tag{4}$$

$$(0P)^2 = s^2 = y^2\,(1 + \tan^2 \varphi + \tan^2 \psi) = y^2\,\theta^2, \quad \text{say}, \tag{5}$$

and

$$(PR)^2 = r^2 = \left(y^2 + 2y \frac{A}{\theta} + \frac{x^2 + z^2}{\theta^2} \right) \theta^2 \tag{6}$$

where $A = -\dfrac{x \tan \varphi + z \tan \psi}{\theta}$. (7)

Hence,

$$\Delta v_x = \frac{E\,(x - y\tan\varphi)\,dy}{4\pi\,\theta^2 \left(y^2 + 2y\dfrac{A}{\theta} + \dfrac{x^2+z^2}{\theta^2} \right)^{3/2}} \tag{8}$$

and

$$\Delta v_z = \frac{E\,(z - y\tan\psi)\,dy}{4\pi\,\theta^2 \left(y^2 + 2y\dfrac{A}{\theta} + \dfrac{x^2+z^2}{\theta^2} \right)^{3/2}} \tag{9}$$

Integrating (8) and (9) yields the induced velocities due to the infinite kinked source line

$$v_x(x, 0, z) = \frac{E}{2\pi\,\theta^2} \int_0^\infty \frac{(x - y\tan\varphi)\,dy}{\left(y^2 + 2y\dfrac{A}{\theta} + \dfrac{x^2+z^2}{\theta^2} \right)^{3/2}}$$

$$= \frac{E}{2\pi\,\theta^2} \left\{ - \int_0^\infty \frac{\tan\varphi \left(y + \dfrac{A}{\theta} \right) dy}{\left(y^2 + 2y\dfrac{A}{\theta} + \dfrac{x^2+z^2}{\theta^2} \right)^{3/2}} + \int_0^\infty \frac{\left(x + \dfrac{A}{\theta}\tan\varphi \right) dy}{\left(y^2 + 2y\dfrac{A}{\theta} + \dfrac{x^2+z^2}{\theta^2} \right)^{3/2}} \right\}$$

$$= \frac{E}{2\pi\,\theta^2} \left\{ \tan\varphi \left[\frac{1}{\left(y^2 + 2y\dfrac{A}{\theta} + \dfrac{x^2+z^2}{\theta^2} \right)^{1/2}} \right]_0^\infty + \left(x + \frac{A}{\theta}\tan\varphi \right) \int_{A/\theta}^\infty \frac{d\left(y + \dfrac{A}{\theta} \right)}{\left\{ \left(y + \dfrac{A}{\theta} \right)^2 + \dfrac{x^2+z^2-A^2}{\theta^2} \right\}^{3/2}} \right\}$$

$$= \frac{E}{2\pi\,\theta^2} \left[-\frac{\theta\tan\varphi}{\sqrt{x^2+z^2}} + \frac{\theta^2 \left(x + \dfrac{A}{\theta}\tan\varphi \right)}{x^2+z^2-A^2} \left(1 - \frac{A}{\sqrt{x^2+z^2}} \right) \right]$$

$$= \frac{E}{2\pi}\left[\frac{x+\frac{A}{\theta}\tan\varphi}{x^2+z^2-A^2}\left(1-\frac{A}{\sqrt{x^2+z^2}}\right)-\frac{\tan\varphi}{\theta\sqrt{x^2+z^2}}\right] \tag{10}$$

$$v_y(x, 0, z) = 0$$

By comparing (8) and (9), and using equation (10)

$$v_z(x, 0, z) = \frac{E}{2\pi}\left[\frac{z+\frac{A}{\theta}\tan\psi}{x^2+z^2-A^2}\left(1-\frac{A}{\sqrt{x^2+z^2}}\right)-\frac{\tan\psi}{\theta\sqrt{x^2+z^2}}\right] \tag{11}$$

3.2. *Streamwise Distribution of Source Lines.*

Consider a distribution of parallel source lines in $z' = 0$.
Let $q(x)$ be the source strength per unit area. Then

$$E\,ds = q\,ds\,dx\cos\Phi\,,$$

therefore

$$E = q\,dx\cos\Phi\,. \tag{12}$$

Equation (10) can be rewritten in the form

$$v_x(x, 0, z) = \frac{E}{2\pi}\left[\frac{x\,\theta^2-\tan\varphi\,(x\tan\varphi+z\tan\psi)}{\theta^2(x^2+z^2-A^2)}\left(1+\frac{x\tan\varphi+z\tan\psi}{\theta\sqrt{x^2+z^2}}\right)-\frac{\tan\psi}{\theta\sqrt{x^2+z^2}}\right]$$

$$= \frac{E}{2\pi}\left[\frac{x\sec^2\psi-z\tan\psi\tan\varphi}{x^2\sec^2\psi-2x\,z\tan\psi\tan\varphi+z^2\sec^2\varphi}-\right.$$

$$\left.-\frac{\theta\,(z^2\tan\varphi-x\,z\tan\psi)}{(x^2\sec^2\psi-2x\,z\tan\psi\tan\varphi+z^2\sec^2\varphi)\sqrt{x^2+z^2}}\right]. \tag{13}$$

Replacing E by $q(x')\,dx'\cos\Phi$, and x by $x-x'$, integration with respect to x' gives the velocities induced by the streamwise distribution of source lines between the leading edge, $x' = 0$, and the trailing edge, $x' = 1$. Hence

$$v_x(x, 0, z) = \int_0^1 \frac{q(x')\cos\Phi}{2\pi}\,\frac{(x-x')\sec^2\psi-z\tan\psi\tan\varphi}{(x-x')^2\sec^2\psi-2(x-x')\,z\tan\psi\tan\varphi+z^2\sec^2\varphi}\,dx' -$$

$$-\int_0^1 \frac{q(x')\cos\Phi}{2\pi}\,\frac{\theta\,[z^2\tan\varphi-(x-x')\,z\tan\psi]}{[(x-x')^2\sec^2\psi-2(x-x')\,z\tan\psi\tan\varphi+z^2\sec^2\varphi]\sqrt{(x-x')^2+z^2}}$$

$$= I_1 - I_2\,.$$

The velocities are evaluated at $z' = 0$. In particular, at the centreline, (the x-axis), $z' = z = 0$ and, as shown in Appendix A,

$$I_1(x, 0, 0) = \int_0^1 \frac{q(x')}{2\pi} \frac{dx'}{x-x'} \cos \Phi \tag{14}$$

$$= v_x^* \cos \Phi$$

where v_x^* is the streamwise velocity increment induced by a distribution of unswept two-dimensional source lines. $I_1(x, 0, 0)$ is therefore the same as the streamwise velocity increment induced by a planar distribution of infinite source lines swept at an angle Φ, i.e. the 'infinite sheared wing'[3].

As shown in Appendix B,

$$I_2(x, 0, 0) = \frac{q(x)}{2\pi} \cos \Phi \log\left(\frac{1+\sin \Phi}{1-\sin \Phi}\right). \tag{15}$$

Hence the total chordwise velocity induced by the distribution of source lines is

$$v_x(x, 0, 0) = \cos \Phi \left\{ \int_0^1 \frac{q(x')}{2\pi} \frac{dx'}{x-x'} - \frac{q(x)}{2\pi} \log\left(\frac{1+\sin \Phi}{1-\sin \Phi}\right) \right\}. \tag{16}$$

Thus it can be seen that since ψ does not appear in equation (16), v_x is not affected by dihedral.

