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Torsional Vibration in Aircraft Power Plants: Methods of Calculation

Part I. Introduction and General Comments
Part II. Practical Treatment of the General Problem
Part III. Practical Calculations for a Typical 12-Cylinder Vee engine

By

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Parts I, II and III

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Summary.—Introduction.—The object of this report is to assist designers of aircraft power plants to avoid harmful torsional vibration of the crankshaft-airscrew system.

Arrangement of the Report.—The report is divided into three parts as follows :—

Part I. Introduction and general comments.

Part II. Practical treatment of the general problem.

Part III. Practical calculation for a typical 12-cylinder Vee-engine.

PART I

Introduction and General Comments

Resonant torsional vibration can be so destructive that the problem of avoiding its harmful effects has to be given careful attention during the design of any new power plant for aircraft. The object of this report is to provide assistance to those who are concerned with this problem.

The manner in which resonant torsional vibration comes about will appear from the following : Dynamically, a crank shaft-airscrew system is in effect a system of flywheels connected by elastic shafting of negligible inertia. It so happens that this system has natural frequencies of vibration in torsion of the same order as the frequencies of important constituent harmonics of the piston torque impulses. The former frequencies are unaffected by shaft rotation (with a virtually rigid airscrew) whereas the latter are proportional to engine speed. There are thus particular speeds at which harmonic torque impulses have the same frequency as a natural mode of vibration of the system : these are referred to as criticals.

* L.A.8 Report E.3586, received 27th April, 1938.

The main synchronous speeds for a selection of direct-drive and geared engines are given in the following tables :—

DIRECT-DRIVE AERO-ENGINES

Main Synchronous Speeds (M.S.S.)

Horse Power	No. of cyls.	Type	Single node : cycles per sec	M.S.S. r.p.m.	M.S.S. ÷ Normal Speed	Remarks
100	5	Radial	233	5,600	2·54	Observed
125	4	Line	194	5,820	2·77	Observed
180	7	Radial	143	2,450	1·51	Calculated
186	6	Line	176	3,530	1·68	Observed
330	10	Radial	135	1,625	0·81	Observed
370	14	Radial	128	1,100	0·65	Observed
400	12	45° Vee	98	(980)*	0·60	M.S.S. for 60° Vee
408	9	Radial	157	2,100	1·24	Observed
470	9	Radial	169	2,250	1·27	Observed
510	12	Double Vee	157	1,570	0·67	Observed
600	8	Line	62	930	0·98	Comp. Ign.
560	8	Line	73	1,100	1·25	Comp. Ign.
900	9	Radial	120	1,600	0·64	Racing Engine

* The 45° Vee engine has no main synchronous speed : the figure relates to 60° Vee-engine.

GEARED AERO-ENGINES

Main Synchronous Speeds (M.S.S.)

Horse Power	No. of cyls.	Type	Single node: cycles per sec.	M.S.S. crank r.p.m.	M.S.S. ÷ Normal Speed	Gear Ratio
450	12	Double Vee	*100	1,000	0·5	0·53 : 1
480	12	60° Vee	*80	800	0·4	0·48 : 1
490	9	Radial	*87	1,160	0·52	0·66 : 1
510	12	Double Vee	*92	920	0·39	0·66 : 1
565	9	Radial	*68	900	0·45	0·5 : 1
565	9	Radial	*79	1,050	0·53	0·66 : 1
580	9	Radial	*79	1,055	0·53	0·66 : 1
650	14	Radial	*81	695	0·35	0·65 : 1
670	14	Radial	*87·6	750	0·35	0·594 : 1
575	12	Vee	†105	1,050	0·40	0·553 : 1
360	16	H engine	*123	925	0·26	0·391 : 1

* Observed. † Calculated.

Criticals may be classified in three groups; those severe enough to cause very rapid failure, those which cause failure by fatigue after a period of use, and those not severe enough to produce stresses exceeding fatigue limits. At what are called 'major' criticals, harmonic impulses from all the pistons act in unison* to force vibration in the corresponding mode. At such criticals, serious torsional vibration must occur involving large torque variations unless the harmonic impulses are small or the damping in the system is large. Actually, the internal damping is usually so small that, in order to obviate torsional resonance troubles, it is necessary to arrange the design so that no major critical with a large forcing harmonic occurs in the range of operating speeds, or, alternatively, to introduce extra damping by some means. If the natural frequency of the system for torsional oscillation with a single node between the crank throws and the airscrew happens to be the same as the frequency of the explosion impulses of the engine at a full throttle operating speed, the amplification of torque variation caused by resonance will be such that shaft or aircrew failure will occur after a very short period of operation at this speed, possibly a matter of minutes (R. & M. 1303[1]).

The single node major critical of firing impulse frequency is sometimes referred to as the 'Main synchronous speed' because of its severity when it occurs at large throttle openings. As, however, it may be well beyond the maximum engine speed or in a region of small throttle opening, the term 'main' is not to be taken to imply that this critical is the most severe. It forms a convenient datum for reckoning the speeds of other single-node criticals: values for a number of direct-drive and geared engines have already been given in the preceding tables.

* Harmonic impulses of the same frequency produced by different cylinders may be represented by vectors which have defined phase relationships for a given system operating under specified conditions.

Major criticals are not the only ones to be avoided ; what are termed ' minor ' criticals may also be important. For these criticals the corresponding forcing harmonics do not all operate in unison but offset each other to a greater or less extent according to their phase relationships and their points of application in the system.

The fact that minor criticals can cause serious trouble was demonstrated by the Graf Zeppelin engine failures ; these were due to two superposed minor criticals of different modes of vibration, and they occurred in spite of the fact that the engines were fitted with torsional vibration dampers.[2] Failures, due in large measure to minor critical torsional vibration, have occurred in 12-cylinder *Liberty* engines (R. & M. 1304[3]).

In some text books, curves of resultant torque are given which have been derived by adding together, with due regard to phase, the curves of applied torque for the several cylinders. It will be rendered clear from what follows that these resultant curves are quite misleading as regards the torque variation in the final drive. They represent this variation only for crankshafts of infinite stiffness, and furthermore they do not necessarily represent the character of the torque reaction of the engine on its mounting, because resonance effects may be present here also, amplifying particular harmonic components. For example, in the 12-cylinder 60° Vee, articulated-rod engine considered in Part III, the variation of airscrew driving torque with infinitely rigid shafting and airscrew would be composed mainly of 3rd and 6th order harmonics and the combined range would be about 60 per cent. of the mean torque. The corresponding limits of torque oscillation are indicated by the chain-dotted lines in Fig. 7, Part III for comparison with the limits calculated for the actual system.

To arrange a design so that torsional vibration troubles shall be avoided entails having the power to predict the critical speeds from design data and also to predict the severity of the criticals that come within the running range, having regard to the various conditions of operation. As an outcome of research, critical speeds can, as a rule, be predicted with sufficient accuracy for practical purposes. The prediction of the severity of the several criticals is more difficult and less precise. Criticals that must be avoided are easily identified, and those which may be neglected for smooth running conditions may be discovered by a fuller analysis. This fuller analysis may also reveal what may be termed border-line criticals, the importance of which cannot be assessed accurately enough to place them in either of the other two categories. The uncertainty arises from limited knowledge concerning damping, extent of unequal operation of cylinders under operating conditions, permissible stress ranges in material, and possible effects of airscrew-blade flexibility. The degree of uncertainty is being reduced as investigation proceeds and the number of cross-checks between theory and observation increases.

In Part II is a description of a process for calculating the torsional resonance characteristics of an aircraft power plant.

To illustrate the method, there is given in Part III complete calculation for a typical geared 12-cylinder Vee engine. This calculation is made in accordance with a regularised method in which calculations for different engines are arranged with as much as possible in common as regards notation, paragraphing, headings and sub-headings, figure and table numbering, and quantities plotted. In this way, comparison of both detailed working and results is much facilitated and the difficulty of learning to do the calculations is reduced.

Throughout, the airscrew is treated as being a rigid body. This assumption has, in general, given results which agree well with torsiograph observations, but anomalies have been noted which may be due to airscrew-blade vibration, and it may be expected that blade flexibility will be taken into consideration to an increasing extent as knowledge grows : this matter has an important bearing on fatigue failure in airscrew blades but its study lies outside the scope of this report.[4]

The torsional vibration characteristics of an engine running on a dynamometer test bed are not examined here but the same general processes can be applied when particulars of the system are known. The characteristics may be very different from those when an airscrew is attached, and it is important that cognizance be taken of this in dealing with engine testing.

Relevant references are given in each part of the report.

The author has drawn freely on the work of those colleagues at the Royal Aircraft Establishment whose names appear in the references, and he desires to acknowledge the assistance in the preparation of this report given by L. E. Caygill, B.Sc., E. M. Butcher, B.Sc., W. J. Evans, B.Sc., and H. P. Baker, B.Sc.

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PART II

Practical Treatment of the General Problem

It has been mentioned in Part I that, for the purpose of examining torsional vibration characteristics, a crankshaft-airscrew system may be regarded as being virtually a flywheel system. The characteristics of the vibration under specified harmonic impulses are not affected by steady mean transmitted torque nor by steady rotation : thus at any specified speed and power of operation the system may be treated as having no rotation and the vibration as occurring about the condition of the shaft as strained by the mean transmitted torque.

The dynamic system comprises the airscrew, crankshaft, reciprocating masses, big-end and balance masses, main reduction gearing (if any), the system upon which the moving parts react, and driven auxiliaries.

In general, driven auxiliaries such as camshafts, pumps, and magnetos may be ignored. Where there is a positively driven fan or supercharger, the drive is usually so flexible that the vibration of the crankshaft-airscrew system is unaffected, but it is important to examine this point and to include this item in the system if this appears to be necessary (R. & M. 1053[1]).

The manner of taking accounting of reciprocating masses will be described later. In treating any particular problem one proceeds in definite stages, and part or all of the investigation is carried out according to the needs of the case. The stages involved are shown by the following index to the description of methods, the main sections of which correspond to those of the worked example given in Part III.

Index to Description of Methods

1. *Data.*—The data required comprise: Details of crankshaft-airscrew transmission system, the piston system, compression ratio, performance characteristics, conditions of operation, and experience of similar engines where available.

2. *Moments of Inertia.*—2a. *Crankshaft.*—As a general rule, the quantities involved in aircraft engine systems are such that, in deriving the virtual system, a sufficient approximation is obtained by ignoring the inertia of the shafting and substituting equivalent rigid flywheel masses for the principal inertia masses. It is a straight-forward matter to compute the polar moments of inertia of the main masses concerned: the effects of piston and connecting rod masses will now be considered,

A simple connecting-rod is dynamically equivalent to a system having point masses at the gudgeon pin and crankpin axes, giving the same mass centre and the same total mass, plus a massless inertia moment which, when added to the inertia moment of the point masses about the mass centre, gives the value that holds for the actual system (*see* end of section 7a.3).

When on dead centre, a crank undergoing small angular oscillation does not impart appreciable movement to the corresponding piston-mass, but, when the crank is at right angles to this position, the piston-mass has the same vibratory motion as the crankpin axis. The average flywheel effect of the piston-mass is approximately that of a particle of half the mass located at the crankpin axis, and this half-mass is added to the point mass representing the big-end to obtain the total virtual addition to the crank inertia. Goldsborough has examined the effects of reciprocating masses[2,3] and one of his conclusions is that the error involved in the foregoing simplifying assumption is small.

2b. *Gears.*—Where gears are involved, speed effects have to be taken into account. It will be shown later (Section 4) that the equivalent inertia at a selected common speed (usually that of the crankshaft) is obtained by multiplying the actual polar moment of inertia of the geared mass by the square of its speed ratio.

2c. *Airscrew.*—The relationship of airscrew and crankshaft speeds needs to be taken account of in the same way as for gears. To a first approximation, the airscrew may be treated as a rigid mass : the polar moment of inertia of the airscrew is very large in relation to that of the remaining masses. For consideration of the effects of blade flexibility, *see* R. & M. 1758[4] and Ref. 5.

3. *Flexibilities and Stiffnesses.*—Information concerning crankthrow stiffness is given in Ref. 6 and R. & M. 1201[7]. Experience has shown that the empirical formula of Ref. 6 gives results of sufficient accuracy for practical purposes for shafts of normal design. This formula is given in Fig. 1. In the notation and units of this figure, the flexibility of length '*a*' inches of straight steel shafting of changing diameter and bore may be obtained by integration, using the formula :

$$F = 0\cdot86 \times 10^{-6} \int_0^a \frac{dx}{D^4 - d^4} \text{ radn/lb in.} \qquad (61)$$

The integration may be effected graphically by plotting $1/(D^4 - d^4)$ along the shaft axis, as is shown in Fig. 3b of Part III.

There is elastic yielding between the airscrew boss and the airscrew shaft and allowance requires to be made accordingly. This quantity is not amenable to calculation but some tests have been made to obtain data from which its value may be assessed in particular designs. In each instance the flexibility from the run out of the splines to the airscrew boss has been determined as the equivalent length of plain parallel shafting having the diameter and bore of the airscrew shaft at the 'run out' section. It has been found that the ratio of this length to the diameter is in the region of unity : actual values are given below :—

Where illustrated	Remarks	Equivalent flexibility of connection
	Wooden airscrew, parallel splines	$0\cdot96D$
	Wooden airscrew, taper splines	$0\cdot84D$
	Wooden airscrew, parallel splines	$1\cdot13D$
Fig. 2A ..	Wooden airscrew, parallel splines	$1\cdot04D$
Fig. 2B ..	Wooden airscrew, parallel splines	$0\cdot82D$
Fig. 2C ..	Metal airscrew, taper splines	$0\cdot45D$

This method of expressing the flexibility of connection is considered to be much better than the method adopted hitherto of stating the virtual point of rigid attachment between shaft and hub.

3a. *Ungeared Systems.*—The flexibility of the shafting between adjacent crank-masses in the virtual system is normally that of the crankshaft between the middle points of adjacent crankpins. Instead of replacing a whole crank-throw by a single flywheel, separate equivalent flywheels may be adopted for each crank-cheek (with balance-masses, if any) and for the crankpins with the attachments equivalent to the piston-connecting-rod system : then the flexibility of the shafting between the adjacent flywheels for the crank-throw is that of the crank-throw from the middle of the crankpin to the corresponding crank-cheek.

3b. *Geared Systems.*—Elastic yielding in gearing systems is a source of uncertainty in torsional vibration calculations as is not always amenable to complete calculation.

The gear yielding that can be calculated with fair accuracy comprises three portions :—

(*a*) depressions at the surface of tooth contact,

(*b*) bending of the teeth as cantilevers, and

(*c*) winding of the disc between boss and rim.

Information concerning (*a*) and (*b*) is given in Refs. 8 and 9 : the relevant equations will be given using the following notation.

P	The load on the gear in lb per inch face width
b	Length of flat in inches
ν	Poisson's ratio $= 0 \cdot 3$
h	Tooth thickness at pitch line
h_0	Tooth thickness at base
L	Distance from base of tooth to apex of triangle approximating tooth outline
x	Distance from apex of triangle to point where line of action intersects centre-line of tooth
r_1 and r_2	The radii of the pinion and gear
α	Pressure angle
z	Number of teeth
θ	Angle between the radial line through centre of tooth and the line drawn from centre of tooth to the line of action
ϕ	Angle between line of action and line perpendicular to centre-line of tooth : it can be shown to be equal to $\tan\theta - \frac{\pi}{2z} - \tan\alpha + \alpha$

3b.1 *Depression at Contact*

$$= \varepsilon_c = \frac{2P}{\pi E}(1 - \nu^2)\left[\frac{2}{3} + \log_e \frac{4r_1}{b} + \log_e \frac{4r_2}{b}\right] \quad \ldots \quad (62)$$

$$\text{where } b = 3 \cdot 04\left(\frac{P}{E} \times \frac{r_1 r_2}{r_1 + r_2}\right)^{1/2}$$

3b.2 *Deflection of gear wheel tooth due to bending*

$$= (\varepsilon_b)_g = \frac{12P L^3 \cos\phi}{E h_0^3}\left(\frac{3}{2} - \frac{x}{2c}\right)\left(\frac{x}{c} - 1\right) + \log_e \frac{L}{x} + \frac{4P(L - x)(1 + \nu)\cos\phi}{E(h + h_0)}$$

A similar equation is used to obtain $(\varepsilon_q)_p$ for the pinion wheel tooth.

The total tooth deflection along the line of action is the sum of the deflection due to depression at contact, gear tooth bending and pinion tooth bending.

$$\varepsilon = \varepsilon_c + (\varepsilon_b)_g + (\varepsilon_b)_p$$

By virtue of the properties of the involute curve, the total deflection given above, corresponding to a load P, is equivalent to a movement of the same amount at either one of the base circles, if the other is held stationary. Corresponding torsional flexibilities can readily be derived.

3b.3 *Winding of wheel or pinion disc.*—This is given by the formula :—

$$F = \frac{1}{4\pi\, t\, G}\left(\frac{1}{R_2^2} - \frac{1}{R_1^2}\right)$$

where F is flexibility in radn/lb in.
t disc thickness in inches
G modulus of rigidity in lb sq in.
R_1 external radius of disc in inches
R_2 internal radius of disc in inches

3b.4 *Additional flexibilities.*—To assist in estimating what allowance should be made for flexibility beyond the items that can be calculated, static twisting measurements have been made on three types of aero-engine reduction gearing : the results are given and analysed in Refs. 1, 2 and 3 at the end of the report, and an application of the investigation is given in Section 3.6 of Part III.

The investigation shows that, as regards engines similar to those examined, for plain spur gearing and for epicyclic *bevel* bearing, the extra flexibility which should be introduced into the equivalent system (*see* Section 4*b*) to take account of yielding in the gearing is some 30 per cent of the total virtual flexibility calculated between the airscrew boss and the first crank mass : further that, for epicyclic *spur* reduction gearing there is no appreciable extra flexibility because the engine nose is very stiff (being approximately cylindrical) and is subjected only to small torque reactions represented by the difference between crankshaft and airscrew shaft torques.

The flexibility of the epicyclic bevel reduction gearing decreased appreciably with loading, so that a mean requires to be taken : the yielding of the other gearing was linear.

4. *Dynamic System.*

4a. *Ungeared Systems.*—Here the equivalent system is the same as the virtual system, that is, a series flywheel system.

4b. *Geared Systems.*—When the airscrew shaft is driven through gearing located at one end of the crankshaft, it is convenient to adopt an equivalent ungeared system having the same frequencies. This step is possible as an approximation because the engine mass is relatively large. Strictly the engine mass and the flexibility of the mounting connection to other masses should be taken into account. A general tabular method of treatment for such systems is described later (Section 5b.2).

In deriving an equivalent ungeared system, it is usual to adopt crankshaft speed as basis because the torque impulses come upon the cranks. Consider, for example, the geared system shown in Fig. 3. Let it be assumed that the gear housing is rigid and rigidly attached to a mass large enough for its angular motion to be ignored. Imagine the crankshaft to be extended beyond the pinion as indicated where inertias and stiffnesses are R^2 times corresponding values for the airscrewshaft system, where R is the ratio of airscrew speed to crankshaft speed. Imagine also, that at any time t the vibration displacement at any point in the extension region is $1/R$ times the value for the corresponding point in the airscrew shaft system : this must clearly hold for the flywheel representing the gear wheel.

In the notation given in the figure, we have ;

$$\theta_A = \frac{\theta_a}{R}, \quad \frac{d\theta_A}{dt} = \frac{1}{R}\frac{d\theta_a}{dt}.$$

It follows that the kinetic energy of flywheel I_A is equal to that of flywheel I_a at the same instant, for :

$$I_A\left(\frac{d\theta_A}{dt}\right)^2 = I_a\left(\frac{d\theta_a}{dt}\right)^2.$$

Again, the strain energy in the corresponding shafts are the same, for :

$$C_{GA}\,(\theta_G - \theta_A)^2 = C_{ga}\,(\theta_g - \theta_a)^2.$$

It is easy to show that the frequencies are the same for the corresponding systems and that vibration torques in the airscrew shaft systems are $1/R$ times corresponding torques in the substitute system. Thus, any torques acting upon the airscrew system should be represented by torques of R times the value in the substitute system.

The same rule of changing moments of inertia and stiffnesses in proportion to the square of the reduction ratio applies for any number of flywheels and connecting shafts, and, where there are intermediate gears, as in an epicyclic system, the inertia effects can be taken into account by working with the appropriate speed ratios.

5. *Natural Frequencies, and Displacement Curves.*—5a. Having derived an equivalent simple flywheel system, the next step is to determine the important natural frequencies.

The system being elastic and having polar moment of inertia, can undergo free torsional vibration in a number of modes and for each mode there is a corresponding natural frequency and a definite number of points at which no vibrational displacement occurs : these points of uniform rotation are the nodes and the fewer nodes the lower is the corresponding natural frequency. Natural undamped vibration in any one mode may be regarded as a transformation in each swing of strain energy into kinetic energy and back again into strain energy, the interchange taking place in such a manner that the total energy is the same at every instant. At the extremes of swing, all the energy is stored as strain energy (as the entire mass comes to rest at the same moment) and at mid-swing all the energy is in kinetic form (as the system is then unstrained throughout and the angular velocity at each point has its maximum value). For each natural mode of vibration, a graph of definite form results from plotting the extreme angular deflection of the system at each point : for larger or smaller swings the ordinates of the graph are all greater or less in the same ratio. If the form of the angular deflection graph be known for a natural mode of vibration, the strain energy can be computed for any arbitrary amplitude of swing and, by equating this to an expression denoting the total kinetic energy at mid-swing, the natural frequency for the mode may be computed. In the present problem, however, the form of deflection graph is one of the things to be determined and frequencies are obtained otherwise.

Where the crank masses are equal, and likewise the stiffnesses between them, the single-node and two-node frequencies for an infinite airscrew inertia can be calculated. If the finite inertia of the airscrew requires to be taken into account, a value for the flexibility to the node near the airscrew is assumed and the frequency of the system on either side of this node is determined : the values are equal when the node position has been correctly assessed. To assist in such assessment frequency discrepancy should be plotted against assumed node position : the point at which the curve cuts the base gives the true node position.

For systems comprising unequal flywheels coupled by shafts of various stiffnesses, an approximation to the frequency in any one mode may be obtained by adopting an approximately equivalent system.

5b. *Tabular Method.*—A tabular method of determining frequencies is described below which involves adopting an approximate value for the frequency at the beginning. The equivalent system taken in getting this approximation may be obtained by rough compounding or averaging inertia moments and flexibilities.

This method [10, 11,12] is particularly useful because it is quite general, it gives the form of the deflection curve for any natural mode of vibration, and it forms a step towards the treatment of forced vibrations.

5b.1 *Treatment for a series flywheel equivalent system.*—Let there be a system of m flywheels $I_1, I_2, I_3 \ldots\ldots I_m$ in series and let the stiffnesses of the shafts connecting them be $C_{1,2}, C_{2,3}, C_{3,4} \ldots\ldots C_{(m-1),m}$. Let the system be vibrating torsionally in a natural mode with amplitude θ_1 at I_1 and let the frequency be $f = r/2\pi$. The value taken for θ_1 is arbitrary, as it does not affect the frequency, and for convenience it is usually taken to be one radian. At the extreme of the swing when all the flywheels are at rest, the inertia torque for I_1 is $\theta_1 I_1 r^2$ and the twist in the shaft $C_{1,2}$ is therefore $\theta_1 I_1 r^2/C_{1,2}$.

The amplitude of vibration of I_2 is thus $\theta_1 (1 - I_1 r^2/C_{1,2})$ and the corresponding inertia torque is $I_2 r^2$ times this. By subtracting the torque $\theta_2 I_2 r^2$ we get the torque in the shafting $C_{2,3}$ and thence the twist and so determine the amplitude of vibration of I_3.

In this manner, having first assumed values for r and θ_1 we may tabulate the quantities and work along to the end of the shaft, when we obtain a value for the torque beyond the last flywheel. If this value is zero, it indicates that the system can vibrate naturally with the assumed frequency and, by plotting the amplitude obtained for the motion of the several flywheels and joining the points by straight lines, we obtain the displacement curve. This reveals the number of nodes and their positions which is a check on the frequency assumed being appropriate to the mode of vibration under consideration.

If the value for the torque beyond the last flywheel is not zero, a different frequency is assumed and a new value obtained. From these two values it is usually possible to infer the true frequency for the third trial with sufficient accuracy to give practically zero for the criterion value.

