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Summary.—The general equations of the steady motion of a non-viscous fluid are given in tensor notation. It is then assumed that one family of co-ordinate surfaces, x^α = constant, are characteristic surfaces, *i.e.*, surfaces on which the transverse derivatives of the flow-variables are not determined by their values on the surface itself. The condition for this is given by the relation $(u^\alpha)^2 = a^2 g^{\alpha\alpha}$ which can be interpreted to give the well-known result that the velocity normal to the surface is sonic. The relation which must then hold between the variables on the surface itself is also determined (characteristic equation).

The special cases of axisymmetric and two-dimensional flow are also considered and the results interpreted to give the well-known relationships. As an example, the flow in a simple wave, *i.e.*, a flow in which one family of characteristic lines are straight, is treated in detail.

While no new results have been obtained, the authors feel that the extra simplicity resulting from the use of quite general co-ordinates gives a deeper insight into the behaviour of such flows.

Introduction.—In Ref. 1 Dr. Meyer gives a novel method of developing the fundamental properties of the ' characteristics ' of the equations of motion of a gas in steady two-dimensional supersonic flow. He refers the equations to general orthogonal co-ordinates, one set of which are afterwards identified with one system of characteristic lines ; he then bases his definition of a characteristic on the fundamental property that the conditions on the curve fail to determine the rate of change of velocity and density in passing away from the curve. This property leads directly to the condition that the component velocity normal to the curve is sonic and also establishes a differential equation which must be satisfied along a characteristic.

In R. & M. 2615[2] Mr. C. K. Thornhill develops the theory of the general quasi-linear second order partial differential equation in three variables and derives the characteristic equations and curves, together with the partial differential equations holding on them. These results are then applied to steady supersonic flow in three dimensions and unsteady flow in two dimensions.

It seemed to the authors that the use of orthogonal co-ordinates was not sufficiently far-reaching, and that it might be more informative to use quite general curvilinear co-ordinates so that when one family of characteristics is made a co-ordinate family, the other co-ordinate family is left to the discretion of the user. The work of Mr. Thornhill encouraged us to apply the results to three-dimensional flow, and this was done without any serious increase of complexity and with some refinement of technique. Once the notation of the tensor calculus has been mastered, and it is hoped that Appendix I may be of some use to this end, then the whole development is very simple and gives considerable insight into the nature of the flow.

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2. *Reduction of the Equations*.—The reader unfamiliar with tensor notation is referred directly to Appendix 1 where the notation as used in this paper is explained in as simple a manner as possible. The majority of the results are there stated without proof. Where proof is necessary reference should be made to Refs. 3, 4 and 5.

A distinction is made between Greek and Roman suffixes. Greek suffixes α, β and γ are held to have fixed values, *i.e.*, they each refer to one of the three co-ordinate families (in three dimensions) and once they have been assigned to one family their meaning is unaltered. They do not obey the summation convention (*see* Appendix 1). Roman suffixes, on the other hand, are the usual dummy suffixes of tensor notation and obey the summation convention.

The further convention will also be adopted that the suffixes l, m and n refer only to β and γ, and never to α. Thus

$$g_{\alpha m}u^m = g_{\alpha\beta}u^\beta + g_{\alpha\gamma}u^\gamma.$$

We shall consider steady rotational flow (so that the entropy S may vary from one streamline to another) but without shock-waves or other discontinuities of the physical variables. The pressure is assumed to be a known function of the density and entropy, thus $p = p(\rho, S)$. The velocity of sound is defined as usual as a, where

$$a^2 = \frac{\partial p}{\partial \rho}.$$

We shall assume that all the variables are known on a co-ordinate surface x^α = constant, and hence also their derivatives with respect to x^β and x^γ. We shall then consider the equations of motion as equations to determine the derivatives with respect to x^α.

The equations of motion may be written in terms of co-variant velocity components, as follows:—

$$u^c u_{a,c} + \frac{1}{\rho}\frac{\partial p}{\partial x^a} = 0 \quad \ldots \quad (1) \text{ to } (3)$$

$$g^{bc}u_{b,c} + \frac{u^b}{\rho}\frac{\partial \rho}{\partial x^b} = 0 \quad \ldots \quad (4)$$

$$u^b \frac{\partial S}{\partial x^b} = 0 \quad \ldots \quad (5)$$

and the equation of state is

$$p = p(\rho, S)$$

$$\frac{\partial p}{\partial x^b} = a^2 \frac{\partial \rho}{\partial x^b} + \frac{\partial p}{\partial S}\frac{\partial S}{\partial x^b}. \quad \ldots \quad (6)$$

Multiplying equation (6) by u^b and using equation (5), we get

$$u^b \frac{\partial p}{\partial x^b} = a^2 u^b \frac{\partial \rho}{\partial x^b}. \quad \ldots \quad (7)$$

Substituting for $\frac{\partial \rho}{\partial x^b}$ in equation (4), we get

$$a^2 g^{bc}u_{b,c} + \frac{u^b}{\rho}\frac{\partial p}{\partial x^b} = 0. \quad \ldots \quad (8)$$

It should be noted that equation (5) determines $\partial S/\partial x^\alpha$ in terms of the values of S and its derivatives in the plane x^α = constant, while equation (6) determines $\partial \rho/\partial x^\alpha$ in terms of the derivatives of p and S.

We shall now write equations (1) to (3) and (8) as simultaneous linear equations in the four variables $u_{\alpha,\alpha}$ $u_{\beta,\alpha}$ $u_{\gamma,\alpha}$, $\partial p/\partial x^\alpha$. Thus

$$u^\alpha u_{\alpha,\alpha} + \frac{1}{\rho}\frac{\partial p}{\partial x^\alpha} = - u^m u_{\alpha,m} \quad \ldots \quad (9)$$

$$u^\alpha u_{n,\alpha} = - u^m u_{n,m} - \frac{1}{\rho}\frac{\partial p}{\partial x_n} \quad \ldots \quad (10), (11)$$

$$a^2 \left\{ g^{\alpha\alpha} u_{\alpha,\alpha} + g^{n\alpha} u_{n,\alpha} \right\} + \frac{u^\alpha}{\rho}\frac{\partial p}{\partial x^\alpha} = - a^2 g^{bm} u_{b,m} - \frac{u^n}{\rho}\frac{\partial p}{\partial x^m} \quad \ldots \quad (12)$$

where m, n can only take the values β and γ. Equations (10), (11) are sufficient to determine $u_{\beta,\alpha}$ and $u_{\gamma,\alpha}$ in terms of conditions on the plane provided that u^α does not vanish.