Equation (11) can be rewritten in the form

$$v_z(x, 0, z) = \frac{E}{2\pi} \left\{ \frac{z\,\theta^2 - \tan\psi\,(x \tan\varphi + z \tan\psi)}{\theta^2\,(x^2+z^2-A^2)} \left(1 + \frac{x \tan\varphi + z \tan\psi}{\theta\sqrt{x^2+z^2}}\right) - \frac{\tan\psi}{\theta\sqrt{x^2+z^2}} \right\}$$

$$= \frac{E}{2\pi} \left\{ - \frac{\tan\varphi \tan\psi}{\sec^2\psi} \cdot \frac{x \sec^2\psi - z \tan\varphi \tan\psi}{x^2 \sec^2\psi - 2x\,z \tan\varphi \tan\psi + z^2 \sec^2\varphi} - \right.$$

$$- \frac{\theta \tan\psi}{\sec^2\psi \sqrt{x^2+z^2}} + \frac{\theta^2}{\sec^2\psi} \frac{z}{(x^2 \sec^2\psi - 2x\,z \tan\varphi \tan\psi + z^2 \sec^2\phi)} +$$

$$\left. + \frac{\theta\,[z^2 \sec^2\varphi \tan\psi + x\,z \tan\varphi\,(1-\tan^2\psi)]}{\sec^2\psi \sqrt{x^2+z^2}\,(x^2 \sec^2\psi - 2x\,z \tan\varphi \tan\psi + z^2 \sec^2\varphi)} \right\}. \tag{17}$$

the terms being arranged to facilitate integration.

Again replacing E by $q(x')\,dx' \cos\Phi$ and x by $x-x'$, and integrating with respect to x'

$$v_z(x,0,z) = \frac{\cos\Phi}{2\pi}\left\{ -\frac{\tan\varphi\tan\psi}{\sec^2\psi}\int_0^1 \frac{q(x')\,(\overline{x-x'}\sec^2\psi - z\tan\varphi\tan\psi)\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\varphi\tan\psi + z^2\sec^2\varphi}\right\} -$$

$$-\frac{\theta\tan\psi}{\sec^2\psi}\int_0^1 \frac{q(x')\,dx'}{\sqrt{(x-x')^2+z^2}}+$$

$$+\frac{\theta^2}{\sec^2\psi}\int_0^1 \frac{q(x')\,z\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\varphi\tan\psi + z^2\sec^2\varphi}\,dx'+$$

$$+\frac{\theta}{\sec^2\psi}\int_0^1 \frac{q(x')\,[z^2\sec^2\varphi\tan\psi + (x-x')\,z\tan\varphi\,(1-\tan^2\psi)]\,dx'}{\sqrt{(x-x')^2+z^2}\,(\overline{x-x'}^2\sec^2\psi - 2\overline{x-x'}\,z\tan\varphi\tan\psi + z^2\sec^2\varphi)} \tag{18}$$

$$= I_3 + I_4 + I_5 + I_6\,.$$

Once again the integrals have to be evaluated in the limit $z \to 0$.

$$I_3(x,0,0) = -\frac{\cos\Phi\tan\varphi\tan\psi}{2\pi\sec^2\psi} \times \lim_{z\to 0}\int_0^1 \frac{q(x')\,(\overline{x-x'}\sec^2\psi - z\tan\varphi\tan\psi)\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\varphi\tan\psi + z^2\sec^2\varphi}$$

$$= -\sin\Phi\sin\psi\int_0^1 \frac{q(x')\,dx'}{2\pi\,(x-x')} \qquad \text{as in Appendix A}$$

$$= -\sin\Phi\sin\psi\; v'_x \tag{19}$$

I_4 has to be evaluated at a non-zero value of z, because it has a logarithmically infinite value for $z = 0$

$$I_4(x,0,z) = -\frac{\cos\Phi}{2\pi}\,\frac{\theta\tan\psi}{\sec^2\psi}\int_0^1 \frac{q(x')\,dx'}{\sqrt{(x-x')^2+z^2}}$$

$$= -\sin\psi\int_0^1 \frac{q(x')}{2\pi}\,\frac{dx'}{\sqrt{(x-x')^2+z^2}} \tag{20}$$

$$I_5(x,0,0) = \frac{\cos\Phi}{2\pi}\,\frac{\theta^2}{\sec^2\psi}\lim_{z\to 0}\int_0^1 \frac{q(x')\,z\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\varphi\tan\psi + z^2\sec^2\varphi}$$

$$= -\frac{\cos\Phi}{2\pi}\frac{q(x)}{\cos^2\Phi}\lim_{z\to 0}\int_{x-x'=+\varepsilon}^{x-x'=-\varepsilon}\frac{z\,d(x-x')}{(x-x')^2\sec^2\psi-2(x-x')z\tan\varphi\tan\psi+z^2\sec^2\varphi}$$

where ε is small compared to 1 but large compared to z. We take the limit $z\to 0$ and $\varepsilon\to 0$.

$$= -\frac{q(x)}{2\pi\cos\Phi}\lim_{\substack{z\to 0\\ \varepsilon\to 0}}\left[\frac{1}{\sqrt{\sec^2\psi\sec^2\varphi-\tan^2\psi\tan^2\varphi}}\times\right.$$

$$\left\{\tan^{-1}\frac{-\sec^2\psi-\frac{z}{\varepsilon}\tan\varphi\tan\psi}{\frac{z}{\varepsilon}\sqrt{\sec^2\psi\sec^2\varphi-\tan^2\varphi\tan^2\psi}}\right.$$

$$\left.\left.-\tan^{-1}\frac{\sec^2\psi-\frac{z}{\varepsilon}\tan\varphi\tan\psi}{\frac{z}{\varepsilon}\sqrt{\sec^2\psi\sec^2\varphi-\tan^2\varphi\tan^2\psi}}\right\}\right]$$

as given in Ref. 4

$$= \pm\frac{q(x)}{2\pi\cos\Phi}\frac{\pi}{\sqrt{\sec^2\psi\sec^2\varphi-\tan^2\psi\tan^2\varphi}}$$

the positive sign being used if $z\to 0$ from the positive side, and *vice versa*.
From the relation $\cos\psi\tan\varphi = \tan\Phi$,

$$I_5(x,0,0) = \pm\frac{q(x)}{2}\cos\psi \tag{21}$$

$$I_6(x,0,0) = \frac{\cos\Phi}{2\pi}\frac{\theta}{\sec^2\psi}\lim_{z\to 0}\int_0^1\frac{q(x')\left[z^2\sec^2\varphi\tan\psi+(x-x')z\tan\varphi(1-\tan^2\psi)\right]dx'}{\sqrt{(x-x')^2+z^2}\,(\overline{x-x'}^2\sec^2\psi-2\,\overline{x-x'}\,z\tan\varphi\tan\psi+z^2\sec^2\varphi)}.$$

This is the type considered in Appendix B, with $\alpha = \sec^2\varphi\tan\psi$ and $\beta = \tan\varphi(1-\tan^2\psi)$.

