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Summary.—A method is developed for the calculation of the pressure distribution and the aerodynamic forces and moments on a wing performing harmonic pitching and heaving oscillations. The calculation is based on the assumption of inviscid potential flow without shock waves and is restricted to small incidence, so that the linearized theory is valid.

In contrast to other work in the field the theory applies to all Mach numbers. It is restricted to small values of the reduced frequency and should be valid for the usual range of short periods occurring at present in flight. The formal solution yields two integral equations for the parts of the load, which are in phase and go out of phase with the oscillation; these are of the same form as the corresponding equation in steady flow.

The way is thus opened for solutions over the whole Mach number range at small frequencies, if the corresponding steady solutions can be found. The calculation is in fact easiest for $M = 1$ and has been done here for Delta wings to supplement a previous supersonic calculation[23], made on different frequency assumptions, which broke down near $M = 1$. It appears from the two sets of results that the short-period oscillation will be unstable near $M = 1$, if the apex angle of the Delta wing is greater than about 60 deg. This confirms a now generally recognised trend.

Such results near $M = 1$ must of course be invalidated to an unknown extent by thickness viscosity and shock waves at their maximum effect. Nevertheless it is unlikely that these factors will remove the critical nature of the transonic damping as calculated by this method. With all its obvious limitations this method, when extended to other plan-forms, should provide a useful tool in studying the effect of geometrical parameters on the stability of an aircraft at transonic speeds.

NOTATION

V		Undisturbed velocity
x, y, z		Cartesian co-ordinates
u, v, w		Perturbation velocities, caused by the wing
t		Time
ρ		Air density
M	=	V/a_∞, Mach number
p		Static pressure

† R.A.E. Report Aero. 2468, received 4th November, 1952.

I	=	$(p - p_\infty)/\rho$, Enthalpy (acceleration potential)
Φ		Velocity potential
n	=	$2\pi f$, Frequency
ω	=	$n\bar{c}/V$, Reduced frequency
h, H, r, σ		See equations (11) and (18)
b		Wing span
A		Aspect ratio
S		Wing area
S^*		Part of wing area in forward Mach cone of pivotal point (x, y)
$\bar{c}$		Mean chord
s	=	$s(x)$, Equation of leading edge (local half span)
Λ_e		Sweep angle of leading edge
h	=	$x_0/\bar{c}$, Axis position, measured from apex in terms of mean chord (positive, if pointing backwards)
q		Angular velocity about y-axis
ϑ^*		Amplitude of rotary oscillation
w^*		,, of heaving oscillation
l	=	$2\Delta p/\rho V^2$, Load coefficient

Suffixes $_{r,\ i}$ real and imaginary part

Aerodynamic coefficients (Fig. 1b) :

Heaving oscillation : $\dfrac{w}{V} = \dfrac{w^*}{V}\,\mathrm{e}^{int}$

$$\frac{-Z}{\rho V^2 S} = \frac{w^*}{V}\,(z_w + i\omega z_{\dot{w}})\mathrm{e}^{int}$$

$$\frac{-M}{\rho V^2 S\bar{c}} = \frac{w^*}{V}\,(m_w + i\omega m_{\dot{w}})\mathrm{e}^{int}\,.$$

Rotary oscillation : $\dfrac{w}{V} = -\vartheta^*\left(1 + i\omega\,\dfrac{x - x_0}{\bar{c}}\right)\mathrm{e}^{int}$

$$\frac{-Z}{\rho V^2 S} = -\vartheta^*(z_\vartheta + i\omega z_{\dot{\vartheta}})\mathrm{e}^{int}$$

$$\frac{-M}{\rho V^2 S\bar{c}} = -\vartheta^*(m_\vartheta + i\omega m_{\dot{\vartheta}})\mathrm{e}^{int}\,.$$

Steady pitch : $\dfrac{w}{V} = -\dfrac{q(x - x_0)}{V} = -\dfrac{q\bar{c}}{V}\cdot\dfrac{x - x_0}{\bar{c}}$

$$\frac{-Z}{\rho V^2 S} = -\frac{q\bar{c}}{V}z_q$$

$$\frac{-M}{\rho V^2 S\bar{c}} = -\frac{q\bar{c}}{V}m_q\,.$$

1. *Introduction.*—Experience with modern aircraft has shown that there may be a marked decrease of the damping of the so-called short-period longitudinal oscillation at transonic speeds[1, 2]. This may be so serious that an aircraft, which behaves quite well at subsonic or even supersonic speeds, becomes dynamically unstable at transonic speeds. Having regard to the importance of this problem for the design of high-speed aircraft and the difficulties involved in experimental research in the sonic range, both in wind tunnels and in free flight, it seems worthwhile to supplement the knowledge on this subject by theoretical calculations although these are necessarily based on certain restricting assumptions. The theory can help to show the important parameters which influence the problem and underline at least general trends.

The theory presented in this paper is based on the assumption of a compressible flow without friction or shock waves. All disturbances, caused by the oscillating wing, are assumed to be so small that quadratic terms can be neglected and the usual linearized theory can be adopted. The results of a sonic theory which neglects shock waves and friction must clearly be used with some care. It will be seen later that for Delta wings the theory predicts for some cases a severe instability near $M = 1$. It is unlikely that the neglected factors will change the critical nature of the transonic results and so it is suggested that those plan-forms which show up badly on this theory should be avoided in design and those which seem better should be tested. The theory, incomplete as it is, should be a useful tool in sorting out geometrical wing parameters in the transonic region.

Stability calculations for aircraft at subsonic speeds were usually based on stability derivatives derived from a quasi-steady calculation or steady experiments. Recently the importance of the 'unsteady' derivatives, based on the calculation of the pressure distribution over an oscillating wing, has been stressed[1, 2]. In flutter work, where the reduced frequency ω is high, derivatives have always been calculated from a simple harmonic motion of the wing, usually only for sectional properties, where the two-dimensional solution is assumed to be sufficient. In a recent paper Neumark[3] has summarized the results of the two-dimensional theory for subsonic speeds, as far as smaller frequencies are concerned, as they occur in the so-called 'short-period oscillations'. The same two-dimensional stability derivatives have been used by Pinsker[4] and by Statler[5] to calculate the longitudinal stability of an aircraft. The finite aspect ratio of the wing has been allowed for in these calculations by a simple conversion formula, derived from the steady case, whereas Reissner[6] treated this problem in a more elaborate way. He succeeded in separating the problem into two parts, one of them being the problem of two-dimensional oscillatory flow, the other being the problem of the spanwise circulation distribution. W. P Jones[7] reduced the calculation of the load of an oscillating wing, at least for smaller frequencies, to a similar problem for a related plan-form in an incompressible flow and solved the latter by the vortex-lattice method[8, 9] introducing the two-dimensional oscillatory solution for each chordwise section.

