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Summary.—The design of two-dimensional converging channels is considered, with special reference to (i) the lengths of the channels and (ii) the occurrence or absence of unfavourable velocity gradients at the walls. It is shown that it is not possible to have a short channel unless the velocity at the wall decreases at the beginning (the upstream end) of the channel; and it is further shown how a series of channels may be designed of decreasing lengths with increasingly unfavourable velocity gradients at the walls.

Introduction.—In recent publications[1,2,3] it has been shown that it is possible to design numerically straight contracting passages of circular section, such as may occur immediately upstream of the 'working section' of a wind-tunnel, in such a way that the velocity along the wall is continually increasing, or a negative velocity gradient, if one occurs, does not exceed a certain specified amount. The aim of the designs is to avoid boundary-layer separation; the calculations are carried out on potential theory, and, if the design aim is achieved, such a theory should provide a good guide, the only modification necessary in the theoretical results being a comparatively small allowance for the displacement thickness of the boundary layer.

Contractions so designed, however, are all rather long. It is clearly desirable to be able to design as short a contraction as possible. Three-dimensional motion, even potential motion symmetrical about an axis, does not lend itself readily to general mathematical analysis, and the authors who have considered these designs have all resorted to the computation of special cases at an early stage in their work. If, however, we wish to gain any insight into the reasons for the considerable length of these contractions, and into the possibilities of shortening them without too great a danger of boundary-layer separation, it is clear that some quite general mathematical analysis would be more valuable than arithmetical solutions for a number of special cases. Such general analysis is easily carried out by standard methods[4] for the case of two-dimensional flow, and the general analysis of the two-dimensional case may provide at least valuable hints on the points to be considered in the design of actual three-dimensional contractions. Whether it can provide more, in the sense that in practice there can be devised some simple rule for an approximate connection between the two- and three-dimensional cases, at any rate sufficiently far upstream and sufficiently far downstream, remains to be considered. It should be borne in mind that in actual wind-tunnels the sections of the contracting passages are, in many cases, more likely to be square, rectangular or octagonal, than circular; in some cases, at any rate, calculations carried out for the case of axial symmetry would have to be applied with considerable caution; and there will probably even be cases, for example where the contraction is carried out in one direction only from a square to a narrow rectangular section, where two-dimensional calculations would be at least as applicable as calculations for the axially-symmetrical case.

Analysis for the Two-Dimensional Case.—We are concerned only with a symmetrical channel. With a usual notation

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \; v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x},$$

$$w = \phi + i\psi, \; \frac{dw}{dz} = u - iv = q\,\mathrm{e}^{-i\vartheta}$$

$$\Omega = \log \frac{dw}{dz} = \log q - i\vartheta$$

The plane of the motion is taken as the z-plane, which is shown in Fig. 1, where a, b, V, U are defined; clearly

$$aV = bU$$

Take $\psi = 0$ on A′B′C′, $\psi = aV$ on ABC. Along the middle line of the channel $\psi = \frac{1}{2}\,aV$.

ϕ goes from $-\infty$ at A and A′ to $+\infty$ at C and C′. In the z-plane the middle line of the channel is taken as axis of x, and the origin at the point O on that line where $\phi = 0$. In Fig. 1, B, B′ are the points on the channel walls where $\phi = 0$. The corresponding region in the w-plane is the infinite strip shown in Fig. 2, which is conformally transformed into the upper half of the t-plane by the transformation

$$t = \exp\,(\pi w/aV)$$

the boundaries corresponding as shown in Fig. 3. The middle line of the channel corresponds with the positive imaginary axis in the t-plane, and B′OB ($\phi = 0$) with the upper half of the unit circle $|\,t\,| = 1$.

The upper half of the t-plane is transformed into the inside of unit circle, with the origin at the centre, in the ζ-plane, by the transformation

$$\zeta = \frac{t - i}{t + i}$$

$$\left(\text{so that} \qquad t = i\,\frac{1 + \zeta}{1 - \zeta}\right),$$

the boundaries corresponding as shown (Fig. 4). The origins in the z- and ζ-planes correspond; the middle line of the channel corresponds with the diameter AOC (or A′OC′) along the real axis in the ζ-plane and BOB′ ($\phi = 0$) with the diameter along the imaginary axis. Then

$$dw = \frac{aV}{\pi}\,\frac{dt}{t} = \frac{2aV}{\pi}\,\frac{d\zeta}{1 - \zeta^2}.$$

But

$$dz = \mathrm{e}^{-\Omega}\,dw.$$

Hence

$$dz = \frac{2aV}{\pi}\,\frac{\mathrm{e}^{-\Omega}}{1 - \zeta^2}\,d\zeta, \qquad \ldots \qquad (1)$$

and for the point on the boundary corresponding with $\zeta = \exp\,(i\theta)$

$$z = \frac{2aV}{\pi}\int_0^{\exp(i\theta)} \frac{\mathrm{e}^{-\Omega}}{1 - \zeta^2}\,d\zeta. \qquad \ldots \qquad (2)$$

We thus have a formula for the co-ordinates of the points of the channel boundary for any assumed relation between Ω and ζ. Actual computations near the middle of the channel are most easily carried out by the method used by Lighthill, by integrating from B along the circle. In the integrand in equation (2) put $\zeta = \exp i\,(\frac{1}{2}\pi - \alpha)$, so

$$\frac{d\zeta}{1-\zeta^2} = \frac{d\alpha}{2\cos\alpha}$$

and

$$dz = \frac{aV}{\pi}\,\frac{\exp(i\vartheta)\;d\alpha}{q\cos\alpha},$$

$$x - x_B = \frac{aV}{\pi}\int_0^{\pi/2-\theta}\frac{\cos\vartheta\;d\alpha}{q\cos\alpha}, \quad y - y_B = \frac{aV}{\pi}\int_0^{\pi/2-\theta}\frac{\sin\vartheta\;d\alpha}{q\cos\alpha},$$

where $\log q$ and $-\vartheta$ are the real and imaginary parts of the function Ω of ζ for $\zeta = \exp(\frac{1}{2}\pi i - \alpha i)$, and ϑ is negative along ABC.