Therefore

$$I_6(x,0,0) = \frac{\cos\psi}{2\pi}q(x)\left[2\psi+\frac{\tan\varphi\tan\psi}{\theta}\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right)\right]$$

$$= q(x)\cos\psi\frac{\psi}{\pi}+\frac{q(x)}{2\pi}\sin\Phi\sin\psi\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right). \tag{22}$$

Hence,

$$v_z(x,0,0) = -\sin\varphi\sin\psi\int_0^1 \frac{q(x')\,dx'}{2\pi(x-x')} - \sin\psi\int_0^1 \frac{q(x')}{2\pi}\frac{dx'}{\sqrt{(x-x')^2+z^2}} +$$

$$+\frac{q(x)}{2}\cos\psi\frac{\psi}{\pi/2} + \frac{q(x)}{2\pi}\sin\Phi\sin\psi\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right) \pm \frac{q(x)}{2}\cos\psi \tag{23}$$

For zero dihedral, $\psi = 0$, and

$$[v_z(x,0,0)]_{\psi=0} = \pm\frac{q(x)}{2}$$

For zero sweep, $\Phi = \varphi = 0$, and

$$[v_z(x,0,0)]_{\Phi=\varphi=0} = -\sin\psi\int_0^1 \frac{q(x')}{2\pi}\frac{dx'}{\sqrt{(x-x')^2+z^2}} + \frac{q(x)}{2}\cos\psi\frac{\psi}{\pi/2} \pm \frac{q(x)}{2}\cos\psi\,.$$

4. *Velocities Induced by a Distribution of Parallel Infinite Kinked Vortex Lines.*

4.1. *Single Vortex Line.*

Using the same notation as for source lines, the velocity at point $R(x,0,z)$ due to an element of the vortex line at $P(|y|\tan\varphi, y, |y|\tan\psi)$ is

$$\Delta\mathbf{v} = \frac{\Gamma}{4\pi}\frac{d\mathbf{s}\times\mathbf{r}}{r^3} \tag{24}$$

where Γ is the strength per unit length of the vortex line. The components of $\Delta\mathbf{v}$ are

$$\left.\begin{aligned} \Delta v_x &= \frac{\Gamma z\,dy}{4\pi r^3} \\ \Delta v_y &= \frac{\Gamma(x\tan\psi - z\tan\varphi)\,dy}{4\pi r^3} \\ \Delta v_z &= -\frac{\Gamma x\,dy}{4\pi r^3} \end{aligned}\right\}. \tag{25}$$

Substituting for r from equation (6) and integrating over the complete vortex line,

$$v_x(x,0,z) = \frac{\Gamma}{2\pi\theta^3}\int_0^\infty \frac{z\,dy}{\left(y^2+2y\frac{A}{\theta}+\frac{x^2+z^2}{\theta^2}\right)^{3/2}}$$

$$= \frac{\Gamma z}{2\pi\theta}\frac{1}{x^2+z^2-A^2}\left(1-\frac{A}{\sqrt{x^2+z^2}}\right) \tag{26}$$

as in the derivation of equation (10).

$$v_y(x,0,z) = 0$$

and, by analogy with (26)

$$v_z(x,0,z) = -\frac{\Gamma x}{2\pi\theta}\frac{1}{x^2+z^2-A^2}\left(1-\frac{A}{\sqrt{x^2+z^2}}\right). \tag{27}$$

4.2. *Streamwise Distribution of Vortex Lines.*

Consider a distribution of parallel vortex lines in $z' = 0$.
Let $\gamma(x)$ be the vortex strength per unit area. Then

$$\Gamma\, ds = \gamma\, ds\, dx \cos\Phi$$

therefore

$$\Gamma = \gamma\, dx \cos\Phi\,. \tag{28}$$

Equation (26) can be rewritten in the form

$$\begin{aligned} v_x(x,0,z) &= \frac{\Gamma z \theta}{2\pi}\frac{1}{x^2\sec^2\psi-2xz\tan\varphi\tan\psi+z^2\sec^2\varphi}\times\left(1+\frac{x\tan\varphi+z\tan\psi}{\theta\sqrt{x^2+z^2}}\right) \\ &= \frac{\Gamma}{2\pi}\left[\frac{z\theta}{x^2\sec^2\psi-2xz\tan\varphi\tan\psi+z^2\sec^2\varphi}+\right. \\ &\qquad \left.+\frac{xz\tan\varphi+z^2\tan\psi}{\sqrt{x^2+z^2}\,(x^2\sec^2\psi-2xz\tan\varphi\tan\psi+z^2\sec^2\varphi)}\right] \end{aligned} \tag{29}$$

Replacing Γ by $\gamma(x')\,dx'\cos\Phi$, and x by $x-x'$, integration with respect to x' gives the velocities induced by the streamwise distribution of vortex lines between the leading edge, $x' = 0$ and the trailing edge, $x' = 1$.

$$\begin{aligned} v_x(x,0,z) &= \frac{\theta\cos\Phi}{2\pi}\int_0^1 \frac{\gamma(x')\,z\,dx'}{(x-x')\sec^2\psi-2(x-x')z\tan\varphi\tan\psi+z^2\sec^2\varphi} \\ &+\frac{\cos\Phi}{2\pi}\int_0^1 \frac{\gamma(x')\,[(x-x')z\tan\varphi+z^2\tan\psi]\,dx'}{\sqrt{(x-x')^2+z^2}\times[(x-x')^2\sec^2\psi-2(x-x')z\tan\varphi\tan\psi+z^2\sec^2\varphi]} \\ &= I_7+I_8\,. \end{aligned}$$

The integrals are to be evaluated at $z' = 0$ so that at the centreline limiting values as $z\to 0$ are required.
I_7 is evaluated as for I_5, and I_8 as for I_2.
Hence, the total chordwise velocity induced by the distribution of vortex lines is

$$v_x(x,0,0) = \pm\frac{\gamma(x)}{2}\cos\Phi+\gamma(x)\frac{\psi}{\pi}\cos\Phi \tag{30}$$

where the positive sign is used when $z \to 0$ from the positive side, and *vice versa*.

Unlike the chordwise velocity increment in the chordal plane induced by the distribution of sources, the chordwise velocity increment due to the vortex distribution is a function of the dihedral angle ψ.

Equation (27) can be rewritten in the form

$$v_z(x,0,z) = -\frac{\Gamma x \theta}{2\pi}\left[\frac{1}{x^2 \sec^2\psi - 2x z \tan\varphi \tan\psi + z^2 \sec^2\varphi}\right] \times \left(1 + \frac{x \tan\varphi + z \tan\psi}{\theta\sqrt{x^2+z^2}}\right)$$

$$= -\frac{\Gamma}{2\pi}\left[\frac{x\theta}{x^2 \sec^2\psi - 2x z \tan\varphi \tan\psi + z^2 \sec^2\varphi} + \frac{\tan\varphi}{\sec^2\psi\sqrt{x^2+z^2}} + \frac{x z \tan\psi\,(2\tan^2\varphi + \sec^2\psi) - z^2 \sec^2\varphi \tan\varphi}{\sec^2\psi\sqrt{x^2+z^2} \times (x^2\sec^2\psi - 2x z \tan\varphi \tan\psi + z^2\sec^2\varphi)}\right]. \quad (31)$$

the terms being arranged to facilitate integration.

Again replacing Γ by $\gamma(x')\,dx' \cos\Phi$ and x by $x-x'$, integration with respect to x' gives

$$v_z(x,0,z) = -\frac{\theta\cos\Phi}{2\pi}\int_0^1 \frac{\gamma(x')(x-x')\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')z\tan\varphi\tan\psi + z^2\sec^2\varphi}$$

$$-\frac{\tan\varphi\cos\Phi}{2\pi\sec^2\psi}\int_0^1 \frac{\gamma(x')\,dx'}{\sqrt{(x-x')^2+z^2}}$$

$$-\frac{\cos\Phi}{2\pi\sec^2\psi}\int_0^1 \frac{\gamma(x')\,[(x-x')z\tan\psi\,(2\tan^2\varphi+\sec^2\psi) - z^2\sec^2\varphi\tan\varphi]\,dx'}{\sqrt{(x-x')^2+z^2} \times [(x-x')^2\sec^2\psi - 2(x-x')z\tan\varphi\tan\psi + z^2\sec^2\varphi]} \quad (32)$$

$$= I_9 + I_{10} + I_{11}.$$

As before, the integrals are evaluated in the chordal plane as $z \to 0$.

$$I_9(x,0,0) = -\frac{\theta\cos\Phi}{2\pi\sec^2\psi}\left[\lim_{z\to 0}\int_0^1 \frac{[(x-x')\sec^2\psi - z\tan\varphi\tan\psi]\,\gamma(x')\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')z\tan\varphi\tan\psi + z^2\sec^2\psi}\right.$$

$$\left. + \tan\varphi\tan\psi \lim_{z\to 0}\int_0^1 \frac{z\,\gamma(x')\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')z\tan\varphi\tan\psi + z^2\sec^2\psi}\right]$$

These limits can be evaluated in the same way as I_1 and I_5, and therefore

$$I_9(x,0,0) = -\frac{\theta\cos\Phi}{\sec^2\psi}\int_0^1 \frac{\gamma(x')}{2\pi}\,\frac{dx'}{(x-x')} \mp \frac{\gamma(x)}{2}\sin\Phi\sin\psi \quad (33)$$

the negative sign being used as $z \to 0$ from the positive side.