However, it seems by no means certain, that this approach leads to the correct answer for small ω's, since the term $\log \omega$, which occurs in the two-dimensional calculation, becomes more and more unimportant as the aspect ratio decreases (compare Reissner and Stevens[10], Miles[11]). Without resorting to two-dimensional results Multhopp in an unpublished paper succeeded in reducing the problem, for small values of ω, to the solution of two integral equations of the same form as occurs in the load calculation for a lifting surface in steady flow, the unsteadiness being allowed for by additional downwash terms. As in most other papers, where the numerical work has been kept within reasonable bounds, he introduced the parameter $\omega/(1 - M^2)$, which is assumed to be small. Thus the results become invalid as the sonic range ($M \sim 1$) is approached.

On the supersonic side a number of papers exist which deal with the oscillatory flow problem in two dimensions or with wings with a 'supersonic' leading edge only (Garrick and Rubinov[12, 13] or the flutter report[14], W. P. Jones[15]). From the point of view of sonic flight the case of 'subsonic' leading edges is more interesting. Only a few trends from the large output in this field can be mentioned here.

Robinson[16] outlined a method, valid for all frequencies ω, without producing numerical results. The numerical work is considerably simplified by restricting the calculation to small frequencies, so that $\omega/(M^2 - 1)$ is small or equivalent assumptions (Robinson[17], W. P. Jones[14], Ribner and Malvestuto[18], Miles[19], Acum[20], Watkins[21], Moeckel[22], Mangler[23] or Temple[24]). Again by virtue of these assumptions the results become invalid near the sonic speed.

Furthermore an attempt has been made to extend the conception of the wing of a very small aspect ratio which was so successful in steady flight conditions[25,26,27,28,29] to unsteady flight (Ribner[27], Garrick (App. B of Ref. 14)). Unfortunately the range of validity of these unsteady results becomes more and more restricted as the sonic speed is approached. As we shall see later on, a small increase of the aspect ratio at sonic speeds has a great effect on the results.

Under these circumstances it seems only natural to try another attack on this problem of oscillatory flight by going back to the beginning and devising a method which is valid for the whole Mach number range. The differential equation, which governs this problem, is related to the general wave equation for acoustic waves (section 2.1). By using the fundamental solutions of this equation, which correspond to oscillatory sources and doublets, the problem can be shown to require the solution of an integral equation for the doublet strength (section 2.3). In section 2.4 this equation is given in a more convenient form valid for any Mach number and any frequency.

Since for stability calculations we are only interested in small values of ω, this equation can be reduced for subsonic and supersonic flow to two integral equations for the real and imaginary part of the load distribution (section 2.5). These equations are of the same type as the corresponding steady flow problem of a lifting surface, the unsteadiness being allowed for by an additional downwash term in the equation for the imaginary part. Since this procedure restricts the frequency but not the Mach number, we may go to the limit of sonic flow (section 3). Here the corresponding problem for steady flow becomes particularly simple[29]. In subsonic flow the steady problem has not yet found a satisfactory solution valid for any aspect ratio, although Multhopp's method[31] seems to meet most of the requirements for such a calculation.

As an application results for a Delta wing at sonic speeds are obtained in section 4. All stability derivatives, required for the calculation of the short period longitudinal oscillation of a Delta wing, are compiled in Table 1. Strictly speaking these derivatives are based on a sustained harmonic oscillation and thus are valid only near the stability boundary. But it is felt that these derivatives can be employed in connection with the usual equations of motion even for damped oscillations or near $M = 1$, where some of the derivatives cannot be regarded as constants, but are functions of the reduced frequency.

The two-dimensional case of an oscillating wing at sonic speed, which has been dealt with by Rott[30] and Jordan (unpublished paper), has been left outside the scope of this report. It is remarkable that the limiting case of steady flow does not produce finite derivatives in two-dimensional flow, whereas the three-dimensional case contains the steady flow as a limit ($\omega \to 0$). An extension of the theory to higher frequencies (flutter calculations) is not attempted, although it may be feasible.

2. *Principles of the Theory of an Oscillating Wing.*—2.1. *The Equations of Motion.*—We consider a thin wing at a small incidence in a parallel flow and assume that all perturbations caused by the presence of the (oscillating) wing are so small that the linearised potential theory can be applied. Then a velocity potential ϕ exists so that the perturbation velocities u, v, w are given by†:

$$u = \frac{\partial \phi}{\partial x}, \; v = \frac{\partial \phi}{\partial y}, \; w = \frac{\partial \phi}{\partial z} . \qquad (1)$$

† We use a Cartesian system of co-ordinates with x in the direction of the undisturbed velocity V and z directed upwards (Fig. 1a)

The linearized Euler equations, which connect the velocities with the pressure p or enthalpy I, are

$$\left.\begin{aligned}\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)u&=-\frac{\partial I}{\partial x}\\\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)v&=-\frac{\partial I}{\partial y}\\\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)w&=-\frac{\partial I}{\partial z}\end{aligned}\right\}\quad\ldots\quad(2)$$

where V is the velocity of undisturbed flow and

$$I=I_\infty+\int_\infty^p\frac{dp}{\rho}=I_\infty+\frac{p-p_\infty}{\rho_\infty}\quad\ldots\quad(3)$$

since the density ρ in the denominator can be replaced by its value ρ_∞ in the undisturbed flow. The continuity equation can be linearised as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}+\frac{M^2}{V^2}\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)I=0\quad\ldots\quad(4)$$

where $M=V/a_\infty$ is the Mach number of the undisturbed flow. By differentiating this equation again along a streamline ($y=$ const, $z=$ const) (using the operator $\partial/\partial t+V\,\partial/\partial x$) and introducing the derivatives of u, v, w from (2) we find

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)I=M^2\left(\frac{1}{V}\frac{\partial}{\partial t}+\frac{\partial}{\partial x}\right)^2 I\,.\quad\ldots\quad(5)$$

This is the general wave equation which must be satisfied by the enthalpy I. By using Bernoulli's equation

$$\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)\Phi+I=I_\infty\quad\ldots\quad(6)$$

which is a consequence of (1) and (2), and by inserting (1) into (4), we can prove that the velocity potential ϕ also satisfies the wave equation (5).