Multiplying equations (10) and (11) by $a^2 g^{n\alpha}$ and equation (12) by u^α and subtracting gives

$$a^2 g^{\alpha\alpha} u^\alpha u_{\alpha,\alpha} + \frac{(u^\alpha)^2}{\rho}\frac{\partial p}{\partial x^\alpha} = a^2 (g^{\alpha n} u^m u_{n,m} - g^{bm} u^\alpha u_{b,m}) + (a^2 g^{n\alpha} - u^\alpha u^n) \frac{1}{\rho}\frac{\partial p}{\partial x^n} . \quad \ldots \quad (13)$$

Equations (9) and (13) are then sufficient to determine $u_{\alpha,\alpha}$ and $\partial p/\partial x^\alpha$ unless

$$\text{either} \qquad u^\alpha = 0 \quad \ldots \quad (14)$$

$$\text{or} \qquad a^2 g^{\alpha\alpha} = (u^\alpha)^2 . \quad \ldots \quad (15)$$

If neither of these exceptional cases occur, all the transverse derivatives can be determined from conditions on the surface itself. If equation (14) is true, the surface x^α = constant is a stream surface and this possibility is considered in detail later on. If equation (15) is true, the surface x^α = constant is known as a characteristic surface. On such a surface the derivatives of u_α, p and ρ are indeterminate. In this latter case equations (13) and (9) are only consistent if

$$(u^\alpha)^2 u^m u_{\alpha,m} + a^2 (g^{\alpha n} u^m u_{n,m} - g^{bm} u^\alpha u_{b,m}) + (a^2 g^{n\alpha} - u^\alpha u^n) \frac{1}{\rho}\frac{\partial p}{\partial x^n} = 0$$

or, using equation (15),

$$u^\alpha (g^{\alpha b} u^m - g^{bm} u^\alpha) u_{b,m} + (g^{n\alpha} u^\alpha - g^{\alpha\alpha} u^n) \frac{1}{\rho}\frac{\partial p}{\partial x^n} = 0 \quad \ldots \quad (16)$$

or

$$u^\alpha (u^m u^\alpha{}_{,m} - u^\alpha u^m{}_{,m}) + (g^{m\alpha} u^\alpha - g^{\alpha\alpha} u^m) \frac{1}{\rho}\frac{\partial p}{\partial x^m} = 0 . \quad \ldots \quad (16a)$$

This equation must hold on the characteristic surface and is known as the 'characteristic equation'.

We have now established the following properties of a characteristic surface.

(a) If the surface x^α = constant is a characteristic surface then a knowledge of values of the dependent variables and their β and γ derivatives in the surface does not determine the values of u, p and ρ on a neighbouring surface and in particular their derivatives may change discontinuously at the surface in a manner consistent with the determinate values of $u_{\beta,\alpha}$, $u_{\gamma,\alpha}$. Since a discontinuity of a derivative may be considered as an infinitesimal disturbance in the flow, this will mean that such disturbances can exist and will be propagated along and only along characteristic surfaces.

(b) $$(g^{\alpha\alpha})^{1/2}\, a = \pm u^{\alpha} \qquad (17)$$

at all points of the surface. It is proved in Appendix 2 that this relation is equivalent to

$$v_{n\alpha} = \pm a \, ,$$

where $v_{n\alpha}$ is the component velocity normal to the surface.

It follows from geometrical considerations that the characteristic surfaces passing through a given point P, envelope a conical surface (conoid) and in particular that they touch a right-cone with vertex at P and semi-angle μ where

$$\sin \mu = a/w \, .$$

(c) $$u^{\alpha}(u^{m}u^{\alpha}_{,m} - u^{\alpha}u^{m}_{,m}) + (g^{m\alpha}u^{\alpha} - g^{\alpha\alpha}u^{m}) \frac{1}{\rho}\frac{\partial p}{\partial x_{m}} = 0$$

at all points of the characteristic surface.

3. *Two-dimensional Flows.*—For two-dimensional flow we let $x^{\gamma} =$ constant be a plane on which

$$\left.\begin{aligned} g_{\gamma\alpha} = g_{\beta\gamma} &= 0 \\ \text{and} \qquad g_{\gamma\gamma} &= 1 \end{aligned}\right\} \qquad (18)$$

u_{γ}, u^{γ} and all derivatives with respect to x^{γ} also vanish.

The equations of motion then have the same form as before, but the suffixes can now only take one of the two values α or β. It will also be convenient to use the suffixes r, s and t as holding over these two values, only, and we shall write

$$G = g_{\alpha\alpha}g_{\beta\beta} - g_{\alpha\beta}^{\ 2} \, .$$

Then the equations of motion are

$$u^{r}u_{s,r} + \frac{1}{\rho}\frac{\partial p}{\partial x^{s}} = 0 \qquad (19, 20)$$

$$g^{rs}u_{s,r} + \frac{u^{s}}{\rho}\frac{\partial p}{\partial x^{s}} = 0 \, . \qquad (21)$$

The equation of state remains

$$p = f(\rho, S) \, . \qquad (22)$$

The condition that the curve $x^{\alpha} =$ constant, should be a characteristic curve remains

$$\left.\begin{aligned} a^{2}g^{\alpha\alpha} &= (u^{\alpha})^{2} \\ \text{or} \qquad v_{n\alpha} &= \pm a \end{aligned}\right\} \qquad (23)$$

and the characteristic equation becomes

$$u^{\alpha}(u^{\beta}u^{\alpha}_{,\beta} - u^{\alpha}u^{\beta}_{,\beta}) + (g^{\beta\alpha}u^{\alpha} - g^{\alpha\alpha}u^{\beta}) \frac{1}{\rho}\frac{\partial p}{\partial x^{\beta}} = 0$$

or

$$u^{\alpha}(u^{\beta}u^{\alpha}_{,\beta} - u^{\alpha}u^{\beta}_{,\beta}) + (g^{\beta\alpha}g^{\beta\alpha} - g^{\alpha\alpha}g^{\beta\beta}) \frac{u_{\beta}}{\rho}\frac{\partial p}{\partial x^{\beta}} = 0$$

or

$$u^{\alpha}(u^{\beta}u^{\alpha}_{,\beta} - u^{\alpha}u^{\beta}_{,\beta}) - \frac{u_{\beta}}{\rho G}\frac{\partial p}{\partial x^{\beta}} = 0 \, . \qquad (24)$$

It is shown in Appendix 2 that

$$w^2 \frac{\partial \theta}{\partial x^\beta} = G^{1/2} (u^\beta u^\alpha{}_{,\beta} - u^\alpha u^\beta{}_{,\beta}) \qquad (25)$$

and that when equation (23) holds

$$u_\beta{}^2 = g_{\beta\beta}(w^2 - a^2) \,. \qquad (26)$$

Thus the characteristic equation becomes

$$u^\alpha w^2 \frac{\partial \theta}{\partial x^\beta} - \frac{g_{\beta\beta}{}^{1/2}(w^2 - a^2)^{1/2}}{\rho G^{1/2}} \frac{\partial p}{\partial x_\beta} = 0 \,.$$

Putting $u^\alpha = \pm (g^{\alpha\alpha})^{1/2} a$

and remembering that $g^{\alpha\alpha} = (g_{\beta\beta}/G)$

we get

$$w^2 \frac{\partial \theta}{\partial x^\beta} \pm \frac{(w^2 - a^2)^{1/2}}{\rho a} \frac{\partial p}{\partial x^\beta} = 0 \,. \qquad (27)$$

This is the well-known form of the characteristic equation.