For the present, however, we are more concerned with the rapidity of the approach of y to its final values a and b, *i.e.*, with asymptotic expressions for y in terms of x as ζ tends to 1 and -1 along the circle.

To join computed values with these asymptotic expressions, it may be useful to note that

$$x_B = -\frac{2aV}{\pi}\int_0^1\frac{\sin\vartheta\;d\eta}{q(1+\eta^2)}, \quad y_B = \frac{2aV}{\pi}\int_0^1\frac{\cos\vartheta\;d\eta}{q(1+\eta^2)},$$

where $\log q$ and $-\vartheta$ are the real and imaginary parts of Ω for $\zeta = i\eta$.

We remark that $\vartheta = 0$ along the middle line of the channel, which corresponds with the real axis in the ζ-plane, so Ω is real for real ζ.

When $\zeta = 1$, at C and C′, $q = U$, $\vartheta = 0$, $\Omega = \log U$; when $\zeta = -1$, at A and A′, $q = V$, $\vartheta = 0$, $\Omega = \log V$. Along ABC ϑ is negative and the imaginary part of Ω is positive. Hence if we write

$$\Omega - \tfrac{1}{2}\log UV = KZ, \qquad \ldots \qquad (3)$$

where

$$K = \tfrac{1}{2}\log\frac{U}{V} = \tfrac{1}{2}\log\frac{a}{b} \qquad \ldots \qquad (4)$$

and

$$Z = f(\zeta), \qquad \ldots \qquad (5)$$

then f is a real function,

$$f(1) = 1, \quad f(-1) = -1, \qquad \ldots \qquad (6)$$

and the curve in the Z-plane which corresponds with the unit circle in the ζ-plane passes through $Z = \pm 1$, and is symmetrical about the real axis of Z. If the velocity q is continually to increase along the walls, this curve, which has tangents parallel to the imaginary axis at $Z = \pm 1$, must have such ‘ vertical ’ tangents at no other points: the real part of Z is a maximum at 1 and a minimum at -1, and must have no other stationary values on the curve (Fig. 5).

It will usually be advisable to restrict $|\vartheta|$ to be not greater than $\pi/2$: if larger values occur we have a re-entrant channel, which, without suction, seems undesirable not only on the grounds of difficulty of manufacture but because in such a channel boundary-layer effects near the re-entrant portion are difficult to predict, and may cause an appreciable departure from potential flow in the sense that, if the re-entrant portion lies entirely in the boundary layer (Fig. 6), the effective shape of the channel will be very different from the design shape.

When we substitute the value of Ω from equation (3) into equation (2), we find that

$$z = \frac{2a}{\pi}\left(\frac{V}{U}\right)^{1/2}\int_0^{\exp(i\theta)} \frac{e^{-KZ}\, d\zeta}{1-\zeta^2}$$

$$= \frac{2\,(ab)^{1/2}}{\pi}\int_0^{\exp(i\theta)} \frac{\exp\,[-K\, f\,(\zeta)]\, d\zeta}{1-\zeta^2} \qquad \text{(7)}$$

Suppose now that $0<\theta<\pi/2$, so that equation (7) gives the co-ordinates of any point on BC. For the point on AB corresponding with $\zeta = \exp\,[i\,(\pi - \theta)] = -\exp\,(-i\theta)$, we have, by changing the sign of ζ in the integral, that

$$z = -\frac{2\,(ab)^{1/2}}{\pi}\int_0^{\exp(-i\theta)} \frac{\exp\,[K\,F\,(\zeta)]\, d\zeta}{1-\zeta^2}, \qquad \text{(8)}$$

where

$$F\,(\zeta) = -f\,(-\,\zeta). \qquad \text{(9)}$$

If f is an odd function of ζ, *i.e.*, if the curve in the Z-plane is symmetrical about the imaginary axis as well as about the real axis, f and F are the same function. In such a case, the expression for $z/(ab)^{1/2}$ at the point on AB corresponding with $(\pi - \theta)$ is minus the conjugate of the expression for $z/(ab)^{1/2}$ at the point on BC corresponding with θ, with the sign of K changed; *i.e.*, the expressions for $x/(ab)^{1/2}$, $y/(ab)^{1/2}$ on AB are obtained from the corresponding expressions on BC by changing the sign of K (and changing the sign of x). If f is not an odd function of ζ we must also change f into F, where F is given by equation (9).

These results may also be obtained from physical reasoning, since the potential motion along the channel is reversible.