2.2. *Boundary Conditions.*—We consider a thin flat wing performing heaving or rotary oscillations in a steady flow V. Then all perturbations must vanish at a great distance from the wing ($M<1$) or along the Mach cone extending from the foremost point (apex) of the wing ($M>1$). The motion of the wing surface may be described (in the usual complex notation) by

$$z=z_0=\bar{z}_0\,(x,y)\,.\,e^{int},$$

i.e., we consider a wing performing harmonic oscillations of the frequency $n(=2\pi f)$. The downwash inside the wing surface S is prescribed by the wing motion

$$w(x,y,0\,;t)\equiv\bar{w}(x,y,0)\,.\,e^{int}=\left(\frac{\partial}{\partial t}+V\frac{\partial}{\partial x}\right)z_0$$

$$=V\,e^{int}\left(\frac{in\bar{z}_0}{V}+\frac{\partial\bar{z}_0}{\partial x}\right)\equiv V\left(w_r+\frac{in\bar{c}}{V}w_i\right)e^{int}.$$

Here $n\bar{c}/V = \omega(\bar{c} =$ mean chord) denotes the reduced frequency. (Letters with bars denote amplitudes ; suffix $_r$ refers to real part, suffix $_i$ to imaginary part of a complex function.) Thus we have for a *heaving oscillation*

$$\bar{z}_0 = z_0{}^* = \text{const} = -\frac{w^*}{in}$$

the boundary condition

$$\frac{\bar{w}}{V}(x, y, 0) = -\frac{w^*}{V} = \text{const}, \; w_r = -\frac{w^*}{V}, \; w_i = 0 \quad \ldots \qquad (7a)$$

and for a *rotary oscillation* $\bar{z}_0 = -\vartheta^*(x - x_0)$ about the axis $x = x_0$:

$$\frac{\bar{w}}{V}(x, y, 0) = -\vartheta^*\left(1 + \frac{in(x - x_0)}{V}\right), \; w_r = -\vartheta^* = \text{const},$$

$$w_i = -\vartheta^*. \frac{x - x_0}{\bar{c}}. \quad \ldots \qquad (7b)$$

These conditions have to be satisfied for all points inside the area S of the wing.

In order to transform the conditions (7) for the downwash w into conditions for the enthalpy

$$I = \bar{I}(x, y, z) \,.\, e^{int}$$

we integrate the first equation (2), which reads for a harmonic oscillation :

$$\left(in + V\frac{\partial}{\partial x}\right)w + \frac{\partial I}{\partial z} = 0$$

along a streamline ($y =$ const, $z =$ const) and obtain

$$\bar{w}(x, y, z) = w(x, y, z, t).\, e^{-int}$$

$$= -\frac{1}{V}\int_{-\infty}^{x} \frac{\partial \bar{I}}{\partial z}(\xi, y, z)\, e^{-in(x-\xi)/V}\, d\xi\,. \quad \ldots \qquad (8)$$

The amplitude $\bar{I}$ of the enthalpy must satisfy the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\bar{I} = M^2\left(\frac{in}{V} + \frac{\partial}{\partial x}\right)^2 \bar{I} \quad \ldots \qquad (9)$$

as follows by inserting $I(x, y, z, t)$ into (5).

2.3. *Solution of the Wave Equation.*—We try to solve the wave equation (5) or (9) for the functions I or $\bar{I}$ respectively by a distribution of pressure doublets over the wing area. In accordance with the assumptions of the linearized theory these singularities will be placed in the plane $z = 0$ instead of the actual wing surface. They produce a pressure discontinuity between both faces of the wing and therefore a load distribution. The strength l of this load will be chosen in such a way that the resultant downwash according to equation (8) satisfies the boundary conditions (7), prescribed by the motion of the wing.

We obtain a pressure doublet by superposing a source and a sink of a harmonically oscillating strength. The source is given by (*see* Refs. 7, 13, 16, 24, 32, 33 and Fig. 10) :

$$\frac{1}{r'}\, e^{in(t - MR/V)}$$

with

$$r' = \sqrt{\{(x - x')^2 + (1 - M^2)((y - y')^2 + (z - z')^2)\}}\,,$$

$$R = \frac{-M(x - x') + \varepsilon r'}{1 - M^2} \geqslant 0 \quad (\varepsilon = \pm 1)\,.$$

R is the radius of the sphere of influence (Fig. 11), affected by a disturbance, which was originated in the point (x', y', z') a certain time MR/V ago. Since the centre of this sphere travels with the velocity V and the disturbance with the velocity V/M, every point (x, y, z) enters this sphere of influence but never leaves it in subsonic flow. Thus we have $\varepsilon = +1$ for $M < 1$. In supersonic flow every point (x, y, z) inside the after-cone ($r' = 0$, $x' < x$) experiences this disturbance twice, once when entering and once when leaving this sphere of influence. Thus we have $\varepsilon = \pm 1$ for $M > 1$ and both solutions have to be added. We have to remember that every source affects only points inside its after-cone ($x' < x, r' = 0$) and every point (x, y, z) is affected only by singularities inside its fore-cone.

For convenience we shall retain the solution given above with ε, but we have to remember the meaning of ε, which is $\varepsilon = +1$ for $M < 1$ and $\varepsilon = \pm 1$ for $M > 1$ with both solutions added. Later on we shall see that there is only one solution for $M = 1$, which can be obtained by a suitable limit $M \rightarrow 1$, taken either from the subsonic or the supersonic side.

From the solution for a source or sink, we obtain the solution for a doublet in the point $(x', y', 0)$ by differentiating $(\partial/\partial z')$ and putting $z' = 0$ afterwards:

$$\frac{\partial}{\partial z'}\left(\frac{1}{r'}\,e^{in(t - MR/V)}\right)_{z'=0} = -\frac{\partial}{\partial z}\left(\frac{1}{r'}\,e^{in(t - MR/V)}\right)_{z'=0}$$

$$= \frac{z(1 - M^2)}{r^3}\left(1 + \varepsilon\,\frac{inr}{V}\,\frac{M}{1 - M^2}\right)e^{int - ih} \quad \ldots \quad (10)$$

with

$$\left.\begin{aligned} r &= \sqrt{\{(x - x')^2 + (1 - M^2)\sigma^2\}},\ \sigma^2 = (y - y')^2 + z^2, \\ h &= \frac{nM}{V(1 - M^2)}\,\{-(x - x')M + \varepsilon r\} = \frac{nMR}{V} \end{aligned}\right\} \quad \ldots \quad (11)$$

A distribution of doublets of the strength

$$l(x, y, t) = \bar{l}(x, y).\,e^{int} \quad \ldots \quad (12)$$

over the plan-form S of the wing results in an enthalpy function

$$\bar{I}(x, y, z) = -\frac{zV^2(1 - M^2)}{8\pi}\iint\limits_S \bar{l}\,(x', y')\left(1 + \varepsilon\,\frac{inr\,M}{V(1 - M^2)}\right)e^{-ih}\,\frac{dx'\,dy'}{r^3}. \quad \ldots \quad (13)$$

The strength $\bar{l}$ must be chosen in such a way that equations (7) and (8) are satisfied inside the plan-form of the wing.