By way of example, consider the system shown in Fig. 4. To a first approximation, this is equivalent to a system of six flywheels of inertia moment $I = 111$ lb/in.² connected to an infinitely large flywheel through a small flywheel which may be ignored.

The single-node frequency is 105 per sec. The actual value will be somewhat lower on account of the approximations made. Values are tabulated below for two assumed frequencies, namely 100 and 95 per sec.

	Units	lb in.sec²	lb in.	Radian	lb in.	lb in.	Radn./lb in	Radian
	Mass No.	$\frac{I}{g}$	$\frac{r_1^2 I}{g}$	θ	$\frac{r_1^2 I\theta}{g}$	$\sum \frac{r_1^2 I\theta}{g}$	F	$\sum\left(\frac{r_1^2 I\theta \times F}{g}\right)$
$f = 100\,;\ r^2 = (2\pi f)^2$			$10^6\times$		$10^6\times$	$10^6\times$	$10^{-6}\times$	
	1	0·288	0·114	1·0	0·114	0·114	0·1745	0·0199
	2	0·288	0·114	0·9801	0·1119	0·2259	0·1745	0·0394
	3	0·288	0·114	0·9407	0·1071	0·3330	0·1745	0·0582
	4	0·288	0·114	0·8825	0·1005	0·4335	0·1745	0·0756
	5	0·288	0·114	0·8069	0·0920	0·5255	0·1745	0·0916
	6	0·288	0·114	0·7153	0·0816	0·6071	0·89	0·541
	7	0·404	0·159	0·1743	0·0278	0·6349	0·493	0·312
	8	23·60	9·325	—0·1377	—1·285	—0·6501	—	—
$f = 95\,;\ r^2 = (2\pi f)^2$			$10^6\times$		$10^6\times$	$10^6\times$	$10^{-6}\times$	
	1	0·288	0·103	1·0	0·103	0·103	0·1745	0·0179
	2	0·288	0·103	0·9821	0·1011	0·2041	0·1745	0·0356
	3	0·288	0·103	0·9465	0·0975	0·3016	0·1745	0·0526
	4	0·288	0·103	0·8939	0·0921	0·3937	0·1745	0·0687
	5	0·288	0·103	0·8252	0·0850	0·4787	0·1745	0·0835
	6	0·288	0·103	0·7417	0·0765	0·5552	0·89	0·493
	7	0·404	0·144	0·2487	0·0359	0·5911	0·473	0·291
	8	23·60	8·430	—0·0423	—0·3570	0·2341	—	—

By plotting the remainders as shown in Fig. 6, the value 96·3 is obtained for no remainder, and this is a close approximation to the true value.

To obtain a first approximation to the frequency in two-node vibration, it suffices to assume a node near the middle of the crank system and calculate the frequency of the system between this node up to and including Mass No. 1, taking infinite inertia at the node : the two-node quantities in the table in section 7b.2 have been obtained by the same general process as that described for single-node vibration.

The deflection curves for single and two-node vibration are plotted in Fig. 5.

5b.2 *Treatment which takes account of engine mass and elastic support.*—The method described in section 5b.1 suffices for most practical purposes, but special problems arise for which more precise treatment is needed. The system is analogous to the dynamic system shown in full lines in Fig. 7 and is of the general kind concerned in the study of the vibration of vehicles[13].

The restraint given by the gearing imposes the condition that the forces operating between the gear masses and the rigid rod shall have no resultant force or couple : they may thus be represented by $+T$, $-T/R$ and $T(1/R-1)$ as shown. Furthermore, the displacements ϕ_1, ϕ_2 and ϕ_3 must satisfy the gearing equation.

$$\phi_3 = R\phi_1 + (1-R)\,\phi_2 \qquad (64)$$

The following elaboration of the tabular method may be employed :

Imagine the crank mass I_1 to be vibrating with a selected frequency $r/2\pi$ with unit amplitude. Work through the table and determine the force T_1 between G_1 and the rod and the amplitude ϕ_1. Assume unit amplitude at I_3 and work along to the corresponding force between G_3 and the rod. Then take the pro rata amplitude ϕ_3 that makes the force between G_1 and the rod equal to $-T_1/R$. Again, work along the table from I_4 and find ϕ_2, the amplitude at G_2 which will make the force between G_2 and the rod equal to $T_1(1/R-1)$. The conditions imposed by the gearing as regards forces are then satisfied and the criterion that the assumed frequency is a natural frequency of the system is that the gear displacement relationship of equation (64) shall hold. Where an end mass is taken to be infinite, the procedure requires to be somewhat different. For example, if I_4 is infinite, the displacement ϕ_2 is taken to be that which satisfies equation (64), and the table for the masses G_2 I_2 and I_4 is worked from the G_2 end. The criterion that the frequency taken is a natural frequency of the system is that the displacement at I_4 is zero.

It is evident that the same method will apply if there are additional masses and flexibilities as, for example, those shown by the dotted lines in Fig. 7.

6. *Orders of Vibration and Critical Speeds.*—As the explosion impulses repeat after two crankshaft revolutions in a four-stroke engine, the harmonic torques occur : $\frac{1}{2}$, 1, $1\frac{1}{2}$, . . . *etc.* times per revolution of the crankshaft. Resonant torsional vibration in any particular mode of vibration occurs when one of these impulses has a frequency equal to the natural frequency of the system in this mode. The crankshaft speeds at which this happens are called ' criticals ' and the number of complete vibrations per crankshaft revolution in any particular instance is called the ' order ' of the critical—this is clearly 60 times the frequency per second divided by the crankshaft r.p.m.

A chart which expresses the relationships between frequency, order, and critical speed is given in Fig. 8. If ordinates are drawn on this chart representing the extreme values of the operating speed range of the engine, and if horizontal lines are drawn representing single-node and two-node natural frequencies, the criticals that may be important are clearly indicated.

7. *Estimation of Relative Magnitudes of Various Criticals.*—Here we have to consider the effects of gas torques, inertia torques, connecting-rod articulation, and different arrangements of cranks and cylinders: the subject matter has been divided into a series of sections and subsections accordingly.

7a. *Periodic Torques Acting on the System.*—These are of two kinds, active and passive. The active torques force torsional vibration and are due to gas and inertia forces. The passive torques are produced by the vibration and they oppose it in the kind of vibration we are considering : they are the damping torques—to which further reference is made in Section 8.

7a.1 *Gas-torques : general treatment.*—For a four-stroke engine, Fourier analysis of the gas torque for any particular cylinder gives the mean torque and an infinite series of harmonic torques repeating : once, twice, three times, ... *etc.* per *two revolutions* of the crankshaft. Thus, gas harmonic impulses occur at $\frac{1}{2}$, 1, $1\frac{1}{2}$, ... *etc.* times per crankshaft revolution. An indicated gas-torque curve for an aero-engine cylinder is shown in Fig. 9 and, beneath it to the same scale two of the component harmonics are shown in full lines. The components may be regarded as being projections in elevation of corresponding vectors OV as shown. (It will be seen later that such rotating vectors afford a convenient means of compounding periodic torques of the same frequency, produced by different cylinders). Each component may be expressed as a sine curve or as a cosine curve by assigning the phase angle that gives it the correct location in relation to the firing top-dead-centre.

From the general identity:

$$A \sin (\theta + \alpha) = (A \sin \alpha) \cos \theta + (A \cos \alpha) \sin \theta \qquad (65)$$

it is clear that each component may be expressed as the sum of a cosine and a sine curve, both having zero phase angle, as indicated by the dotted curves.

Thus the indicated gas-torque for a single-cylinder, whether with a master or articulated connecting-rod, can be expressed mathematically by the following series :—

$$\begin{aligned} q_g = a_0 &+ a_0 \cos \lambda + a_2 \cos 2\lambda + \ldots a_n \cos n\lambda + \ldots . \\ &+ b_1 \sin \lambda + b_2 \sin 2\lambda + \ldots b_n \sin n\lambda + \ldots \end{aligned} \qquad (66)$$

where, for convenience q_g is taken to represent the gas-torque per unit piston area per unit crank-throw: λ is the half-speed shaft angle and equals $\omega t/2$, where ωt is the angular position of the crankpin from the firing top-dead-centre after time interval 't'. The constant a_0 represents the indicated mean torque for the cylinder and is the corresponding mean effective pressure divided by 2π.

It is important to note that if the 'order' of an harmonic torque component is taken to signify the number of complete cycles per revolution, the nth harmonic in the series above is the pth order, where $p = n/2$.

The mean torque applied to the airscrew (and the B.M.E.P.) is connected with the engine speed and the running conditions by the characteristics of the airscrew. The corresponding average indicated M.E.P. is inferred by assessing the mechanical efficiency of the engine which is usually in the region of 90 per cent. for full-power operation.

7a.2 *Gas-torques for a single cylinder and direct connecting rod.*—Two methods of determining gas-torque harmonics* will be described:

(I) An indicator diagram appropriate to the conditions of working is adopted and the gas-torque for each number of crank positions is computed from exact formulae. A Fourier analysis may then be made of the values so obtained.

(II) An approximate method, in which gas-torque harmonic coefficients are obtained from tables which have been calculated for a series of representative conditions in respect of both electrical ignition and compression ignition engines.

* Only *varying* friction can damp torsional vibration : the mean friction of pistons and bearings has no damping effect. In view of this, indicated torques are taken when examining forcing effects.

Method I.—For any crank angle ζ and corresponding gas force F_g the instantaneous gas-torque is:

$$Q_g = - F_g \frac{dz}{d\zeta} \quad \ldots \quad (67)$$

where z is the distance between the gudgeon-pin and crankshaft centres: that this is so is evident from the principle of 'Virtual work.'

If R denotes the crank radius and L the connecting rod length, the following exact relationships hold:

$$z = R \cos \zeta + L \left(1 - \frac{R^2}{L^2} \sin^2 \zeta\right)^{1/2} \quad \ldots \quad (68)$$

$$Q_g = RF_g \left\{\sin \zeta + \frac{R}{2L} \sin 2\zeta \left(1 - \frac{R^2}{L^2} \sin^2 \zeta\right)^{-1/2}\right\} \quad \ldots \quad (69)$$

For particular values of ζ the corresponding values of z can be computed from (68) and thence may be found the corresponding values of F_g for given conditions of operation. Values of Q_g can then be computed from (69). When there are no other cylinders operating on the same pin, these Q_g values are subjected to harmonic analysis on the basis of the angle λ: the latter operation can conveniently be performed by Runge's tabular method. It is frequently convenient to performed this analysis on the values for torque per unit piston area per unit crankthrow: then the coefficients correspond to those of equation (66).

Method II.—If D denote the cylinder bore and R the crankthrow, both in inches, the cosine and sine coefficients a and b of equation (66) require to be multiplied by $(\pi/4)D^2R$ to obtain actual values. Thus, if M_c and M_s denote the amplitudes of the cosine and sine components of the actual gas-torque for the pth order

$$M_c = \frac{\pi}{4} D^2 R a_{2p} \quad \ldots \quad (70)$$

$$M_s = \frac{\pi}{4} D^2 R b_{2p} \quad \ldots \quad (71)$$

and the indicated mean torque (in pound inches) is:

$$Q_0 = \frac{\pi}{4} D^2 R a_0 \quad \ldots \quad (72)$$

In Tables 1 and 2 values of the cosine and sine coefficients are given for an aero-engine cylinder operating at different indicated mean effective pressures: these have been obtained by Method I. By the same method, the coefficients given in Tables 3 and 4 have been obtained for a typical compression-ignition aero-engine.

7a.3 *Inertia torques for a single cylinder and direct connecting rod.*—The inertia torque produced by the reciprocating mass concerned with any one cylinder varies with crank position and repeats itself in *one revolution.* The Fourier components comprise an infinite series of harmonic forcing torques which occur 1, 2, 3, 4 . . . . times per revolution and which, for any one engine speed, may be combined with the corresponding gas-torque forcing harmonics, having due regard to phase.

For a cylinder whose axis intersects the crankshaft axis and which operates a normal piston, connecting rod and crank, there are sine terms only in the inertia-torque analysis, and no fractional orders.

Let M_s' denote the coefficient of the pth order sine term in pound-inch units, then:

$$M_s' = \frac{W}{g} \omega^2 R^2 t_p \qquad \text{(73)}$$

where W is the weight of the reciprocating mass in pounds, g is 386 in./sec², R is the crankthrow in inches, ω is the crankshaft speed in radians per second, and t_p is a numerical coefficient which has the values tabulated below for various obliquity ratios:

Order	Sign	Ratio: Connecting-rod/crank						
		3	3·43	4	4·2	4·4	4·6	4·8
1	+	0·0857	0·07453	0·06351	0·06038	0·05757	0·0550	0·05266
2	—	0·5004	0·50024	0·50012	0·50012	0·50008	0·50007	0·50006
3	—	0·2607	0·22606	0·19217	0·18246	0·17385	0·16601	0·15886
4	—	0·0304	0·02295	0·01613	0·01458	0·01325	0·0120	0·0111
5	+	0·006	0·00418	0·00258	0·00222	0·00193	0·00167	0·00148
6	+	0·0012	0·00073	0·00014	0·00012	0·00009	0·00008	0·00007
7	—	0·000168	0·00008	0·00004	0·00003	0·00002	0·00002	0·00002
8	—	0·000044	0·00002	0·00001	0·00001	0	0	0

The angular momentum of the connecting-rod has an inertia-torque effect which is additional to that obtained by replacing the piston and connecting-rod masses by rotating and reciprocating masses in the usual manner giving the same mass-centre position. This gives another series of sine terms and these are zero for all odd and fractional orders.

If M_s'' denote the coefficient of the pth order term (lb/in.):

$$M_s'' = S_p \frac{I_0}{386} \omega^2 p/2 \qquad \text{(74)}$$

where I_0 is the difference between the moment of inertia of the rod about its C.G. and the moment of inertia of the point masses about the rod C.G. expressed in lb/in.² units. (Ref. 14.)*

Values of S_p are given in Fig. 10 for various obliquity ratios. Normally, the values of M_s'' are very small and may be neglected in in-line engines but must be taken into account in radial engines owing to the large polar moment of inertia of the big-ends.

7a.4 *Gas-torques and inertia-torques for an articulated-rod cylinder.*—In. Fig. 11 there is shown a system in which one connecting rod is articulated on the big-end of another: the cylinder axes pass through the crankshaft axis. The following are relevant symbols for the system, at time t.

- ζ Angle between the crank and its position on top-dead-centre for the master cylinder
- λ $=\zeta/2$ and relates to a camshaft at half crankshaft speed
- ω Angular velocity of the crankshaft $= d\zeta/dt$
- ϕ Angular pitch of cylinder.
- ψ Angle between centre-lines of master-rod and link

* The sign convention changes at this point in Ref. 14: the plus sign is correct for the convention of this report.

α	Angular displacement of master-rod
β	Angular displacement of auxiliary-rod
δ	Angle between auxiliary-rod and its link
$\zeta/2$	Crankthrow
J	Length of anchor link
L	Length of master-rod
L_2	Length of auxiliary-rod
y	Distance between centres of crankshaft and auxiliary piston gudgeon-pin
F	Force along cylinder axis, at a gudgeon-pin, due to gas or inertia (suffixes g and i respectively)
W	Weight of reciprocating mass, in pounds
g	386 in./sec²

Inch-pound-second units are used throughout and angles are in radian measure.

Corresponding to equation (67), we have:

$$Q = -F\frac{dy}{d\zeta} \cdot \qquad (67')$$

The application of this formula to the determination of gas and inertia harmonic forcing torques will now be examined.

Piston position.—It is clear from Fig. 11 that the position of the auxiliary piston at time t is given by:

$$y = \frac{S}{2}\cos(\zeta - \phi) + J\cos(\phi - \psi + \alpha) + L_1 \cos\beta. \qquad (75)$$

Where
$$\sin\alpha = \frac{S}{2L}\sin\zeta \qquad (76)$$

and
$$\sin\beta = \frac{S}{2L_1}\sin(\zeta - \phi) - \frac{J}{L_1}\sin(\phi - \psi + \alpha) \qquad (77)$$

whence y can be determined as a function of ζ.

The following equivalent equations are sometimes more convenient to use:

$$y = \frac{S}{2}\cos(\zeta - \phi) + k_1(1 - \sin^2\alpha)^{1/2} - k_2 \sin\alpha + L_1(1 - \sin^2\beta)^{1/2} \qquad (75')$$

$$\sin\beta = \frac{S}{2L_1}\sin(\zeta - \phi) - \frac{k_1}{L_1}\sin\alpha - \frac{k_2}{L_1}(1 - \sin^2\alpha)^{1/2} \qquad (77')$$

where:
$$k_2 = J\cos(\phi - \psi)$$
$$k_1 = J\sin(\phi - \psi)$$

Gas torque.—The gas force F_g for any value of y can be found in the same manner as for a single cylinder and direct connecting rod (section 7a.1). By resolution of forces it can be shown that the corresponding torque is:

$$Q_g = F_g \frac{S}{2} \sec \beta \left\{ \sin (\zeta - \phi + \beta) - \frac{J \sin \delta \cos \zeta}{L \cos \alpha} \right\} \quad \ldots \quad (78)$$

where $$\delta = \psi - \phi + \beta - \alpha \quad \ldots \quad (79)$$

whence Q_g can be found in terms of crank angle, and its harmonics obtained if required.

An articulated rod always operates with a master rod, and there may be other rods articulated on the same big end: unnecessary computation is avoided by summing (algebraically) the values of Q_g for the several cylinders, for each crank position taken, and applying Fourier analysis only to the resultant values (*see* section 7a.5).

Inertia torque.—From equation (67′) the inertia torque is:

$$Q_i = - F_i \frac{dy}{d\zeta} \quad \ldots \quad (67'')$$

where $$F_i = \frac{W}{g} \times \frac{d^2y}{dt^2} = \frac{W\omega^2}{g} \times \frac{d^2y}{d\zeta^2} . \quad \ldots \quad (80)$$

Here ω is taken to be sensibly constant.

To obtain inertia torques up to the pth order, the piston position curves computed from equations (75), (76) and (77) should be analysed to the $(p + 2)$ order. Thus we get the coefficients in the following equation for y:

$$y = \left\{ \begin{array}{l} c_0 + c_1 \cos \zeta + c_2 \cos 2\zeta + \ldots + c_p \cos p\zeta + \ldots\ldots \\ \quad + d_1 \sin \zeta + d_2 \sin 2\zeta + \ldots + d_p \sin p\zeta + \ldots\ldots \end{array} \right\} . \quad \ldots \quad (81)$$

For most purposes it suffices to analyse to the 8th term as inertia torques beyond the sixth order are small. The first and second derivatives with respect to ζ are:

$$\frac{dy}{d\zeta} = \left\{ \begin{array}{l} - c_1 \sin \zeta - 2c_2 \sin 2\zeta - \ldots - pc_p \sin p\zeta - \\ + d_1 \cos \zeta + 2d_2 \cos 2\zeta + \ldots + pd_p \cos p\zeta + \end{array} \right\} \begin{array}{l} \ldots \\ \ldots \end{array} \quad \ldots \quad (82)$$

$$\frac{d^2y}{d\zeta^2} = \left\{ \begin{array}{l} - c_1 \cos \zeta - 4c_2 \cos 2\zeta - \ldots - p^2c_p \cos p\zeta - \\ - d_1 \sin \zeta - 4d_2 \sin 2\zeta - \ldots - p^2d_p \sin p\zeta - \end{array} \right\} \begin{array}{l} \ldots \\ \ldots \end{array} \quad \ldots \quad (83)$$

From equations (67″) and (80) the inertia torque is minus $W\omega^2/386$ times the product of the series given in equations (82) and (83): multiplying out and simplifying by using the identities:

$$\cos p\zeta \times \sin q\zeta = \tfrac{1}{2} \{\sin (p + q)\zeta - \sin (p - q)\zeta\}$$
$$\cos p\zeta \times \cos q\zeta = \tfrac{1}{2} \{\cos (p + q)\zeta + \cos (p - q)\zeta\}$$

we get the coefficients in the following equation for the inertia torque Q_i:

$$Q_i = \frac{W\omega^2}{772} \left[\begin{array}{l} C_1 \cos \zeta + C_2 \cos 2\zeta + \ldots + C_p \cos p\zeta + \ldots \\ + D_1 \sin \zeta + D_2 \sin 2\zeta + \ldots + D_p \sin p\zeta + \ldots \end{array} \right] \quad \ldots \quad (84)$$

The coefficients are as follows :—

Order		Inertia Torque Coefficients
1	$C_1 =$	$2d_1c_2 + 6d_2c_3 + 12d_3c_4 + 20d_4c_5 + \ldots$
		$- 2c_1d_2 - 6c_2d_3 - 12c_3d_4 - 20c_4d_5 - \ldots$
	$D_1 =$	$2d_1d_2 + 6d_2d_3 + 12d_3d_4 + 20d_4d_5 + \ldots$
		$+ 2c_1c_2 + 6c_2c_3 + 12c_3c_4 + 20c_4c_5 + \ldots$
2	$C_2 =$	$2d_1c_1$
		$+ 6d_1c_3 + 16d_2c_4 + 30d_3c_5 + 48d_4c_6 + \ldots$
		$- 6c_1d_3 - 16c_2d_4 - 30c_3d_5 - 48c_4d_6 - \ldots$
	$D_2 =$	$d_1^2 - c_1^2$
		$+ 6d_1d_3 + 16d_2d_4 + 30d_3d_5 + 48d_4d_6 + \ldots$
		$+ 6c_1c_3 + 16c_2c_4 + 30c_3c_5 + 48c_4c_6 + \ldots$
3	$C_3 =$	$6d_1c_2 + 6d_2c_1$
		$+ 12d_1c_4 + 30d_2c_5 + 54d_3c_6 + 84d_4c_7 + \ldots$
		$- 12c_1d_4 - 30c_2d_5 - 54c_3d_6 - 84c_4d_7 - \ldots$
	$D_3 =$	$6d_1d_2 - 6c_1c_2$
		$+ 12d_1d_4 + 30d_2d_5 + 54d_3d_6 + 84d_4d_7 + \ldots$
		$+ 12c_1c_4 + 30c_2c_5 + 54c_3c_6 + 84c_4c_7 + \ldots$
4	$C_4 =$	$12d_1c_3 + 16d_2c_2 + 12d_3c_1$
		$+ 20d_1c_5 + 48d_2c_6 + 84d_3c_7 + 128d_4c_8 + \ldots$
		$- 20c_1d_5 - 48c_2d_6 + 84c_3d_7 - 128c_4d_8 - \ldots$
	$D_4 =$	$12d_1d_3 + 8d_2^2 - 12c_1c_3 - 8c_2^2$
		$+ 20d_1d_5 + 48d_2d_6 + 84d_3d_7 + 128d_4d_8 + \ldots$
		$+ 20c_1c_5 + 48c_2c_6 + 84c_3c_7 + 128c_4c_8 + \ldots$
5	$C_5 =$	$20d_1c_4 + 20c_1d_4 + 30d_2c_3 + 30d_3c_2$
		$+ 30d_1c_6 + 70d_2c_7 + 120d_3c_8 + 180d_4c_9 + \ldots$
		$- 30c_1d_6 - 70c_2d_7 - 120c_3d_8 - 180c_4d_9 - \ldots$
	$D_5 =$	$20d_1d_4 + 30d_2d_3 - 20c_1c_4 - 30c_2c_3$
		$+ 30d_1d_6 + 70d_2d_7 + 120d_3d_8 + 180d_4d_9 + \ldots$
		$+ 30c_1c_6 + 70c_2c_7 + 120c_3c_8 + 180c_4c_9 + \ldots$
6	$C_6 =$	$30d_1c_5 + 30d_5c_1 + 48d_2c_4 + 48d_4c_2 + 54d_3c_3$
		$+ 42d_1c_7 + 96d_2c_8 + 162d_3c_9 + 240d_4c_{10} + \ldots$
		$- 42c_1d_7 - 96c_2d_8 - 162c_3d_9 - 240c_4d_{10} - \ldots$

Order	Inertia Torque Coefficients	
6	D_6	$30d_1d_5 + 48d_2d_4 + 27d_3^2$
		$- 30c_1c_5 - 48c_2c_4 - 27c_3^2$
		$+ 42d_1d_7 + 96d_2d_8 + 162d_3d_9 + 240d_4d_{10} + \ldots\ldots$
		$+ 42c_1c_7 + 96c_2c_8 + 162c_3c_9 + 240c_4c_{10} + \ldots\ldots$
p	$C =$	$d_1c_{p-1}(p - 1) + d_2c_{p-2}(\overline{4p - 2}) + d_3c_{p-3}(\overline{9p - 3}) + d_{p-1}c_1(p - 1)^2 \ldots\ldots$
		$+ c_1d_{p-1}(p - 1) + c_2d_{p-2}(\overline{4p - 2}) + c_3d_{p-3}(\overline{9p - 3}) + c_{p-1}d_1(p - 1)^2 \ldots\ldots$
		$+ d_1c_p + 1(p^2 + p) + d_2c_{p+2}(2p^2 + 4p) + d3c_{p+3}(3p^2 + 9p) + \ldots\ldots$
		$- c_1d_{p+1}(p^2 + p) - c_2d_{p+2}(2p^2 + 4p) - c_3d_{p+3}(3p^2 + 9p) - \ldots\ldots$
	$D_p =$	$d_1d_{p-1}(p - 1) + d_2d_{p-2}(\overline{4p - 2}) + d_3d_{p-3}(\overline{9p - 3}) + d_{p-1}d_1(p - 1)^2 \ldots\ldots$
		$- c_1c_{p-1}(p - 1) - c_2c_{p-2}(\overline{4p - 2}) - c_3c_{p-3}(\overline{9p - 3}) - c_{p-1}c_1(p - 1)^2 \ldots\ldots$
		$- d_1d_{p+1}(p^2 + p) + d_2d_{p+2}(2p^2 + 4p) + d_3d_{p+3}(3p^2 + 9p) + \ldots\ldots$
		$+ c_1c_{a+1}(p^2 + p) + c_2c_{p+2}(2p^2 + 4p) + c_3c_{d+3}(3p^2 + 9p) + \ldots\ldots$

In a radial engine having N cylinders equally spaced about one crankpin, if the master cylinder is No. 1 and the inertia torque of No. 2 cylinder is represented by:

$$\frac{W\omega^2}{772} \times \sum_{p=1}^{p=\infty} (C_p \cos p\zeta - D_p \sin p\zeta)$$

The inertia torque of No. N cylinder is:

$$\frac{W\omega^2}{772} \times \sum_{p=1}^{p=\infty} (- C_p \cos p\zeta - D_p \sin p\zeta)$$

There is the same relation between the inertia torques for cylinders Nos. 3 and $(N - 1)$: 4 and $(N - 2)$: . . . . . . *etc.*, so that in the summation for all the auxiliary cylinders operating on the one crankpin, the cosine terms cancel out: it is therefore only necessary to evaluate the sine harmonics in this instance and the computation need only be made for one side of the master rod.