The simplest example in which q increases continually along the walls is that worked out by Cheers, in which the curve in the Z-plane is a circle, so that

$$f\,(\zeta) = \zeta. \qquad \text{(10)}$$

In this case the maximum value of $|\,\vartheta\,|$ is K, and if this maximum value is not to exceed $\pi/2$, the contraction ratio a/b $(= e^{2K})$ must not exceed 23·14. On the other hand, for values of K less than $\pi/2$, shorter channels may be obtained by increasing the maximum value of $|\,\vartheta\,|$. Thus for $K < \pi/2$ we want to be able to take, as the curves in the Z-plane, a series of ovals with maximum ordinates greater than 1, and for $K > \pi/2$ a series of ovals with maximum ordinates less than 1. (This latter series will necessarily lead to longer channels than the circle does, but not to re-entrant channels, such as will be obtained from the circle.) A suitable transformation for the former series is

$$Z = f(\zeta) = \frac{\tan^{-1}\alpha\,\zeta}{\tan^{-1}\alpha} = \frac{1}{2i\,\tan^{-1}\alpha}\log\frac{1+i\,\alpha\,\zeta}{1-i\,\alpha\,\zeta} \qquad \text{(11)}$$

where α is real and $0 \leqslant \alpha < 1$. Since

$$f'\,(\zeta) = \frac{\alpha}{\tan^{-1}\alpha}\;\frac{1}{1+\alpha^2\zeta^2},$$

$f'\,(\zeta)$ has no zeros or singularities inside the unit circle $|\,\zeta\,| = 1$, and the transformation is conformal. On the boundary $\zeta = e^{i\theta}$,

$$\log q = \tfrac{1}{2} \log U V + \frac{K}{2 \tan^{-1} \alpha} \tan^{-1} \frac{2 \alpha \cos \theta}{1 - \alpha^2},$$

$$-\vartheta = \frac{K}{4 \tan^{-1} \alpha} \log \frac{1 + 2 \alpha \sin \theta + \alpha^2}{1 - 2 \alpha \sin \theta + \alpha^2}.$$

q continually increases along ABC. The oval in the Z-plane is, in fact, symmetrical about both axes. The maximum value of $| \vartheta |$ is

$$\frac{K}{2 \tan^{-1} \alpha} \log \frac{1 + \alpha}{1 - \alpha} = K \frac{\tanh^{-1} \alpha}{\tan^{-1} \alpha}$$

and so, by a proper choice of α, may have any value greater than K (corresponding with $\alpha = 0$). If we abandon the restriction $| \vartheta | \leqslant \pi$, we see that as $\alpha \rightarrow 1 - 0$, $| \vartheta |_{max} \rightarrow \infty$; as $\alpha \rightarrow 1 - 0$, the relevant region in the Z-plane becomes an infinite strip bounded by the parallels to the imaginary axis through $Z = \pm 1$ (Fig. 7); the velocity on AB becomes constant and equal to V, and on BC becomes constant and equal to U. We thus recover the case considered (as an expanding channel) by Hughes[5]. For $\alpha = 0$ we have again the case of the circle, and as α increases from 0 to 1 we have a sequence of ovals between the circle and the infinite strip.

It is unlikely that contraction ratios greater than 23·14 will be required, so the second series of ovals mentioned above will probably not be needed. If it is, it can be obtained from

$$f(\zeta) = \frac{\tanh^{-1} \alpha \zeta}{\tanh^{-1} \alpha} = \frac{1}{2 \tanh^{-1} \alpha} \log \frac{1 + \alpha \zeta}{1 - a \zeta}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

with α real and $0 < \alpha < 1$.

Here, too, $f'(\zeta)$ has no zeros or singularities for $| \zeta | \leqslant 1$, and the transformation is conformal. $\alpha = 0$ again corresponds with a circle in the Z-plane. On the boundary $\zeta = e^{i\theta}$,

$$\log q = \tfrac{1}{2} \log UV + \frac{K}{4 \tanh^{-1} \alpha} \log \frac{1 + 2\alpha \cos \theta + \alpha^2}{1 - 2\alpha \cos \theta + \alpha^2},$$

$$-\vartheta = \frac{K}{2 \tanh^{-1} \alpha} \tan^{-1} \frac{2 \alpha \sin \theta}{1 - \alpha^2}.$$

The oval in the Z-plane is again symmetrical about both its axes; q continually increases along ABC and the maximum value of $| \vartheta |$ is

$$K \frac{\tan^{-1} \alpha}{\tanh^{-1} \alpha},$$

and so, by a proper choice of α, may have any value less than K.

Simpler transformations than equations (11) and (12) may, of course, be found in special cases, but they are all subject to somewhat narrow restrictions and are not by any means as general as equations (11) and (12). For example, symmetrical ovals with maximum ordinates greater than 1 may be obtained by putting

$$f(\zeta) = \frac{\zeta - \alpha \zeta^3}{1 - \alpha},$$

with α positive, but $\alpha < \frac{1}{3}$ is necessary if the transformation is to be conformal, and $\alpha \leqslant \frac{1}{9}$ if q is continually to increase along ABC. The maximum value of $| \vartheta |$ is

$$K \frac{1 + \alpha}{1 - \alpha}.$$

Similarly, asymmetrical ovals with maximum ordinates greater than 1 are obtained by putting

$$f(\zeta) = \zeta - \tfrac{1}{2}\alpha(1 - \zeta^2)$$

with α positive. The transformation is conformal for $\alpha < 1$, but $\alpha \leqslant \frac{1}{2}$ is necessary if q is continually to increase along ABC. For $\alpha = \frac{1}{2}$, the maximum value of $|\vartheta|$ is $1 \cdot 101K$.