It can be shown that for points inside the plan-form

$$I(x, y, \pm 0\,;t) = \bar{I}(x, y, \pm 0)\,e^{int}$$

$$= \mp\frac{V^2}{4}\,\bar{l}(x, y)e^{int} \quad \ldots \quad (14)$$

so that the pressure is discontinuous along the wing surface. The local pressure coefficient c_p or the local load is obtained from

$$c_p = \frac{\Delta p}{\frac{1}{2}\rho V^2} = \frac{\Delta I}{\frac{1}{2}V^2} = \frac{4I(x, y, -0\,;t)}{V^2}. = \bar{l}\,(x, y)\,.\,e^{int} \quad \ldots \quad (15)$$

When introducing (13) into (8) we obtain the following integral equation for the unknown load coefficient $\bar{l}$:

$$\frac{\bar{w}}{V}(X, y, 0) = \lim_{z \to 0} \left\{ \frac{\partial}{\partial z} \frac{z(1 - M^2)}{8\pi} \int_{-\infty}^{x} \left[\iint_S \bar{l}(x', y') e^{-iH} \left(1 + i\varepsilon \frac{nrM}{V(1 - M^2)} \right) \frac{dx'\, dy'}{r^3} \right] dx \right\} \quad (16)$$

where

$$H = h + \frac{n(X - x)}{V} = \frac{nM}{V(1 + M)} \left[\frac{1 + M}{M} (X - x) + (x - x') - \frac{x - x' - \varepsilon r}{1 - M} \right] \quad (17)$$

or

$$H = \frac{nM}{V(1 + M)} \left[\frac{1 + M}{M} (X - x) + (x - x') + \frac{(1 + M)\sigma^2}{(x - x') + \varepsilon r} \right]. \quad \ldots \quad (18)$$

This last form for H is most important, since it will enable us to consider the case of sonic flow $M = 1$. This would not be possible if the usual procedure of introducing a 'modified' velocity potential or acceleration potential (*see* App. II and III) had been followed. When we approach $M = 1$ from the subsonic side ($\varepsilon = +1$) and remember that a point x is only influenced by singularities x' in the fore-cone ($x' < x$), we find that $r \to |x - x'|$ for $M \to 1$ and H tends to a finite limit. This will be explained in more detail later on.

It may be pointed out, that for supersonic flow ($M > 1$) the integral in (13) and (16) is to be extended only over that part S^* of the wing area S, which is inside the fore-cone $X - x' \geqslant \sqrt{(M^2 - 1)}\, |\sigma|$ extending from the point (X, y). Since r tends to zero along this fore-cone, the integral does not converge in the usual sense. As has been shown previously (compare Refs. 34, 35, 36), the 'principal value' of the integral or its 'finite part' has to be taken in this case (Hadamard)†. In the next section we shall derive another form for the integral equation (16), which will be more convenient for all Mach numbers.

2.4. *Simplification of the Integral Equation for $\bar{l}$.*—In order to simplify the complicated integral equation (16) for $\bar{l}$, we make use of the following two relations which can be proved for any integer number m and $\varepsilon^2 = 1$:

$$-\frac{\partial}{\partial x'} \left[e^{-iH} \left(\varepsilon - \frac{x' - x}{r} \right) r^{1-m} \right] = e^{-iH} r^{-m} \left\{ \frac{\sigma^2}{r^2} \left(m(1 - M^2) + i\varepsilon \frac{nMr}{V} \right) + \left(\varepsilon - \frac{x' - x}{r} \right) \left(\varepsilon(1 - m) - i \frac{nMr}{V(1 + M)} \right) \right\} \quad (19)$$

$$\frac{\partial}{\partial x} \left[e^{-iH} \left(\varepsilon - \frac{x' - x}{r} \right) r^{1-m} \right] = e^{-iH} r^{-m} \left\{ \frac{\sigma^2}{r^2} \left(m(1 - M^2) + i\varepsilon \frac{nMr}{V} \right) + \left(\varepsilon - \frac{x' - x}{r} \right) \left(\varepsilon (1 - m) + i \frac{nr}{V(1 + M)} \right) \right\}. \quad (20)$$

The proof of these relations follows by differentiation of the left-hand side. Their difference is

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \right) \left[e^{-iH} \left(\varepsilon - \frac{x' - x}{r} \right) r^{1-m} \right] = \frac{in}{V} e^{-iH} \left(\varepsilon - \frac{x' - x}{r} \right) r^{1-m}. \quad \ldots \quad \ldots \quad (21)$$

† It has been shown in Ref. 37 that Green's method for subsonic flow can be extended to supersonic flow and leads to the same results as Hadamard's approach.

From (19) we have for $m = 1$:

$$\frac{e^{-iH}}{r^3}\left(1 - M^2 + i\varepsilon\frac{nrM}{V}\right) = \left(-\frac{\partial}{\partial x'} + \frac{inM}{V(1+M)}\right)\left[\frac{e^{-iH}}{\sigma^2}\left(\varepsilon - \frac{x'-x}{r}\right)\right],$$

which enables us to express the influence function in equation (16) in a simpler form. Thus we have instead of (16) ;

$$\frac{\bar{w}}{V}(X, y, 0) = \lim_{z\to 0}\frac{\partial}{\partial z}\left\{\frac{z}{8\pi}\int_{-\infty}^{X}\iint_S l(x', y')\left(-\frac{\partial}{\partial x'} + \frac{inM}{V(1+M)}\right)\left[e^{-iH}\left(\varepsilon - \frac{x'-x}{r}\right)\right]\frac{dx'\,dy'}{\sigma^2}\,dx\right\}. \quad (22)$$

We integrate by parts (with respect to x') the first term of the integral and obtain ($x' = x_l(y')$ and $x' = x_t(y')$ are the equations of the leading edge and the trailing edge, b = wing span) :

$$\frac{\bar{w}}{V}(X, y, 0) = \lim_{z\to 0}\frac{\partial}{\partial z}\left\{-\frac{z}{8\pi}\int_{-\infty}^{X}\int_{-b/2}^{b/2}\left(l(x', y')\,e^{-iH}\left(\varepsilon - \frac{x'-x}{r}\right)\right)_{x'=x_l}^{x'=x_t}\frac{dy'}{\sigma^2}\,dx\right.$$

$$\left. + \frac{z}{8\pi}\int_{-\infty}^{X}\iint_S\left(\frac{\partial}{\partial x'}l(x', y') + \frac{inM}{V(1+M)}l(x', y')\right)e^{-iH}\left(\varepsilon - \frac{x'-x}{r}\right)\frac{dx'\,dy'}{\sigma^2}\,dx\right\}. \quad (22a)$$

In order to perform the integration with respect to x in the last term by means of equation (21), we define now a new quantity $L(x, y)$ by the relation :

$$L(x, y) = l(x, y) - \frac{in}{V(1+M)}\int_{x_l}^{x} e^{\frac{in}{V}(x-\xi)}\,l(\xi, y)\,d\xi. \quad (23)$$

L agrees with l in case of a steady flow. Its relation to the velocity potential is explained in Appendix I. We differentiate equation (23) with respect to x and obtain the important relation :

$$\frac{\partial L}{\partial x} + \frac{in}{V}L = \frac{\partial l}{\partial x} + \frac{in}{V}\frac{M}{1+M}l \quad (24)$$

and finally, by solving this equation for l :

$$l(x, y) = L(x, y) + \frac{in}{V(1+M)}\int_{x_l}^{x} e^{\frac{-inM}{V(1+M)}(x-\xi)}\,L(\xi, y)\,d\xi. \quad (25)$$

Inserting this into equation (23) leads to an identity as required.