4. *Symmetry about an Axis.*—In this case one can take x^γ to be the angular co-ordinate about the axis, so that if r is the distance from the axis the fundamental metric becomes

$$ds^2 = g_{rs}\, dx^r\, dx^s + r^2(dx^\gamma)^2 \qquad (28)$$

so that

$$\left. \begin{aligned} g_{\alpha\gamma} &= g_{\beta\gamma} = 0 \\ g_{\gamma\gamma} &= r^2 \end{aligned} \right\} . \qquad (29)$$

As in two-dimensional flow u_γ, u^γ and all scalar derivatives with respect to x^γ are zero. All covariant derivatives with respect to x^γ are not zero, however, for

$$u_{a,\gamma} = \frac{\partial u_a}{\partial x^\gamma} - \Gamma^b_{a\gamma}\, u_b \qquad (30)$$

$$\Gamma^b_{a\gamma}\, u_b = \tfrac{1}{2} g^{bc} \left\{ \frac{\partial g_{c\gamma}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^\gamma} - \frac{\partial g_{a\gamma}}{\partial x^c} \right\} u_b \,.$$

Inspection then shows that

$$\left. \begin{aligned} u_{\gamma,\gamma} &= \frac{1}{2} \frac{\partial g_{\gamma\gamma}}{\partial x^c} u^c \\ u_{r,\gamma} &= 0 \end{aligned} \right\} . \qquad (31)$$

Since $g_{\gamma\gamma} = r^2$, we may write

$$\begin{aligned} u_{\gamma,\gamma} &= r \frac{\partial r}{\partial x^c} u^c \\ &= r \frac{\partial r}{\partial t} = r\bar{V} \end{aligned} \qquad (32)$$

where $\bar{V}$ is the velocity component normal to the axis.

Thus the equations of motion may be written

$$u^r u_{s,r} + \frac{1}{\rho} \frac{\partial p}{\partial x_s} = 0 \qquad (33), (34)$$

$$g^{rs} u_{s,r} + \frac{\bar{V}}{r} + \frac{1}{\rho} \frac{\partial p}{\partial x^s} = 0 \,. \qquad (35)$$

The condition for a characteristic is unaltered, and the characteristic equation becomes

$$w^2 \frac{\partial \theta}{\partial x^\beta} \pm \frac{(w^2 - a^2)^{1/2}}{\rho a} \frac{\partial p}{\partial x^\beta} \pm \frac{g_{\beta\beta}{}^{1/2} a \bar{V}}{r} = 0 . \qquad (36)$$

Introducing $\sin \mu = a/w$ .. .. (37)

and $j = \begin{cases} 0 & \text{for two-dimensional flow} \\ 1 & \text{for axisymmetric flow ,} \end{cases}$

this may be written

$$w^2 \frac{\partial \theta}{\partial x^\beta} \pm \cot \mu \frac{1}{\rho} \frac{\partial p}{\partial x^\beta} \pm j \frac{\sin \theta}{\sin \mu \sin(\theta \pm \mu)} \frac{dr}{r} = 0 \qquad (38)$$

since

$$g_{\beta\beta}{}^{1/2}\, dx^\beta \sin(\theta \pm \mu) = dr .$$

5. *Streamlines*.—If $u^\alpha = 0$, so that the surfaces $x^\alpha =$ constant are stream surfaces, some simplification is introduced into the equations of motion. It is more convenient, however, to deal with streamlines rather than surfaces, and we shall consider the case where the γ-lines are streamlines, so that

$$u^\alpha = u^\beta = 0$$

and

$$w^2 = g_{\gamma\gamma}(u^\gamma)^2 .$$

We shall also use the convention that the suffixes r, s and t refer to α and β only. The last two equations of motion, equations (4) and (5), express the fact that the entropy and the mass-flow are constant along stream-tubes and consequently no new information can be expected from putting $u^\alpha = u^\beta = 0$. The two equations become in fact

$$u^\gamma \frac{\partial S}{\partial x^\gamma} = 0 \qquad (39)$$

$$\frac{\partial}{\partial x^\gamma}(\rho g^{1/2} u^\gamma) = 0 , \qquad (40)$$

where equation (40) is obtained by expressing equation (4) in its simplest form (Appendix 1).

Equations (1) to (3) may be written

$$g_{bc} u^\gamma u^c{}_{,\gamma} + \frac{1}{\rho}\frac{\partial p}{\partial x^b} = 0 .$$

For equation (3), $b = \gamma$ and this becomes

$$g_{c\gamma} u^\gamma u^c{}_{,\gamma} + \frac{1}{\rho}\frac{\partial p}{\partial x^\gamma} = 0 . \qquad (41)$$

Now

$$\frac{\partial (w^2)}{\partial x^\gamma} = \frac{\partial}{\partial x^\gamma}(g_{bc} u^b u^c) = u^b{}_{,\gamma} u_b + u^b u_{b,\gamma}$$

$$= 2 g_{bc} u^b u^c{}_{,\gamma} .$$

Since $u^\alpha = u^\beta = 0$, equation (41) becomes

$$\frac{1}{2}\frac{\partial (w^2)}{\partial x^\gamma} + \frac{1}{\rho}\frac{\partial p}{\partial x^\gamma} = 0 \qquad (42)$$

which is Bernouilli's equation for flow along streamlines.

When $b \neq \gamma$, equations (1) and (2) become

$$g_{cr} u^{\gamma} u^{c}{}_{,\gamma} + \frac{1}{\rho} \frac{\partial p}{\partial x^{r}} = 0$$

or

$$g_{cr} u^{\gamma} \left[\frac{\partial u^{c}}{\partial x^{\gamma}} + \Gamma^{c}_{\gamma\gamma} u^{\gamma} \right] + \frac{1}{\rho} \frac{\partial p}{\partial x_{r}} = 0 . \quad \ldots \quad (43)$$

Now it is shown in Appendix 2 that if R is a vector representing in magnitude and direction the curvature of the streamline $dx^{\alpha} = dx^{\beta} = 0$,

then

$$\Gamma^{c}_{\gamma\gamma} = g_{\gamma\gamma} R^{c} + g^{c}_{\gamma} \frac{1}{(g_{\gamma\gamma})^{1/2}} \frac{\partial (g_{\gamma\gamma})^{1/2}}{\partial x^{\gamma}} \quad \ldots \quad (44)$$

so that equation (43) may now be written

$$g_{r\gamma} u^{\gamma} \frac{\partial u^{\gamma}}{\partial x^{\gamma}} + g_{r\gamma} (u^{\gamma})^{2} R_{r} + g_{r\gamma} (u^{\gamma})^{2} \frac{1}{(g_{\gamma\gamma})^{1/2}} \frac{\partial (g_{\gamma\gamma})^{1/2}}{\partial x^{\gamma}} + \frac{1}{\rho} \frac{\partial p}{\partial x^{r}} = 0$$

or

$$\frac{g_{r\gamma} u^{\gamma}}{(g_{\gamma\gamma})^{1/2}} \frac{\partial}{\partial x^{\gamma}} \left\{ (g_{\gamma\gamma})^{1/2} u^{\gamma} \right\} + g_{\gamma\gamma} (u^{\gamma})^{2} R_{r} + \frac{1}{\rho} \frac{\partial p}{\partial x_{r}} = 0 . \quad \ldots \quad (45)$$