Asymptotic Expressions for y for Large Values of $|x|$.—To study the approach of the ordinate y of a point on the channel wall to its final values as $x \to \infty$ we write equation (7) in the following form for $x > 0$.

$$z = \frac{(ab)^{1/2}}{\pi}\left\{-e^{-K}\log(1-e^{i\theta}) + \int_0^{\exp(i\theta)} \frac{2\exp[-Kf(\zeta)] - e^{-K}(1+\zeta)}{1-\zeta^2}\,d\zeta\right\}$$

$$= \frac{b}{\pi}\left[-\log(1-e^{i\theta}) + \int_0^{\exp(i\theta)} \frac{2\exp\{K[1-f(\zeta)]\}-1-\zeta}{1-\zeta^2}\,d\zeta\right]$$

With

$$D = \int_0^1 \frac{2\exp\{K[1-f(t)]\}-1-t}{1-t^2}\,dt, \qquad \ldots \quad (13)$$

we have

$$\frac{\pi z}{b} = -\log(1-e^{i\theta}) + D + \int_1^{\exp(i\theta)} \frac{2\exp\{K[1-f(\zeta)]\}-1-\zeta}{1-\zeta^2}\,d\zeta, \qquad \ldots \quad (14)$$

Similarly, for $x < 0$,

$$\frac{\pi z}{a} = \log(1-e^{-i\theta}) - E - \int_1^{\exp(-i\theta)} \frac{2\exp\{-K[1+f(-\zeta)]\}-1-\zeta}{1-\zeta^2}\,d\zeta, \qquad \ldots \quad (15)$$

where

$$E = \int_0^1 \frac{2\exp\{-K[1+f(-t)]\}-1-t}{1-t^2}\,dt \qquad \ldots \quad (16)$$

[See the remarks following equation (9).]

The logarithms are defined by

$$\log(1-e^{i\theta}) = -\int_0^{\exp(i\theta)} \frac{d\zeta}{1-\zeta}, \quad \log(1-e^{-i\theta}) = \int_0^{\exp[i(\pi-\theta)]} \frac{d\zeta}{1+\zeta},$$

so on ABC

$$\log(1-e^{i\theta}) = \log(2\sin\tfrac{1}{2}\theta) - i(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta),$$

$$\log(1-e^{-i\theta}) = \log(2\sin\tfrac{1}{2}\theta) + i(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta).$$

Now suppose to start with that near $\zeta = \pm 1$, $f(\zeta)$ is expansible in the forms

$$f(\zeta) = 1 + a_1(\zeta-1) + a_2(\zeta-1)^2 + \ldots,$$

$$f(\zeta) = -1 + b_1(\zeta+1) + b_2(\zeta+1)^2 + \ldots,$$

where $a_1 = f'(1)$, $a_2 = \dfrac{1}{2!}f''(1), \ldots, b_1 = f'(-1)$, $b_2 = \dfrac{1}{2!}f''(-1), \ldots,$

the dashes denoting derivatives.

In the integrand in equation (14) for large values of x, *i.e.*, for small values of θ, write

$$\zeta = e^{i\alpha}$$

so

$$\frac{d\zeta}{1-\zeta^2} = -\frac{d\alpha}{2\sin\alpha},$$

$$1-\zeta = -i\alpha + O(\alpha^2),$$

$$1-f(\zeta) = a_1(1-\zeta) - a_2(1-\zeta)^2 + \ldots = -i a_1 \alpha + O(\alpha^2),$$

$$2\exp\{K[1-f(\zeta)]\} - 1 - \zeta = -i\alpha - 2iKa_1\alpha + O(\alpha^2),$$

and the integral in equation (14) is

$$\int_0^\theta [\tfrac{1}{2} + iKa_1 + O(\alpha)]\, d\alpha = \frac{i\theta}{2} + iKa_1\theta + O(\theta^2).$$

The term in θ^2 is, in fact, real and

$$\frac{\pi x}{b} - D = -\log(2\sin\tfrac{1}{2}\theta) + O(\theta^2) = -\log\theta + O(\theta^2),$$

$$\frac{\pi y}{b} - \frac{\pi}{2} = -\tfrac{1}{2}\theta + \tfrac{1}{2}\theta + Ka_1\theta + O(\theta^3).$$

Hence $\theta = e^D e^{-\pi x/b} + O(e^{-3\pi x/b})$

and $y - \frac{1}{2}b = \dfrac{Kb}{\pi} f'(1)\, e^D e^{-\pi x/b} + O(e^{-3\pi x/b})$. .. .. (17)

In exactly the same way from equation (15), or by the remarks following equation (9), for large negative values of x,

$$\frac{1}{2}a - y = \frac{Ka}{\pi} f'(-1)\, e^E e^{-\pi|x|/a} + O(e^{-3\pi|x|/a}) \quad \ldots \quad (18)$$

The approach of y to its final value is, therefore, as was to be expected, exponential: but whereas $y - \frac{1}{2}b$ becomes very small downstream as soon as πx is fairly large compared with b, $\frac{1}{2}a - y$ does not become very small upstream until $\pi|x|$ is fairly large compared with a, and a is large compared with b, the ratio a/b being the contraction ratio. We see that, if there is to be no reverse velocity gradient at the walls, the beginning of the contraction will appear to be very gradual compared with those designed by current practice. Measured from the section where, say, the velocity on the axis is the geometric mean of the exit and entry velocities, or from any section near that one, the upstream length will be greater than the downstream length roughly in the ratio of the entry to the exit widths, the exact circumstances depending slightly on the constants in equations (17) and (18), whose values will be considered later for a few typical cases.