I_{10}, like I_4, can only be evaluated on the aerofoil surface,
I_{11} is of the type considered in Appendix B with $\alpha = -\sec^2\varphi\tan\varphi$
and $\beta = \tan\psi\,(2\tan^2\varphi + \sec^2\psi)$.

Hence

$$I_{11}(x,0,0) = -\frac{\cos\Phi}{2\pi\sec^2\psi}\gamma(x)\left\{2\psi\tan\varphi\tan\psi - \theta\log\frac{1+\sin\Phi}{1-\sin\Phi}\right\}. \tag{34}$$

Hence

$$\begin{aligned}
v_z(x,0,0) = & -\frac{\theta\cos\Phi}{\sec^2\psi}\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{(x-x')} \mp \frac{\gamma(x)}{2}\sin\Phi\sin\psi \\
& -\frac{\tan\varphi\cos\Phi}{\sec^2\psi}\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{\sqrt{(x-x')^2+z^2}} \\
& -\frac{\cos\Phi}{\sec^2\psi}\frac{\gamma(x)}{2\pi}2\psi\tan\psi\tan\varphi \\
& +\frac{\cos\Phi}{\sec^2\psi}\frac{\gamma(x)}{2\pi}\theta\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right) \\
= & -\cos\psi\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{(x-x')} \mp \frac{\gamma(x)}{2}\sin\Phi\sin\psi \\
& -\sin\Phi\cos\psi\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{\sqrt{(x-x')^2+z^2}} \\
& -\frac{\gamma(x)}{2}\frac{\psi}{\pi/2}\sin\Phi\sin\psi \\
& +\frac{\gamma(x)}{2\pi}\cos\psi\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right).
\end{aligned} \tag{35}$$

For zero dihedral, $\psi = 0$, and

$$[v_z(x,0,0)]_{\psi=0} = -\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{(x-x')} - \sin\Phi\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{\sqrt{(x-x')^2+z^2}} + \frac{\gamma(x)}{2\pi}\cos\psi\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right).$$

For zero sweep, $\Phi = \varphi = 0$, and

$$[v_z(x,0,0)]_{\Phi=\varphi=0} = -\cos\psi \int_0^1 \frac{\gamma(x')}{2\pi} \frac{dx'}{(x-x')}.$$

Far away from the plane of symmetry, $y = 0$, we have the condition of a sheared wing with dihedral, having a thin section and a lift distribution represented by a distribution of vorticity. If z' is the co-ordinate perpendicular to this sheared wing

$$v_x(x,\infty,z') = \int_0^1 \frac{\gamma(x')\, z'\, dx'}{2\pi\,[(x-x')^2 + z'^2 \sec^2\Phi]}$$

where the local leading edge is assumed to be $x' = 0$. In the chordal plane $z' = 0$,

$$v_x(x,\infty,0) = \lim_{z'\to 0} \int_0^1 \frac{\gamma(x')\, z'\, dx'}{2\pi\,[(x-x')^2 + z'^2 \sec^2\Phi]}$$

$$= -\frac{\gamma(x)}{2\pi}\cos\Phi \lim_{z'\to 0} \int_{x-x'=+\varepsilon}^{x-x'=-\varepsilon} \frac{z'\sec\Phi\, d(x-x')}{(x-x')^2 + z'^2\sec^2\Phi}$$

$$= \pm\frac{\gamma(x)}{2}\cos\Phi,$$

the positive sign being used when $z' \to 0$ from the positive side.
Similarly

$$v_z'(x,\infty,z') = -\int_0^1 \frac{\gamma(x')}{2\pi} \frac{(x-x')\, dx'}{[(x-x')^2 + z'^2\sec^2\Phi]}$$

and as $z' \to 0$

$$v_z'(x,\infty,0) = -\int_0^1 \frac{\gamma(x')}{2\pi} \frac{dx'}{x-x'}.$$

For convenience the induced velocities derived in this section are listed in Appendix D.

5. *Discussion.*

The use of distributions of infinite source and vortex lines of constant strength along their span to represent thick, lifting, swept wings in incompressible flow has been studied in detail by, for example, Küchemann[1] and Weber[2]. This Report has carried the study one stage further to consider the case where the source (and vortex) distribution on one side of the kink is not coplanar with that on the other side, i.e. a distribution with dihedral as well as sweep. In the preceding sections the velocities induced by distributions of such vortices and sources in their own plane(s) have been derived, and are listed in Appendices C and D. Particular attention has been paid to conditions in the plane of symmetry, that is, the centre section, where strong dihedral effects can be expected. Far away from the centre section, of course, the flow field is the same as that induced by a planar distribution.

The outstanding feature of the results in Appendices C and D is that, at the centre section, a distribution of sources with constant spanwise strength gives rise to a component of downwash, v_z, which has the same direction on both sides of the chordal plane: and a distribution of vortex lines with constant spanwise strength gives rise to a component of downwash which has opposite signs on either side of the chordal plane. That is, part of the downwash due to the sources has a characteristic one associates with a vortex distribution, and part of the downwash due to the vortices has a characteristic one associates with a source distribution. In addition, the chordwise velocity increment, v_x, due to the vortices has a term proportional to the local vortex strength and with the same sign above and below the chordal plane; i.e. it is of the form usually associated with a source distribution. Thus the great simplification that linear theory brings to the study of planar wings, namely the ability to treat thickness and lifting effects separately, no longer applies when the wing has a dihedral angle.

Consider an unswept wing of infinite span and constant chord, with the same symmetrical section shape at all spanwise positions. If it is at zero incidence, the velocities induced by this wing can be calculated in linearised theory by representing the wing by a distribution of infinite source lines in the chordal plane, the source lines having constant strength spanwise. At any chordwise position, x, the induced velocity perpendicular to the chordal plane, $v_z(x)$, is equal to $\pm\frac{q(x)}{2}$, where $q(x)$ is the local source strength. If we consider an infinite wing which is swept on each side of a particular section—the 'centre section'—the induced velocity v_z everywhere is still equal to the local source strength (*see* equation (C.2) with $\psi = 0$). Therefore a wing with a uniform symmetrical section can still be represented by a distribution of source lines having constant strength spanwise.

If now we consider a distribution of source lines of infinite span which has both sweep and dihedral, the results in Appendix C show that the above result for plane wings—that the induced velocity perpendicular to the chordal plane is equal to $\pm\frac{q}{2}$ —is now true only at points far away from the centre section. At the centre section, v_z contains a number of terms which have the same sign above and below the chord line and which are therefore characteristic of the velocity induced by a distribution of vortices. Therefore to represent a swept, dihedralled wing of uniform symmetrical section at zero incidence, a distribution of source lines of constant strength is not sufficient. A distribution of vorticity is also needed, to cancel out the unwanted terms in v_z. Since the 'centre effect' disappears as $y \to \infty$, this distribution of vorticity will not be infinite spanwise, nor of constant strength. The equations of Appendix D cannot therefore be used to determine this vortex distribution.