Finally, let K_{r} be the actual component of geodesic curvature of the streamline in the x^{r} direction, so that

$$(g_{rr})^{1/2} K_{r} = R .$$

Remembering that

$$g_{\gamma\gamma} (u^{\gamma})^{2} = w^{2}$$

and

$$\frac{g_{r\gamma}}{(g_{rr} g_{\gamma\gamma})^{1/2}} = \cos \psi_{r\gamma} ,$$

where ψ_{ab} is the angle between the a and b axes,the equation (45) reduces to

$$\frac{w}{(g_{\gamma\gamma})^{1/2}} \frac{\partial w}{\partial x^{\gamma}} \cos \psi_{r\gamma} + w^{2} K_{r} + \frac{1}{\rho (g_{rr})^{1/2}} \frac{\partial p}{\partial x^{r}} = 0 . \quad \ldots \quad (46)$$

The first term (which vanishes when the r-axis is orthogonal to the streamlines) gives the component of the stream acceleration in the direction of the r-axis, whilst the second term gives a measure of the centripetal acceleration.

6. *Simple Wave.*—As an example of the application of the tensor notation we shall apply it to the case of a 'simple wave', in which one family of characteristics are straight lines. This family will be represented by the β-lines, the streamlines by the α-lines, and we consider the case in which $v_{n\beta} = + a$.

It follows from the previous analysis that

$$u^{\beta} = 0 \quad \ldots \quad (47)$$

$$(u^{\alpha})^{2} = g^{\alpha\alpha} a^{2} \quad \ldots \quad (48)$$

$$w^{2} \frac{\partial \theta}{\partial x^{\alpha}} - \frac{(w^{2} - a^{2})^{1/2}}{a\rho} \frac{\partial p}{\partial x^{\beta}} = 0 \quad \ldots \quad (49)$$

and the equations of motion are

$$\frac{1}{2}\frac{\partial(w^2)}{\partial x^\alpha}+\frac{1}{\rho}\frac{\partial p}{\partial x^\alpha}=0 \quad \ldots \quad (50)$$

$$g_{c\beta}u^\alpha u^c{}_{,\alpha}+\frac{1}{\rho}\frac{\partial p}{\partial x^\beta}=0 \quad \ldots \quad (51)$$

$$\frac{\partial}{\partial x^\alpha}\,(\rho g^{1/2}u^\alpha)=0 \quad \ldots \quad (52)$$

$$\frac{\partial S}{\partial x^\alpha}=0\,. \quad \ldots \quad (53)$$

We shall make the further assumption that the total energy and the entropy are the same on all streamlines and we shall limit the discussion to the case of a perfect gas for which the equation of state may be written

where

$$\left.\begin{array}{l} p\;=K\rho^\gamma \\ K=\exp\,(S/c_v)\end{array}\right\} \quad \ldots \quad (54)$$

and c_v is constant. γ is constant throughout the fluid, and the same is true of S and K. From equations (50) and (54), and writing

$$E=\tfrac{1}{2}w^2+\frac{\gamma}{\gamma-1}\frac{p}{\rho} \quad \ldots \quad (55)$$

for the total energy we have

$$\frac{\partial E}{\partial x^\alpha}=0 \quad \ldots \quad (56)$$

along a streamline, and so E is constant throughout the fluid. From the definition of μ, the angle which a characteristic makes with a streamline at any point,

$$w^2\sin^2\mu=\frac{\gamma p}{\rho}\,. \quad \ldots \quad (57)$$

Differentiating equations (54) and (57) with respect to x^β

$$\left.\begin{array}{ll} \dfrac{2}{w}\dfrac{\partial w}{\partial x^\beta}+2\cot\mu\,\dfrac{\partial\mu}{\partial x^\beta}=\dfrac{1}{p}\dfrac{\partial p}{\partial x^\beta}-\dfrac{1}{\rho}\dfrac{\partial\rho}{\partial x^\beta} \\ \text{and} \qquad \dfrac{1}{p}\dfrac{\partial p}{\partial x^\beta}=\dfrac{\gamma}{\rho}\dfrac{\partial\rho}{\partial x^\beta}\,. \end{array}\right\} \quad \ldots \quad (58)$$

Hence,

$$\begin{aligned}\frac{\partial E}{\partial x^\beta}&=w\,\frac{\partial w}{\partial x^\beta}+\frac{1}{\rho}\frac{\partial p}{\partial x^\beta}\\ &=\{1+\tfrac{1}{2}(\gamma-1)\operatorname{cosec}^2\mu\}\,\frac{1}{\rho}\frac{\partial p}{\partial x^\beta}-w^2\cot\mu\,\frac{\partial\mu}{\partial x^\beta}\\ &=0\,. \quad \ldots \quad (59)\end{aligned}$$

We also note that equation (49) may be written

$$\frac{1}{\rho}\frac{\partial p}{\partial x^\beta}=w^2\tan\mu\,\frac{\partial\theta}{\partial x^\beta}\,. \quad \ldots \quad (60)$$

Eliminating $\partial p/\partial x^\beta$ from equations (59) and (60)

$$\left(1 + \frac{\gamma - 1}{2} \operatorname{cosec}^2 \mu\right) \frac{\partial \theta}{\partial x^\beta} - \cot^2 \mu \frac{\partial \mu}{\partial x^\beta} = 0 . \qquad (61)$$

But since the β characteristic is straight

$$\frac{\partial}{\partial x^\beta} (\theta - \mu) = 0$$

or

$$\frac{\partial \theta}{\partial x^\beta} - \frac{\partial \mu}{\partial x^\beta} = 0 . \qquad (62)$$

Equations (61) and (62) are only compatible if $\partial \theta/\partial x^\beta$ and $\partial \mu/\partial x^\beta$ both vanish, and hence (in view of equations (49) and (58)) on a straight characteristic

$$\frac{\partial P}{\partial x^\beta} = 0 \qquad (63)$$

for $P = p, \rho, w, a, \mu$ and θ, *i.e.*, all the flow variables are constant along the characteristic.

Since

$$\frac{\partial \theta}{\partial x^\beta} = \frac{g^{1/2}}{w} (u^\alpha u^\beta{}_{,\beta} - u^\beta u^\alpha{}_{,\beta}) = 0$$

and $u^\beta = 0$, it follows that

$$u^\beta{}_{,\beta} = u^\alpha \Gamma^\beta_{\alpha\beta} = 0 . \qquad (64)$$

In order to find a relation between θ and μ which holds along the streamlines we shall eliminate $\partial u^\alpha/\partial x^\alpha$ between the two equations of motion (51) and (52).