7a.5 *Combination of gas and inertia torques for one cylinder.*—Extending part of the notation given in sections 7a.2 and 7a.3 so as to cover articulated-rod cylinders as well as master-rod cylinders, let the pth order total applied torque be:

$$Q_{pu} = (M_c + M_c') \cos p\omega t + (M_s + M_s' + M_s'') \sin p\omega t \qquad (85)$$

where suffix u signifies one cylinder; M_c and M_s are due to gas forces; M_c', M_s', and M_s'' to inertia forces. The term M_c' is zero for a master cylinder and, in general, M_s'' may be neglected —it is small and it would be difficult to work out its value for an articulated rod. We may write:

$$Q_{pu} = A_p \sin \{p(\omega t + \xi_p)\} \qquad (86)$$

where: $A_p = \{ (M_c + M_c')^2 + (M_s + M^{s'} + M_s'')^2 \}^{1/2}$ .. .. (87)

The angle of lead is $p\xi_p$ and:

$$\sin p\xi_p = \frac{M_c + M_c'}{A_p} \quad : \quad \cos p\xi_p = \frac{M_s + M_s' + M_s''}{A_p} \qquad \text{(88)}$$

Thus the resultant applied torque per cylinder is:—

$$Q_u = \frac{\pi}{4} D^2 Ra_0 + \Sigma \left\{ A_p \sin p(\omega t + \xi_p) \right\} \qquad \text{(89)}$$

where $p = \frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots\ldots$ *etc.* (*See* equations 66 and 86)

7a.6 *Combination of gas and inertia torques for two or more cylinders operating on the same crankpin.—General.*—The combination is made separately for each order of component torque. If these components are derived for each cylinder with reference to a common datum crank position, the cosine components can be added algebraically, and likewise the sine components, to get the resultant coefficients for the crankpin. However, as mentioned under the heading of 'gas torque' in sub-section 7a.4, it is usually more convenient to sum the gas-torque values for the several cylinders operating on the one crankpin and then to analyse the resultant gas torque. Whichever method is adopted, the resultant pth order torque on the crankpin may be expressed as:

$$\begin{aligned} Q_p &= \{\Sigma(M_c + M_c')\} \cos p\omega t + \{\Sigma(M_s + M_s' + M_s'')\} \sin p\omega t \\ &= ZA_p \sin p(\omega t + \xi_p') \qquad \text{(90)} \end{aligned}$$

where A_p relates to the datum cylinder, and the quantity Z is a non-dimensional factor which expresses the ratio of the resultant pth order torque to that of the datum cylinder: where there are articulated connecting rods, the datum cylinder is taken to be the master cylinder. Clearly:

$$ZA_p = [\{\Sigma (M_c + M_c')\}^2 + \{\Sigma (M_s + M_s' + M_s'')\}^2]^{1/2} \qquad \text{(91)}$$

In particular arrangements of engine, Z can be determined in such a manner as to simplify the working; in all arrangements, ZA_p will be used to signify the magnitude of the resultant pth order torque for one crank.

Two Cylinder Vee Arrangement with Both Connecting Rods Acting Directly on the Crankpin.—Consider the torques on one crank of a four-stroke Vee engine having forked connecting rods and a Vee angle ϕ. Let the cylinders be denoted by L and R, where the crank passes the top-dead-centre for L before that for R. The sequence of firing is usually such that R fires $(2\pi - \phi)$ radians of crank rotation after L. From section 7a.5, the pth order resultant torque for L has the value at time t,

$$Q_{puL} = A_p \sin \{p(\omega t + \xi_p)\} \qquad \text{(86')}$$

and the crank has turned through ωt radians since the top-dead-centre for the datum explosion in L. As the crank has turned through $\omega t - (2\pi - \phi)$ radians since the top-dead-centre for the succeeding explosion in R, it follows that the pth order resultant torque for R has the value at time t,

$$Q_{puR} = A_p \sin [p \{\omega t - (2\pi + \psi) + \xi_p\}] \qquad \text{(92)}$$

Thus Q_{puL} and Q_{puR} are respectively the projections in elevation of the vectors:

$$\overrightarrow{A_p \ , \ p(\omega t + \xi_p)}$$

and:

$$\overrightarrow{A_p \ , \ p(\omega t + \xi_p) - p(2\pi + \phi)}$$

shown in Fig. 12.

The resultant pth order torque on the crank at time 't' is clearly the projection in elevation of the vector:

$$\overrightarrow{ZA_p \ , \ p(\omega t + \xi_p) \ - \frac{p}{2}(2\pi + \phi)}$$

where $Z = 2\cos\left\{ \frac{p}{2}(2\pi + \phi) \right\}$ .. .. (93)

and must have a value between -2 and $+2$.

It is convenient to treat Z as a positive quantity and change the direction of the vector by π when $\cos\left\{ \frac{p}{2}(2\pi + \phi) \right\}$ is negative.

Numerical values of Z for different orders are tabulated below for some particular values of ϕ:

Order	30°	45°	60°	90°	135°	180°
½	0·261	0·390	0·517	0·765	1·111	1·414
1	1·932	1·848	1·732	1·414	0·797	0
1½	0·765	1·111	1·414	1·834	0·356	1·414
2	1·732	1·414	1·0	0	1·414	2·0
2½	1·218	1·663	1·932	1·834	1·962	1·414
3	1·414	0·765	0	1·414	1·873	0
3½	1·565	1·961	1·932	0·765	1·663	1·414
4	1·0	0	1·0	2·0	0	2·0
4½	1·834	1·961	1·414	0·765	1·663	1·414
5	0·508	0·765	1·732	1·414	1·873	0
5½	1·987	1·663	0·517	1·834	1·962	1·414
6	0	1·414	2·0	0	1·414	2·0

Equally Spaced Radial Cylinders with Connecting Rods Acting Directly on a Common Crankpin.— If there are N cylinders, the crank rotation between successive explosions is $(4\pi)/N$ so that, at time 't', the crank position in relation to the firing top deadcentre for the cylinder that fires next after the datum cylinder is $\left(\omega t - \frac{4\pi}{N} \right)$.

Thus, at this instant, the vector representing the pth order torque impulse for this cylinder is:

$$\overrightarrow{A_p \ , \ p(\omega t + \xi_p) \ - \frac{4\pi p}{N}}$$

shown as line 1–2 in Fig. 13a for which $N = 5$.

Similarly, vectors 2–3, 3–4, ... *etc.*, may be drawn in firing sequence for the remaining cylinders, the direction of each lagging $(4\pi p)/N$ radians behind the preceding one. This lag is always an integral multiple of $2\pi/N$ because 2_p is always an integer. Now $2\pi/N$ is the external angle of a regular polygon of N sides and thus, when $p = \frac{1}{2}$, the vectors from a closed polygon (as shown in Fig. 13a for $N = 5$) giving zero as resultant vector: for any other value of p, each vector is parallel to a side of this polygon and a closed polygon is obtained except when $(4\pi p)/N = m \times 2$, where m is an integer, that is, when p is an integral multiple of $N/2$. The resultant vector is then:

$$\overrightarrow{NA_p\ ,\ p(\omega t + \xi_p)}$$

so that $ZA_p = NA_p$ in this instance (as shown in Fig. 13c for $N = 5$). This condition gives rise to ' Major criticals '.

7b. *Forced Torsional Vibration.*—7b.1 *General.*—An irregular disturbance of the system gives rise to natural vibrations which are quickly damped out. The vibrations forced by continued harmonic disturbances are those to be considered.

If the crankshaft and the airscrew drive were infinitely stiff, no yielding would occur under the applied torque and there would be no storing of strain energy: thus no elastic vibration would occur and the applied torques would be transmitted unchanged to the airscrew. A slight fluctuation of angular velocity of the whole system would result from its acting as a rigid oscillating flywheel of large but finite polar moment of inertia.

If the crankshaft in the region of the cranks is very stiff relative to the airscrew drive, the crank mass as a whole operates as an oscillating flywheel with an amplitude so small (because the torques alternate so rapidly) that there is little torque variation produced in the airscrew drive. It will be understood that in this extreme instance only single node vibration can occur—as there are, in effect, only two flywheel masses, the airscrew and the total crank mass. The conditions are such that the natural frequency is very low in relation to the frequency of any forcing harmonic.

The conditions for aircraft power plants lie between these extremes, and in examining what happens, two simplifying principles are applied, namely:

(*a*) The vibratory movements superposed upon the steady rotation of the system are the same as would occur under the same variation of applied torque if the shaft were not rotating and they are unaffected by the mean transmitted torque.

(*b*) With no damping, or with damping torques proportional to the angular velocity of vibration at each instant, the resultant motion produced by any number of harmonically varying torques, applied to the system at any points, is the vector sum of the motions corresponding to each of the several harmonic torques acting alone.

As regards (*a*), the speed of rotation determines the frequency of the applied harmonic torques and thus comes into consideration although it does not affect the natural frequencies of the system.

As regards (*b*), it is usual to assume that the damping present is of the kind mentioned, *i.e.*, that it is ' viscous-friction ' damping. This assumption is made for the sake of simplicity and, in general, it suffices for practical purposes. Thus the effects of each of the harmonic components of the torque for each cylinder or crank may be considered separately and the resultant effect of any group of such forcing torques may be derived by super-position with due regard to phase.

7b.2 *Multi-crank engines in general: undamped forced vibration away from resonance.*—Picture a series-system of flywheel masses as shown in Fig. 14 undergoing undamped forced vibration due to any harmonic torque $B_m \sin (rt + \beta_m)$ acting at m*. It can be shown that the frequency of the vibration will be that of the forcing torque, and that each mass will be at one or other of its extremes of displacement at the same instant as all other masses as in free vibration.

Let $| \alpha_m |$ denote the amplitude of motion at the point m and λ_m its angle of lag behind the forcing torque. Then the displacement at m at time t is:

$$\alpha_m = | \alpha_m | \sin (rt + \beta_m - \lambda_m) \qquad \text{(94)}$$

The input work per cycle is clearly $\pi B_m | \alpha_m | \sin \lambda_m$. This must be zero, because there is no damping.† It follows that $\lambda_m = 0$ or π unless $| \alpha_m |$ is zero, that is, unless 'm' is at a node, a particular condition which will be considered later.

Substituting 0 and π for λ_m in equation (94), we get: $\alpha_m = \pm | \alpha_m | \sin (rt + \beta_m)$, with the plus and minus sign respectively. Thus all displacements have extreme values when $(rt + \beta_m) = \pi/2$, that is, when the forcing torque reaches its maximum positive value B_m. The system being momentarily at rest at this instant, the magnitudes and signs of the displacements of the several masses may be determined by working through a table similar to that in section 5b.1, but starting with unknown extreme displacement α_1 at mass 1 and adding in the torque B_m at the appropriate place, as shown in the table on the next page.

The value obtained for the torque beyond the last mass is of the form $(a\alpha_1 + bB_m)$ where a and b are positive or negative numerals. By equating this torque to zero, we get:

$$\alpha_1 = -\frac{b}{a} B_m$$

which gives the magnitude and sign of α_1 when $rt + \beta_m = \pi/2$.

The extreme displacements of the several masses at this instant may be evaluated for both magnitude and sign by substituting for α_2 in column 4 of the table: whether λ_m is 0 or π may be determined if desired by substituting in equation (94) remembering that $| \alpha_m |$ is positive by definition. It will be found that the sign of the inertia force for each mass is appropriate to the corresponding sign of displacement.

Two examples are worked out in the following table and the corresponding resultant deflection curves are given in Fig. 14. It will be noted that the deflection for $f = 75 \cdot 8$ does not correspond to a natural mode of vibration: this is because the frequency taken for the applied torque is non-resonant. The node always so locates itself that the natural frequency about it of the part-system on the side remote from m is that of the forcing torque.

When the node coincides with the point m, one part of the system remains stationary and the other part vibrates in antiphase to the applied torque. The second example in the following table has been chosen to illustrate this: note the corresponding deflection curve in Fig. 14. The torque variation in the shafting one side of m has then the same magnitude as the applied torque, and the latter is virtually the reaction torque that would exist in an infinite flywheel if it were added at the node: adding such a flywheel would not affect the motion. This critical condition of m coinciding with a node is revealed in the investigations of R. & M. 1053[1]. It is utilised in the 'Harmonic Balancer' incorporated in motor car engines by General Motors Ltd., U.S.A.; design problems in relation to this balancer may be treated by the general methods outlined here.

When several harmonic torques of known amplitudes, having the same frequency and in phase, operate at various points in the system, the resulting displacements when $rt + \beta_m = \pi/2$ can be obtained by the same tabular method, the several torques being added into the table

* m is usually at a mass but if not, a flywheel of zero inertia may be taken to exist at m, and to reckon as one of the masses. This device enables the analysis to be made in this instance without modifying the form of procedure.

† Except on resonance, in which instance the amplitude of motion increases indefinitely. In dealing with resonant conditions, however, damping is not neglected: this reservation need not trouble us here.

	Mass No.	I/g	$\frac{Ir^2}{g} \times 10^{-3}$	α	$\frac{Ir^2 \alpha}{g} \times 10^{-3}$	$\sum \frac{Ir^2 \alpha}{g} \times 10^{-3}$	$F/10^{-3}$	$\sum \frac{Ir^2 \alpha F}{g}$
Single Node: $f = 75 \cdot 8$/sec, $r^2 = (2\pi f)^2 = 0 \cdot 227 \times 10^6$, $\alpha_1 = \frac{9 \cdot 42}{4251 \cdot 2} = 0 \cdot 0022$	1	0·59	134	α_1	134 α_1	134 α_1	0·000222	0·0298 α_1
	2	0·59	134	0·9702 α_1	130 α_1	264 α_1	0·000222	0·0586 α_1
	3	0·59	134	0·9116 α_1	122·1 α_1	386·1 α_1	0·000222	0·0857 α_1
	4	0·59	134	0·8259 α_1	110·8 α_1	496·9 α_1	0·000222	0·1102 α_1
	5	0·59	134	0·7157 α_1	96 α_1 + 1·585	592·9 α_1 + 1·585	0·000222	0·1316 α_1 + 0·000352
	6	0·59	134	0·5841 α_1 — 0·000352	78·3 α_1 — 0·0472	671·2 α_1 + 1·5378	0·0004	0·2685 α_1 + 0·000615
	7	50·0	11,350	0·3156 α_1 — 0·000967	3580 α_1 — 10·96	4251·2 α — 9·42	—	—
Two Node: f=153/sec, r^2=4 $(2\pi f)^2$=922×10^6, $\alpha_1 = -\frac{40 \cdot 5}{40{,}022} = -0 \cdot 00101$	1	0·59	544	α_1	544 α_1	544 α_1	0·000222	0·121 α_1
	2	0·59	544	0·879 α_1	479 α_1	1023 α_1	0·000222	0·227 α_1
	3	0·59	544	0·652 α_1	355 α_1	1378 α_1	0·000222	0·306 α_1
	4	0·59	544	0·346 α_1	188 α_1	1566 α_1	0·000222	0·346 α_1
	5	0·59	544	0	+ 1·585	1566 α_1 + 1·585	0·000222	0·346 α_1 + 0·000352
	6	0·59	544	—0·346 α_1 — 0·000352	—188 α_1 — 0·192	1378 α_1 + 1·393	0·0004	0·551 α_1 + 0·000557
	7	50·0	46,100	—0·897 α_1 — 0·000909	—41,400 α_1 — 41·9	—40,022 α_1 — 40·5		

at the appropriate places. Separate tables require to be worked out for torques of the same frequency operating at different points in different phases, because the corresponding motions have different phases also. The resultant motion is obtained by combining vectorially the results of the separate tables.

The application of this to the pth order torque component in a multi-crank engine, for a speed at which some other order is resonating, neglecting pth order damping, will be apparent. In R. & M. 1304[15] the matter is examined very fully in regard to the 12-cylinder Liberty engine. In most problems, however, the effects of the resonating torque predominate and, in consequence, we are usually concerned only with resonating torques at their several corresponding critical speeds, the effects of non-resonating harmonics being ignored.

7b.3. *Multi-crank engines in general: damped forced vibration on resonance.*—Consider a series-system of flywheel masses, as shown in Fig. 14, undergoing damped resonant forced vibration due to any harmonic torque $B_m \sin rt$ acting at any point m. The frequency of the vibration will be that of the forcing torque and each mass will be at one or other of its extremes of displacement at the same instant as all the other masses—assuming that the damping forces are not large and are proportional to vibration velocity at corresponding points.

As in section 7b.2, the motion of point m is:

$$\alpha_m = |\alpha_m| \sin (rt + \beta_m - \lambda_m) \qquad (94)$$

and the input work per cycle is $\pi B_m |\alpha_m| \sin \lambda_m$.

At resonance, however, the phase of the motion of the point of application of the forcing torque is such that the input work is the maximum possible, that is to say: $\lambda_m = \pi/2$.

The amplitude of motion is such that the work done per cycle against the damping torque is equal to the input work per cycle, that is, to: $\pi B_m |\alpha_m|$.

If all the damping were effected at point m, the damping torque would at each instant counter-balance the forcing torque at its point of application and the system would swing in the natural mode corresponding to the frequency concerned. The deflection curve would thus correspond proportionately to the deflection curve derived when determining the natural frequency in this mode; the actual amplitude at m would be that which causes the maximum damping torque to have magnitude B_m.

If h denote the damping torque per unit of vibrational velocity at m, we have:

$$\text{Damping torque} = -h\frac{d\alpha_m}{dt} = -hr|\alpha_m| \cos\left(rt + \beta_m - \frac{\pi}{2}\right)$$

$$= -hr |\alpha_m| \sin (rt + \beta_m).$$

So that when $(rt + \beta_m) = \pi/2$ the damping torque is: $-hr |\alpha_m|$ which is equal to $-B_m$. Thus:

$$|\alpha_m| = \frac{B_m}{hr}$$

and the damping energy per cycle is: $\pi hr |\alpha_m|^2$.

Also, at this instant, the displacement of m is $-|\alpha_m|$ and both the forcing torque and the damping torque are zero.

If damping torques act at various points in the system, being: $-h_1 r |\alpha_1| \sin (rt + \beta_1)$; $-h_2 r |\alpha_2| \sin (rt + \beta_2) \ldots\ldots -h_m r |\alpha_m| \sin (rt + \beta_m) \ldots$ *etc.*, the curve of maximum deflection will approximate very closely to that for free vibration at the resonant frequency concerned: the forcing and damping torques are zero at the extremes of the motion, and shaft torques balance inertia torques then, as in free vibration. At any instant, the torque at any point is that due to free swinging motion plus the resultant of the forcing and damping torques.

The phase difference between damping torques and motion is $\pi/2$ and thus the total work done per cycle by the system against damping is:

$$\pi r \ \Sigma\{h \mid \alpha \mid^2\} = \pi B_m \mid \alpha_m \mid$$

where Σ signifies algebraic summation.

If θ_1, θ_2, ... *etc.*, be ordinates of free deflection corresponding to $\mid \alpha_1 \mid$, $\mid \alpha_2 \mid$, ... *etc.*, then:

$$\frac{\mid \alpha_1 \mid}{\theta_1} = \frac{\mid \alpha_2 \mid}{\theta_2} \cdots\cdots\cdots = \frac{\mid \alpha_m \mid}{\theta_m} = \cdots\cdots\cdots \frac{\mid \alpha \mid}{\theta} \cdots$$

and by substituting in the above energy equation we get the following expression for the movement at mass 1.

$$\mid \alpha_1 \mid = \frac{\theta_1 \theta_m B_m}{r \ \Sigma \ (h\theta^2)}$$

Thus the amplitude of vibration produced by B_m for given values of h and r is proportional to B_m and also to θ_m. The magnitude of the product $B_m \, \theta_m$ is thus a relative criterion of the effectiveness of an harmonic torque in forcing resonant vibration in the corresponding mode.

It should be noted that:

$$\alpha_1 = \mid \alpha_1 \mid \sin\left(rt + \beta_m - \frac{\pi}{2}\right)$$

which is the projection in elevation of the vector:

$$\overrightarrow{\frac{\theta_1 B_m \theta_m}{r \ \Sigma \ (h\theta^2)} , \left(rt + \beta_m - \frac{\pi}{2}\right)} .$$

Where several harmonic torques of the same frequency and known relative phases act at various points in the system, the resultant vibration at No. 1 mass is obtained by means of vector summation or its trigonometrical equivalent, whichever may happen to be the more convenient.

Thus, if $B_n \sin (rt + B_n)$ denote the resultant forcing torque of frequency $r/(2\pi)$ acting on mass n, the resultant displacement α_{1n} of No. 1 mass due to this forcing torque is represented by the projection in elevation of the vector:

$$\overrightarrow{\frac{\theta_1 B_n \theta_n}{r \ \Sigma \ (h\theta^2)} , \left(rt + \beta_n - \frac{\pi}{2}\right)}$$

and so on for other crank masses.

Usually it is only necessary to determine the maximum amplitude of the resultant motion of No. 1 mass, which leads to simplification of procedure:

As the quantity $\theta_1 / r \, \Sigma (h\theta^2)$ is a common factor in the magnitudes of the several vectors, and as $(rt - \pi/2)$ is a common part of their directions, the magnitudes of the resultant is $\theta_1 \ \Sigma \ (B\theta) / r \ \Sigma \ (h\theta^2)$ where $\Sigma \ (B\theta)$ denotes the magnitude of the closing side of the polygon formed by the vectors:

$$\overrightarrow{B\theta, \ \beta}$$

for all the crank masses.

As in section 7a.5, let the pth order forcing harmonic on any crank mass be represented by

$$ZA_p \sin \{p(\omega t + \varepsilon_p')\}$$

Then, by comparison, we put ZA_p for B: $p\omega$ for r: and $p\varepsilon_p'$ for β. Assume that the damping torque per unit vibrational velocity has the same value h_c at each crank mass, and the value h_a at the airscrew: let θ_c denote values for displacement curves at the several cranks and θ_a the value at the airscrew—when θ_1 is the displacement for No. 1 mass on the displacement curve.