Subsonic wind tunnel tests (unpublished) on a constant chord wing of aspect ratio 5, 45° sweep and 32° dihedral, with a constant symmetrical aerofoil section, have shown that, at and near the centre section, there is indeed an appreciable lift force at zero incidence.

The distribution of sources represents a wing which is cambered and twisted at, and near, the centre section, to conform, within linearised theory, to the geometry implied by equation (C.2). The camber line of this section is determined by the v_z terms other than $\pm\frac{q}{2}\cos\psi$. Since v_x is the same above and below the plane of the sources, ΔC_p, the local normal force, is everywhere zero and, in particular there is no lift on the cambered centre section. The source distribution considered here therefore represents a wing with zero lift but this is not a wing with uniform symmetrical section shape.

Consider now a thin, unswept wing without dihedral, of infinite span and constant chord. If it has the same streamwise camber and the same incidence at all spanwise positions, the velocities induced by this wing can be calculated in linearised theory by representing the wing by a distribution of infinite vortex lines in the chordal plane, the vortex lines having constant strength spanwise. At any chordwise position, x, the induced velocity parallel to the chordal plane or x-axis, (the incidence is assumed to be small), v_x, is equal to $\pm\frac{\gamma(x)}{2}$ and the induced velocity perpendicular to the chordal plane, $v_z(x)$, is given by $\int_{\text{chord}} \frac{\gamma(x')}{2\pi}\frac{dx'}{(x-x')}$, so that vortex strength can be related to the geometry of the thin wing section. If we consider a distribution of vortex lines of infinite span which has sweep but no dihedral the velocity, v_z, at the centre section cannot be calculated on the chord line due to the presence of a singularity in an integral, and v_z calculated off the chordline (e.g. on the surface of a wing of finite thickness) is not the same as at sections far outboard (*see* equation (D.2) with $\psi = 0$). This means that the distribution of constant strength vortices represents a wing cambered and twisted near the centre section to give the same distribution of lift, $\Delta C_p(x)$, at all spanwise positions.

If we now have a distribution of infinite, constant strength vortices having both sweep and dihedral, this distribution represents a different camber and twist at the centre section compared with the swept, planar wing (*see* equation (D.2)). As both v_x and v_z at the centre section contain terms characteristic of sources (v_z has opposite signs above and below the plane) this distribution of vorticity cannot represent an infinitely thin wing.

6. *Conclusions.*

Equations have been presented for the velocities induced in incompressible flow by distributions of kinked source and vortex lines, having both sweep and dihedral. These have shown that the dihedral makes it impossible to separate the effects of wing thickness and wing load distribution, even when linear theory assumptions are retained. To calculate the velocity distribution on a wing of given shape is therefore much more complicated than for planar wings.

LIST OF SYMBOLS

$0X, 0Y, 0Z$	Right handed set of axes, the origin being at the kink of a source line or vortex line. $0X$ is positive in the downstream direction, $0Y$ positive to starboard and $0Z$ positive upwards. For a chordwise distribution of singularities the unit of length is the wing chord (assumed constant). (*See* Fig. 1).
$0Y', 0Z'$	Axes at right angles to $0X$, along and perpendicular to the plane of the singularity distribution, respectively. For integrations far away from the plane of symmetry the origin is taken at the local leading edge, for convenience.
x, y, z, y', z'	Co-ordinates with respect to the above axes
x'	Variable of integration
v_x, v_y, v_z, v_z'	Velocity increments in the x, y, z and z' directions
v_x^*	Velocity increment in the x-direction induced by a distribution of two-dimensional singularities
V_0	Free-stream velocity
Φ	Angle of sweep in the plane of a half-wing (sweepback is positive)
φ	Projected angle of sweep in the plane $z = 0$ (sweepback is positive)
ψ	Projected angle of dihedral on the plane $x = 0$ (dihedral is positive)
$\cos\psi$	$= \dfrac{\tan\Phi}{\tan\varphi}$
θ	$1+\tan^2\varphi+\tan^2\psi$
A	$-\dfrac{x\tan\varphi+z\tan\psi}{\theta}$
P	General point in the plane of symmetry
R	General point on a source or vortex line
r	Distance between P and R
s	Distance along a source or vortex line
E	Source strength per unit length
Γ	Vortex strength per unit length
$q(x)$	Source strength per unit area
$\gamma(x)$	Vortex strength per unit area
u	Variable of integration
$\alpha, \beta, a, b, A, B, C$	Constants used in evaluating integrals
μ, v, t, T, X	Variables used in evaluating integrals

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APPENDIX A

Evaluation of $I_1(x, 0, 0)$.

(*see* Section 3.2).

$$I_1(x,0,0) = \frac{\cos\Phi}{2\pi} \lim_{z\to 0} \int_0^1 \frac{q(x')\,\{(x-x')\sec^2\psi - z\tan\psi\tan\varphi\}\,dx'}{(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\psi\tan\varphi + z^2\sec^2\varphi}$$

The integrand has a singularity when $z = 0$, $x = x'$. We divide the range of integration into three parts, $0 \leqslant x' \leqslant x-\varepsilon$, $x-\varepsilon \leqslant x' \leqslant x+\varepsilon$, and $x+\varepsilon \leqslant x' \leqslant 1$. where $\varepsilon \to 0$ but $\varepsilon >> z$. In the first and third of these ranges, $x \neq x'$, and so no singularity occurs when z is put equal to zero.

In the second range of integration we have

$$-\frac{\cos\Phi}{2\pi} \lim_{z\to 0} \int_{x-x'=\varepsilon}^{x-x'=-\varepsilon} \frac{q(x')\,\{(x-x')\sec^2\psi - z\tan\psi\tan\varphi\}\,d(x-x')}{(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\psi\tan\varphi + z^2\sec^2\varphi}.$$

It is assumed that $q(x')$ is a continuous function and can be replaced in this range by a mean value, say $q(x)$. The integral then becomes

$$-\frac{\cos\Phi}{2\pi}\,q(x) \lim_{\substack{z\to 0\\ \varepsilon\to 0}} \frac{1}{2}\log\Big[(x-x')^2\sec^2\psi - 2(x-x')\,z\tan\psi\tan\varphi + z^2\sec^2\varphi\Big]_{x-x'=\varepsilon}^{x-x'=-\varepsilon}$$

$$= \frac{q(x)\cos\Phi}{4\pi} \lim_{\substack{z\to 0\\ \varepsilon\to 0}} \log \frac{\sec^2\psi + 2\dfrac{z}{\varepsilon}\tan\psi\tan\varphi + \dfrac{z^2}{\varepsilon^2}\sec^2\varphi}{\sec^2\psi - 2\dfrac{z}{\varepsilon}\tan\psi\tan\varphi + \dfrac{z^2}{\varepsilon^2}\sec^2\varphi}$$

APPENDIX A

$$= 0, \quad \text{since} \quad z << \varepsilon .$$

Hence,

$$I_1(x,0,0) = \frac{\cos\Phi}{2\pi}\left\{ \lim_{\varepsilon\to 0} \int_0^{x'=x-\varepsilon} \frac{q(x')\,dx'}{x-x'} + \lim_{\varepsilon\to 0} \int_{x'=x+\varepsilon}^{x'=1} \frac{q(x')\,dx'}{x-x'} \right\}$$

$$= \int_0^1 \frac{q(x')}{2\pi}\,\frac{dx'}{x-x'}\cos\Phi .$$

APPENDIX B

Evaluation of $I_2(x,0,0)$.