The only way in which these conclusions may be modified is if $f'(-1) = 0$. If $f'(-1) = 0$, we may suppose that, near $\zeta = -1$,

$$Z = f(\zeta) = -1 + (1+\zeta)^{1+m}\, p(\zeta), \quad m > 0, \quad p(-1) \neq 0.$$

Near $\zeta = -1$, $\arg[f(\zeta) + 1] = (1+m)\arg(1+\zeta)$ ($p(-1)$ is real), and as $\arg(1+\zeta)$ changes by π as we go round $\zeta = -1$ along the inside of the circle in the z-plane, $\arg(1+Z)$ changes by $\pi(1+m)$. The curve in the Z-plane is therefore re-entrant, as shown in Fig. 8. If $f(\zeta)$ is analytic at $\zeta = -1$, with $m = 1$, we have a re-entrant cusp, but in any case we see that *we cannot have a rapid approach of y to its final value upstream unless the velocity at the wall decreases at the beginning of the channel.*

We shall, therefore, consider the approach of y to its final value upstream in some cases in which q decreases at the beginning, as in Fig. 8. Before passing to the case of a re-entrant cusp ($m = 1$), we may briefly set out the circumstances for a fractional value of m. We therefore put, near $\zeta = -1$, $f(\zeta) = -1 + (1 + \zeta)^{1+m}\{b_1 + b_2(1 + \zeta) + \ldots\}$. $(0 < m < 1, \quad b_1 \neq 0)$. In the integrand of the integral in equation (15) we now put $\zeta = e^{-i\alpha}$, and expand the integrand in powers of α, as before. It will be found that

$$\frac{d\zeta}{1 - \zeta^2} = -\frac{d\alpha}{2 \sin \alpha}$$

and

$$\frac{2 \exp\{-K[1 + f(-\zeta)]\} - 1 - \zeta}{2 \sin \alpha} = \frac{i}{2} - iKb_1\, e^{\frac{1}{2}m\pi i}\, \alpha^m + \frac{\alpha}{4} + O(\alpha^{1+m}),$$

whence

$$\frac{\pi z}{a} = \log(2 \sin \tfrac{1}{2}\, \theta) + i(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta) - E + \tfrac{1}{2} i\theta - \frac{i\,K\,b_1}{1+m} e^{\frac{1}{2}m\pi i}\, \theta^{1+m} + \frac{\theta^2}{8} + O(\theta^{2+m}),$$

$$-\frac{\pi\,|x|}{a} = \log \theta - E + O(\theta^{1+m}),$$

so that

$$\theta = e^E\, e^{-\pi\,|x|\,/a} + 0\,(e^{-(2+m)\pi\,|x|\,/a}),$$

and

$$\frac{\pi y}{a} = \frac{\pi}{2} - \frac{K\,b_1}{1+m} \cos(\tfrac{1}{2}m\pi)\, \theta^{1+m} + O(\theta^{2+m}),$$

so

$$\tfrac{1}{2}\,a - y = \frac{K\,a}{\pi\,(1+m)}\, b_1 \cos(\tfrac{1}{2}m\,\pi) e^{(1+m)E}\, e^{-(1+m)\pi\,|x|\,/a} + O(e^{-(2+m)\pi\,|x|\,/a}). \quad \ldots \quad (19)$$

The approach of y to its final value $\frac{1}{2}a$ is certainly more rapid than before, because of the factor $1 + m$ in the exponential. The gain, however, is not very large. Thus if $m = \frac{1}{2}$, the exponential term is now as small as that in equation (18) when $|x|$ has two-thirds of its previous value. Thus, although such cases may be fairly simply worked out in detail, for example by putting

$$f(\zeta) = 2^{-m}(1 + \zeta)^{1+m} - 1, \quad \ldots \quad (20)$$

we shall proceed at once to the case of a re-entrant cusp.

This case is studied in the same way by putting near $\zeta = -1$,

$$f(\zeta) = -1 + b_2\,(1 + \zeta)^2 + b_3\,(1 + \zeta)^3 + \ldots.$$

Proceeding in exactly the same way, we find that

$$\frac{\pi z}{a} = \log(2 \sin \tfrac{1}{2}\, \theta) + i(\tfrac{1}{2}\pi - \tfrac{1}{2}\theta) - E + \tfrac{1}{2}\, i\, \theta + \frac{\theta^2}{8}(1 + 4\,K\,b_2)$$
$$- \frac{i\,K\,\theta^3}{3}(b_2 - b_3) + O(\theta^4),$$

whence

$$\theta = e^E\, e^{-\pi\,|x|\,/a} + O(e^{-3\pi\,|x|\,/a}),$$

$$\tfrac{1}{2}a - y = \frac{K a}{3\pi}(b_2 - b_3)\, e^{3E}\, e^{-3\pi\,|x|\,/a}) + O(e^{-5\pi\,|x|\,/a}) \quad \ldots \quad (21)$$

The approach of the exponential factor to zero is now three times as rapid.

The simplest example of this case is obtained by putting $b_2 = \frac{1}{2}$, $b_3 = b_4 = \ldots = 0$,

$$f(\zeta) = \tfrac{1}{2}\zeta^2 + \zeta - \tfrac{1}{2}, \qquad (22)$$

for which

$$\log q = \tfrac{1}{2} \log UV + K(\tfrac{1}{2} \cos 2\theta + \cos \theta - \tfrac{1}{2}),$$
$$-\vartheta = K(\tfrac{1}{2} \sin 2\theta + \sin \theta).$$

$|\vartheta|$ is a maximum when $\theta = \frac{1}{3}\pi$; the maximum value is $\dfrac{3\sqrt{3}}{4} K$, which does not exceed $\frac{1}{2}\pi$ if

$$K \leqslant \frac{2\pi}{3\sqrt{3}} = 1\cdot208, \qquad \frac{a}{b} = \frac{U}{V} \leqslant 11\cdot2.$$

The minimum value of $\log q$, namely $\log V - \frac{1}{8} \log (U/V)$, occurs when $\theta = 2\pi/3$. For large values of $|x|$, *i.e.*, for values of θ near to π,

$$\log q = \log V - \tfrac{1}{2}\, e^{2E}\, e^{-2\pi |x|/a} + O(e^{-4\pi |x|/a}),$$

but this formula will not provide a good approximation all the way to $\theta = 2\pi/3$, and numerical computation will be necessary to determine the adverse gradient of q.