The amplitude of the pth order vibration at No. 1 crank is clearly:

$$\alpha_{1p} = \frac{\theta_1 \, \Sigma \, (ZA_p\theta_c)}{p\omega[h_c \, \Sigma \, (\theta_c^2) + h_a\theta_a^2]} \qquad (95)$$

where the summation in the numerator is vectorial and that in the denominator is arithmetic.

The vectors concerned in determining $\Sigma \, (ZA_p\theta_c)$ are:

$$\overrightarrow{ZA_p\theta_c \, , \, p\varepsilon_p'}$$

in respect of the several cranks.

Conditions repeat after two crankshaft revolutions of a four-stroke engine: therefore the differences of ε_p' add up to 4π radians for the complete engine. The fact that half these differences add up to 2π is represented by the ' Phase diagram ' shown in Fig. 15. The $\frac{1}{2}$ order resultant vectors for the several cranks are thus parallel to the radiating lines shown and the corresponding vector diagram is as indicated.

The intervals between the vectors for pth order vibrations are $2p$ times the corresponding angles for the $\frac{1}{2}$ order: phase diagrams can be drawn accordingly (as indicated in the figure) and used for deriving vector diagrams or for getting the magnitudes and signs of the projections in plan and elevation of the several vectors. Denote these projections by H and V respectively: then the algebraic summations Σ(H) and Σ(V) are respectively the horizontal and vertical components of the resultant of the vector diagram, so that the magnitude of the resultant is:

$$\sqrt{[\{\Sigma\,(\mathrm{H})\}^2 + \{\Sigma\,(\mathrm{V})\}^2]}$$

It will be noted that $\Sigma \, (ZA_p\theta_c)$ is a relative measure of the magnitudes of criticals in a given mode when h_c and h_a are unvarying but not necessarily otherwise known: when the pth order harmonic torques on the cranks all have the same value, the quantity ZA_p may be put in front of the summation sign.

8. *Amplified Torques and Stress Ranges.*—To determine the absolute values of α_{1p} from equation (95) values require to be assigned to the damping quantities h_c and h_a. For a *rigid* airscrew so attached to the shaft that there is little mechanical hysteresis in the connection, the value of $h_a\theta_a^2$ is small in comparison with $h_c \Sigma \, (\theta_c^2)$, because there is a node near the airscrew and in this case the aerodynamic damping is small (R. & M. 1304[15]). Values of h_a are deduced for particular airscrews in R. & M. 1562[16] and 1557[17] but the deduction is made from very limited data and until the matter has been explored more completely, values of h_c are better taken on the basis that h_a is zero. With a wooden airscrew, relative movement between flanges and boss may give h_a an important value, but then charring of the boss is likely to occur and the condition is not one to be reckoned with for normal operation.

The quantity h_c depends upon engine design and engine size[18] and some progress is made towards setting limits to the uncertainty of its value if the size factor can be separated out.

Comparison of results of a number of torsiograph experiments has suggested the empirical rule that h_c varies as the fourth power of linear dimensions. According to this, we may write:

$$h_c = E_c \left(\frac{I_c}{386} \right)^{4/5} \text{ lb/in. per radn/sec} \quad \ldots \quad (96)$$

where I_c is the polar moment of inertia per crank in pounds inches squared and E_c is a coefficient for which the following values have been obtained from analysis of torsiograph observations.

Engine	h_c	E_c	E_c divided by the number of cylinders per crank
12-cylinder ungeared Vee	25	42	21
14-cylinder two-throw ungeared radial ..	234	164	23
9-cylinder geared radial	1800	360	40
14-cylinder two-throw geared radial ..	271	104	15
16-cylinder two-crank geared H engine ..	5·5	56	28

The last column indicates that the value of E_c is roughly proportional to the number of cylinders per crank: close agreement cannot be expected. There is evidence that in a given system E_c increases with forcing torque (*see* Fig. 16).

Having assigned a value to E_c by comparison with experimentally determined values for similar engines, the corresponding value of h_c can be inserted in equation (95) to obtain α_{1p} values for the several criticals of interest and for appropriate conditions of running.

The corresponding torque oscillations in any part of the system can then be deduced, as the angles of twist between adjacent masses are known: the mean torque transmitted through the part may be superposed. To deduce the corresponding stress limits in the region of the cranks is a matter which does not come within the province of this report but stress ranges in the drive due to torsion only can be computed having due regard to stress concentration factors. Torque variation in the airscrew drive corresponding to the various orders of vibration may conveniently be shown in graphical form on a base of engine speed. This has been done in Fig. 7 of Part III. The peak heights relative to the indicated mean torque line have been computed from α_{1p} values and the shapes of the curves leading to the peaks have been obtained as follows:—

Let x denote the ratio of the speed taken to the critical speed concerned

y denote the corresponding amplitude of the superposed airscrew torque variation for the order concerned

y_m denote the value of y at the critical

R the ratio: (airscrew r.p.m.)/(crankshaft r.p.m.)

$\frac{1}{\Delta}$ denote the ratio of torque amplification at the peak, that is $\frac{1}{\Delta} = y_m \Big/ \frac{ZA_p \; \Sigma\theta_c \text{ (on resonance)}}{R}$

Then:

$$y = \frac{ZA_p \; \Sigma \; (\theta_c)}{R\sqrt{\{(1 - x^2)^2 + \Delta^2 x^2\}}} \quad \ldots \quad (97)$$

where $ZA_p \; \Sigma \; (\theta_c)$ must be taken for the off-resonance condition defined by x: for practical purposes it suffices to adopt the $\Sigma \; (\theta_c)$ value that applies at resonance.

The method of obtaining these curves by applying the above formula is worked out by way of example in section 8.3 of Part III. The practical application is limited to engines with crankshafts having mirror symmetry.

To estimate the stress ranges it is necessary to take account of the effect of all the orders operating at the speed selected. Ideally these effects should be added with due regard to their phase relationship at the particular speed but when regard is taken of the uncertainty connected with the assumption of the damping factor and the value taken for the stress concentration factor, such an extensive computation is not justified. It is therefore considered sufficiently accurate to add the effects arithmetically, this will of course give results which as a rule will be greater than the true value. In the engine considered in Part III, the third order critical is the most important and here it will be seen that the side bands from the other criticals give but a small addition to the torque oscillation so that, at this critical, the error involved by arithmetic summation is of the second order of magnitude.

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PART II. TABLE 1

Harmonic Coefficients

Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3\cdot43}$ Compression Ratio 5·3:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Cosine Terms												
½	a_1	10·50	11·25	12·10	13·00	14·00	15·00	16·10	17·20	18·30	19·55	20·80	22·20	23·55
1	a_2	4·25	4·60	5·00	5·40	5·80	6·25	6·70	7·20	7·60	8·05	8·50	9·10	9·80
1½	a_3	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15
2	a_4	−0·90	−1·20	−1·50	−1·80	−2·10	−2·40	−2·70	−3·00	−3·30	−3·60	−3·90	−4·20	−4·45
2½	a_5	−3·05	−3·30	−3·50	−3·75	−4·00	−4·25	−4·50	−4·70	−4·95	−5·15	−5·40	−5·60	−5·90
3	a_6	−2·40	−2·60	−2·75	−3·00	−3·20	−3·40	−3·60	−3·75	−4·00	−4·20	−4·40	−4·60	−4·80
3½	a_7	−2·20	−2·40	−2·65	−2·90	−3·15	−3·40	−3·65	−3·90	−4·20	−4·40	−4·65	−4·90	−5·15
4	a_8	−2·50	−2·55	−2·60	−2·75	−2·85	−2·95	−3·05	−3·25	−3·50	−3·75	−4·05	−4·35	−4·65
4½	a_9	−2·05	−2·10	−2·15	−2·25	−2·35	−2·45	−2·60	−2·75	−2·95	−3·20	−3·40	−3·70	−3·95
5	a_{10}	−0·75	−0·95	−1·15	−1·35	−1·55	−1·75	−2·00	−2·20	−2·40	−2·60	−2·80	−3·00	−3·25
5½	a_{11}	−0·45	−0·60	−0·75	−0·95	−1·15	−1·30	−1·45	−1·65	−1·80	−2·00	−2·15	−2·35	−2·50
6	a_{12}	−0·25	−0·40	−0·55	−0·75	−0·90	−1·05	−1·20	−1·35	−1·50	−1·65	−1·80	−2·00	−2·10

Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
½	a_1	25·00	26·50	27·95	29·40	30·80	32·25	33·60	35·00	36·40	37·80	39·25	40·55	42·00
1	a_2	10·45	11·20	11·95	12·70	13·50	14·25	14·90	15·50	16·10	16·75	17·50	18·25	19·20
1½	a_3	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15	−0·15
2	a_4	−4·70	−4·90	−5·10	−5·35	−5·55	−5·80	−6·00	−6·20	−6·40	−6·55	−6·70	−6·80	−6·90
2½	a_5	−6·10	−6·35	−6·60	−6·80	−7·10	−7·30	−7·50	−7·75	−8·00	−8·25	−8·50	−8·70	−8·90
3	a_6	−5·00	−5·20	−5·35	−5·55	−5·75	−5·90	−6·10	−6·35	−6·55	−6·70	−6·80	−6·85	−6·90
3½	a_7	−5·40	−5·60	−5·90	−6·10	−6·40	−6·60	−6·85	−7·10	−7·40	−7·60	−7·80	−8·10	−8·25
4	a_8	−4·90	−5·15	−5·45	−5·75	−6·00	−6·30	−6·60	−6·90	−7·20	−7·50	−7·75	−8·00	−8·25
4½	a_9	−4·20	−4·45	−4·70	−4·95	−5·20	−5·40	−5·70	−5·95	−6·20	−6·40	−6·70	−6·90	−7·20
5	a_{10}	−3·45	−3·65	−3·85	−4·05	−4·25	−4·50	−4·70	−4·90	−5·10	−5·25	−5·45	−5·65	5·85
5½	a_{11}	−2·65	−2·80	−3·00	−3·20	−3·35	−3·50	−3·70	−3·85	−4·05	−4·20	−4·40	−4·55	−4·75
6	a_{12}	−2·25	−2·40	−2·55	−2·70	−2·85	−3·05	−3·20	−3·35	−3·50	−3·65	−3·80	−4·00	−4·20

Electric Ignition Engine

PART II. TABLE 1a

Harmonic Coefficients

Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3 \cdot 43}$ Compression Ratio 5·3:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Cosine Terms												
6½	a_{13}	—0·25	—0·40	—0·50	—0·65	—0·75	—0·90	—1·00	—1·10	—1·20	—1·30	—1·40	—1·50	—1·60
7	a_{14}	—0·25	—0·30	—0·40	—0·50	—0·55	—0·65	—0·75	—0·80	—0·90	—0·95	—1·05	—1·10	—1·20
7½	a_{15}	0·00	—0·05	—0·15	—0·25	—0·30	—0·40	—0·50	—0·55	—0·60	—0·70	—0·75	—0·85	—0·95
8	a_{16}	0·10	0·00	—0·10	—0·20	—0·25	—0·35	—0·40	—0·50	—0·60	—0·70	—0·75	—0·85	—0·90
8½	a_{17}	—0·05	—0·10	—0·20	—0·25	—0·35	—0·40	—0·50	—0·55	—0·65	—0·70	—0·75	—0·85	—0·90
9	a_{18}	0·05	0·00	—0·05	—0·15	—0·20	—0·30	—0·35	—0·45	—0·50	—0·55	—0·65	—0·70	—0·80
9½	a_{19}	0·05	0·00	—0·05	—0·10	—0·15	—0·20	—0·25	—0·30	—0·35	—0·40	—0·45	—0·50	—0·50
10	a_{20}	0·50	0·00	0·00	—0·05	—0·10	—0·10	—0·15	—0·20	—0·25	—0·30	—0·35	—0·40	—0·45
10½	a_{21}	0·20	0·15	0·10	0·05	0·00	—0·05	—0·05	—0·10	—0·15	—0·20	—0·25	—0·25	—0·30
11	a_{22}	0·20	0·15	0·10	0·05	0·00	—0·05	0·05	—0·10	—0·10	—0·15	—0·15	—0·20	—0·25
11½	a_{23}	0·20	0·15	0·15	0·10	0·05	0·05	0·00	—0·05	—0·10	—0·10	—0·15	—0·20	—0·25
12	a_{24}	0·20	0·20	0·15	0·15	0·10	0·10	0·05	0·05	0·00	0·00	—0·05	—0·05	—0·10
Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
6½	a_{13}	—1·75	—1·90	—2·05	—2·20	—2·40	—2·60	—2·80	—3·00	—3·20	—3·40	—3·60	—3·80	—4·10
7	a_{14}	—1·30	—1·45	—1·60	—1·75	—1·90	—2·05	—2·25	—2·40	—2·60	—2·80	—3·05	—3·30	—3·60
7½	a_{15}	—1·05	—1·14	—1·25	—1·35	—1·50	—1·65	—1·80	—1·95	—2·10	—2·30	—2·50	—2·75	—3·00
8	a_{16}	—1·00	—1·15	—1·25	—1·35	—1·50	—1·60	—1·75	—1·90	—2·05	2·20	—2·35	—2·55	—2·75
8½	a_{17}	—0·95	—1·05	—1·10	—1·15	—1·20	—1·30	—1·35	—1·45	—1·50	—1·60	—1·65	—1·75	—1·85
9	a_{18}	—0·85	—0·90	—0·95	—1·05	—1·10	—1·15	—1·20	—1·30	—1·40	—1·45	—1·50	—1·60	—1·70
9½	a_{19}	—0·55	—0·60	—0·65	—0·65	—0·70	—0·75	—0·80	—0·85	—0·90	—0·95	—0·00	—1·05	—1·10
10	a_{20}	—0·50	—0·50	—0·55	—0·60	—0·65	—0·70	—0·75	—0·80	—0·85	—0·90	—0·95	—1·00	—1·05
10½	a_{21}	—0·30	—0·35	—0·40	—0·45	—0·50	—0·55	—0·60	—0·65	—0·70	—0·75	—0·80	—0·85	—0·90
11	a_{22}	—0·30	—0·30	—0·35	—0·35	—0·40	—0·45	—0·50	—0·50	—0·55	—0·60	—0·65	—0·70	—0·75
11½	a_{23}	—0·30	—0·35	—0·40	—0·45	—0·45	—0·50	—0·55	—0·60	—0·65	—0·70	—0·75	—0·80	—0·85
12	a_{24}	—0·10	—0·15	—0·15	—0·20	—0·20	—0·25	—0·25	—0·30	—0·30	—0·35	—0·35	—0·40	—0·40

Electric Ignition Engine

PART II. TABLE 2

Harmonic Coefficients

Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3\cdot 43}$ Compression Ratio 5·3:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Sine Terms												
½	b_1	7·15	7·90	8·75	9·60	10·05	11·45	12·50	13·50	14·60	15·65	16·75	17·90	19·00
1	b_2	13·95	14·75	15·50	16·50	17·55	18·70	19·85	21·20	22·60	24·20	25·75	27·40	29·05
1½	b_3	13·90	14·65	15·35	16·25	17·15	18·20	19·30	20·45	21·75	23·10	24·50	25·90	27·40
2	b_4	10·60	11·05	11·50	12·10	12·75	13·45	14·20	15·00	15·80	16·70	17·60	18·60	19·65
2½	b_5	8·00	8·25	8·55	8·90	9·30	9·70	10·15	10·60	11·20	11·80	12·40	13·05	13·75
3	b_6	6·00	6·10	6·25	6·40	6·60	6·85	7·15	7·50	7·90	8·30	8·75	9·25	9·75
3½	b_7	4·00	4·10	4·25	4·40	4·60	4·80	5·00	5·25	5·50	5·85	6·25	6·60	7·00
4	b_8	2·50	2·55	2·60	2·75	2·90	3·00	3·10	3·25	3·40	3·60	3·80	4·00	4·25
4½	b_9	1·50	1·55	1·60	1·70	1·80	1·90	1·95	2·00	2·05	2·10	2·15	2·25	2·55
5	b_{10}	0·80	0·85	0·85	0·85	0·85	0·90	0·90	0·90	0·95	0·95	1·00	1·00	1·05
5½	b_{11}	0·30	0·30	0·30	0·30	0·35	0·35	0·35	0·35	0·40	0·40	0·45	0·45	0·50
6	b_{12}	−0·10	−0·10	−0·10	−0·10	−0·10	−0·10	−0·10	−0·10	−0·05	−0·05	−0·05	0	0·05

Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
½	b_1	20·10	21·25	22·35	23·50	24·70	25·75	26·80	27·90	29·00	30·10	31·10	32·15	33·15
1	b_2	30·75	32·40	34·05	35·75	37·40	39·00	40·60	42·30	43·90	45·50	47·20	48·80	50·40
1½	b_3	28·90	30·45	32·00	33·50	35·05	36·60	38·10	39·60	41·10	42·60	44·15	45·60	47·15
2	b_4	20·70	21·80	22·90	23·95	25·00	26·10	27·20	28·25	29·30	30·40	31·55	32·70	33·85
2½	b_5	14·45	15·15	15·85	16·55	17·30	18·00	18·75	19·50	20·20	20·90	21·60	22·35	23·05
3	b_6	10·25	10·80	11·40	11·90	12·40	12·90	13·40	13·90	14·35	14·80	15·35	15·95	16·55
3½	b_7	7·35	7·75	8·10	8·50	8·90	9·25	9·60	10·00	10·30	10·75	11·20	11·70	12·20
4	b_8	4·45	4·65	4·90	5·15	5·40	5·60	5·90	6·10	6·40	6·65	7·10	7·60	8·20
4½	b_9	2·45	2·55	2·70	2·85	3·00	3·10	3·30	3·45	3·60	3·90	4·25	4·60	5·10
5	b_{10}	1·15	1·30	1·40	1·50	1·60	1·75	1·85	2·00	2·10	2·40	2·75	3·15	3·70
5½	b_{11}	0·55	0·60	0·70	0·80	0·90	1·00	1·10	1·13	1·50	1·80	2·10	2·40	2·80
6	b_{12}	0·05	0·10	0·15	0·25	0·35	0·45	0·50	0·55	0·60	0·80	1·05	1·50	2·10

Electric Ignition Engine

PART II. TABLE 2a

Harmonic Coefficients

Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3 \cdot 43}$ Compression Ratio 5·3:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Sine Terms												
6½	b_{13}	—0·35	—0·35	—0·35	—0·35	—0·30	—0·30	—0·30	—0·25	—0·25	—0·25	—0·25	—0·25	—0·20
7	b_{14}	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·45	—0·45	—0·45	—0·45
7½	b_{15}	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·50	—0·45	—0·45	—0·45	—0·45
8	b_{16}	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35
8½	b_{17}	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35
9	b_{18}	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35
9½	b_{19}	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30
10	b_{20}	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30
10½	b_{21}	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25
11	b_{22}	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20
11½	b_{23}	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15
Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
6½	b_{13}	—0·20	—0·20	—0·15	—0·15	—0·15	—0·10	—0·10	—0·10	—0·05	—0·05	—0·05	0·00	0·00
7	b_{14}	—0·45	—0·45	—0·45	—0·45	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40
7½	b_{15}	—0·45	—0·45	—0·45	—0·45	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40	—0·40
8	b_{16}	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30
8½	b_{17}	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35
9	b_{18}	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35	—0·35
9½	b_{19}	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30
10	b_{20}	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30	—0·30
10½	b_{21}	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25	—0·25
11	b_{22}	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20	—0·20
11½	b_{23}	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15	—0·15

Electric Ignition Engine

(99681) c

PART II. TABLE 3

Harmonic Coefficients

C.I. Engine. Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3 \cdot 46}$ Compression Ratio 13·5:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Cosine Terms												
½	a_1	9·00	10·40	11·85	13·50	14·70	16·15	17·60	19·00	20·40	21·80	23·25	24·70	26·15
1	a_2	4·00	4·85	5·70	6·60	7·45	8·30	9·25	10·10	10·95	11·85	12·70	13·60	14·40
1½	a_3	0·20	0·45	0·75	1·00	1·25	1·50	1·75	2·10	2·30	2·60	2·80	3·10	3·35
2	a_4	0·75	0·70	0·65	0·60	0·55	0·50	0·45	0·40	0·35	0·30	0·25	0·20	0·15
2½	a_5	−0·80	−1·00	−1·25	−1·50	−1·70	−1·95	−2·15	−2·40	−2·60	−2·80	−3·00	−3·25	−3·50
3	a_6	−0·25	−0·50	−0·70	−0·95	−0·20	−1·40	−1·65	−1·90	−2·15	−2·40	−2·70	−3·00	−3·30
3½	a_7	−0·75	−1·00	−1·15	−1·40	−1·60	−1·85	−2·10	−2·35	−2·60	−2·85	−3·15	−3·50	−3·85
4	a_8	−0·60	−0·85	−1·05	−1·25	−1·50	−1·75	−2·00	−2·25	−2·55	−2·80	−3·10	−3·40	−3·70
4½	a_9	−0·55	−0·80	−1·00	−1·25	−1·50	−1·75	−2·00	−2·25	−2·45	−2·70	−2·90	−3·15	−3·40
5	a_{10}	−0·65	−0·90	−1·10	−1·35	−1·55	−1·75	−1·95	−2·20	−2·40	−2·60	−2·85	−3·10	−3·35
5½	a_{11}	−0·60	−0·85	−1·05	−1·30	−1·50	−1·60	−1·85	−2·10	−2·35	−2·55	−2·70	−2·90	−3·10
6	a_{12}	−0·75	−0·90	−1·10	−1·25	−1·40	−1·60	−1·80	−1·95	−2·15	−2·35	−2·50	−2·65	−2·85
Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
½	a_1	27·50	28·90	30·35	31·80	33·20	34·65	36·05	37·40	38·85	40·20	41·65	43·10	44·50
1	a_2	15·30	16·15	17·05	17·90	18·75	19·60	20·40	21·35	22·15	23·00	23·90	24·75	25·60
1½	a_3	3·60	3·90	4·20	4·40	4·70	4·90	5·15	5·40	5·70	5·90	6·20	6·45	6·70
2	a_4	0·05	−0·05	−0·15	−0·25	−0·35	−0·50	−0·65	−0·80	−1·00	−1·25	−1·50	−1·80	−2·15
2½	a_5	−3·75	−4·00	−4·25	−4·50	−4·80	−5·10	−5·40	−5·70	−6·00	−6·35	−6·70	−7·10	−7·50
3	a_6	−3·60	−3·90	−4·20	−4·60	−4·90	−5·30	−5·70	−6·15	−6·65	−7·20	−7·75	−8·35	−9·00
3½	a_7	−4·20	−4·60	−4·90	−5·25	−5·70	−6·10	−6·55	−7·00	−7·50	−8·10	−8·70	−9·30	−10·00
4	a_8	−4·00	−4·30	−4·60	−5·00	−5·40	−5·75	−6·20	−6·60	−7·20	−7·75	−8·40	−9·00	−9·70
4½	a_9	−3·65	−3·95	−4·25	−4·55	−4·85	−5·15	−5·50	−5·80	−6·10	−6·50	−6·85	−7·25	−7·70
5	a_{10}	−3·60	−3·85	−4·10	−4·35	−4·60	−4·90	−5·20	−5·50	−5·80	−6·14	−6·50	−6·85	−7·25
5½	a_{11}	−3·35	−3·55	−3·75	−4·00	−4·25	−4·50	−4·75	−5·00	−5·25	−5·55	−5·80	−6·10	−6·40
6	a_{12}	−3·05	−3·25	−3·40	−3·65	−3·85	−4·10	−4·35	−4·60	−4·85	−5·10	−5·40	−5·70	−6·00