(*see* Section 3.2)

$$_2(x,0,0) = \frac{\theta\cos\Phi}{2\pi}\lim_{z\to 0}\int_0^1 \frac{q(x')\,[z^2\tan\varphi-(x-x')\,z\tan\psi]\,dx'}{[(x-x')^2\sec^2\psi-2(x-x')\,z\tan\psi\tan\varphi+z^2\sec^2\varphi]\times\sqrt{(x-x')^2+z^2}}$$

Let $u = x-x', \alpha = \tan\varphi, \beta = -\tan\psi,^{i} \quad X') = f(x-u)$.
Then the integral is

$$\lim_{z\to 0} -\int_{u=x}^{u=x-1} \frac{f(x-u)\,(\alpha z^2+\beta u z)\,du}{\sqrt{u^2+z^2}\,(u^2\sec^2\psi-2u z\tan\psi\tan\varphi+z^2\sec^2\varphi} \tag{B.1}$$

$$\sqrt{u^2+z^2} = 0 \quad \text{only when} \quad u = z = 0 .$$

$$u^2\sec^2\psi-2u z\tan\psi\tan\varphi+z^2\sec^2\varphi$$

$$= u^2(1+\tan^2\psi)-2u z\tan\psi\tan\varphi+z^2(1+\tan^2\varphi)$$

$$= u^2+z^2+(u\tan\psi-z\tan\varphi)^2 .$$

Since all the terms are positive, this expression is zero only when $u = z = 0$. Therefore the denominator of the integrand is zero only when $u = z = 0$. Substituting $z = 0$ in the integral, the integrand is therefore zero except when $u = 0$ (i.e. when $x' = x$). Since $0 \leqslant x \leqslant 1, u = 0$ within the range of integration. The integral may therefore be written

$$\lim_{z\to 0} f(x)\int_{-\varepsilon}^{\varepsilon} \frac{(\alpha z^2+\beta u z)\,du}{\sqrt{u^2+z^2}\,(u^2\sec^2\psi-2u z\tan\psi\tan\varphi+z^2\sec^2\varphi}$$

where ε will $\rightarrow 0$, but $\varepsilon >> z$, and assuming $f(x-u) = \text{constant} = f(x)$ in the range of integration.

Substitute $u = \dfrac{\mu t + \nu}{t+1}$ where μ, ν are roots of the equation

$$\xi^2 + \xi z \frac{\tan^2\psi - \tan^2\varphi}{\tan\psi \tan\varphi} - z^2 = 0$$

i.e.

$$\mu = z\frac{\tan\varphi}{\tan\psi}$$

$$\nu = -z\frac{\tan\psi}{\tan\varphi}$$

$$u = \frac{z\left(\dfrac{\tan\varphi}{\tan\psi}t - \dfrac{\tan\psi}{\tan\varphi}\right)}{t+1}$$

$$t = \frac{u-\nu}{\mu-u} = \frac{u + z\dfrac{\tan\psi}{\tan\varphi}}{z\dfrac{\tan\varphi}{\tan\psi} - u}$$

$$du = \frac{\mu-\nu}{(t+1)^2}dt = \frac{z(\tan^2\psi + \tan^2\varphi)}{\tan\varphi\tan\psi(1+t)^2}dt\,.$$

Studying the behaviour of t in the range of integration:
When

$$u = -\varepsilon,\, t = -\frac{\varepsilon - z\dfrac{\tan\psi}{\tan\varphi}}{\varepsilon + z\dfrac{\tan\varphi}{\tan\psi}}$$

$$< 0, \quad \text{since} \quad z << \varepsilon\frac{\tan\varphi}{\tan\psi} \text{ assuming } \varphi \text{ and } \psi > 0$$

when

$$u = 0\,, t = \frac{\tan^2\psi}{\tan^2\varphi} > 0$$

when

$$u = +\varepsilon\,,\, t = -\frac{\varepsilon + z\dfrac{\tan\psi}{\tan\varphi}}{\varepsilon - z\dfrac{\tan\varphi}{\tan\psi}}$$

$$< 0 \quad \text{since} \quad z << \varepsilon\frac{\tan\psi}{\tan\varphi}.$$

As $u \to z\dfrac{\tan\varphi}{\tan\psi}$ from below, $t \to \infty$.

As $u \to z\dfrac{\tan\varphi}{\tan\psi}$ from above, $t \to -\infty$.

Therefore t increases from $-\dfrac{\varepsilon - z\dfrac{\tan\psi}{\tan\varphi}}{\varepsilon + z\dfrac{\tan\varphi}{\tan\psi}}$ at $u = -\varepsilon$ to $+\infty$ at $u = z\dfrac{\tan\varphi}{\tan\psi}$: jumps to $-\infty$ and increases again to $-\dfrac{\varepsilon + z\dfrac{\tan\psi}{\tan\varphi}}{\varepsilon - z\dfrac{\tan\varphi}{\tan\psi}}$ at $u = +\varepsilon$.

The integral becomes

$$\lim_{z\to 0} f(x) \int_{u=-\varepsilon}^{u=+\varepsilon} \frac{\left[\alpha(t+1) + \beta\left(\dfrac{\tan\varphi}{\tan\psi}t - \dfrac{\tan\psi}{\tan\varphi}\right)\right](\tan^2\psi + \tan^2\varphi)\,dt}{\tan\varphi\tan\psi\sqrt{(t+1)^2 + \left(\dfrac{\tan\varphi}{\tan\psi}t - \dfrac{\tan\psi}{\tan\varphi}\right)^2} \times \left[\begin{array}{l}\left(\dfrac{\tan\varphi}{\tan\psi}t - \dfrac{\tan\psi}{\tan\varphi}\right)^2 \sec^2\psi + (1+t)^2\sec^2\varphi \\ -(1+t)\,2\tan\psi\tan\varphi\left(\dfrac{\tan\varphi}{\tan\psi}t - \dfrac{\tan\psi}{\tan\varphi}\right)\end{array}\right]}$$

$$= \lim_{z\to 0} f(x) \int_{u=-\varepsilon}^{u=+\varepsilon} \frac{\left\{\left(\alpha + \beta\dfrac{\tan\varphi}{\tan\psi}\right)t + \left(\alpha - \beta\dfrac{\tan\psi}{\tan\varphi}\right)\right\}\tan^2\psi\,dt}{\tan\varphi\sqrt{\tan^2\varphi + \tan^2\psi}\sqrt{t^2 + \dfrac{\tan^2\psi}{\tan^2\varphi}}\left(t^2 + \theta^2\dfrac{\tan^2\psi}{\tan^2\varphi}\right)}$$

$$= \lim_{z\to 0} f(x)\left\{\frac{\tan\psi\,(\alpha\tan\psi+\beta\tan\varphi)}{\tan\varphi\sqrt{\tan^2\varphi+\tan^2\psi}}\int_{u=-\varepsilon}^{u=+\varepsilon}\frac{t\,dt}{\sqrt{t^2+\frac{\tan^2\psi}{\tan^2\varphi}}\left(t^2+\theta^2\frac{\tan^2\psi}{\tan^2\varphi}\right)}+\right.$$

$$\left.+\frac{\tan^2\psi\,(\alpha\tan\varphi-\beta\tan\psi)}{\tan^2\varphi\sqrt{\tan^2\varphi+\tan^2\psi}}\int_{u=-\varepsilon}^{u=+\varepsilon}\frac{dt}{\sqrt{t^2+\frac{\tan^2\psi}{\tan^2\varphi}}\left(t^2+\theta^2\frac{\tan^2\psi}{\tan^2\varphi}\right)}\right\}. \tag{B.2}$$