From (51),

$$g_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^\alpha} = - g_{\alpha\beta} \Gamma^\alpha_{\alpha\alpha} u^\alpha$$

and from (52)

$$\frac{\partial u^\alpha}{\partial x^\alpha} = - \left[\frac{1}{\rho} \frac{\partial \rho}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\alpha}\right] u^\alpha .$$

For these to be compatible we must have

$$g_{\alpha\beta} \left[\frac{1}{\rho} \frac{\partial \rho}{\partial x^\alpha} + \Gamma^\alpha_{\alpha\alpha}\right] = g_{\beta\alpha} \Gamma^\alpha_{\alpha\alpha}$$

or

$$g_{\alpha\beta} \frac{1}{\rho} \frac{\partial \rho}{\partial x^\alpha} = g_{\beta\beta} \Gamma^\beta_{\alpha\alpha} \qquad (65)$$

using equation (64). Again, equations (25), with α and β interchanged, and (47) give

$$w^2 \frac{\partial \theta}{\partial x^\alpha} = g^{1/2} u^\alpha \Gamma^\beta_{\alpha\alpha} u^\alpha , \qquad (66)$$

and since

$$g_{\alpha\beta} = g_{\alpha\alpha}{}^{1/2} g_{\beta\beta}{}^{1/2} \cos \mu$$

equations (65) and (66) give

$$\frac{1}{\rho} \frac{\partial \rho}{\partial x_\alpha} = \frac{1}{\sin \mu \cos \mu} \frac{\partial \theta}{\partial x^\alpha} . \qquad (67)$$

Finally, using equation (56) and differentiating equations (54) and (57) with respect to x^α

$$\frac{1}{\rho}\frac{\partial p}{\partial x^\alpha} = \frac{\cot\mu}{\frac{1}{2}(\gamma-1)+\sin^2\mu}\frac{\partial\mu}{\partial x^\alpha} \qquad (68)$$

so that

$$\frac{\partial\theta}{\partial x^\alpha} = \frac{\cos^2\mu}{\frac{1}{2}(\gamma-1)+\sin^2\mu}\frac{\partial\mu}{\partial x^\alpha}. \qquad (69)$$

Integrating we find that the following condition is satisfied along a streamline

$$\theta = -\mu + \chi/\lambda + \text{constant} \qquad (70)$$

with

$$\tan\chi = (1/\lambda)\tan\mu, \quad \lambda^2 = (\gamma-1)/(\gamma+1). \qquad (71)$$

This is the fundamental equation for a simple wave.

The other quantities w, p and ρ may be determined by substitution in equations (54), (55), (57) in the form

$$\left.\begin{aligned} w &= (2E)^{1/2}\cos\chi\sec\mu \\ \frac{p}{\rho} &= \frac{2E}{\gamma}\lambda^2\sin^2\chi \\ &= K^{\frac{1}{\gamma}}p^{\frac{\gamma-1}{\gamma}} \\ &= K\rho^{\gamma-1}. \end{aligned}\right\} \qquad (72)$$

To complete the geometrical determination of the arbitrary streamline in terms of a standard streamline it is necessary to derive an equation for the variation of $g_{\beta\beta}$ along a streamline. Putting $\psi = \mu$ in equation (52), and using (70), (71) and (72)

$$\begin{aligned} g_{\beta\beta}{}^{1/2} &= \text{constant}/\rho w\sin\mu \\ &= \text{constant}\,(\sin\chi)^{\,1/\lambda^2} \end{aligned} \qquad (73)$$

along a streamline. Let dr be the element of length along a straight characteristic, then

$$dr = g_{\beta\beta}{}^{1/2}\,dx^\beta.$$

But $x^\beta =$ constant along a streamline, and so if r is measured from a standard streamline as datum we have

$$r = \text{constant}\,(\sin\chi)^{-1/\lambda^2}.$$

Writing

$$\phi = \theta - \mu \qquad (74)$$

for the inclination of the straight characteristic to a fixed direction, r and ϕ may be considered as quasi-polar co-ordinates of an arbitrary streamline referred to a fixed streamline as datum and are determined by equations (70), (71) and (74) as functions of the single parameter μ along the fixed streamline. Since we have not stipulated the significance of x^α, and $\phi\ (= \theta - \mu)$ is constant along the characteristic we might write $x^\alpha = \phi$. In the case of a centred simple wave the datum streamline may be taken as a fixed point (the centre), and r and ϕ become true polar co-ordinates satisfying the same relations as for the general simple wave.

APPENDIX I

Fundamental Results.—For simplicity we shall consider only the two-dimensional case. We use a perfectly arbitrary set of oblique curvilinear co-ordinates x^α, x^β. Let P be an arbitrary point (x^α, x^β) and let PQ be an arbitrary infinitesimal vector $\underline{A}$. Let the co-ordinate lines through

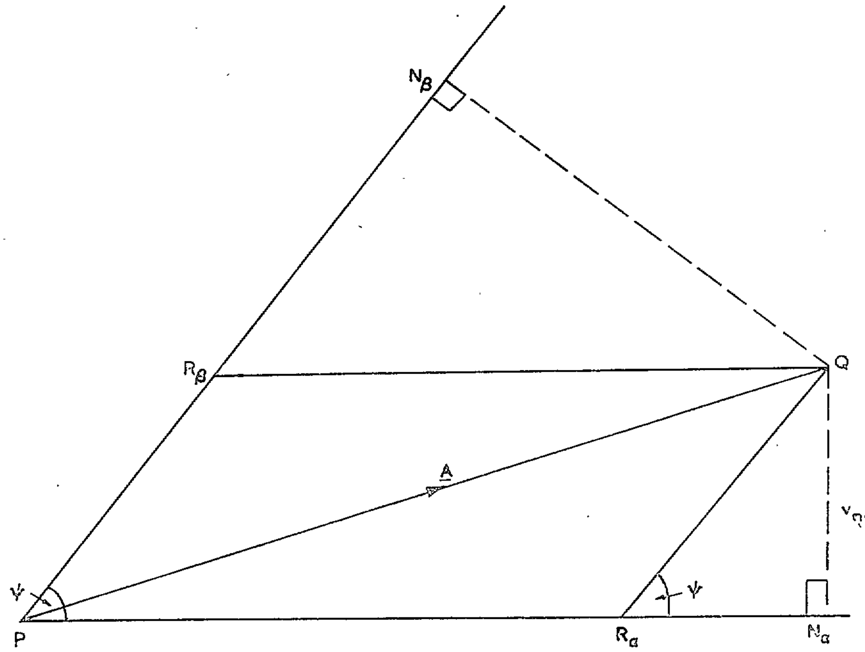

FIG. 1.