Compression Ignition Engine

PART II. TABLE 3a

Harmonic Coefficients

C.I. Engine. Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3\cdot46}$ Compression Ratio 13·5:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Cosine Terms												
6½	a_{13}	−0·80	−1·00	−1·15	−1·35	−1·50	−1·65	−1·85	−2·00	−2·15	−2·35	−2·50	−2·60	−2·75
7	a_{14}	−1·05	−1·20	−1·35	−1·45	−1·60	−1·70	−1·85	−2·00	−2·10	−2·25	−2·40	−2·50	−2·65
7½	a_{15}	−1·15	−1·25	−1·35	−1·45	−1·55	−1·65	−0·75	−1·85	−1·95	−2·05	−2·15	−2·25	−2·35
8	a_{16}	−1·10	−1·20	−1·35	−1·45	−1·55	−1·65	−1·75	−1·85	−2·00	−2·10	−2·20	−2·30	−2·40
8½	a_{17}	−1·25	−1·35	−1·45	−1·55	−1·65	−1·75	−1·85	−1·05	−2·05	−2·15	−2·25	−2·35	−2·45
9	a_{18}	−1·15	−1·25	−1·35	−1·45	−1·55	−1·65	−1·75	−1·85	−1·95	−2·05	−2·15	−2·25	−2·35
9½	a_{19}	−1·20	−1·30	−1·35	−1·45	−1·50	−1·60	−1·65	−1·75	−1·85	−1·95	−2·05	−2·15	−2·20
10	a_{20}	−1·15	−1·25	−1·35	−1·45	−1·50	−1·60	−1·70	−1·80	−1·90	−2·00	−2·10	−2·20	−2·30
10½	a_{21}	−1·25	−1·35	−1·45	−1·55	−1·60	−1·70	−1·80	−1·90	−2·00	−2·05	−2·15	−2·25	−2·35
11	a_{22}	−1·10	−1·35	−1·55	−1·70	−1·85	−2·00	−2·15	−2·25	−2·35	−2·40	−2·45	−2·50	−2·55
11½	a_{23}	−1·15	−1·40	−1·60	−1·80	−1·95	−2·10	−2·20	−2·25	−2·35	−2·40	−2·50	−2·55	−2·60
12	a_{24}	−1·60	−0·70	−0·80	−0·90	−1·00	−1·10	−1·15	−1·20	−1·25	−1·30	−1·30	−1·35	−1·35

Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
6½	a_{13}	−2·90	−3·10	−3·20	−3·35	−3·50	−3·65	−3·85	−4·00	−4·15	−4·30	−4·50	−4·65	−4·80
7	a_{14}	−2·75	−2·90	−3·00	−3·10	−3·25	−3·40	−3·55	−3·70	−3·85	−4·00	−4·10	−4·25	−4·40
7½	a_{15}	−2·45	−2·55	−2·65	−2·75	−2·85	−2·95	−3·05	−3·15	−3·25	−3·35	−3·45	−3·55	−3·65
8	a_{16}	−2·50	−2·60	−2·75	−2·85	−3·00	−3·10	−3·20	−3·35	−3·45	−3·55	−3·70	−3·80	−3·90
8½	a_{17}	−2·55	−2·60	−2·70	−2·80	−2·90	−3·00	−3·10	−3·20	−3·30	−3·40	−3·50	−3·60	−3·70
9	a_{18}	−2·45	−2·55	−2·65	−2·75	−2·85	−2·95	−3·00	−3·10	−3·20	−3·30	−3·40	−3·50	−3·60
9½	a_{19}	−2·30	−2·40	−2·50	−2·55	−2·60	−2·70	−2·80	−2·85	−2·90	−3·00	−3·10	−3·15	−3·20
10	a_{20}	−2·40	−2·45	−2·55	−2·65	−2·70	−2·80	−2·90	−3·00	−3·05	−3·10	−3·20	−3·30	−3·40
10½	a_{21}	−2·45	−2·55	−2·60	−2·70	−2·80	−2·90	−3·00	−3·05	−3·15	−3·20	−3·30	−3·40	−3·50
11	a_{22}	−2·60	−2·60	−2·60	−2·65	−2·65	−2·65	−2·65	−2·65	−2·75	−2·75	−2·75	−2·75	−2·75
11½	a_{23}	−2·65	−2·65	−2·65	−2·65	−2·70	−2·70	−2·70	−2·70	−2·70	−2·70	−2·70	−2·70	−2·70
12	a_{24}	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40	−1·40

Compression Ignition Engine

PART II. TABLE 4

Harmonic Coefficients

C.I. Engine. Single Cylinder 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3\cdot46}$ Compression Ratio 13·5:1

Net I.M.E.P. lb per sq in.		25	30	35	40	45	50	55	60	65	70	75	80	85
Order No.	Harmonic Coefft.	Sine Terms												
½	b_1	13·70	14·40	15·15	15·85	16·55	17·25	18·00	18·75	19·45	20·15	20·85	21·60	22·30
1	b_2	23·20	24·35	25·50	26·70	27·80	28·90	30·00	31·00	32·00	32·95	33·90	34·90	36·00
1½	b_3	25·70	26·85	28·00	29·15	30·30	31·45	32·55	33·65	34·65	35·60	36·55	37·55	38·60
2	b_4	23·00	24·10	25·10	26·00	26·85	27·65	28·45	29·20	29·95	30·75	31·60	32·50	33·35
2½	b_5	20·15	20·90	21·65	22·40	23·15	23·85	24·50	25·15	25·65	26 20	26·70	27·20	27·70
3	b_6	16·75	17·25	17·75	18·25	18·75	19·25	19·75	20·20	20·70	21·20	21·70	22·20	22·70
3½	b_7	13·80	14·15	14·55	14·90	15·25	15·60	15·95	16·30	16·65	17·00	17·35	17·70	18·10
4	b_8	11·25	11·50	11·75	12·00	12·25	12·50	12·75	12·95	13·20	13·45	13·70	13·95	14·20
4½	b_9	9·45	9·60	9·75	9·90	10·05	10·20	10·35	10·50	10·65	10·80	10·95	11·10	11·25
5	b_{10}	7·55	7·65	7·75	7·85	7·95	8·10	8·20	8·30	8·40	8·50	8·60	8·70	8·80
5½	b_{11}	6·25	6·30	6·40	6·45	6·50	6·55	6·60	6·70	6·75	6·85	6·90	7·00	7·05
6	b_{12}	5·15	5·20	5·25	5·30	5·35	5·40	5·45	5·50	5·50	5·55	5·60	5·65	5·70

Net I.M.E.P. lb per sq in.		90	95	100	105	110	115	120	125	130	135	140	145	150
½	b_1	23·00	23·75	24·45	25·15	25·85	26·60	27·30	28·00	28·75	29·45	30·15	30·85	31·55
1	b_2	37·15	38·35	39·50	40·75	42·00	43·20	44·40	45·60	46·85	48·15	49·40	50·70	52·00
1½	b_3	39·75	40·90	42·10	43·40	44·70	46·00	47·35	48·75	50·10	51·50	52·85	54·25	55·60
2	b_4	34·30	35·20	36·10	37·10	38·15	39·25	40·35	41·50	42·75	44·00	45·40	46·85	48·40
2½	b_5	28·25	28·85	29·50	30·15	30·90	31·70	32·60	33·60	34·65	35·75	37·00	38·30	39·70
3	b_6	23·20	23·70	24·20	24·70	25·15	25·65	26·15	26·65	27·15	27·65	28·15	28·65	29·15
3½	b_7	18·45	18·80	19·15	19·50	19·85	20·20	20·55	20·90	21·25	21·60	21·95	22·35	22·70
4	b_8	14·40	14·65	14·90	15·15	15·35	15·60	15·85	16·10	16·35	16·55	16·80	17·05	17·30
4½	b_9	11·40	11·55	11·70	11·85	11·95	12·10	12·25	12·40	12·55	12·70	12·85	13·00	13·15
5	b_{10}	8·90	9·00	9·10	9·20	9·30	9·40	9·55	9·65	9·75	9·85	9·95	10·05	10·15
5½	b_{11}	7·10	7·15	7·25	7·35	7·40	7·45	7·50	7·55	7·60	7·70	7·75	7·85	7·90
6	b_{12}	5·75	5·80	5·85	5·90	5·95	6·00	6·00	6·05	6·10	6·15	6·20	6·25	6·30

Compression Ignition Engine

PART II. TABLE 4a

Harmonic Coefficients

C.I. Engine. Single Cylinder, 1 sq in. Piston Area and 1 in. Crankthrow

Obliquity Ratio $\frac{R}{L} = \frac{1}{3 \cdot 46}$ Compression Ratio 13·5:1

Order No.	Harmonic Coefft.	Net I.M.E.P. lb per sq in. 25	30	35	40	45	50	55	60	65	70	75	80	85
		Sine Terms												
6½	b_{13}	3·80	4·10	4·35	4·50	4·60	4·70	4·75	4·75	4·75	4·75	4·75	4·75	4·75
7	b_{14}	3·00	3·25	3·50	3·70	3·80	3·90	4·00	4·00	4·00	4·00	4·00	4·00	4·00
7½	b_{15}	2·65	2·75	2·85	2·90	2·95	3·00	3·00	3·05	3·10	3·15	3·15	3·20	3·20
8	b_{16}	2·10	2·20	2·30	2·35	2·40	2·45	2·50	2·55	2·60	2·60	2·65	2·65	2·65
8½	b_{17}	1·80	1·85	1·90	1·95	2·00	2·05	2·10	2·15	2·20	2·25	2·25	2·25	2·25
9	b_{18}	1·50	1·55	1·60	1·65	1·70	1·75	1·80	1·85	1·90	1·95	1·95	2·00	2·00
9½	b_{19}	1·30	1·35	1·40	1·40	1·45	1·45	1·50	1·50	1·55	1·55	1·60	1·60	1·60
10	b_{20}	1·10	1·10	1·15	1·20	1·25	1·25	1·30	1·30	1·35	1·35	1·35	1·35	1·30
10½	b_{21}	0·90	0·90	0·85	0·85	0·85	0·85	0·85	0·85	0·85	0·85	0·85	0·85	0·85
11	b_{22}	0·60	0·60	0·60	0·60	0·60	0·60	0·60	0·65	0·65	0·65	0·65	0·70	0·70
11½	b_{23}	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25

Order No.	Harmonic Coefft.	Net I.M.E.P. lb per sq in. 90	95	100	105	110	115	120	125	130	135	140	145	150
6½	b_{13}	4·75	4·75	4·75	4·75	4·75	4·70	4·60	4·50	4·40	4·25	4·00	3·75	3·50
7	b_{14}	4·00	3·95	3·90	3·85	3·80	3·70	3·60	3·50	3·35	3·35	3·15	2·90	2·35
7½	b_{15}	3·15	3·15	3·10	3·10	3·05	3·00	2·90	2·75	2·60	2·40	2·15	1·90	1·60
8	b_{16}	2·65	2·65	2·60	2·55	2·50	2·45	2·35	2·20	2·00	1·80	1·50	1·25	0·90
8½	b_{17}	2·25	2·20	2·20	2·15	2·10	2·05	2·00	1·90	1·75	1·60	1·35	1·10	0·75
9	b_{18}	2·00	1·95	1·95	1·90	1·85	1·75	1·65	1·55	1·40	1·15	0·90	0·60	0·25
9½	b_{19}	1·55	1·55	1·50	1·50	1·45	1·40	1·35	1·30	1·20	1·10	0·95	0·80	0·60
10	b_{20}	1·30	1·25	1·20	1·15	1·10	1·05	1·00	0·90	0·80	0·65	0·50	0·30	0·10
10½	b_{21}	0·80	0·80	0·80	0·80	0·75	0·70	0·65	0·60	0·50	0·40	0·30	0·20	0·10
11	b_{22}	0·70	0·70	0·65	0·65	0·60	0·55	0·50	0·40	0·35	0·25	0·15	0·05	0·00
11½	b_{23}	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·25	0·20	0·15	0·10	0·05	0·00

Compression Ignition Engine

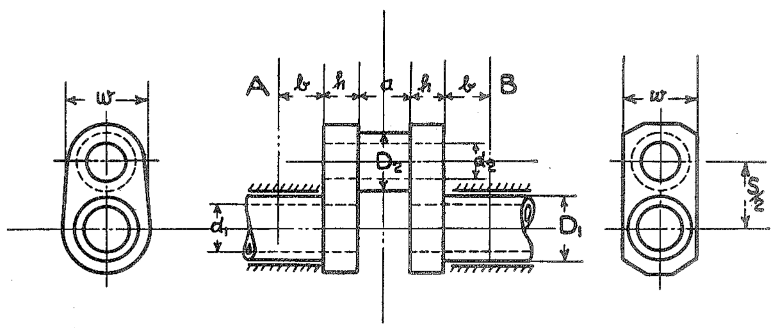


FIG. 1. Empirical formula for crankshaft. Stiffness in bearings.

Equivalent length from A to B in terms of journal shafting is :—

$$l = (2b + 0{\cdot}8h) + \frac{3}{4}\left(\frac{D_1^4 - d_1^4}{D_2^4 - d_2^4}\right) a + \frac{3}{4}\left(\frac{D_1^4 - d_2^4}{hw^3}\right) S$$

The flexibility from A to B is $\frac{l}{GJ}$ where G is $11{\cdot}8 \times 10^6$ lb/sq in.

for steel, and l and J in inch units. J being $\frac{\pi}{32}(D_1^4 - d_1^4)$

It follows that the flexibility is $\frac{0{\cdot}86l}{(D_1^4 - d_1^4) \times 10^6}$ radn per lb in.

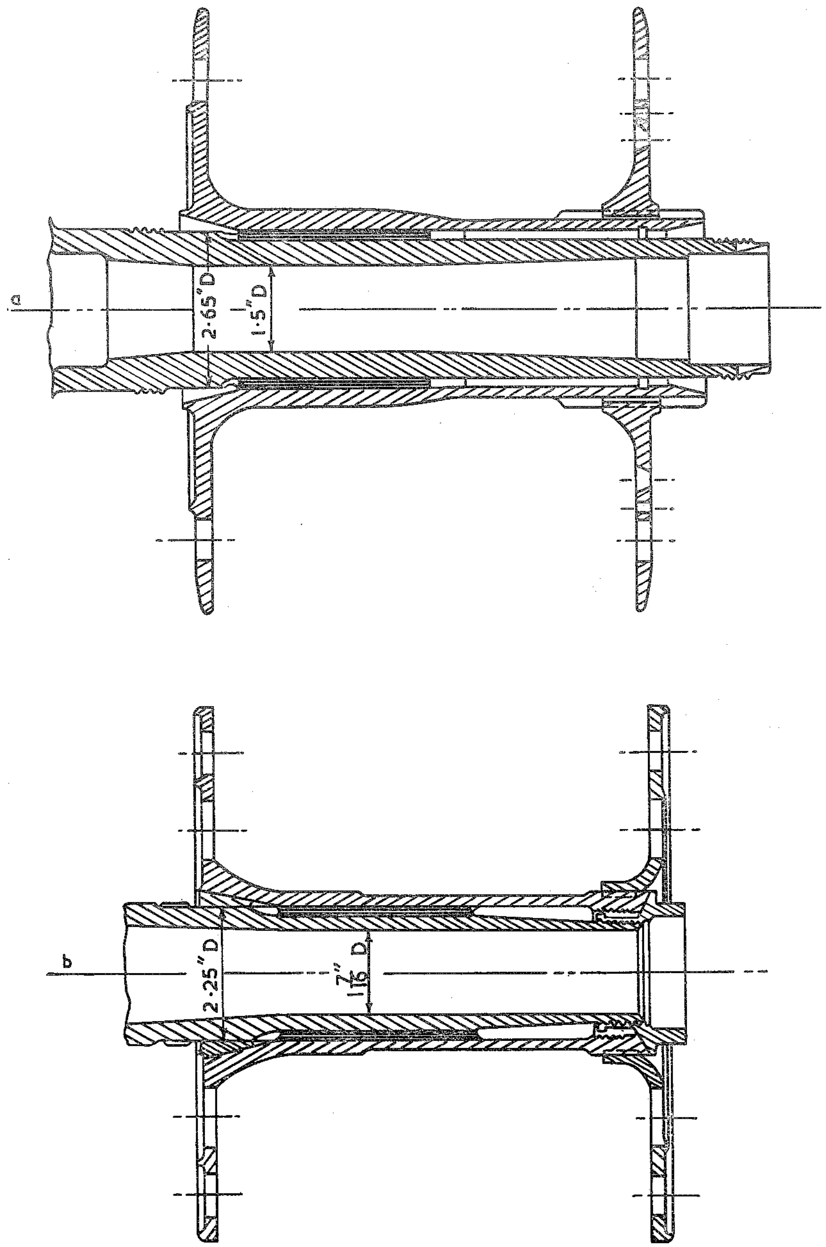


FIG. 2a and b. Airscrew connections.

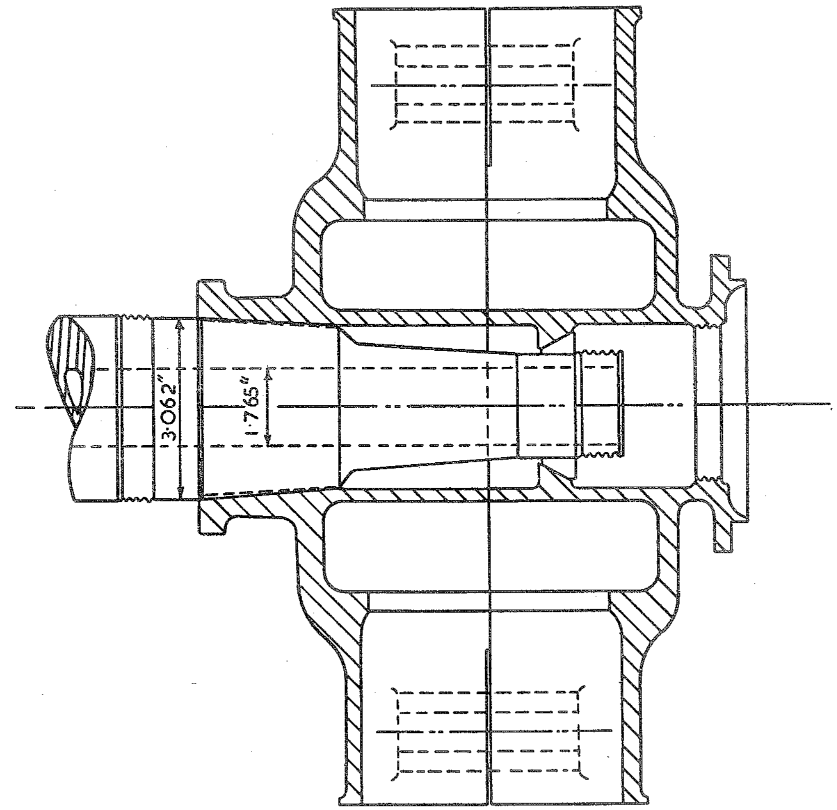


FIG. 2c. Airscrew connections.

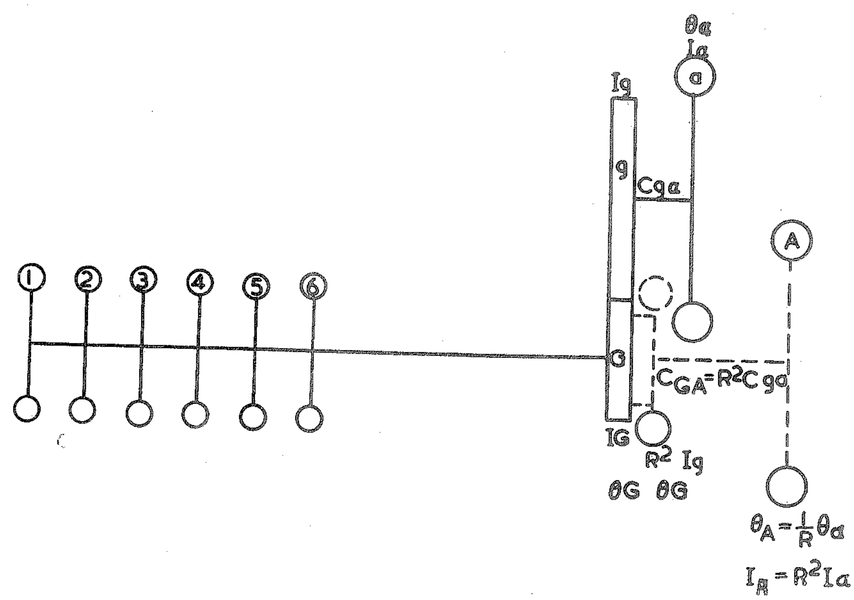


FIG. 3. Virtual engine system.

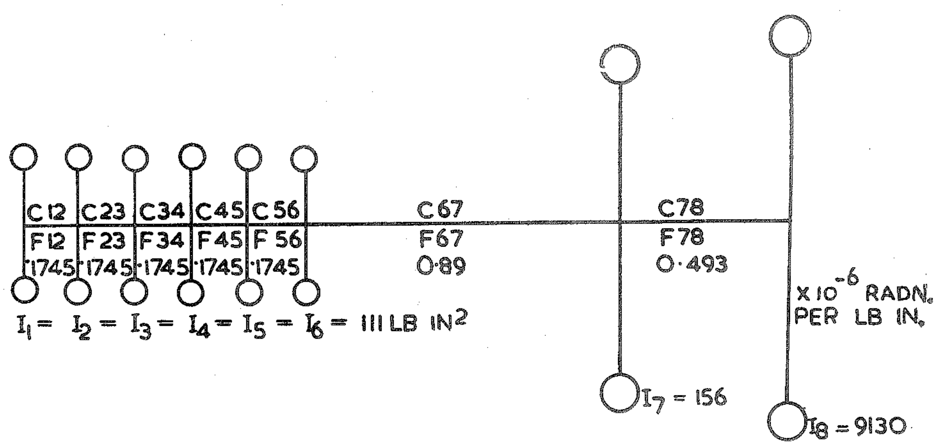


FIG. 4. Equivalent engine system.

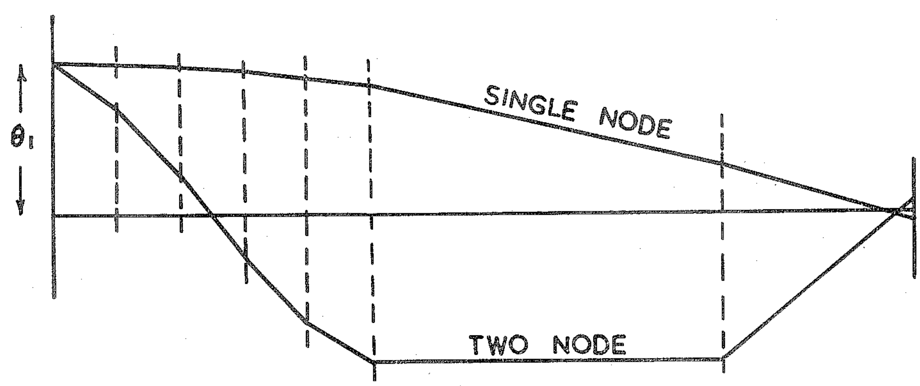


FIG. 5. Deflection curves.

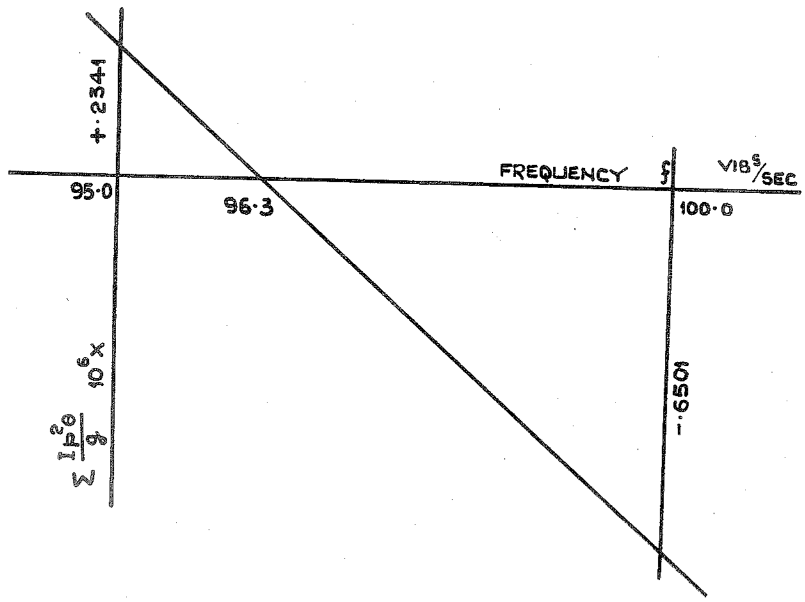


FIG. 6. Frequency interpolation diagram.