Consider firstly

$$\int_{u=-\varepsilon}^{u=+\varepsilon}\frac{t\,dt}{\sqrt{t^2+\frac{\tan^2\psi}{\tan^2\varphi}}\left(t^2+\theta^2\frac{\tan^2\psi}{\tan^2\varphi}\right)}$$

$$\frac{1}{2}\int_{u=-\varepsilon}^{u=+\varepsilon}\frac{dT}{\sqrt{T+A}\,(T+B)},\ \text{on substituting } T=t^2.$$

This is of the form

$$\int\frac{dx}{(x-c)\sqrt{a\,x+b}}$$

of which the solution is given in Ref. 4.
Here $x = T$, $a = 1$, $b = A$ and $c = -B$

$$a\,\beta+b = A-B = \frac{\tan^2\psi}{\tan^2\varphi}(1-\theta^2) < 0 \text{ since } \theta^2 > 1\,.$$

The integral is therefore

$$\frac{-2}{\sqrt{-(a\,\beta+b)}}\tan^{-1}\sqrt{\frac{-(a\,\beta+b)}{a\,x+b}}+\text{constant}$$

$$=\frac{-2\tan\varphi}{\tan\psi\sqrt{\tan^2\varphi+\tan^2\psi}}\tan^{-1}\sqrt{\frac{\frac{\tan^2\psi}{\tan^2\varphi}(\theta^2-1)}{t^2+\frac{\tan^2\psi}{\tan^2\varphi}}}+\text{constant.}$$

The first part of (B.2) is therefore

$$\lim_{z\to 0} f(x)\left\{\frac{\tan\psi\,(\alpha\tan\psi+\beta\tan\varphi)}{\tan\varphi\sqrt{\tan^2\varphi+\tan^2\psi}}\,\frac{1}{2}\,\frac{-2\tan\varphi}{\tan\psi\sqrt{\tan^2\varphi+\tan^2\psi}}\times\left[\tan^{-1}\sqrt{\frac{\frac{\tan^2\psi}{\tan^2\varphi}(\theta^2-1)}{t^2+\frac{\tan^2\psi}{\tan^2\varphi}}}\right]_{u=-\varepsilon}^{u=+\varepsilon}\right\}$$

$$= \lim_{z\to 0} f(x)\left\{-\frac{\alpha\tan\psi+\beta\tan\varphi}{\tan^2\varphi+\tan^2\psi}\left[\tan^{-1}\sqrt{\frac{(z\tan\varphi-u\tan\psi)^2}{u^2+z^2}}\right]_{u=-\varepsilon}^{u=+\varepsilon}\right\}.$$

Since $z\tan\varphi-u\tan\psi$ and $\sqrt{u^2+z^2}$ are physical distances, take the positive square root.

$$\lim_{z\to 0} f(x)\left\{-\frac{\alpha\tan\psi+\beta\tan\varphi}{\tan^2\varphi+\tan^2\psi}\left[\tan^{-1}\left|\frac{z\tan\varphi-u\tan\psi}{\sqrt{u^2+z^2}}\right|\right]_{u=-\varepsilon}^{u=+\varepsilon}\right\}$$

$$= \lim_{z\to 0} f(x)\left\{-\frac{\alpha\tan\psi+\beta\tan\varphi}{\tan^2\varphi+\tan^2\psi}\right\}\left\{\tan^{-1}\frac{z\tan\varphi-\varepsilon\tan\psi}{\sqrt{\varepsilon^2+z^2}}-\tan^{-1}\frac{z\tan\varphi+\varepsilon\tan\psi}{\sqrt{\varepsilon^2+z^2}}\right\}.$$

Remembering that $\varepsilon >> z$, as $z\to 0$, the limit is

$$f(x)\left[-\frac{\alpha\tan\psi+\beta\tan\varphi}{\tan^2\varphi+\tan^2\psi}\{\tan^{-1}(-\tan\psi)-\tan^{-1}(\tan\psi)\}\right]$$

$$= 2f(x)\,\psi\,\frac{\alpha\tan\psi+\beta\tan\varphi}{\tan^2\varphi+\tan^2\psi}.$$

Consider secondly

$$\int_{u=-\varepsilon}^{u=+\varepsilon}\frac{dt}{\sqrt{t^2+\frac{\tan^2\psi}{\tan^2\varphi}}\left(t^2+\theta^2\frac{\tan^2\psi}{\tan^2\varphi}\right)}$$

$$= \int_{u=-\varepsilon}^{u=+\varepsilon}\frac{dX}{\sqrt{X^2+A^2}\,(X^2+C^2)}\quad \text{where } X=t,\ \text{and } C^2>A^2.$$

From Ref. 4 the solution is therefore

$$\frac{1}{2C\sqrt{C^2-A^2}}\log\left\{\text{constant}\,\frac{C\sqrt{X^2+A^2}+X\sqrt{C^2-A^2}}{C\sqrt{X^2+A^2}-X\sqrt{C^2-A^2}}\right\}$$

$$= \frac{\tan^2\varphi}{2\theta\tan^2\psi\sqrt{\tan^2\varphi+\tan^2\psi}}\log\left\{\text{constant}\,\frac{\dfrac{\theta}{\sqrt{\tan^2\varphi+\tan^2\psi}}+\dfrac{t}{\sqrt{t^2+\dfrac{\tan^2\psi}{\tan^2\varphi}}}}{\dfrac{\theta}{\sqrt{\tan^2\varphi+\tan^2\psi}}-\dfrac{t}{\sqrt{t^2+\dfrac{\tan^2\psi}{\tan^2\varphi}}}}\right\}$$

The second part of (B.2) is therefore

$$\lim_{z\to 0} f(x)\left\{\frac{\alpha\tan\varphi-\beta\tan\psi}{2\theta(\tan^2\varphi+\tan^2\psi)}\left[\log\frac{\dfrac{\theta}{\tan\varphi}+\dfrac{u+z\dfrac{\tan\psi}{\tan\varphi}}{\sqrt{u^2+z^2}}}{\dfrac{\theta}{\tan\varphi}-\dfrac{u+z\dfrac{\tan\psi}{\tan\varphi}}{\sqrt{u^2+z^2}}}\right]_{u=-\varepsilon}^{u=+\varepsilon}\right\}.$$

As before $\sqrt{u^2+z^2} > 0$ and $\varepsilon >> z$. Taking the limit as $z\to 0$

$$\lim_{z\to 0} f(x)\left\{\frac{\alpha\tan\varphi-\beta\tan\psi}{2\theta(\tan^2\varphi+\tan^2\psi)}\left[\log\frac{1+\dfrac{u+z\dfrac{\tan\psi}{\tan\varphi}}{\sqrt{u^2+z^2}}\sin\Phi}{1-\dfrac{u+z\dfrac{\tan\psi}{\tan\varphi}}{\sqrt{u^2+z^2}}\sin\Phi}\right]_{u=-\varepsilon}^{u=+\varepsilon}\right\}$$

$$=f(x)\left[\frac{\alpha\tan\varphi-\beta\tan\psi}{2\theta(\tan^2\varphi+\tan^2\psi)}\left\{\log\frac{1+\sin\Phi}{1-\sin\Phi}-\log\frac{1-\sin\Phi}{1+\sin\Phi}\right\}\right]$$

$$=f(x)\frac{\alpha\tan\varphi-\beta\tan\psi}{\theta(\tan^2\varphi+\tan^2\psi)}\log\frac{1+\sin\Phi}{1-\sin\Phi}.$$

Therefore

$$I_2(x,0,0)=\frac{\theta\cos\Phi}{2\pi}q(x)\left\{2\psi\frac{\alpha\tan\psi+\beta\tan\varphi}{\tan^2\varphi+\tan^2\psi}+\frac{\alpha\tan\varphi-\beta\tan\psi}{\theta(\tan^2\varphi+\tan^2\psi)}\times\log\frac{1+\sin\Phi}{1-\sin\Phi}\right\}$$

$$=\frac{q(x)}{2\pi}\cos\Phi\log\frac{1+\sin\Phi}{1-\sin\Phi}.$$

APPENDIX C

Summary of Equations of Velocities Induced by Source Distributions of Infinite Span and Constant Strength Spanwise.