Now we apply equation (24) in the last term of equation (22a), introducing L instead of l, and find after another integration by parts (with respect to x') :

$$\frac{\bar{w}}{V}(X, y, 0) = \lim_{z\to 0}\frac{\partial}{\partial z}\left\{\frac{z}{8\pi}\int_{-\infty}^{x}\int\left(\{L(x', y') - l(x', y')\}\,e^{-iH}\left(\varepsilon - \frac{x'-x}{r}\right)\right)_{x'=x_l}^{x'=x_t}\frac{dy'\,dx}{\sigma^2}\right.$$

$$\left. + \frac{z}{8\pi}\int_{-\infty}^{x}\iint_S L(x', y')\left(-\frac{\partial}{\partial x'} + \frac{in}{V}\right)\left[e^{-iH}\left(\varepsilon - \frac{x'-x}{r}\right)\right]\frac{dx'\,dy'}{\sigma^2}\,dx\right\}.$$

In the second of these two integrals we apply equation (21) for $m = 1$, which shows that the influence function is the derivative with respect to x of the bracket. Thus the integration over x can be performed. In the first integral we remember that $L = l$ for $x' = x_t$. We obtain finally:

$$\frac{\bar{w}}{V}(X, y, 0) = \lim_{z \to 0} \frac{\partial}{\partial z} \left\{ \frac{z}{8\pi} \iint_S L(x', y') \left[\mathrm{e}^{-iH} \left(\varepsilon - \frac{x' - x}{r} \right) \right]_{x=X} \frac{dx'\, dy'}{\sigma^2} \right.$$

$$\left. + \frac{z}{8\pi} \int_{-\infty}^{X} \int_{-b/2}^{b/2} L_t(y') \left[\mathrm{e}^{-iH} \left(\varepsilon - \frac{x' - x}{r} \right) \right]_{x'=x_t} \frac{dy'\, dx}{\sigma^2} \right\} \quad \ldots \quad \ldots \quad (26)$$

where $L_t(y') \equiv L(x_t, y') - l(x_t, y')$ can easily be obtained from (25).

This integral equation for $L(x', y')$ holds for any frequency n and any Mach number M. We have to remember that $\varepsilon = +1$ for $M < 1$ and $\varepsilon = +1$ or $\varepsilon = -1$ for $M > 1$, where both terms have to be added. For supersonic flow the area S has to be replaced by the part S^* of S, which is inside the Mach fore-cone originating in the point (X, y). The second integral in (26) represents the wake effects as can be seen by introducing the new variable $\xi' = X - x + x_t$ in the chordwise integral, which is then extended over the wake. Strictly speaking the integration by parts, which lead to equation (26), should have been performed for this area S^* instead of S, i.e., the values along this fore-cone $r = 0$ should have been used instead of the values along the trailing edge. But this contribution is zero since the 'finite part' of the integral must be taken as explained before (Hadamard). For subsonic flow the Kutta-Joukowsky condition ($l = 0$) must be satisfied along the trailing edge.

2.5. *Small Values of the Reduced Frequency.*—The solution of equation (26) in its general form for any value of ω and any Mach number M presents considerable mathematical difficulties. Fortunately the frequencies, occurring in stability calculations, are usually small so that terms of higher order in the reduced frequency ω can be neglected. However, an expansion of the solution in powers of ω is not feasible near $M = 1$, as we shall see later on. We shall be forced to introduce terms of the order $\log \omega$ there. Neverthless terms of the order ω^2 and $\omega^2 \log \omega$ will be neglected in this paper.

For the actual calculation we have to split equation (26) into its real and imaginary parts. We introduce ($\bar{c}$ = mean chord, ω = reduced frequency):

$$\left. \begin{aligned} \frac{\bar{w}}{V} &= w_r + i\omega w_i \quad , \qquad l = l_r + i\omega l_i, \\ L &= L_r + i\omega L_i \quad , \quad L_t \equiv L(x_t, y) = L_{tr} + i\omega L_{ti} \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

and find from (23) and (25), if terms of the order ω^2 are neglected:

$$\left. \begin{aligned} L_r(x, y) &= l_r(x, y) \\ L_i(x, y) &= l_i(x, y) - \frac{1}{1 + M} \int_{x_l}^{x} l_r(\xi, y) \frac{d\xi}{\bar{c}} \\ l_i(x, y) &= L_i(x, y) + \frac{1}{1 + M} \int_{x_l}^{x} L_r(\xi, y) \frac{d\xi}{\bar{c}} \end{aligned} \right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

Thus l_r and L_r become equal and l_i and L_i differ by a term which can easily be calculated.

The expression

$$L\,e^{-iH} = (L_r \cos H + \omega L_i \sin H) + i\omega(L_i \cos H - L_r \tfrac{1}{\omega} \sin H)$$

can be simplified by putting

$$H = H_1 + H_{2\varepsilon}$$

with

$$H_1 = \frac{\omega M}{1+M}\left[\frac{1+M}{M}\,\frac{X-x}{\bar{c}} + \frac{x-x'}{\bar{c}}\right] \quad \ldots \quad (29)$$

$$H_{2\varepsilon} = \frac{\omega M}{1+M} \cdot \frac{\varepsilon r - (x-x')}{\bar{c}(1-M)} = \frac{\bar{c}(\varepsilon r + x - x')}{\omega M \sigma^2} .$$

Since $\sin H_1$ and $\cos H_1$ can be expanded, we have for small values of ω :

$$\cos H = \cos H_{2\varepsilon} - H_1 \sin H_{2\varepsilon}$$

$$\sin H = H_1 \cos H_{2\varepsilon} + \sin H_{2\varepsilon}.$$

As will be shown in section 3 (for sonic flow) and in Appendix II (subsonic) and III (supersonic flow), we are entitled to simplify this further and to write

$$\cos H = 1 \qquad \frac{1}{\omega}\sin H = \frac{H_1}{\omega} + \frac{1}{\omega}\sin H_{2\varepsilon} . \quad \ldots \quad (30)$$
$$\omega \sin H = 0$$