P and Q form the curvilinear quadrilateral $PR_\alpha QR_\beta$ (Fig. 1) which is to the first order a parallelogram and is such that x^α has the constant values

$$x^\alpha \text{ and } x^\alpha + \delta x^\alpha = x^\alpha + A^\alpha$$

on PR_β and $R_\alpha Q$ respectively, while x^β has the constant values x^β and $x^\beta + \delta x^\beta = x^\beta + A^\beta$ on PR_α and $R_\beta Q$ respectively. A^α and A^β will be described as the *contra-variant* components of the vector $\underline{A}$. . We shall write

$$PR_\alpha = g_{\alpha\alpha}{}^{1/2}A^\alpha, \qquad PR^\beta = g_{\beta\beta}{}^{1/2}A^\beta\,,$$

and if ψ is the angle between the co-ordinate lines $R_\alpha PR_\beta$ .. .. .. (A1.1)

$$g_{\alpha\beta} = g_{\beta\alpha} = g_{\alpha\alpha}{}^{1/2}g_{\beta\beta}{}^{1/2}\cos\psi\,,$$

where $g_{\alpha\alpha}$, $g_{\alpha\beta}$ and $g_{\beta\beta}$ are all functions of position depending only on the co-ordinates, and completely define the axes of co-ordinates in the neighbourhood of P.

Let N_α and N_β be the feet of the perpendiculars from Q on to PR_β and PR_α respectively. Then

$$A_\alpha = g_{\alpha\alpha}{}^{1/2}\,PN_\alpha \quad \text{and} \quad A_\beta = g_{\beta\beta}{}^{1/2}\,PN_\beta \qquad .. \quad .. \quad .. \quad \text{(A1.2)}$$

will be described as the *co-variant* components of the vector $\underline{A}$.

The following results help to justify these apparently arbitrary definitions and illustrate further the necessity for the two types of component of a vector when the co-ordinates are oblique.

$$|A|^2 = g_{\alpha\alpha}(A^\alpha)^2 + 2g_{\alpha\beta}A^\alpha A^\beta + g_{\beta\beta}(A^\beta)^2$$
$$= \sum_{\alpha,\beta} g_{ab}\,A^a A^b\,.$$

By the 'summation convention':—Any Latin suffix repeated in a given term occurs always once as a lower and once as an upper suffix, and is then understood to be summed for α and β so that the sign of summation becomes unnecessary. Thus we write

$$|A|^2 = g_{ab}A^aA^b . \qquad \text{(A1.3)}$$

Again,

$$A_\alpha = g_{\alpha\alpha}A^\alpha + g_{\alpha\beta}A^\beta$$
$$= g_{\alpha b}A^b$$

and

$$A_\beta = g_{\beta b}A^b,$$

where the Latin suffices obey the summation convention. The two equations will be written as the single equation

$$A_a = g_{ab}A^b . \qquad \text{(A1.4)}$$

This illustrates the convention that a 'floating' (unrepeated) Latin suffix must occur in the same position in every term of an equation and is understood to be replaced by α and β in turn.

We shall write

$$g = \begin{vmatrix} g_{\alpha\alpha} & g_{\alpha\beta} \\ g_{\alpha\beta} & g_{\beta\beta} \end{vmatrix} = g_{\alpha\alpha}g_{\beta\beta} - g_{\alpha\beta}^2 \qquad \text{(A1.5)}$$

and

$$g^{ab} = G_{ab}/g , \qquad \text{(A1.6)}$$

where G_{ab} is the co-factor of g_{ab} in the determinant g. Thus

$$g^{\alpha\alpha} = \frac{g_{\beta\beta}}{g} , \; g^{\beta\beta} = \frac{g_{\alpha\alpha}}{g} \text{ and } g^{\alpha\beta} = -\frac{g_{\alpha\beta}}{g} .$$

Then,

$$A^a = g^{ab}A_b \qquad \text{(A1.7)}$$

and

$$|A|^2 = A^aA_a = g^{ab}A_aA_b . \qquad \text{(A1.8)}$$

In a general change of axes in which the co-ordinates x'^γ, x'^δ are any functions of x^α, x^β the components of a vector are transformed according to the relation

$$A'^a = \frac{\partial x'^a}{\partial x^b} A^b \qquad \text{(A1.9)}$$

$$A'_a = \frac{\partial x^b}{\partial x'^a} A_b$$

The components of a vector are a special first order case of 'tensors' which may be of any order. Thus a tensor T of the second order has contra-variant components T^{ab}, co-variant components T_{ab} or mixed components T^a_b, each of which stands for four numbers obtained by putting a and b equal to α, β in turn. Such a system of numbers of any order is said to be a tensor if it obeys the following law of transformation from the set of co-ordinates $x_1, x_2, x_3 \ldots x_N$ to the new set $y_1, \ldots y_N$

$$T'^{a \ldots e}_{i \ldots k} = \frac{\partial y^a}{\partial x^l} \cdots \frac{\partial y^e}{\partial x^n} \frac{\partial x^r}{\partial y^i} \cdots \frac{\partial x^t}{\partial y^k} T^{l \ldots n}_{r \ldots t} \qquad \text{(A1.10)}$$

g_{ab}, g^{ab} are special cases of tensors of the second order which are functions of the co-ordinate system only. The mixed tensor

$$g^a_b = g^{ac}g_{cb}$$

is often written δ^a_b and has the property of changing a suffix, since $\delta^\alpha_\alpha = \delta^\beta_\beta = 1$ and $\delta^\alpha_\beta = \delta^\beta_\alpha = 0$. For example,

$$A^a = \delta^a_b A^b .$$

Derivative of a Scalar.—The change $\delta\phi$ in a scalar field corresponding to a small displacement from a point P to a point Q considered as a vector $\underline{\delta x}$ with components δx^a is given by the normal type of formula

$$\delta\phi = \phi_Q - \phi_P = \frac{\partial\phi}{\partial x^a}\,\delta x^a \quad \text{(A1.11)}$$

$\partial\phi/\partial x^a$ is the co-variant component of the 'gradient' of the scalar.

Derivative of a Vector.—If $\underline{A}$ is a vector field, we require a formula for $\underline{\delta A}$ in terms of the small displacement $\underline{\delta x}$. In curvilinear co-ordinates it is not true that

$$\delta A^a = \frac{\partial A^a}{\partial x^b}\,\delta x^b$$

and $\partial A^a/\partial x^b$ is not a tensor. It is possible, however, to define a mixed tensor of the second order whose components are denoted by $A^a{}_{,b}$ such that

$$\delta A^a = A^a{}_{,b}\,\delta x^b\,. \quad \text{(A1.12)}$$

It may be shown that

$$A^a{}_{,b} = \frac{\partial A^a}{\partial x^b} + \Gamma^a_{bc}A^c\,,$$

where

$$\Gamma^a_{bc} = g^{ai}\Gamma_{i,bc} \quad \text{(A1.13)}$$

and

$$\Gamma_{i,bc} = \frac{1}{2}\left\{\frac{\partial g_{ib}}{\partial x_c} + \frac{\partial g_{ic}}{\partial x_b} - \frac{\partial g_{bc}}{\partial x_i}\right\}$$

$A^a{}_{,b}$ is known as the co-variant derivative of the contra-variant component A^a of the vector $\underline{A}$.