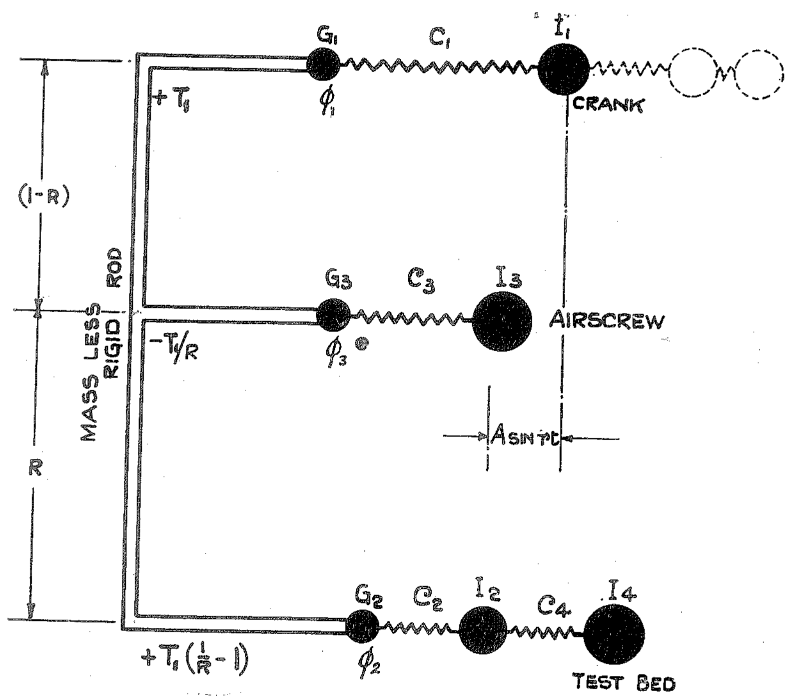


FIG. 7. Virtual dynamic system.

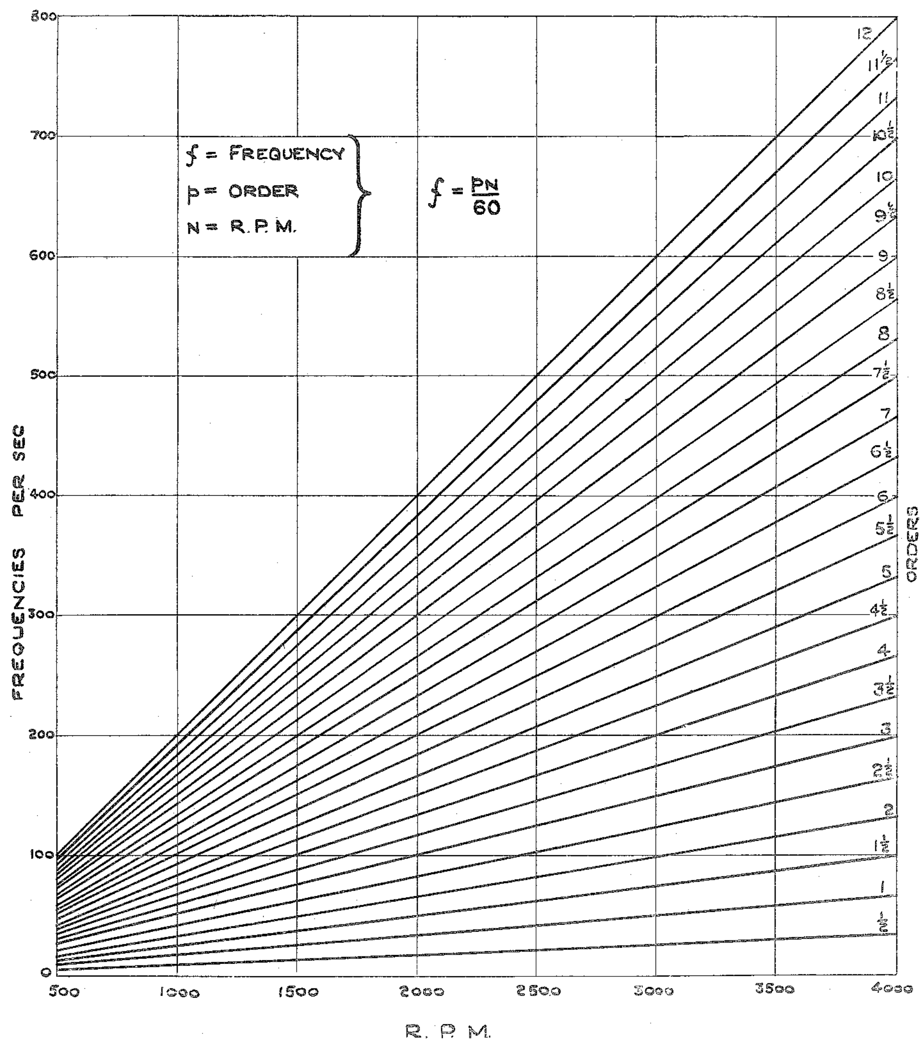


FIG. 8. Critical speed chart.

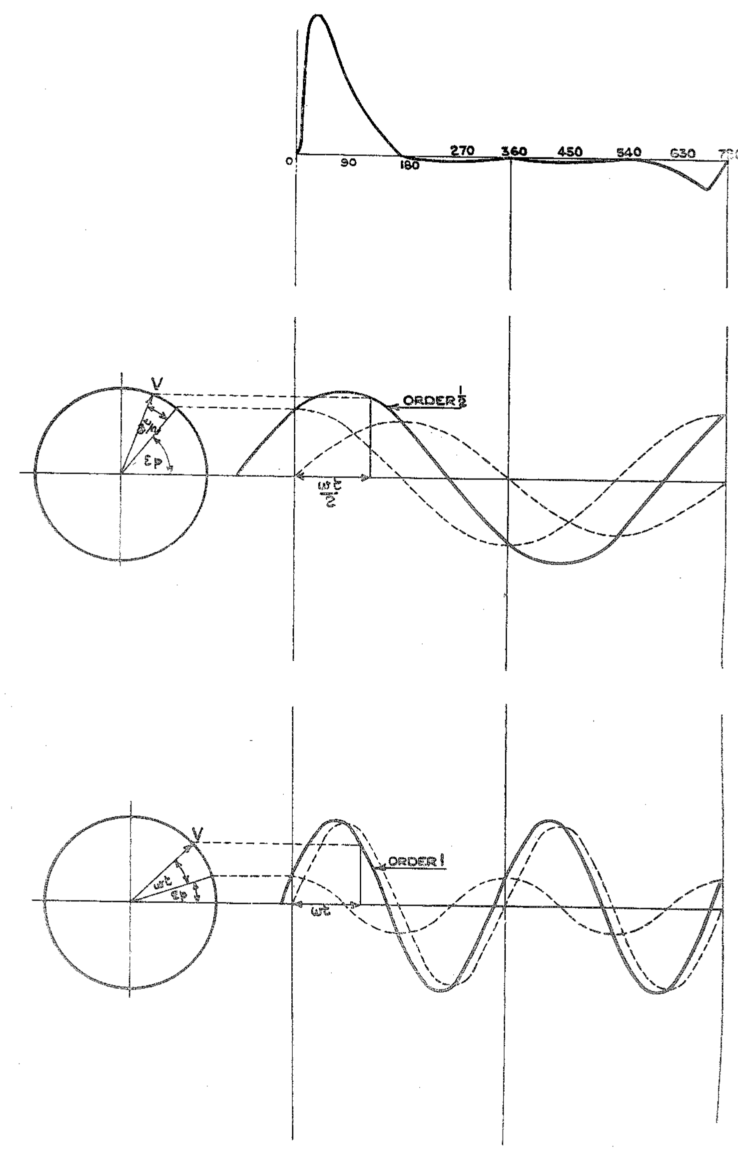


FIG. 9. Torque curve analysis.

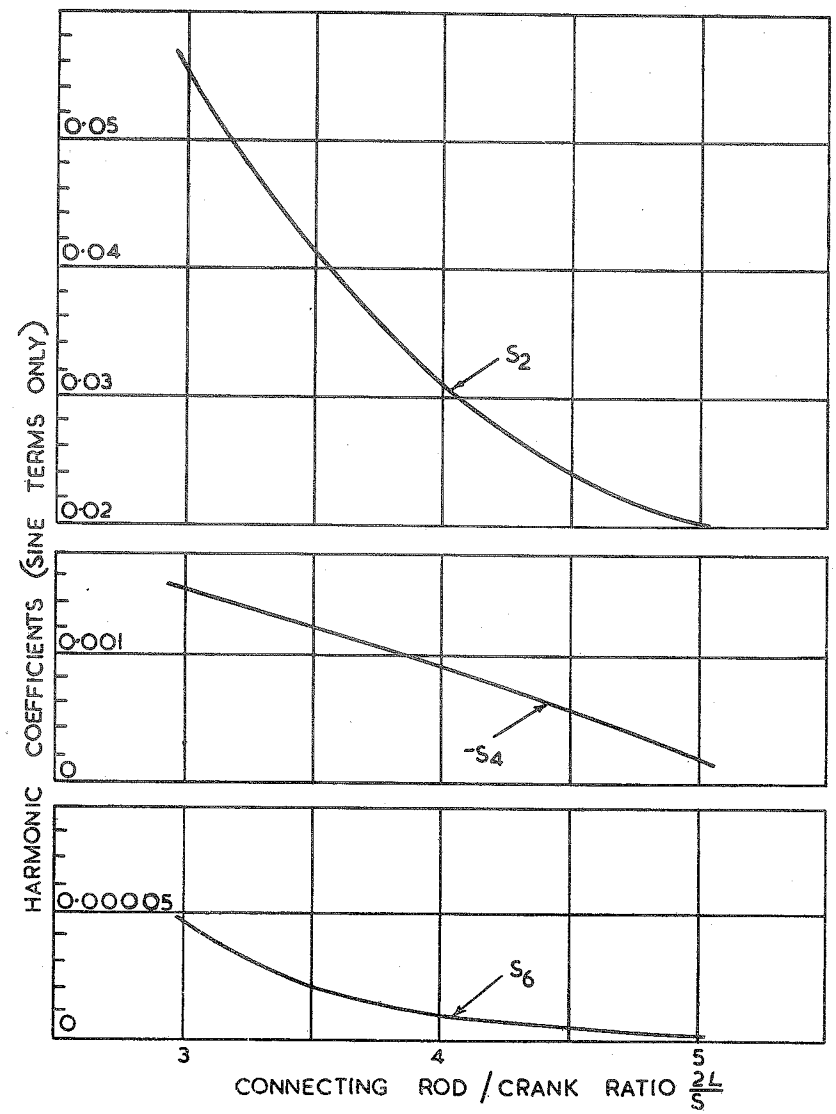


FIG. 10. Harmonic coefficients of crankshaft torque due to inertia. Resistance of connecting rod p^{th} order.

Torque for one rod $= \frac{I_0}{386} \omega^2 \frac{p}{2} S_p \sin p\omega t.$

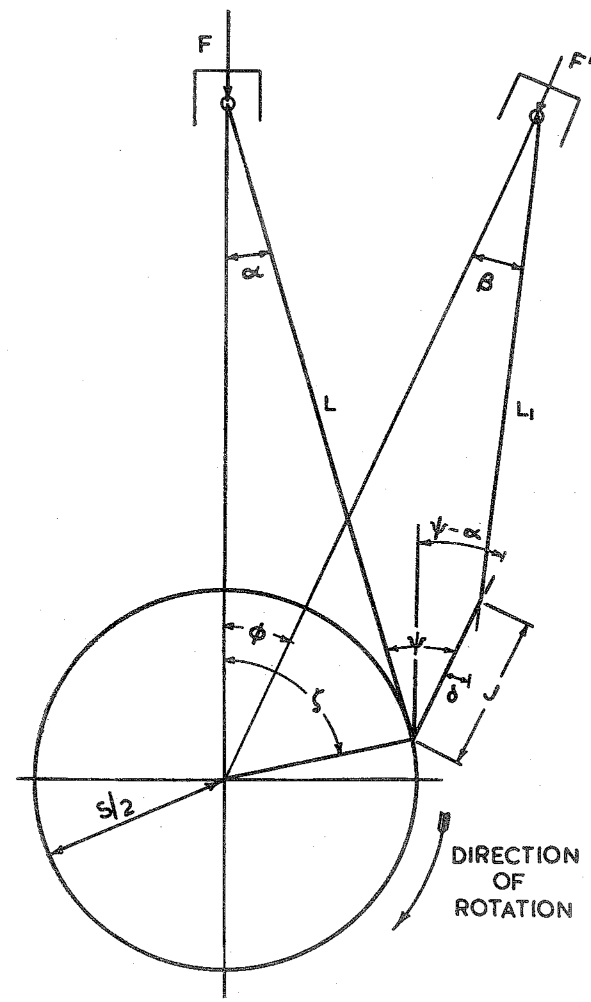
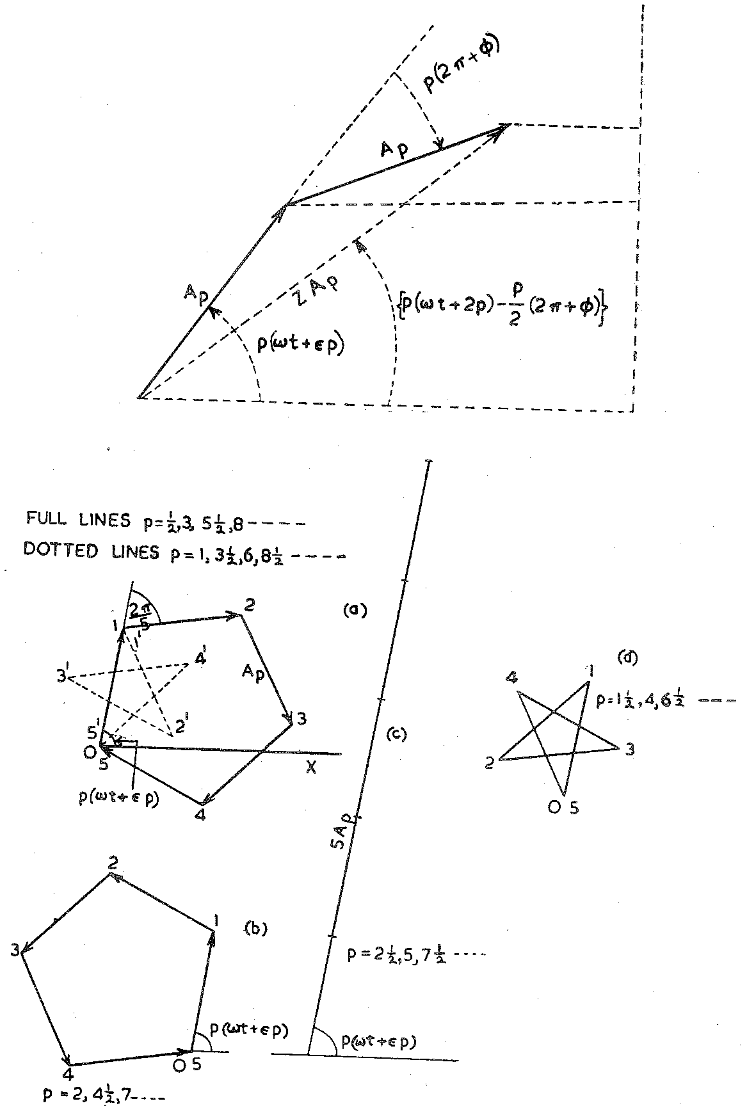


FIG. 11. Connecting rod articulation diagram.



FIGS. 12 and 13. Vector diagrams.

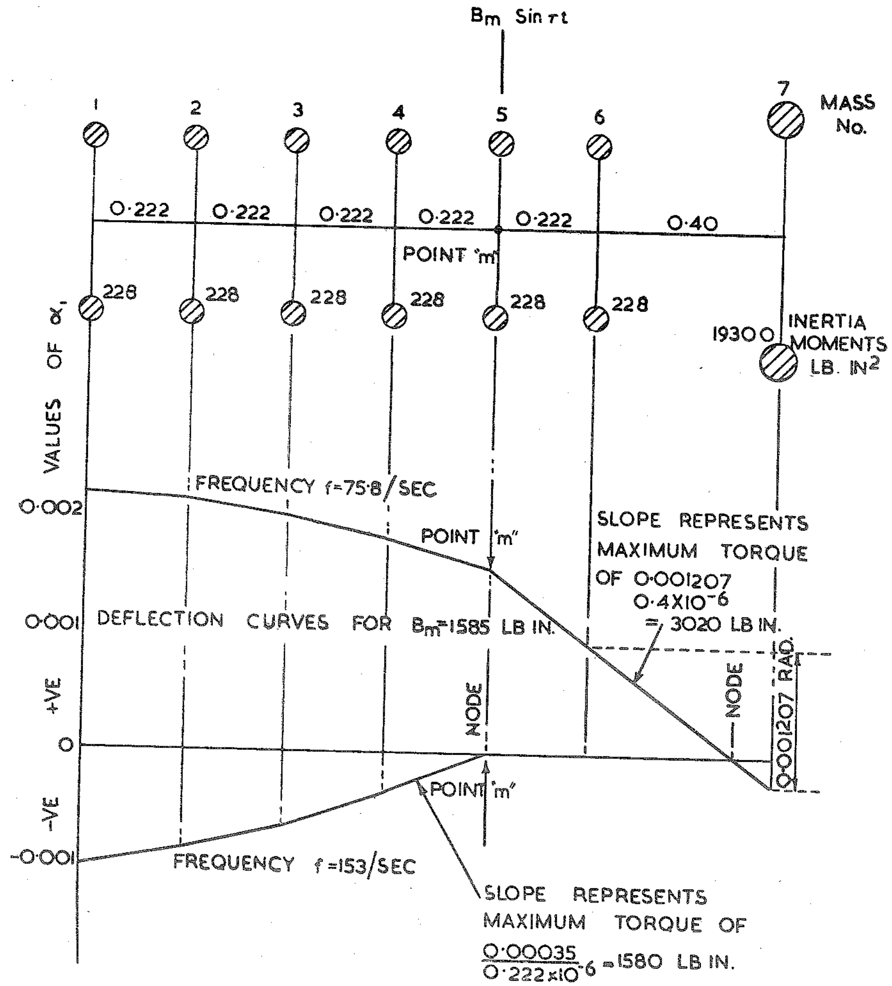


FIG. 14. Equivalent dynamic system and deflection curves.

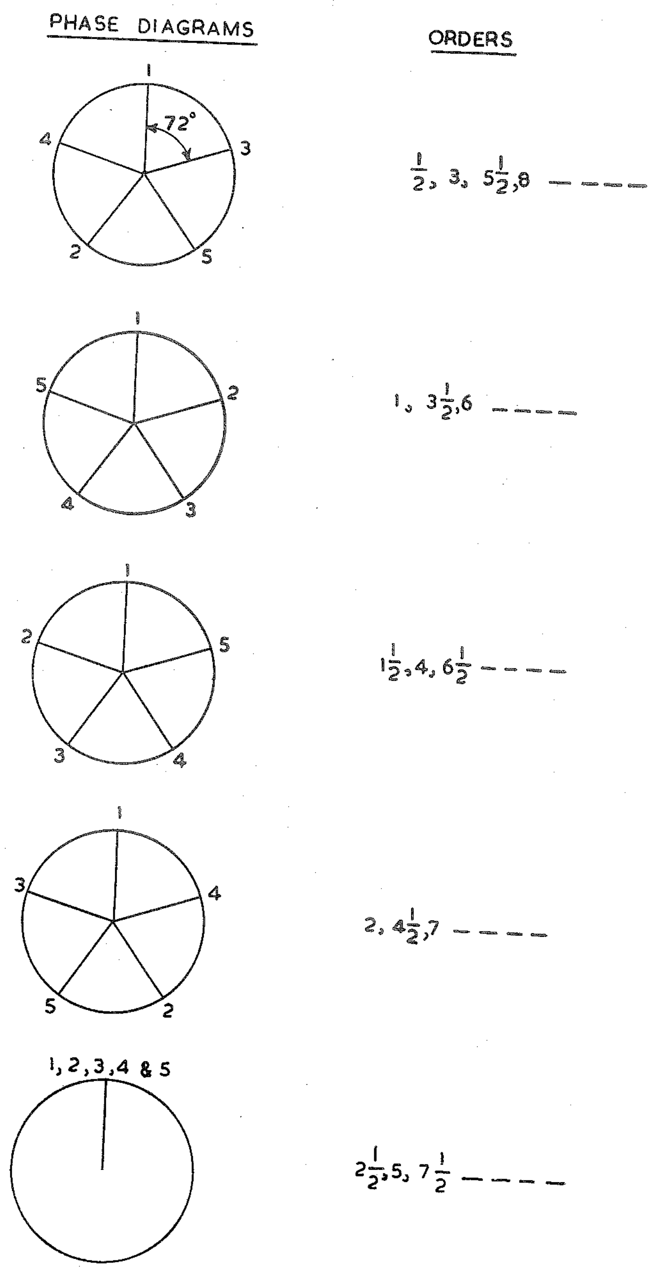


Fig. 15. Phase diagrams.

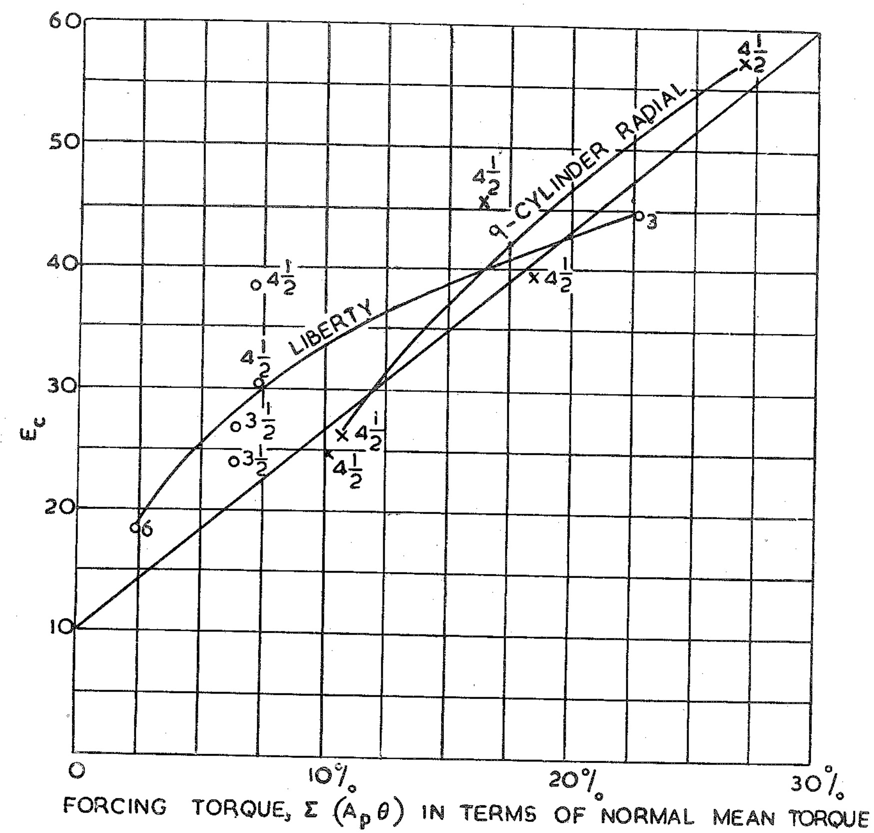


Fig. 16. Effect of forcing torque on damping coefficient.

PART III

Practical Calculations for a Typical Twelve-cylinder Vee Engine

Introduction.—The calculations have been made in conformity with a general scheme of arranging the details in a particular manner and sequence: the main sections correspond to those of Part II, to which reference should be made for explanation beyond that embodied here.

Range of Investigation.—The torsional vibration characteristics of the engine, in respect of both single-node and two-node vibration, have been calculated for the engine running with an airscrew fitted, assuming the airscrew to be virtually rigid.

In determining the critical speeds, the results of gear-yielding investigations made on similar engines have been used, and it is thought that the values obtained are fairly accurate.

The fact that the engine has master and articulated connecting-rods has been taken fully into account. The magnitudes of the criticals have been computed making certain assumptions respecting damping: the magnitudes are tentative on this account, and subject to revision when further progess has been made in the study of damping, but they indicate relative values and show which criticals are liable to prove troublesome.

An index follows to facilitate reference to data and results: it will be noted that general conclusions are given in section 9.