Thus we obtain from equation (26) the following two conditions for L_r and L_i :

$$w_r(X, y) = \frac{1}{8\pi} \iint\limits_S L_r(x', y')\left(\varepsilon - \frac{x'-x}{r}\right)_{\substack{x=X\\z=0}} \frac{dx'\,dy'}{(y-y')^2} \quad \ldots \quad (31a)$$

$$w_i(X, y) = \frac{1}{8\pi} \iint\limits_S L_i(x' - y')\left(\varepsilon - \frac{x'-x}{r}\right)_{\substack{x=X\\z=0}} \frac{dx'\,dy'}{(y-y')^2} - F(x, y) \quad \ldots \quad (31b)$$

where

$$F(X, y) = \frac{1}{8\pi} \iint\limits_S L_r(x', y')\left[\left(\frac{H_1}{\omega} + \frac{1}{\omega}\sin H_{2\varepsilon}\right)\left(\varepsilon - \frac{x'-x}{r}\right)\right]_{\substack{x=X\\z=0}} \frac{dx'\,dy'}{(y-y')^2}$$

$$- \frac{1}{8\pi}\int_{-\infty}^{X}\int_{-b/2}^{b/2} L_{ti}(y')\left(\varepsilon - \frac{x'-x}{r}\right)_{\substack{x'=x_t\\z=0}} \frac{dy'\,dx}{(y-y')^2} \quad \ldots \quad (31c)$$

depends only on L_r, since L_{ti} follows immediately from the third equation (28). We have to remember the meaning of ε, which is $\varepsilon = +1$ for $M < 1$ and $\varepsilon = +1$ or $\varepsilon = -1$ for $M > 1$ with both solutions added. In supersonic flow any point is affected only by points (doublets) in its fore-cone, so that the area of integration S must be replaced by the part S^* of S which is inside the fore-cone originating from (X, y).

Thus the problem of an oscillating wing is reduced (as far as small frequencies are concerned) to a steady flow problem. Both the integral equation (31a) for L_r and the integral equation (31b) for L_i are of the same type as the integral equation for the load distribution of a wing in steady flow. The incidence term in (31b) is modified by a term $F(x, y)$, which can be assumed as known after (31a) has been solved. Since $F(x, y)$ depends on ω, the function L_i and the load l_i will depend on ω.

The actual solution of the problem and the numerical calculation of stability derivatives depends on the possibility of solving this steady flow problem. For subsonic flow Multhopp's method[31] is available. Its general applicability may be restricted to not too small aspect ratios and 'reasonable' incidence distributions. Practical experience must show, whether these restrictions can be overcome. For supersonic flow most calculations have been based on the theory of conical fields and its extensions rather than on a direct solution of the integral equation. For sonic flow the solution of the corresponding steady flow problem[29] is comparatively simple and easy, and so the remainder of this paper is devoted to a solution of the oscillatory wing flow problem at sonic speeds (section 3).

It may be mentioned here that l_i and thus the stability derivatives become independent of the frequency if the Mach number M is either small or big compared to 1. In such a case the function $(1/\omega)\sin H_{2\varepsilon}$ in $F(x, y)$ may further be simplified as

$$\frac{1}{\omega}\sin H_{2\varepsilon} = \frac{1}{\omega}\sin\frac{\omega M(\varepsilon r - (x - x'))}{(1 - M^2)\bar{c}} \simeq \frac{M(\varepsilon r - (x - x'))}{(1 - M^2)\bar{c}} .$$

This assumes that $\omega < < |1 - M^2|$, which excludes the sonic region. Most calculations up to now are based on this additional assumption. A number of authors[7, 8, 9, 11, 15, 17, 18, 19, 21, 23] succeeded in reducing the unsteady case, at least for frequencies $\omega < < |1 - M^2|$, to a steady flow problem, by introducing a 'modified' velocity potential and a 'modified' load distribution. The connections of these methods with the new procedure, outlined in this paper, are explained in Appendix II and III.

3. *Sonic Flow.*—3.1. *The Integral Equation.*—All the results derived in section 2, in particular equation (26), are valid for any Mach number and the results of section 2.4 apply for any frequency ω. Now we consider the case of sonic flow for all values of the frequency as limiting case for a subsonic flow. Then we have $\varepsilon = 1$, $M \leqslant 1$ and

$$\lim_{M \to 1}\left(1 - \frac{x' - x}{r}\right) = \begin{cases} 0 & x' > x \\ & \text{if} \\ 2 & x' < x \end{cases} \qquad \ldots \quad (32a)$$

and from (29) for $x = X$ and $x' < x$:

$$H_0 \equiv \lim_{M \to 1}(H)_{x=X} = \frac{\omega}{2}\left[\frac{x - x'}{\bar{c}} + \frac{\sigma^2}{(x - x')\bar{c}}\right] . \qquad \ldots \quad (32b)$$

Because of (32a) the integration in (26) has to be extended only over the part S^* of S, which is upstream from the point (x, y) $(x' \leqslant x)$. Thus we have the following integral equation for sonic flow and any value of the frequency ω:

$$\frac{\bar{w}}{V}(x, y, 0) = \frac{\partial}{\partial z}\left\{\frac{z}{4\pi}\iint_{S^*} L(x', y')\, e^{-iH_0}\frac{dx'\,dy'}{\sigma^2}\right.$$
$$\left. + \frac{z}{4\pi}\int_{-b/2}^{b/2} L_t(y')\int_{X_t}^{X}\left(e^{-iH_0}\frac{1}{2}\left(1 + \frac{x - x'}{|x - x'|}\right)\right)_{x'=x_t}\frac{dx\,dy'}{\sigma^2}\right\}_{z=0} . \qquad \ldots \quad (33)$$

The Kutta-Joukowsky condition ($\bar{l} = 0$ along the trailing edge) now becomes irrelevant, except for plan-forms (swallow-tailed wings), where parts of the trailing edge are upstream of certain points (x, y) of the wing. After L is determined from (33), the load $\bar{l}$ itself is obtained from (25).

For wings with an unswept trailing edge the integral in (33) may be written as

$$\frac{\bar{w}}{V}(x, y, 0) = \frac{1}{4\pi}\left(\iint_{S^*} L(x', y)\, e^{-iH_0}\frac{dx'\,dy'}{(y - y')^2}\right)_{z=0} \qquad \ldots \quad (33a)$$

if we take the 'finite part' or 'principal value' of this integral (*see* Refs. 31, 34, 35, 36).

It may be mentioned that for $\omega = 0$ ($H_0 = 0$) we obtain from (33) the case of a steady motion. The equation is usually written as

$$\frac{\partial}{\partial x}\,\frac{w}{V}\,(x, y, 0) = \frac{1}{4\pi}\int_{-s(x)}^{s(x)} L(x, y')\,\frac{dy'}{(y - y')^2}$$

($2s(x)$ = local wing span at the point (x, y)). At first we have to solve a two-dimensional problem in the (y, z)-plane, containing the point $(x, y, 0)$, and these 'sectional' solutions are combined afterwards in order to satisfy equation (33a) itself (compare Ref. 29).