We note next that the coefficient of y in equation (21) vanishes if $b_2 = b_3$. If we carry out the analysis with this condition we find that

$$\tfrac{1}{2}a - y = \frac{K a}{5\pi} (b_2 - 2\, b_4 + b_5)\; e^{5E}\, e^{-5\pi |x|/a} + O\,(e^{-7\pi |x|/a}), \qquad (23)$$

and the approach of the exponential factor to zero is now five times as rapid as before.

The simplest example of this case is obtained by putting $b_2 = b_3 = \frac{1}{6}$, $b_4 = b_5 = \ldots = 0$,

$$f(\zeta) = \tfrac{1}{6}\,(\zeta^3 + 4\zeta^2 + 5\zeta - 4), \qquad (24)$$

for which

$$\log q = \tfrac{1}{2} \log UV + \frac{K}{6}\,(\cos 3\,\theta + 4 \cos 2\,\theta + 5 \cos \theta - 4),$$

$$-\vartheta = \frac{K}{6}\,(\sin 3\,\theta + 4 \sin 2\,\theta + 5 \sin \theta).$$

The minimum value of $\log q$ is now $\log V - 0\cdot18 \log (U/V)$ and occurs when θ is about 98 deg, so the unfavourable velocity gradient persists further along the channel than in the previous case, as was to be expected. The mimimum value of $|\vartheta|$, which occurs when $\cos \theta = 2/3$, is $(50\sqrt{5}/81)K$, and does not exceed $\frac{1}{2}\pi$ if

$$K \leqslant \frac{81\,\pi}{100\sqrt{5}} = 1\cdot14, \qquad \frac{a}{b} = \frac{U}{V} \leqslant 9\cdot74.$$

When the connections between $|x|$ and θ have been found, the preceding expressions for y are all easily found from

$$\frac{dy}{dx} = \tan \vartheta\,;$$

when $|x|$ is large, ϑ is small and dy/dx is approximately equal to ϑ; and $-\vartheta/K$ is the imaginary part of $f(\zeta)$. If $f(\zeta)$ is analytic at $\zeta = -1$, $d^n\vartheta/d\theta^n$ vanishes at $\zeta = -1$ when n is even. $f'(-1) = 0$ is the condition that $d\vartheta/d\theta$ should vanish at $\theta = \pi$; when this condition is satisfied, $b_2 = b_3$ is the condition that $d^3\vartheta/d\theta^3$ should vanish; when both these conditions are satisfied, the vanishing of the coefficient on the right in equation (23), namely $b_2 - 2b_4 + b_5 = 0$, is the condition that $d^5\vartheta/d\theta^5$ should vanish at $\theta = \pi$. At $\pi - \theta$, the expansion of ϑ in powers of θ therefore begins generally with a term in θ: if the first condition ($f'(-1) = 0$) is satisfied, the expansion begins with a term in θ^3: if the first two conditions are satisfied, with a term in θ^5, and so on. By making more and more of the odd derivatives of ϑ vanish at $\theta = \pi$, we thus obtain a series of channels, which, for practical purposes, become shorter and shorter at the expense of increasingly unfavourable velocity gradients at the walls.

If we were really designing a two-dimensional channel, the answer to the question which, if any, of the channels so far considered we would choose would depend on the purposes for which it (or rather the wind tunnel of which it may be presumed to be a part) was to be used, and on the space and facilities available for construction. It might, therefore, be of interest to work out details of channels designed, not only according to equation (10) as Cheers has done, but also according to equation (11) with a suitable α, and to equation (22).

Some Numerical Results.—Some rough numerical results are given in the table below. The values of D and E were found by numerical integration from equations (13) and (16).

No.	$f(\zeta)$	$K = 1\cdot2,\ a/b = 11\cdot02$			$K = 1\cdot5,\ a/b = 20\cdot09$		
		$\lvert\vartheta\rvert_{\max}$	D	E	$\lvert\vartheta\rvert_{\max}$	D	E
1	ζ	$1\cdot2$	$3\cdot11_4$	$-0\cdot54_1$	$1\cdot5$	$4\cdot06_6$	$-0\cdot75_3$
2	$\dfrac{\tan^{-1}\alpha\zeta}{\tan^{-1}\alpha}$, $\alpha = 0\cdot6200$, $\tan^{-1}\alpha = 0\cdot5550$	$1\cdot568$	$2\cdot91_8$	$-0\cdot45_2$		not calculated.	
3	$\dfrac{4}{\pi} = \tan^{-1}\zeta$	∞	$2\cdot70_6$	$-0\cdot35_6$	∞	$3\cdot48_8$	$-0\cdot53_9$
4	$\frac{1}{2}\zeta^2 + \zeta - \frac{1}{2}$	$1\cdot559$	$6\cdot06_8$	$+0\cdot29_6$	$1\cdot949$ $(> \frac{1}{2}\pi)$	$8\cdot83_0$	$+0\cdot21_5$
5	$(\zeta^3 + 4\zeta^2 + 5\zeta - 4)/6$	$1\cdot656$ $(> \frac{1}{2}\pi)$	$7\cdot93_3$	$+0\cdot45_5$		not calculated.	