(*See* Section 3.2)

$$v_x(x,0,0) = \cos\Phi \int_0^1 \frac{q(x')}{2\pi}\,\frac{dx'}{x-x'} - \frac{q(x)}{2\pi}\cos\Phi \log\frac{1+\sin\Phi}{1-\sin\Phi} \tag{C.1}$$

$$v_z(x,0,0) = -\sin\Phi\sin\psi \int_0^1 \frac{q(x')}{2\pi}\,\frac{dx'}{x-x'} - \sin\psi \int_0^1 \frac{q(x')}{2\pi}\,\frac{dx'}{\sqrt{(x-x')^2+z^2}}\,^{*}$$

$$+\frac{q(x)}{2}\,\frac{\psi}{\pi/2}\cos\psi + \frac{q(x)}{2\pi}\sin\Phi\sin\psi\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right)$$

$$\pm\frac{q(x)}{2}\cos\psi\,, \tag{C.2}$$

the positive sign being used if $z \to 0$ from the positive side.

$$v_x(x,\infty,0) = \cos\Phi \int_0^1 \frac{q(x')}{2\pi}\,\frac{dx'}{x-x'} \tag{C.3}$$

$$v_z'(x,\infty,0) = \pm\frac{q(x)}{2} \tag{C.4}$$

the positive sign being used if $z \to 0$ from the positive side.

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*This integral cannot be evaluated in the limit as $z \to 0$. A finite value of z must be chosen.

APPENDIX D

Summary o, Equations of Velocities Induced by Vortex Distributions of Infinite Span and Constant Strength Spanwise.

(*See* Section 4.2)

$$v_x(x,0,0) = \pm\frac{\gamma(x)}{2}\cos\Phi + \frac{\gamma(x)}{2}\frac{\psi}{\pi/2}\cos\Phi \qquad \text{(D.1)}$$

the positive sign being used if $z \to 0$ from the positive side.

$$v_z(x,0,0) = -\cos\psi\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{x-x'} - \sin\Phi\cos\psi\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{\sqrt{(x-x')^2+z^2}}\,{}^*$$

$$-\frac{\gamma(x)}{2}\frac{\psi}{\pi/2}\sin\Phi\sin\psi + \frac{\gamma(x)}{2\pi}\cos\psi\log\left(\frac{1+\sin\Phi}{1-\sin\Phi}\right)$$

$$\mp\frac{\gamma(x)}{2}\sin\Phi\sin\psi \qquad \text{(D.2)}$$

the positive sign being used if $z \to 0$ from the negative side.

$$v_x(x,\infty,0) = \pm\frac{\gamma(x)}{2}\cos\Phi, \qquad \text{(D.3)}$$

the positive sign being used if $z \to 0$ from the positive side.

$$v_z'(x,\infty,0) = -\int_0^1 \frac{\gamma(x')}{2\pi}\frac{dx'}{x-x'}. \qquad \text{(D.4)}$$

*This integral cannot be evaluated in the limit as $z \to 0$. A finite value of z must be chosen.

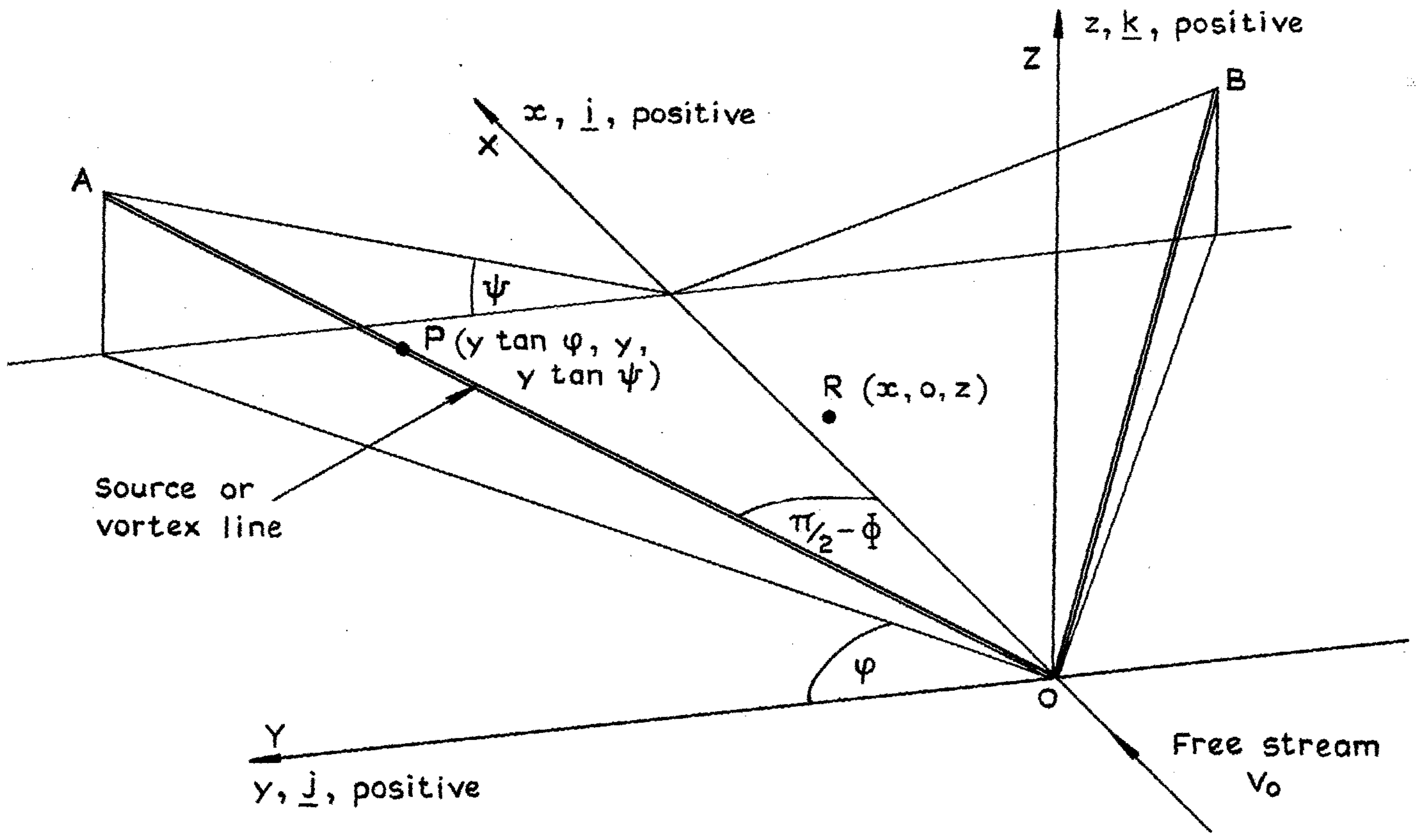

FIG. 1. System of axes and kinked vortex or source line.

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