Unfortunately there is no similar procedure available for the unsteady case. But the general problem of a sonic oscillatory flow can considerably be simplified if we restrict ourselves to the calculation of stability derivatives. Then it is sufficient to consider only small values of the reduced frequency $\omega = n\bar{c}/V$.

As a first step we split equation (33a) in its real and imaginary part. We obtain, using the notations introduced in equation (27):

$$\left.\begin{aligned} w_r(x, y) &= \frac{1}{4\pi}\iint_{S^*}[L_r(x', y')\cos H_0 + \omega L_i(x', y')\sin H_0]_{z=0}\,\frac{dx'\,dy'}{(y - y'^2)} \\ w_i(x, y) &= \frac{1}{4\pi}\iint_{S^*}\left[L_i(x', y')\cos H_0 - \frac{1}{\omega}L_r(x', y')\sin H_0\right]_{z=0}\frac{dx'\,dy'}{(y - y'^2)} \end{aligned}\right\} \quad \ldots \quad (34)$$

If we neglect terms of higher order in $|\omega(x - x')/2\bar{c}| < \omega$ we may write as in section 2.5:

$$\left.\begin{aligned} (\cos H_0)_{z=0} &= \cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')} - \frac{\omega(x - x')}{2\bar{c}}\sin\frac{\omega(y - y')^2}{2\bar{c}(x - x')} \\ (\sin H_0)_{z=0} &= \frac{\omega(x - x')}{2\bar{c}}\cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')} + \sin\frac{\omega(y - y')^2}{2\bar{c}(x - x')} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

and

$$\frac{1}{\omega}(\sin H_0)_{z=0} = \frac{x - x'}{2\bar{c}}\cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')} + \frac{1}{\omega}\sin\frac{\omega(y - y')^2}{2\bar{c}(x - x')}. \quad \ldots \quad \ldots \quad \ldots \quad (35a)$$

Since the functions $\cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')}$ and $\sin\frac{\omega(y - y')^2}{2\bar{c}(x - x')}$ oscillate rapidly between $+1$ and -1 as x' approaches x for constant values of $y - y' \neq 0$, they contribute a great deal to make some of the integrals in (34) convergent. In such a case the function has to be retained and cannot be replaced by a few terms of its Taylor series. If we consider only small values of ω, the second term in the first equation (34) may be omitted, and we may write our system in this way:

$$w_r(x, y) = \frac{1}{4\pi}\iint_{S^*} L_r(x', y')\cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')}\,\frac{dx'\,dy'}{(y - y')^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36a)$$

$$w_i(x, y) + F(x, y) = \frac{1}{4\pi}\iint_{S^*} L_i(x', y')\cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')}\,\frac{dx'\,dy'}{(y - y')^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (36b)$$

with

$$F(x, y) = \frac{1}{4\pi}\iint_{S^*} L_r(x', y')\left[\frac{x - x'}{2\bar{c}}\cos\frac{\omega(y - y')^2}{2\bar{c}(x - x')} + \frac{1}{\omega}\sin\frac{\omega(y - y')^2}{2\bar{c}(x - x')}\right]\frac{dx'\,dy'}{(y - y')^2}. \quad (37)$$

The real part L_r can now be determined from (36a) and the imaginary part L_i follows afterwards from (36b) since $F(x, y)$ does not depend on L_i but on L_r only. Both equations are of the same type. Only the left-hand side is different so that the same method can be applied for the solution in either case.

A further simplification of the equations (36) results from the fact that for wings of finite aspect ratio we may replace $\cos \dfrac{\omega(y-y')^2}{2\bar{c}(x-x')}$ by 1 on the right of equation (36a) and (36b) and in the first term of (37). This is proved in Appendix IV by showing that the terms omitted in this way are at most proportional to the reduced frequency ω. But the frequency must be retained in the second term of (37) since F tends to infinity for $\omega \to 0$ and the load L and l becomes infinite for $\omega \to 0$.

Thus our results will depend on the value of ω although only small values of ω shall be considered in this paper. If bigger values of ω are required, some of the terms which are neglected here, should be taken into account.

3.2. *The Integral Equation for Small Frequencies.*—For a wing with an unswept trailing edge and for small values of ω we obtain from (36), replacing the cosine by 1, the following two equations for L_r and L_i:

$$\left.\begin{aligned} w_r(x, y) &= \frac{1}{4\pi} \iint_{S*} L_r(x', y') \frac{dx'\, dy'}{(y-y')^2} \\ w_i(x, y) + F(x, y) &= \frac{1}{4\pi} \iint_{S*} L_i(x', y') \frac{dx'\, dy'}{(y-y')^2} \end{aligned}\right\} \quad \ldots \quad (38)$$

The expression (37) for the 'incidence' $F(x, y)$ can be simplified by replacing the cosine by 1 in the first term. After an integration by parts with respect to x' (which puts the first term (37) into a form similar to the right-hand side of (38)), we obtain

$$F(x, y) = \frac{1}{4\pi} \iint_{S*} \left[\int_{x\lambda}^{x'} L_r(\xi, y') \frac{d\xi}{2\bar{c}} \right] \frac{dx'\, dy'}{(y-y')^2} + F_1(x, y) \quad \ldots \quad (39)$$

where

$$F_1(x, y) = \frac{1}{4\pi} \iint_{S*} L_r(x, 'y') \frac{1}{\omega} \sin \frac{\omega(y-y')^2}{2\bar{c}(x-x')} \frac{dx'\, dy'}{(y-y')^2} . \quad \ldots \quad (40)$$

Equations (38) to (40) follow from equations (31) for the case of sonic flight. Since the integral equations (38) are linear, we can express L_i as a sum of several terms, each of which corresponds to a certain term on the left-hand side. Thus

$$L_{i0}(x', y') = \int_{x_l}^{x'} L_r(\xi, y') \frac{d\xi}{2\bar{c}} \quad \ldots \quad (41)$$

forms the first contribution to L_i, namely the contribution which arises from the term $F(x, y) - F_1(x, y)$. This follows by comparison of (38) and (39) and we can see that the function $(L_i(x', y') - L_{i0}(x', y'))$ has to satisfy the second equation (38) with $F(x, y)$ replaced by $F_1(x, y)$.

The contribution to L_i, which arises from w_i, is zero for a heaving oscillation (equation (7a)) and corresponds to the load distribution of a cambered wing (incidence proportional to $x - x_0$) for a rotary oscillation (equation (7b)). It shall be denoted by $L_{iq}(x', y')$ and has been calculated in Ref. 29 for a big family of plan-forms.

The greatest difficulty is presented by the calculation of the load contribution to L_i, which arises from the 'incidence' $F_1(x, y)$ as defined by (40). We cannot solve this problem for an arbitrary plan-form in this paper, but shall restrict ourselves to the case of a Delta wing (section 4).