All the channels so far considered are theoretically of infinite length, though practically the exponentially small slopes and differences of the ordinates from their final values become completely negligible for finite values of $|x|$. There is naturally a choice of definitions of the 'length' of each channel; for purposes of illustration we shall find the lengths between the nearest sections where the slopes do not exceed δ, when (1) $\delta = 0\cdot003$ (2) $\delta = 0\cdot03$. [For other values of δ, the 'lengths' are linear functions of $\log \delta$.]

The results are set out below, with the lengths l as multiples of the downstream breadth, b.

Values of l/b

No.	$\delta = 0\cdot003,\ K = 1\cdot2$	$\delta = 0\cdot003,\ K = 1\cdot5$	$\delta = 0\cdot03,\ K = 1\cdot2$	$\delta = 0\cdot03,\ K = 1\cdot5$
1	22·02	38·20	13·21	22·74
2	21·45		12·64	
3	20·80	36·94	12·00	21·48
4	12·02	18·88	8·60	13·25
5	10·02		7·67	

The only large saving in length clearly comes between channels numbers 3 and 4. The saving is all in the 'upstream' length, and, although the choice of section from which these lengths are measured is purely one of mathematical convenience, the results may be of some interest, and the 'upstream' and 'downstream' lengths (l_1 and l_2, respectively) are given as multiples of b in the table below. (For the first three channels, the lengths are measured from the section at which the velocity on the axis is the geometric mean of the entry and exit velocities; in channels numbers 4 and 5, from the sections at which the velocity on the axis is $U^{1/4} V^{3/4}$ and $U^{1/6} V^{5/6}$, respectively.)

No.	$\delta = 0{\cdot}003,\ K = 1{\cdot}2$		$\delta = 0{\cdot}003,\ K = 1{\cdot}5$		$\delta = 0{\cdot}03,\ K = 1{\cdot}2$		$\delta = 0{\cdot}03,\ K = 1{\cdot}5$	
	l_1/b	l_2/b	l_1/b	l_2/b	l_1/b	l_2/b	l_1/b	l_2/b
1	19·12	2·90	34·93	3·27	11·04	2·17	20·20	2·54
2	18·68	2·77			10·60	2·04		
3	18·18	2·62	34·00	2·94	10·11	1·89	19·27	2·21
4	7·23	4·79	13·14	5·74	4·54	4·06	8·24	5·01
5	4·54	5·48			2·93	4·74		

Channels of Finite Length.—As we have remarked, all the channels so far considered are theoretically (though not practically) of infinite length. Channels which are theoretically of finite length may be designed by assuming that $\vartheta = 0$ over finite intervals of θ near $\theta = 0$ and $\theta = \pi$ (or near $\theta = \pi$ only if we seek only a finite length in one direction). From any assumed of ϑ on the unit circle $|\zeta| = 1$, the imaginary part of $f(\zeta)$ is known on that circle, so $f(\zeta)$ may be found at all points in and on the circle by Poisson's integral. (Even if the analysis cannot easily be carried out, the real part of $f(\zeta)$ on the unit circle, and hence $\log q$, may be found numerically, and the values of the co-ordinates of points on the channel walls computed from the general formulae for $x - x_B$, $y - y_B$.) Channels so designed will be shorter, and will have greater unfavourable velocity gradients at the walls, than others considered in these notes.

The connection between x and θ, for large values of $|x|$, will be simlar in such cases to those previously found, but since the imaginary part of $f(\zeta)$ will vanish near $\theta = 0$ and near $\theta = \pi$, the asymptotic expressions for $y - \frac{1}{2}b$, $\frac{1}{2}a - y$ will be identically zero.

As an example, let us take $-\vartheta/K$ to be an odd function of θ, zero for $0 \leqslant \theta \leqslant \theta_1$ and for $\pi - \theta_1 \leqslant \theta \leqslant \pi$, and proportional to $\sin\theta - \sin\theta_1$ for $\theta_1 \leqslant \theta \leqslant \pi - \theta_1$, so that it is continuous. The factor of proportionality is found from the conditions that $f(\zeta)$, whose imaginary part is $-\vartheta/K$, is $+1$ when $\zeta = 1$ and -1 when $\zeta = -1$. From Poisson's integral it is found that

$$\{\pi - 2\theta_1 + 2\sin\theta_1 \log\tan\tfrac{1}{2}\theta_1\}\ f(\zeta) = \frac{1-\zeta^2}{\zeta}\tan^{-1}\left(\frac{\zeta^2 \sin 2\theta_1}{1 - \zeta^2\cos 2\theta_1}\right)$$

$$+ \sin\theta_1 \log\frac{1 - 2\zeta\cos\theta_1 + \zeta^2}{1 + 2\zeta\cos\theta_1 + \zeta^2} + (\pi - 2\theta_1)\,\zeta,$$

and the first term on the right is the same as

$$\frac{1-\zeta^2}{2i\zeta}\log\frac{1 - \zeta^2\exp(-2i\theta_1)}{1 - \zeta^2\exp(2i\theta_1)}$$

The arguments of the various factors $1 \pm \zeta \exp(\pm i\theta_1)$ are all determined at all points inside and on the circle $|\zeta| = 1$, except at $\zeta = \pm \exp(\pm i\theta_1)$, by taking them zero at the origin $\zeta = 0$.