Index to Calculations

1.0. *Data*

Bore and stroke master	$5 \times 5 \cdot 35$ in.
Bore and stroke mean	$5 \times 5 \cdot 46$ in.
Compression ratio	6:1
Rating: Normal	600 h.p. at 2600 r.p.m.
Maximum	695 h.p. at 3000 r.p.m.
Type	12-cylinder 60 deg Vee
Lay-out	*See* Fig. 2

A dimensioned sketch of the crankshaft is given in Fig. 17. Firing sequence: 1A 6B 4A 3B 2A 5B 6A 1B 3A 4B 5A 2B

1.1. *Reduction gear*

Type	Spur
Number of pinion teeth	21
Number of wheel teeth	38
Gear ratio	0·5531
Airscrew rotation	Right-hand tractor
Crankshaft rotation	Opposite sense
Airscrew type	Two-blade, fixed pitch
Airscrew material	Mahogany

1.2. *Supercharger**

Type	Centrifugal, engine driven. Drive contains spring element and centrifugal clutch

1.3. *Connecting-rods*

Master rods in bank A	Articulated rods in bank B
Length of master rod	9·2 in.
Length of articulated rod	6·733 in.
Location of wrist pin	At 2·45 in. link length, 67·5 deg ahead of master rod

1.4. *Remarks*

*Supercharger omitted from calculations because it is assumed to be torsionally insulated from the engine by the flexible drive.

2.0. *Moments of Inertia.*—2.1. *Crankthrow*

(*a*) Reciprocating masses:

	Cylinder	
	Master lb	*Articulated* lb
Piston complete with rings and gudgeon pin	3·25	3·25
Connecting-rod small-end	0·70	0·562
Total per cylinder	3·95	3·812
Equivalent at the crankpin per cylinder	1·975	1·906
Joint equivalent at the crankpin centre	3·881 lb	

(*b*) Rotating masses:

Total big-end mass per throw	4·988 lb

(*c*) Equivalent attached at crankpin centre .. .. .. 8·869 lb

(*d*) Moment of inertia about crankshaft axis, of:

Equilvalent attached mass	63·5 lb/in.^{2}
Crankpin	16·81 lb/in.^{2}
Two crank webs	28·44 lb/in.^{2}
Journal	2·5 lb/in.^{2}
Total per crankthrow	111·25 lb/in.^{2}

2.2. *Reduction Gears*

	Moments of inertia lb in.² Actual	Moments of inertia lb in.² Reduced to basis of crankshaft*
Pinion wheel †	66·5	66·5
Gear wheel ‡	257·3	79·0
Total		145·5
2.3. *Airscrew*	31,000	9,490

3.0. *Flexibilities and Stiffnesses.*—3.1. *Crankthrow.*—In the notation of Fig. 1 of Part II, expressing dimensions in inches; D_1 is 3·0 and d_1 is 2·3 for all journals except B and F—for which it is 2·4. In the following table, quantities are related to both types of journal:

				In the terms of the journals :—		ACD E and G	B and F
Journal	D_1	d_1	$2b$				
	3·0	2·3 and 2·4	1·77	$(2b + 0{\cdot}8h)$	=	2·40	2·40
Crankpin	D_2	d_2	a				
	2·5	1·7	2·30	$\frac{3}{4}\left[\frac{D_1^4 - d_1^4}{D_2^4 - d_2^4}\right] a$	=	2·97	2·68
Two Crankwebs	w	h	s				
	4·2	0·78	5·35	$\frac{3}{4}\left[\frac{D_1^4 - d_1^4}{hw^3}\right] s$	=	3·69	3·32
Equivalent length of throw in inches $= l =$ total					=	9·06	8·40
Flexibility per crankthrow, in mirco-radians per pound-inch, is : $F_1 = 0{\cdot}86\, l/(D_1^4 - d_1^4)$					=	0·147	0·151
Stiffness of crankthrow, in pounds-inches per mirco-radian, is : $C_1 = 1/F_1$					=	6·97	6·61

* The reduced values are obtained by multiplying the actual values by the square of the ratio of speed to crankshaft speed.

† The figures contain the moment of inertia of layshaft, *etc.*, from pinion centre to centre of adjacent journal A.

‡ The figures contain the moment of inertia of 1/3 airscrew shaft measured from gear wheel centre.

The flexibilities between the adjacent crankpin centres are the flexibilities between the crank masses shown in Fig. 20a.

3.2. *Crankthrow Fexibility by Alternative Method.*—No alternative method used.

3.3. *Airscrew Shaft.*—The shaft flexibility may be derived from Fig. 19b where the shaft dimensions are given together with a diagram constructed therefrom giving $1/(D^4 - d^4)$ which is proportional at each section to the flexibility per unit length, and thus the area of the diagram is a measure of the flexibility of the shaft. As mentioned in section 3, Part II, there is uncertainty arising from the flexibility of attachment of the airscrew. In this case the uncertainty is removed by the experimental determination of Ref. 1 at the end of the report, which gives the overall flexibility from the gear wheel centre to the airscrew boss of a similar airscrew shaft as $0 \cdot 132 \times 10^{-6}$ radn. per lb/in. which value is used in these calculations. Fig. 19b is however, included to show the method of determination of the airscrew shaft flexibility when experimental results are not available.

3.4. *Airscrew flexibility*.—The airscrew is taken to be virtually a rigid flywheel.

3.5. *Other flexibilities*

Between centre-line pinion and centre-line journal A = 0·323 micro-radians/lb in.

Between centre-line pinion and centre-line crankpin 6 = 0·397 micro-radians/lb in.

3.6. *Flexibilities of Reduction Gearing*.—Torsional stiffness tests have been made on a similar engine having gearing of the same design. The results show that the major portion of the yielding in the gears and gear housing is not amenable to calculation. The total flexibility from test measured from the centre of the gear wheel to the centre of the pinion referred to airscrew shaft is $0 \cdot 0834 \times 10^{-6}$ radn./lb in.

Of this total certain contributions can be accounted for by calculation, *viz.* :—

	Micro-radians per lb in. *Items*	*Totals*
Flexibility due to bending, shear and surface deformation in the gear teeth	0·0097	
Flexibility due to winding of gear wheel body	0·0057	0·0184
Flexibility due to winding of pinion wheel body	0·0030	
Certain other contributions were provided by the tests, viz :		
Flexibility due to rotation of the gear housing	0·0078	0·0232
Flexibility due to yielding in bearings themselves	0·0154	
Total flexibility accounted for:		0·0416
Total flexibility unaccounted for:		0·042

It is probable that the unaccounted flexibility exists in the supports of the bearings.

	Items
Flexibility between the centre of the gear wheel and airscrew boss reduced to crankshaft datum is:	
$\frac{0 \cdot 132}{(0 \cdot 553)^2} =$	0·431
Flexibility of gears reduced to crankshaft datum is:	
$\frac{0 \cdot 0834}{(0 \cdot 553)^2} =$	0·272
Therefore the total flexibility from the pinion to the airscrew boss reduced to crankshaft datum is:	0·703

The corresponding stiffness is:

1·42 million lb in./radn

4.0. *Dynamic System*.—4.1. The flywheel system corresponding to the foregoing data is represented diagrammatically in Fig. 20a where the distances between the consecutive masses are proportional to the flexibilities of the appropriate connections.

4.2. A diagram is drawn below Fig. 20a representing the system reduced to crankshaft speed datum by modifying the stiffnesses and moments of inertia of the parts which rotate at speeds other than crankshaft speed. This diagram is termed the Equivalent Engine System, Fig. 20b.

5.0. *Natural Frequencies, and Displacement Curves.*—To obtain a first approximation to the frequency in single-node vibration of the system shown in Fig. 20b the system may be simplified into two flywheels and a connecting shaft, as follows:—

Moment of inertia of one flywheel $= I_a = 9{,}490$ lb in.^{2}
Moment of inertia of other flywheel $= I_b = 811$ lb in.^{2}

the first flywheel being the airscrew and the other the remaining mass located at mass 5, say:

The corresponding flexibility of the connecting shafting is $1 \cdot 251 \times 10^{-6}$ radn/lb in. and thus the frequency is:

$$\frac{1}{2\pi}\left\{gC\left(\frac{1}{I_a}-\frac{1}{I_b}\right)\right\}^{1/2} = \frac{1{,}000}{2\pi}\left\{\frac{386}{1 \cdot 251} \times (0 \cdot 000105 - 0 \cdot 001232)\right\}^{1/2} = 102$$

complete vibrations per second.

Starting from this value in using the tabular method, it is found that the single-node frequency for the system of Fig. 20b is 105 vibrations per second, as Table 1 shows; the corresponding displacement curve is given in Fig. 21a.

The two-node frequency is found to be 372 vibrations per second (*see* Table 1); the corresponding displacement curve is shown in Fig. 21b.

6.0. *Orders of Vibration and Critical Speeds.*—From Fig. 22 it is seen that the orders of single-node vibration that give criticals in the region of full throttle speeds are 2, $2\frac{1}{2}$ and 3: these are minor criticals. The orders of two-node vibration that give criticals in this speed region are high, namely: 7 to 9, and the corresponding displacement amplitudes are probably very small. Although the steepness of the displacement curve in the region of the node (Fig. 21b) connotes relatively large torque variation for a given displacement at No. 1 mass, it is doubtful if the criticals in question are important enough to require investigation; in any case, the present calculation does not extend to examination of criticals higher than the 6th order.

7.0. *Estimation of Relative Magnitudes of Various Criticals.*—The symbols used in the ensuing calculations have significance as follows :—

θ_c Amplitude of vibration ($\frac{1}{2}$ total swing) at each crank according to Table 1, where the amplitude crank 1 is taken to be one radian.

θ_a Corresponding amplitude of vibration at the airscrew.

A_p Magnitude of the pth order forcing torque per cylinder.

ZA_p Magnitude of the resultant pth order forcing torque at each crankpin: by its use the engine may be considered to be reduced to an equivalent 6-cylinder engine.

α_{1p} Amplitude (in radian measure) at mass one due to the pth order forcing torques.

$\Sigma\theta_c$ Vector sum of the elastic displacements at the cranks using the phase diagrams appropriate to each order.

$ZA_p\Sigma\theta_c$ Relative measure of severity of criticals for a given mode and constant velocity-coefficient of damping.

7.1. *Amplitudes of Harmonic Forcing Torques : ZA_p.*—The forcing torques due to gas and inertia forces operating on any one crank have been treated separately; gas-pressure curves have been constructed on a base of piston displacement for the I.M.E.P. considered and the corresponding torque curves for master and articulated cylinders deduced. These torque curves have then been added in correct phase and analysed by Runge's method.

Inertia torque coefficients have been found, taking full account of articulation, using the methods described in sections 7a.3 and 7a.4 of Part II.

Gas and inertia torque coefficients for any one crank are given in the following table for a particular condition of operation, namely 2,600 crankshaft r.p.m. and 158 lb/sq in. I.M.E.P.

Order	Gas Harmonics lb/in.		Inertia torque harmonics lb/in.				
			Master	Articulated		Total	
	Cos	Sin	Sin	Cos	Sin	Cos	Sin
½	−1,006	1,051					
1	− 263	4,574	405	−363	46·5	363	451·5
1½	−2,569	−2,100					
2	−1,665	452	−2,720	2,540	1,260	2,540	−1,460
2½	262	−2,590					
3	73	42	−1,230	−396	1,040	−396	− 190
3½	1,290	− 997					
4	176	654	− 125	− 80	−57	− 80	− 182
4½	630	104					
5	− 339	614	23	− 11·4	48·5	− 11·4	72
5½	92	92					
6	− 507	206	4	− 1·5	− 9·1	− 1·5	− 5·1

The resultant harmonic torque per crank (ZA_p) together with the phase leads of the several harmonic components are given in Table 2 for selected engine speeds and mean effective pressures which correspond to speed in the manner indicated.

7.2. *Phase Diagrams and the Evaluation of $\Sigma\theta_c$.*—The next step in making the vector summation $ZA_p\Sigma\theta_c$ is to draw phase diagrams to give the displacement vector directions. Phase diagrams are set out in Table 3 for the firing sequence concerned and for various orders.

The phase diagram for order ½ is simply the firing sequence as it would be if marked upon the end of a shaft running at half crankshaft speed. Thus for the equivalent six-crank engine under consideration, with a firing sequence 1, 5, 3, 6, 2, 4, there is a 60 deg interval between each pair of successive numbers for order ½. This interval is increased by 60 deg for each additional ½ to the order; thus for order 1 it is 120 deg and so on. Starting with the datum crank number, the other cranks are put on the several diagrams with appropriate angular intervals.

Using the phase diagram directions, the vertical components V and the horizontal components H of the several displacements are given in Table 3 for both single-node and two-node vibration; also the values of

$$\Sigma\theta_c = \sqrt{(\Sigma V)^2 + (\Sigma H)^2}$$

7.3. *Relative severity of criticals $ZA_p\Sigma\theta_c$.*—It should be noted that a forcing torque of amplitude $ZA_p\Sigma\theta_c$ acting at mass No. 1 is equivalent in effect to the group of *p*th order forcing torques operating on the cranks; it is thus the resultant *p*th forcing torque referred to No. 1 mass.

Values of $ZA_p\Sigma\theta_c$ are given in row 6, Table 4. It will be noted that, of the single-node criticals, the 3rd order is by far the most severe, and that as regards two-node criticals the investigation does not extend beyond determination of 6th order.

8.0. *Amplified Torques and Resultant Stresses in the Airscrew Shaft.*—The resultant maximum and minimum torques in the airscrew shaft are given by adding to the mean transmitted torque plus and minus the maximum vectorial resultant of the criticals prevailing at the speed concerned. It suffices for practical purposes to adopt arithmetic summation, although this usually over-estimates the stress range.

8.1. *Estimation of Absolute Magnitudes of Various Criticals.*—Referring to section 8 of Part II, the value of the damping quantity E_c will be taken to be 40 having regard to the fact that the engine has reduction gearing (which tends to increase the damping). On this basis:

$$h_c = E_c\left(\frac{I_c}{386}\right)^{4/5} = 40\left(\frac{111}{386}\right)^{4/5} = 14\cdot7 \text{ lb in. per radn/sec.}$$

The magnitudes of the harmonic torques, expressed in terms of α_{1p}, the amplitude of vibration of the free end mass, are derived as follows :—

	Single-Node	Two-Node
Frequency in vibrations per sec.	105	372
$\Sigma(\theta_c^2)$ values from θ_c values in Table 3	4·85	2·93
$\alpha_{1p} = \dfrac{\theta_1 Z A_p \Sigma\theta_c}{2\pi f h_c \Sigma(\theta_c^2)}$ from equation (95), Part II, h_a being neglected ..	$21\cdot2 \times 10^{-6}\, ZA_p \Sigma\theta_c$	$0\cdot82 \times 10^{-6}\, ZA_p\ \Sigma\theta_c$
Twist between airscrew and adjacent mass from Table 1 :— radians	0·5275	1·23
Flexibility between airscrew and adjacent mass :—Radn/lb in. ..	$0\cdot703 \times 10^{-6}$	$0\cdot703 \times 10^{-6}$
Corresponding twist in equivalent shaft :—radians	$0\cdot5275\alpha_{1p}$	$1\cdot23\alpha_{1p}$
Harmonic torque in equivalent airscrew shaft :—lb/in.	$0\cdot75 \times 10^6\alpha_{1p}$	$1\cdot75 \times 10^6\alpha_{1p}$
Gear ratio Airscrew/crankshaft	0·553 : 1	0·553 : 1
Harmonic torque T_H in actual airscrew shaft :—lb/in.	$1\cdot355 \times 10^6\alpha_{1p}$	$3\cdot16 \times 10^6\alpha_{1p}$

Values of α_{1p} and T_H computed from the foregoing are given in rows 7 and 8 respectively of Table 4. The harmonic torque between any two masses in the system can be found by multiplying the harmonic torque in the equivalent airscrew shaft by the ratio of the slope of the displacement curve in the region concerned to the slope for the equivalent airscrew shaft.

8.2. *Brake Mean Torque in Airscrew Shaft.*—The brake mean torque in the airscrew shaft for the conditions of sections 1 and 7.1: is T_m, where

$$T_m = \frac{63{,}020 \times \text{BHP}}{\text{Airscrew R.P.M.}} = \frac{63{,}020 \times 600}{2{,}600 \times 0\cdot553} = 26{,}300 \text{ in. lb.}$$

Below this speed the torque is assumed to vary as the square of the speed and above the torque is assumed constant. Values of brake mean torque are given in row 9 of Table 4.

8.3. *Maximum and Minimum Torques and Shear Stresses in the Airscrew Shaft Splines*.—The cyclical torque changes are shown diagrammatically in Fig. 23. The side bands of these criticals have also been plotted: these have been calculated by use of equation (97) of Part II. The 3rd order is worked out below for guidance.

Resonant r.p.m. 2,100, $T_H = 35{,}600$ from Table 4.

$$\Delta = \frac{1{,}240}{(0 \cdot 553)\ (36{,}600)} = 0 \cdot 063$$

	Gas and inertia torques increasing as the square of engine speed						Gas torques constant		
Engine R.P.M.	1,750	2,000	2,050	2,100	2,150	2,250	2,600	2,750	2,900
Gas M_e	33	43	45	48†	50	55	73*	73	73
Harmonics M_s	19	25	26	27†	29	31	42*	42	42
Inertia M_e'	−180	−235	−247	−259†	−272	−297	−396*	−444	−494
Harmonics M_s'..	− 86	−112	−118	−124†	−130	−142	−190*	−213	−237
Resultant $M_e + M_e'$	−147	−192	−202	−211†	−222	−242	−323	−371	−421
Harmonics $M_s + M_s'$ ..	− 76	− 87	− 92	− 97†	−101	−111	−148	−170	−195
ZA_p	161	211	222	232†	244	266	355	408	464
$ZA_p \Sigma\theta c$	860	1,130	1,190	1,240	1,310	1,430	1,910	2,190	2,490
$\{(1 - x^2)^2 + \Delta^2 x^2\}^{1/2}$	3·22	9·25	12·9	15·9	12·75	16·08	1·86	1·40	1·10
T_H	5,000	18,800	27,800	35,600	30,200	15,700	6,420	5,550	4,950

† From Table 2. * From table in section 7.1.

As mentioned in Part I, a chain-dotted curve has been plotted on Fig. 23 to show the torque variation on the assumption that the shafting and airscrew are rigid. The curve has been constructed from a graphical integration of the torque variations on each crank; it can however, be obtained from this section. It will be seen from the phase diagrams of Table 3 that all the orders balance out (assuming rigid shafting) except the series 3, 6, 9, *etc*. Assuming all orders above the sixth to be small it is only necessary to add the 3rd and 6th orders in correct phase to obtain the resultant torque variation, noting that below 2,600 r.p.m. both gas and inertia torques vary as the square of the speed and that above 2,600 r.p.m., the gas torques do not vary. This method has been applied and found to give substantially the same results.

Reverting to the actual case, the torque range at various speeds has been obtained by adding the ordinates of the full-line curves plotted about the mean torque line in Fig. 23. From these values, the maximum and minimum torques in the airscrew shaft have been deduced in accordance with section 8.0. The results are given in row 12 of Table 4 and thence the shear stresses have been estimated from the resultant torques at a section taken across the splines. No concentration factor has been included. For a ratio of fillet radius to depth of the spline of 1/5 it is estimated that the factor would be 2·5, so that the stresses given in Table 4 would be increased in this proportion.

9.0. *General Conclusions*.—The total elastic yielding in the gear corresponds to a lowering of the single node natural frequency of the system from 119/sec to 105/sec, *i.e.*, by 13·3 per cent.

On the basis of the assumed damping, for the engine with an airscrew fitted, there is one critical of sufficient magnitude to cause torque reversal, namely the single-node critical of order 3 at 2,100 r.p.m. and trouble would be encountered if the engine were run for prolonged periods at this speed. The remaining criticals are quite moderate.

PART III. TABLE 1

Torsional Vibration

NATURAL FREQUENCY TABLE

Single-node frequency $f_1 = 105$ vibrations/sec

$r_1^2 = 4\,\pi^2 f_1^2 = 0{\cdot}436 \times 10^6$

denote $r_1^2 I/g$ by Q_1

6th order synchronous speed = 1050 r.p.m.

Two-node frequency $f^2 = 372$ vibrations/sec

$r_2^2 = 4\,\pi^2 f_2^2 = 5{\cdot}45 \times 10^6$

Denote $r_2^2 I/g$ by Q_2

Mass No.	1	2	3	4	5	6	7	8	Units
I	111	111	111	111	111	111	145	9490	lb in.^{2}
I/g	0·288	0·288	0·288	0·288	0·288	0·288	0·375	24·6	lb sec^2 in.
$F = 10^{-6} \times$ ..	0·151	0·147	0·147	0·147	0·151	0·397	0·703		radn/lb in.
			Single-Node						
$Q_1 = 10^6 \times$ ..	0·1255	0·1255	0·1255	0·1255	0·1255	0·1255	0·1635	10·72	
θ	1	0·9810	0·9444	0·8904	0·8199	0·7319	0·4649	—0·0626	radian
$Q_1\theta = 10^6 \times$ ..	0·1255	0·1230	0·1185	0·1117	0·1028	0·0919	0·0760	—0·6720	lb in.
$\Sigma Q_1\theta = 10^6 \times$ ..	0·1255	0·2485	0·3670	0·4787	0·5815	0·6734	0·7494	0·0774	lb in.
$\Sigma(Q_1\theta F)$ ..	0·0190	0·0366	0·0540	0·0705	0·0880	0·2670	0·5275		radian
			Two-Node						
$Q_2 = 10^6 \times$ ..	1·57	1·57	1·57	1·57	1·57	1·57	2·045	134·1	
θ	1	0·7625	0·3550	—0·135	—0·593	—1·923	—1·213	0·017	radian
$Q_2\theta = 10^6 \times$ ..	1·57	1·198	0·557	—0·212	—0·932	—1·450	—2·480	2·28	lb in.
$\Sigma Q_2\theta = 10^6 \times$ ..	1·57	2·768	3·325	3·113	2·181	0·731	—1·749	0·531	lb in.
$\Sigma(Q_2\theta F)$ ..	0·2375	0·4075	0·490	0·458	0·330	0·290	—1·230		radian

Nodal positions

Single node ..	Flexibility from airscrew	$= 0{\cdot}083 \times 10^{-6}$ radn/lb in.
Two node ..	Flexibility from airscrew	$= 0{\cdot}0098 \times 10^{-6}$ radn/lb in.
	Flexibility from rear mass	$= 0{\cdot}405 \times 10^{-6}$ radn/lb in.

PART III. TABLE 2

Torsional Vibration, 12 cycles per engine

Evaluation of ZA_P

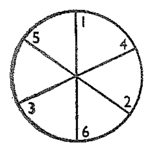
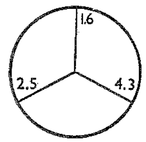
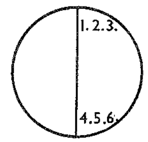
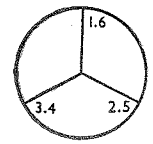
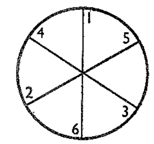
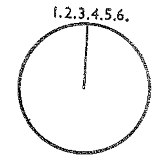
Mode of Vibration		Single-Node							Two-Node	
Crankshaft r.p.m.		3150	2520	2100	1800	1575	1400			3720
Order of Vibration		2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$	6
I.M.E.P. lb/sq in.		158	149	103	76	58	46			158
Total gas torque harmonic components	ΣM_C	−1665	247	48	580	65	183			−507
	ΣM_S	452	−2440	27	−478	240	30			205
Inertia torque harmonics: Master rod M_{SM}'		−4000		−803		−46				8·2
Inertia torque harmonics: Articulated rod	M_C'	3750		−259		−29				−3·1
	M_{SA}'	1850		679		−21				−18·7
Inertia torque harmonics: Resultant inertia	M_C'	3730		−259		−29				−3·1
	$\Sigma M_S' = (M_{SM}'+M_{SA}')$	−2150		−124		−67				−10·5
Resultant harmonic torque	$(\Sigma\, M_C+M_C')$	2065	247	211	580	36	183			−510
	$\Sigma(M_S+M_S')$	−1698	−2440	97	−478	173	30			194
Resultant torque per crank $ZA_P = \{\Sigma(M_C+M_C')^2+\Sigma(M_S+M_S')\}^{1/2}$		2680	2450	232	752	177	185			545
Phase lead $\frac{\Sigma(M_C+M_C')}{\Sigma(M_S+M_S')} = \tan \varepsilon_P$		−1·216	−0·101	2·175	−1·215	0·208	6·1			−2·62
Values of ε_P $\sin^{-1}\frac{(\Sigma M_C+M_C')}{ZA_P} = \cos^{-1}\frac{\Sigma(M_S+M_S')}{ZA_P}$		129° 26′	174° 14′	245° 18′	129° 27′	11° 45′	80° 41′			290° 55′

Bore 5 in. Stroke 5·35 in. I.M.E.P. 158 lb/sq in. At normal r.p.m. of 2,600.
Brake mean airscrew torque at full throttle 26,300 lb/in. Frequency, single-node 105/sec.
Reciprocating mass per cylinder 0·288 lb sec^2/in. Frequency, two-node 372/sec.
Engine throttled from 2,600 r.p.m. I.M.E.P. taken as proportional to (r.p.m.)^2.
Suffixes :—S = Sine. C = Cosine. A = Articulated. M = Master.