The co-variant derivative of the co-variant component of $\underline{A}$ may similarly be defined such that

$$\delta A_a = A_{a,b}\delta x^b \quad \text{(A1.14)}$$

and,

$$A_{a,b} = \frac{\partial A_a}{\partial x^b} - \Gamma^c_{ab}A_c\,. \quad \text{(A1.15)}$$

Co-variant derivatives of tensors of higher orders may be similarly defined. In particular for a mixed tensor of second order we have

$$A^a_{b,c} = \frac{\partial A^a_b}{\partial x^c} + \Gamma^a_{ic}A^i_b - \Gamma^i_{bc}A^a_i\,. \quad \text{(A1.16)}$$

The derivatives of the fundamental tensors are all zero

$$g_{ab,c} = g^{ab}{}_{,c} = g^a_{b,c} = 0\,. \quad \text{(A1.17)}$$

The usual distributive law of differentiation is obeyed by the co-variant derivative; for example

$$\frac{\partial}{\partial x^b}(A^aB_a) = (A^aB_{a,b})$$

$$= A^a{}_{,b}B_a + A^aB_{a,b}\,. \quad \text{(A1.18)}$$

In this last equation it has been found convenient to write the co-variant derivative of a scalar in two forms. Thus

$$(\phi)_{,a} = \frac{\partial\phi}{\partial x^a} \quad \text{(A1.19)}$$

Equations of Motion.—Acceleration.—The contra-variant components f^a of the acceleration $\underline{f}$ which is the rate of change of the velocity vector $\underline{u}$ following the motion of the fluid may be obtained as follows. In time δt the displacement $\overline{\mathrm{PQ}}$ of a particle of fluid is given by

$$\delta x^a = u^a \, \delta t \, .$$

The total change of the velocity u^a of the particle of fluid originally at P is

$$f^a \, \delta t = \frac{\partial u^a}{\partial t} \delta t + u^a{}_{,b} \, \delta x^b$$

$$= \frac{\partial u^a}{\partial t} \delta t + u^a{}_{,b} u^b \, \delta t$$

and so

$$f^a = \left(\frac{Du}{Dt}\right)^a = \frac{\partial u^a}{\partial t} + u^a{}_{,b} u^b \, . \qquad \text{(A1.20)}$$

The general formula

$$\left(\frac{DA}{Dt}\right)^a = \frac{\partial A^a}{\partial t} + A^a{}_{,b} u^b \qquad \text{(A1.21)}$$

will give the rate of change following the motion of the fluid of any vector field $\underline{A}$, while a similar formula applies to a scalar field

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x^b} u^b \, . \qquad \text{(A1.22)}$$

In *steady motion* the acceleration becomes

$$f^a = u^a{}_{,b} u^b$$

and the rate of change of entropy following the motion of the fluid becomes

$$u^a \frac{\partial S}{\partial x^a} \, .$$

The divergence of a vector can, by analogy with the more familiar orthogonal case, be seen to be $A^a{}_{,a}$.

Using these results, a straightforward application of the fundamental physical laws yields the equations of motion of a fluid.

For steady, non-viscous flow we have

1. Constant entropy along streamline

$$\frac{\partial S}{\partial x^a} u^a = 0 \, .$$

2 and 3. Equations of momentum

$$g_{bc} u^a u^c{}_{,a} = -\frac{1}{\rho} \frac{\partial p}{\partial x^b}$$

$$= -\frac{a^2}{\rho} \frac{\partial \rho}{\partial x^b} - \frac{1}{\rho} \frac{\partial p}{\partial S} \frac{\partial S}{\partial x^b}, \quad b = \alpha \text{ or } \beta \, .$$

4. Equation of continuity

$$\operatorname{div}(\rho w) = (\rho u^a)_{,a} = u^a \frac{\partial \rho}{\partial x^a} + \rho u^a_{\ ,a} = 0 .$$

But

$$\frac{\partial}{\partial x^a}(\rho u^a) + \rho u^b \Gamma^a_{ab} = \frac{\partial}{\partial x^a}(\rho u^a) + \rho u^a \Gamma^b_{ab}$$

$$= \frac{\partial}{\partial x^a}(\rho u^a) + \rho u^a \frac{1}{2g}\frac{\partial g}{\partial x^a}$$

$$= g^{-1/2}\frac{\partial}{\partial x^a}(\rho g^{1/2} u^a) .$$

Thus equation (4) is equivalent to $\frac{\partial}{\partial x^a}(\rho g^{1/2} u^a) = 0$.

Since the notation of the tensor calculus remains unaltered however many variables are involved, all results in this appendix that are expressed purely in tensor notation will be found to apply equally well to three dimensions as in two. Such results are the expression for the square of the interval *ds*, the co-variant derivative and all the equations of motion (1) to (4). Proofs of these and the foregoing results can be found in Refs. **3**, **4** and **5** or are immediate results of work found there.

APPENDIX II

Proof of Some Results used in the Text

A2.1. $v_{n\alpha} = u^{\alpha}/(g^{\alpha\alpha})^{1/2}$.

Write
$$w^2 = g^{ab}u_a u_b$$
$$= g^{\alpha\alpha}u_\alpha^{\ 2} + 2g^{\alpha\beta}u_\alpha u_\beta + 2g^{\alpha\gamma}u_\alpha u_\gamma + g^{lm}u_l u_m$$
$$= \frac{1}{g^{\alpha\alpha}}(u^\alpha)^2 + h^{lm}u_l u_m, \qquad \text{(A2.1)}$$
where h is a tensor derived from g, and l, $m \neq \alpha$.

Let, now U component of velocity normal to the β, γ-plane,
V component of velocity parallel to the β, γ-plane

so that
$$\begin{cases} w^2 = U^2 + V^2, \\ V^\alpha = 0, \text{ hence } u^\alpha = U^\alpha, \\ U_\beta = U_\gamma = 0, \text{ hence } u_\beta = V_\beta, \quad u_\gamma = V_\gamma. \end{cases}$$

Thus we may write equation (A2.1)
$$w^2 = \frac{1}{g^{\alpha\alpha}}(U^\alpha)^2 + h^{lm}V_l V_m. \qquad \text{(A2.2)}$$

We obtain $v_{n\alpha}$ the normal component of w, by putting $V = 0$ in equation (A2.2), whence
$$v_{n\alpha}^{\ 2} = \frac{1}{g^{\alpha\alpha}}(U^\alpha)^2 = \frac{(u^\alpha)^2}{g^{\alpha\alpha}}.$$

An alternative, more geometrical proof of this, is as follows.