Then

$$\log q - \tfrac{1}{2} \log UV = \frac{K}{\pi - 2\theta_1 + 2 \sin \theta_1 \log \tan \frac{1}{2} \theta_1} \left\{ (\pi - 2\theta_1) \cos \theta \right.$$

$$\left. + \sin \theta_1 \log \left| \tan \frac{\theta_1 - \theta}{2} \tan \frac{\theta_1 + \theta}{2} \right| + \sin \theta \log \left| \frac{\sin \frac{1}{2} (\theta + V_1)}{\sin \frac{1}{2} (\theta - \theta_1)} \right| \right\}$$

$$- \vartheta = 0 \text{ for } 0 \leqslant \theta \leqslant \theta_1 \text{ and for } \pi - \theta_1 \leqslant \theta \leqslant \pi,$$

$$= \frac{\pi K}{\pi - 2\theta_1 + 2 \sin \theta_1 \log \tan \frac{1}{2}\theta_1} \{\sin \theta - \sin \theta_1\} \text{ for } \theta_1 \leqslant \theta \leqslant \pi - \theta_1.$$

The unfavourable velocity gradient is logarithmically infinite at the point corresponding with $\theta = \theta_1$. This logarithmic infinity does not appear in the velocity gradient if $d\vartheta/d\theta$ is continuous; if $d\vartheta/d\theta$ is continuous and $d^2\vartheta/d\theta^2$ discontinuous, the second derivative of q is logarithmically infinite, and so on. Enough has, however, been said to show how finite channels may be considered.

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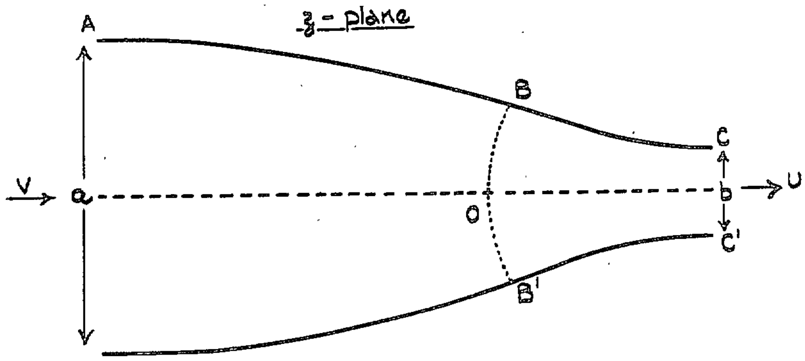

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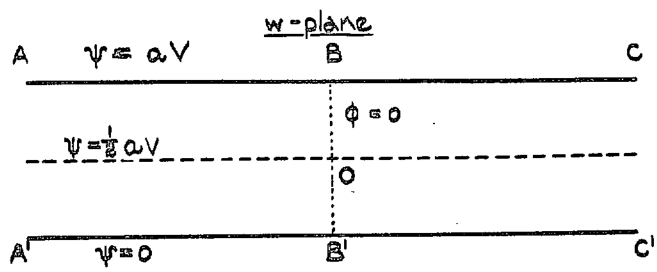

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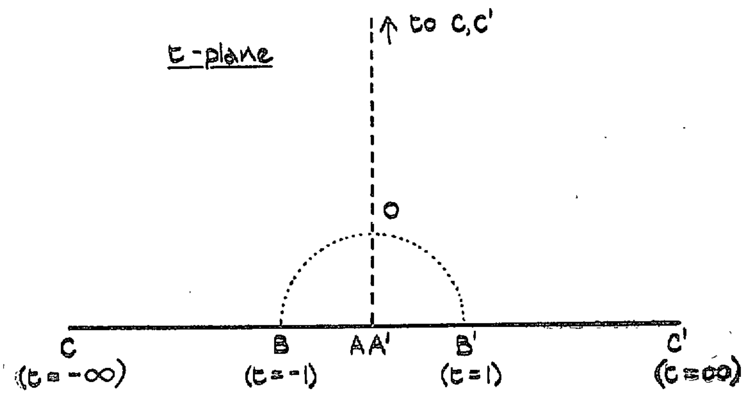
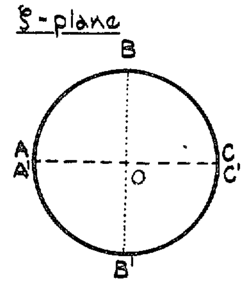


FIG. 3



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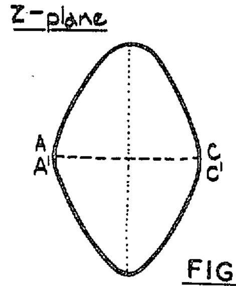

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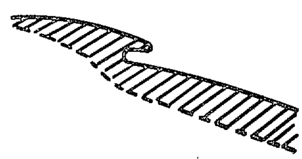

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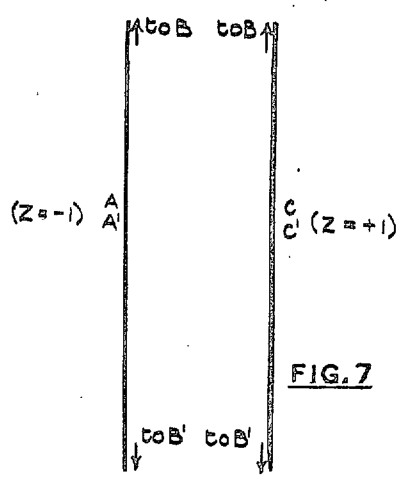
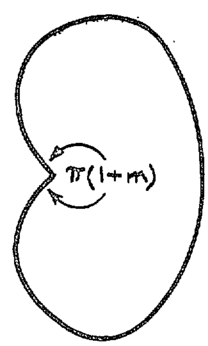


FIG. 7

FIG. 8



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