PART III. TABLE 3

Torsional Vibration

Phase Diagrams and Evaluations of $\Sigma\theta_c$

Orders			$\frac{1}{2}$ $3\frac{1}{2}$ $6\frac{1}{2}$		1 4 7		$1\frac{1}{2}$ $4\frac{1}{2}$ $7\frac{1}{2}$		2 5 8		$2\frac{1}{2}$ $5\frac{1}{2}$ $8\frac{1}{2}$		3 6 9	
Phase diagrams														
Mode of Vibration	Mass No.	θ_c	V	H	V	H	V	H	V	H	V	H	V	H
Single Node	1	1·00	−1·00	0	1·00	0	−0·10	—	1·000	0	−1·0	0	1·00	—
	2	0·981	0·4905	−0·85	−0·4905	−0·85	−0·981	—	−0·4905	0·85	0·4905	0·85	0·981	—
	3	0·944	0·4722	0·818	−0·4722	0·818	−0·944	—	−0·4722	−0·818	0·4722	−0·818	0·944	—
	4	0·8904	−0·4452	−0·771	−0·4452	0·771	0·8904	—	−0·4452	−0·771	−0·4452	0·771	0·8904	—
	5	0·8199	−0·4100	0·71	−0·410	−0·71	0·8199	—	−0·4100	0·71	−0·4100	−0·71	0·8199	—
	6	0·7319	0·7319	0	0·7319	0	0·7319	—	0·7319	0	0·7319	0	0·7319	—
			$\Sigma V=$ −0·1606	$\Sigma H=$ −0·093	$\Sigma V=$ −0·0945	$\Sigma H=$ −0·029	$\Sigma V=$ −0·4832	ΣH	$\Sigma V=$ −0·086	$\Sigma H=$ −0·029	$\Sigma V=$ −0·1606	$\Sigma H=$ −0·093	$\Sigma V=$ −5·3676	ΣH —
$\Sigma\theta_c$	$\{(\Sigma V)^2+(\Sigma H)^2\}^{1/2}$		0·1856		0·0908		0·4832		0·0908		0·1856		5·3676	
Two Node	1	1·00											1.00	—
	2	0·7625											0.7625	—
	3	0·355											0.355	—
	4	−0·135											−0.135	—
	5	−0·593											−0·593	—
	6	−0·923											−0·923	—
			ΣV	ΣH	ΣV	ΣH	ΣV	ΣH	ΣV	ΣH	ΣV	ΣH	$\Sigma V=$ −0·4665	ΣH —
$\Sigma\theta_c$	$\{(\Sigma V)^2+(\Sigma H)^2\}^{1/2}$												−0·4665	

PART III. TABLE 4

Torsional Vibration

Calculation of Harmonic Torques and Resultant Stresses in Airscrew Shaft

1	Mode of Vibration		Single-Node						Two-Node
2	Crankshaft r.p.m.		3150	2520	2100	1800	1575	1400	3720
3	Order of vibration		2	2½	3	3½	4	4½	6
4	ZA_p lb in.		2680	2450	232	752	177	185	545
5	$\Sigma\theta_C$		0·0908	0·1856	5·3676	0·1856	0·0908	0·4832	0·4665
6	$ZA_p\Sigma\theta_C$		243	454	1240	140	16	89	254
7	Amplitude of half swing at rear mass α_{1p}	Radn	0·0051	0·0096	0·0263	0·0030	0·00034	0·00188	0·00275
		Degrees	0·295	0·551	1·5	0·170	0·0194	0·108	0·158
8	Harmonic torque in airscrew shaft T_H lb in.		6970	13050	35600	4060	460	2560	8690
9	Brake mean torque in airscrew shaft T_M lb in.		26300	24800	17100	12650	9630	7650	26300
10	T_H/T_M		0·265	0·526	2·08	0·321	0·048	0·334	0·33
11	ΣT_H (from Fig. 23)		14,000	20,000	38,000	11,000	3,200	4,300	T_H
12	Max. and min. torques in airscrew shaft $(T_M \pm \Sigma T_H)$	Max. lb in.	40,300	44,800	55,100	23,650	12,830	11,950	34,990
		Min. lb in.	12,300	4,800	—20,900	1,650	6,430	3,350	17,610
13	Shear stresses in airscrew shaft	Max. tons sq in.	4·47	4·96	6·10	2·62	1·42	1·32	3·88
		Min. tons sq in.	1·36	0·53	—2·32	0·183	0·712	0·371	1·95

Remarks :—These stresses are calculated for the splined section in the airscrew shaft, where the modulus of section is 4·03 in³. Based on the diameter of uncut shaft.

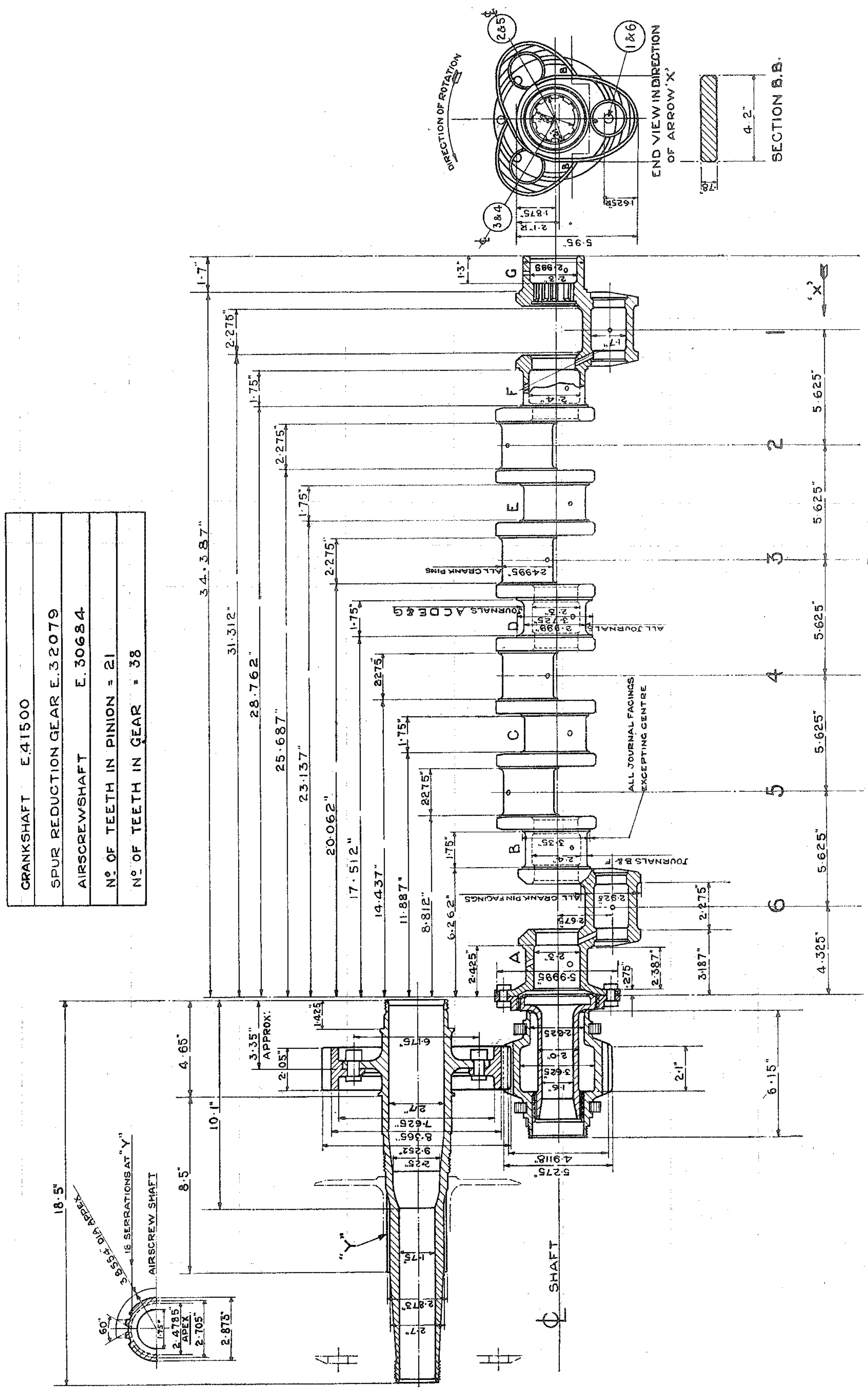


FIG. 17. Crankshaft.

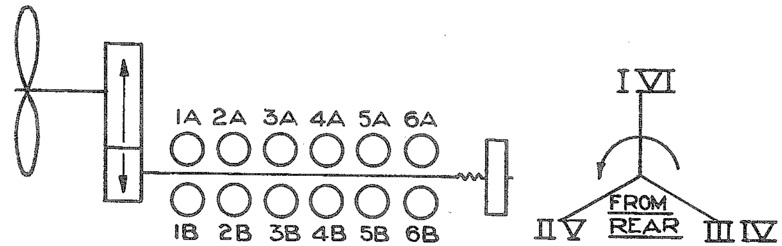


FIG. 18. Layout of crankshaft.

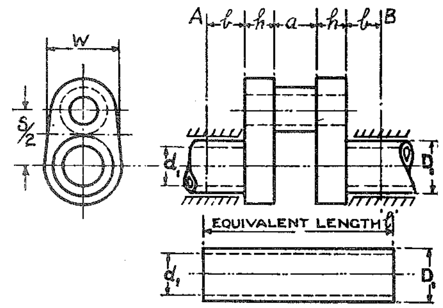


FIG. 19a. Crankthrow stiffness.

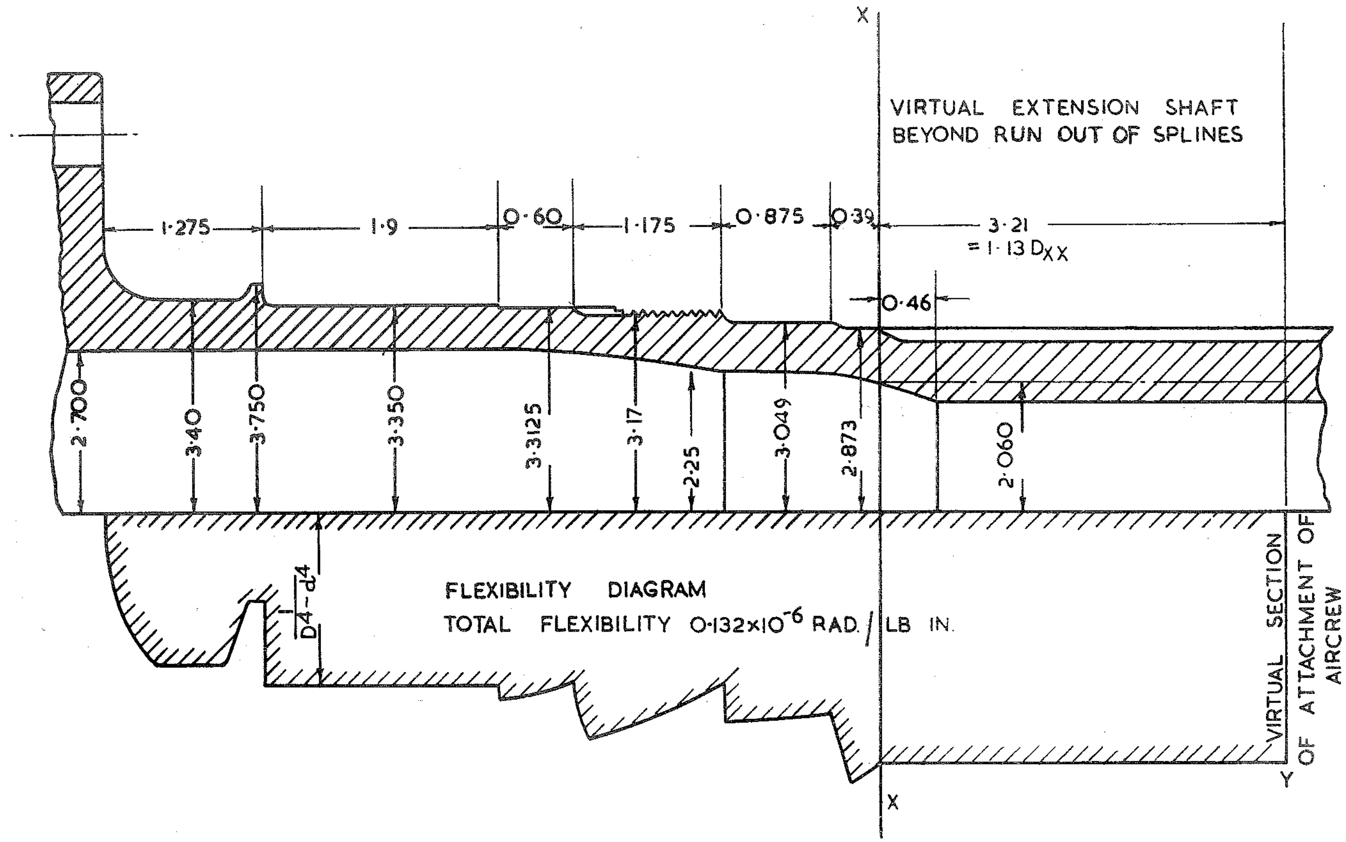


FIG. 19b. Airscrew shaft.

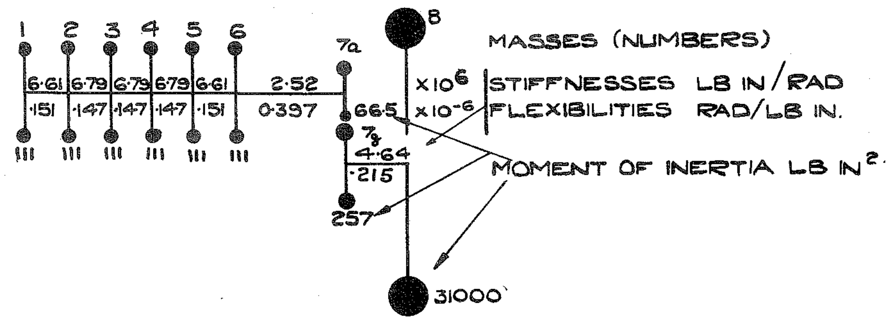


Fig. 20a. Engine system.

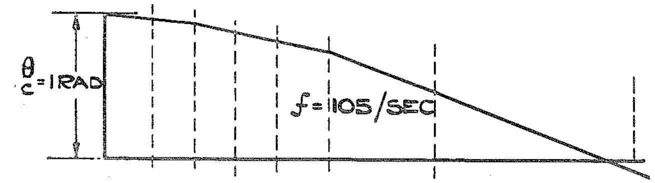


Fig. 21a. Displacement curve single-node vibration.

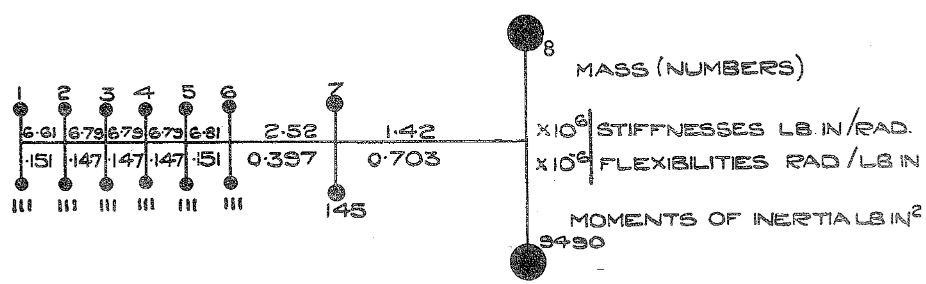


Fig. 20b. Equivalent system.

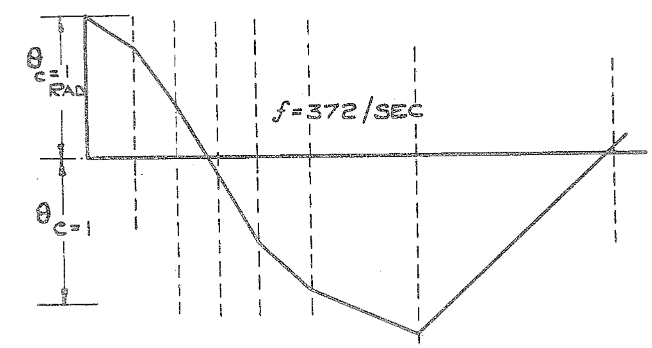


Fig. 21b. Displacement curve two-node vibration.

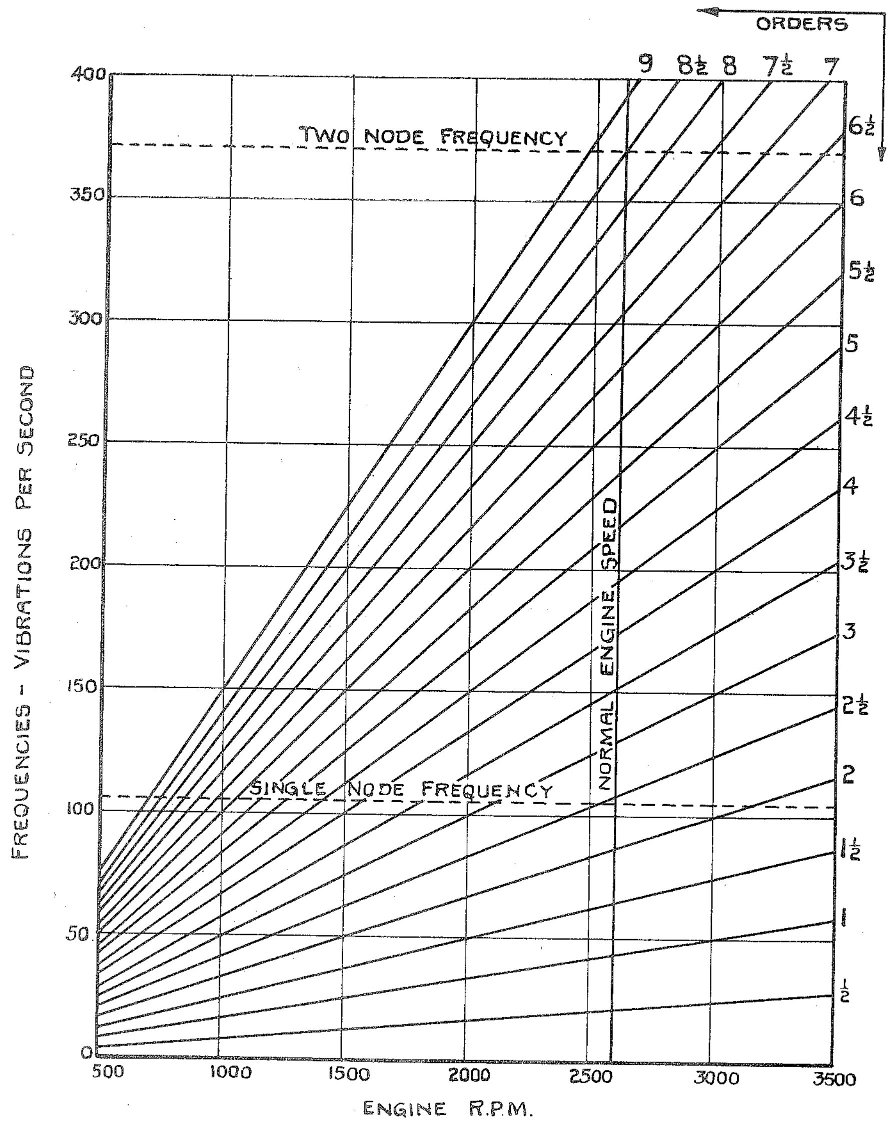


FIG. 22. Frequencies, orders and critical speeds.

Order = complete vibrations per revolution

$$= \frac{60 \times \text{frequency per sec.}}{\text{r.p.m.}}$$

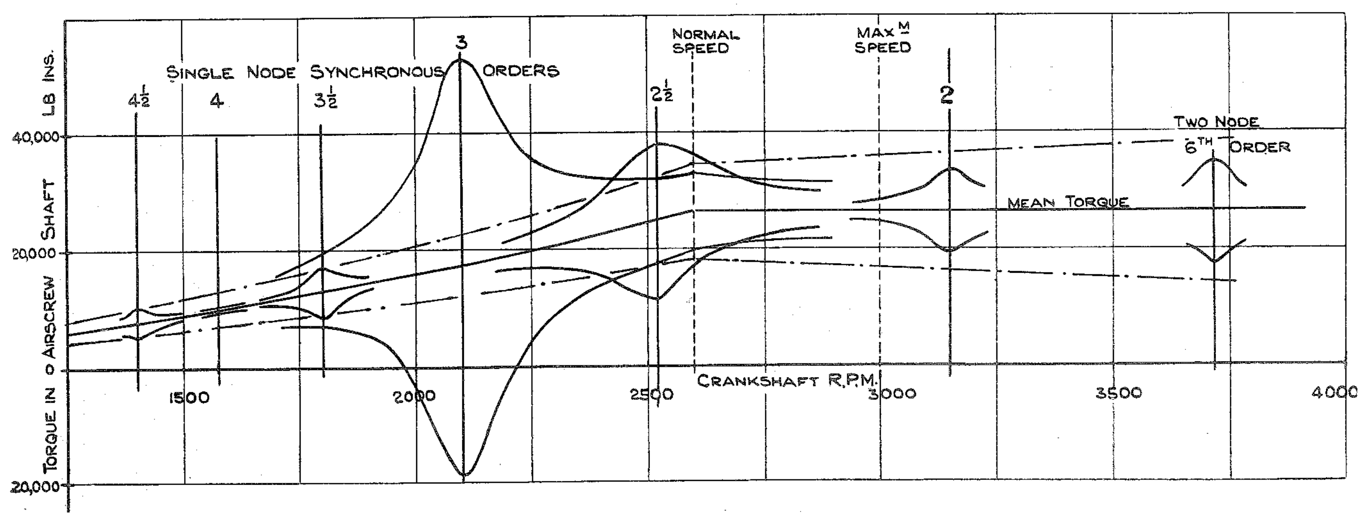


FIG. 23. Curves showing the estimated amplification of torque in airscrew shaft due to resonant torsional vibration.

Damping torque $h_e = 14 \cdot 67$ lb in. per radn per sec, at each crank.

$$= 40\left(\frac{Ic}{g}\right)^{4/5}$$

$$\text{Frequency}\begin{cases}\text{Single node } 105 \text{ per second}\\ \text{Two node } \ 372 \text{ per second}\end{cases}$$

N.B.—The effect of harmonic forcing torques other than the one on resonance is neglected. The form of the curve between the resonance peaks is calculated.

EXTRA REFERENCES

No.	Author	Title, etc.
1	M. A. A. Allfrey	Experimental Investigation of Torsional Flexibilities in the Gearing and Shafting of a 12-Cylinder Vee Aero-Engine with Spur Reduction Gearing. Appendix I to R.A.E. Report E.3586.
2	M. A. A. Allfrey	Experimental Investigation of Torsional Flexibilities in the Gearing and Shafting of an Aero-Engine with Epicyclic Bevel Reduction Gearing. Appendix II to R.A.E. Report E.3586.
3	M. A. A. Allfrey	Experimental Investigation of Torsional Flexibilities in the Gearing and Shafting of a Two-Throw Radial Engine having Epicyclic Spur Reduction Gearing. Appendix III to R.A.E. Report E.3586.
4	The Engine Stress Section, R.A.E. ..	Torsional Vibration of the "R" (Schneider Trophy) 1931 Series Engine. R.A.E. Report No. E.D.O. 137.

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