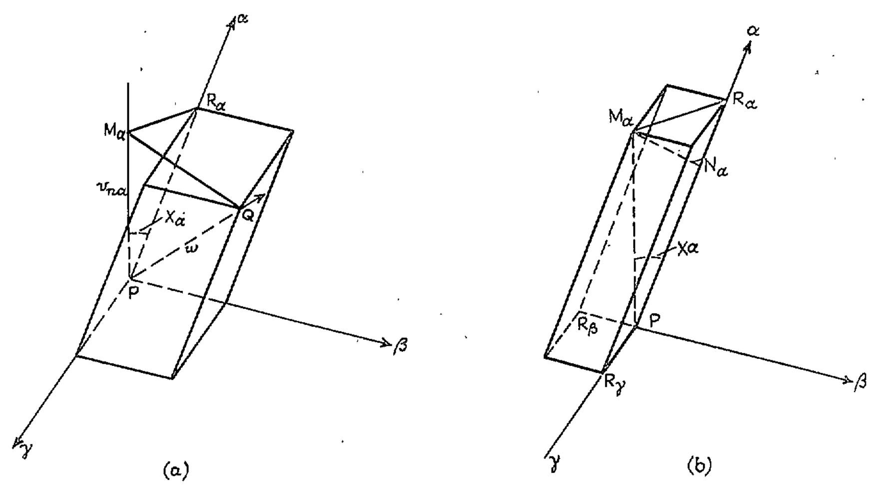

Fig. 2.

(All dimensions in Fig. 2 may be considered as infinitesimal and so all lines and surfaces may be considered as straight lines and planes.)

Let PQ (Fig. 2a) be the vector velocity $\underset{\sim}{u}$, let PM_α be its projection on the normal to the α-surface and let the α-surface through Q cut the α-line at R_α. Then

$$PM_\alpha = v_{n\alpha}$$

and if PM_α is considered as a vector A,

$$\left.\begin{aligned} PR_\alpha &= (g_{\alpha\alpha})^{1/2} u^\alpha \\ &= (g_{\alpha\alpha})^{1/2} A^\alpha \end{aligned}\right\} . \qquad (1)$$

Since $\underline{A}$ is normal to the α-surface

$$A_\beta = A_\gamma = 0$$

and so

$$A^\alpha = g^{\alpha\alpha} A_\alpha . \qquad (2)$$

If χ_α is the angle $R_\alpha PM$, and N_α is the foot of the perpendicular from M_α on PR_α, (Fig. 2*b*),

$$\begin{aligned} PN_\alpha &= (g_{\alpha\alpha})^{-1/2} A_\alpha \\ &= PM_\alpha \cos \chi_\alpha \\ &= PR^\alpha \cos^2 \chi_\alpha \end{aligned}$$

and so by equation (2)

$$\sec^2 \chi_\alpha = g_{\alpha\alpha} g^{\alpha\alpha}$$

thence

$$\begin{aligned} v_{n\alpha} &= (g_{\alpha\alpha})^{1/2} u^\alpha \cos \chi_\alpha \\ &= (g^{\alpha\alpha})^{-1/2} u^\alpha . \end{aligned}$$

A2.2. $\underline{w^2 \dfrac{\partial \theta}{\partial x^\beta} = G^{1/2}(u^\beta u^\alpha{}_{,\beta} - u^\alpha u^\beta{}_{,\beta})}$, *(in two dimensions)* .

Let p and q be any two vectors, then it may be shown that $(p^\beta q^\alpha - p^\alpha q^\beta)$ is an invariant with respect to different sets of axes. If we take axes along p, q so that $p^\beta = q^\alpha = 0$, then

$$\begin{aligned} G^{1/2}(p^\beta q^\alpha - p^\alpha q^\beta) &= - G^{1/2} p^\alpha q^\beta \\ &= \frac{G^{1/2}}{g_{\alpha\alpha}{}^{1/2} g_{\beta\beta}{}^{1/2}} \; |p| \; |q| \\ &= - |p| \; |q| \sin \theta' \end{aligned}$$

where θ' is the angle between the two vectors.

Now, let

$$p = u^b$$

and

$$q^b = u^b + \delta u^b = u^b + u^b{}_{,c}\, \delta x^c$$

so that q^b is the velocity vector at a point displaced a distance δx^c. If θ is the stream direction, $\theta' = \delta\theta$ and $\sin \theta' = \dfrac{\partial \theta}{\partial x^b} \delta x^b$. Then

$$\begin{aligned} G^{1/2}(p^\beta q^\alpha - p^\alpha q^\beta) &= G^{1/2}(u^\beta u^\beta{}_{,b} - u^\alpha u^\beta{}_{,b})\, \delta x^b \\ &= - w^2 \frac{\partial \theta}{\partial x^b} \, \delta x^b \end{aligned}$$

and

$$G^{1/2}(u^\beta u^\alpha{}_{,\beta} - u^\alpha u^\beta{}_{,\beta}) = w^2 \frac{\partial \theta}{\partial x^\beta} .$$

A2.3. $u_\beta^2 = g_{\beta\beta}(w^2 - a^2)$, *along a characteristic line in two dimensions*

$$u_\beta^2 = g_{\beta\beta}^2(u^\beta)^2 + 2g_{\beta\alpha}g_{\beta\beta}u^\alpha u^\beta + g_{\beta\alpha}^2(u^\alpha)^2$$
$$= g_{\beta\beta}\{g_{\beta\beta}(u^\beta)^2 + 2g_{\alpha\beta}u^\alpha u^\beta + g_{\alpha\alpha}(u^\alpha)^2\} + (g_{\alpha\beta}^2 - g_{\alpha\alpha}g_{\beta\beta})(u^\alpha)^2$$
$$= g_{\beta\beta}w^2 - G(u^\alpha)^2$$
$$= g_{\beta\beta}(w^2 - a^2) \text{ since } G(u^\alpha)^2 = g_\beta^\alpha a^2.$$

A2.4. $\Gamma^c_{\gamma\gamma} = g_{\gamma\gamma}R^c + g^c_\gamma \dfrac{1}{(g_{\gamma\gamma})^{1/2}} \dfrac{\partial (g_{\gamma\gamma})^{1/2}}{\partial x^\gamma}$.

The geodesic curvature of a curve is given in magnitude and direction by the relation

$$R^b = \frac{\delta}{\delta s}\left(\frac{dx^r}{ds}\right)$$

or

$$R^b = \frac{d^2(x^b)}{ds^2} + \Gamma^b_{cd}\frac{dx^c}{ds}\frac{dx^d}{ds}.$$

Thus the geodesic curvature of the line $dx^\alpha = dx^\beta = 0$ will be given by

$$R^b = g^b_\gamma \frac{1}{(g_{\gamma\gamma})^{1/2}}\frac{\partial}{\partial x^\gamma}\left(\frac{1}{(g_{\gamma\gamma})^{1/2}}\right) + \frac{1}{g_{\gamma\gamma}}\Gamma^b_{\gamma\gamma}$$

whence

$$\Gamma^b_{\gamma\gamma} = g_{\gamma\gamma}R^b + g^b_\gamma \frac{1}{(g_{\gamma\gamma})^{1/2}}\frac{\partial (g_{\gamma\gamma})^{1/2}}{\partial x^\gamma}.$$

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