Before proceeding to the actual calculation of $F_1(x, y)$ and the corresponding load distribution, we split $F_1(x, y)$ into two terms, each of which permits certain simplifications and thus makes the solution possible. We write†

$$F_1(x, y) = F_{11}(x, y) + F_{12}(x, y) \qquad \text{(42)}$$

where

$$F_{11}(x, y) = \frac{1}{4\pi} \iint_{S^*} [L_r(x', y') - L_r(x, y')] \frac{1}{\omega} \sin \frac{\omega(y - y')^2}{2\bar{c}(x - x')} \frac{dx'\, dy'}{(y - y')^2} \qquad \text{(43a)}$$

and

$$F_{12}(x, y) = \frac{1}{4\pi} \int_{-s}^{s} L_r(x, y') \left[\int_U^\infty \frac{\sin u}{u^2}\, du \right] \frac{dy'}{2\bar{c}} \,. \qquad \text{(43b)}$$

Here the following transformation is used :

$$u = \frac{\omega(y - y')^2}{2\bar{c}(x - x')}\,, \qquad \frac{du}{u^2} = \frac{2\bar{c}}{\omega} \frac{dx'}{(y - y')^2}\,, \qquad U = \frac{\omega(y - y')^2}{2\bar{c}(x - x_e(y'))}\,. \qquad \text{(44)}$$

Since

$$\int_U^\infty \frac{\sin u}{u^2}\, du = \frac{\sin U}{U} + \int_U^\infty \frac{\cos u}{u}\, du = \frac{\sin U}{U} - \log (\gamma U) + \int_0^U \frac{1 - \cos u}{u}\, du \qquad \text{(45)}$$

($\gamma = 1 \cdot 7811$ = Eulerian constant) we may write for small values of U :

$$\int_U^\infty \frac{\sin u}{u^2}\, du = 1 - \log (\gamma U) \qquad \text{(46)}$$

where terms of the order U^2 (*i.e.*, order ω^2) have been neglected. With the same degree of accuracy we may replace the sin-function in (43a) by its argument (compare Appendix IV, section 2) and obtain :

$$F_{11}(x, y) = \frac{1}{4\pi} \int_0^x \int_0^{s(x')} [L_r(x', y') - L_r(x, y')] \frac{dy'}{\bar{c}} \frac{dx'}{x - x'}\,, \qquad \text{(47)}$$

$$F_{12}(x, y) = \frac{1}{4\pi} \int_0^s L_r(x, y') \left[1 - \log \frac{\omega \gamma s^2}{8\bar{c}(x - x_l(y'))} \right] \frac{dy'}{\bar{c}}$$

$$+ \frac{1}{4\pi} \int_{-s}^{s} L_r(x, y') \log \frac{s^2}{4(y - y')^2} \frac{dy'}{\bar{c}} \,. \qquad \text{(48)}$$

The second term in (48) vanishes as is shown in Appendix IV, section 4.

Since

$$\frac{1}{4\pi} \int_0^x \left(\int_0^{s(x)} L_r(x, y') \frac{dy'}{\bar{c}} - \int_0^{s(x')} L_r(x, y') \frac{dy'}{\bar{c}} \right) \frac{dx'}{x - x'}$$

$$= - \frac{1}{4\pi} \int_0^x L_r(x, s') \log \frac{x - x'}{x} \frac{ds'}{dx'} \frac{dx'}{\bar{c}} \qquad \text{(49)}$$

† The original version of this Report contained an error in the evaluation of F_1 as was also pointed out in a private communication by M. Landahl, Stockholm.

as follows by an integration by parts, we find

$$F_{11}(x, y) + F_{12}(x, y) = \frac{1}{4\pi}\int_0^x \left(\int_0^{s(x')} L_r(x', y')\,\frac{dy'}{\bar{c}} - \int_0^{s(x)} L_r\,(x, y')\,\frac{dy'}{\bar{c}}\right)\frac{dx'}{x - x'}\,,$$

$$+ \frac{1}{4\pi}\int_0^{s(x)} L_r\,(x, y')\,\frac{dy'}{\bar{c}}\left[1 - \log\frac{\omega\gamma s^2(x)}{8\bar{c}x}\right]. \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

4. *The Delta Wing.*—4.1. *Evaluation of the Integral* F_1.—For a Delta wing the local span $2s$ is proportional to the distance x from the apex so that the leading edge (Fig. 2) is given by

$$x = x_l = |y| \tan \Lambda_l = s \tan \Lambda_l \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (51)$$

where Λ_l is the sweep angle of the leading edge.

For a Delta wing at a constant incidence the load distribution in a steady sonic flow becomes proportional to $s/\sqrt{(s^2 - y^2)}\,.\,ds/dx$ (compare Ref. 29). Since w_r is constant according to equation (7), we obtain (assuming ω to be small) for the first equation (38) the solution :

$$L_r(x, y) = -\,4w_r\,\frac{\partial}{\partial x}\,\sqrt{(s^2 - y^2)} = -\,4w_r\,\frac{s}{\sqrt{(s^2 - y^2)}}\,\frac{ds}{dx} \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

where w_r is prescribed by the boundary conditions (7a) or (7b) respectively for a wing performing heaving or pitching oscillations. It is easily shown by means of equation (IV, 20) in Appendix IV, that (52) is a solution of (38).

When inserting (52) into (50) we find

$$F_1(x, y) = w_r\,\frac{x}{2\bar{c}K^2}\,\log\frac{\omega\gamma x}{8\bar{c}K^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

with

$$K = \tan \Lambda_l = \frac{dx_l}{dy}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (54)$$

Thus F_1 is independent of y which will simplify the calculation of the corresponding contributions L_{i1} to the load L_i.

4.2. *Solution of the Integral Equation for* L_i.—Now we are in a position to solve the second integral equation (38) for L_i. As mentioned before the solution consists of several terms, which correspond to the various terms on the left-hand side (' incidence ' terms). The first contribution is given by equation (41) and becomes for a Delta wing :

$$L_{i0}(x, y) = -\,2w_r\,\frac{\sqrt{(s^2(x) - y^2)}}{\bar{c}}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (55)$$

The contribution L_{i1} arising from F_1 is easily obtained by means of equations (19) and (20) in Appendix IV. We find

$$L_{i1}(x, y) = -\,\frac{2w_r}{K^2}\,\frac{\partial}{x\partial}\left\{\frac{x}{\bar{c}}\,\sqrt{\{s^2(x) - y^2\}}\,\log\frac{\gamma\omega x}{2\bar{c}K^2}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (56)$$

For a wing, performing *heaving oscillations* we have $w_i = 0$ according to equation (7a). The entire load is given by equation (55) and (56) :

$$L_i \equiv L_{iw} = L_{i0} + L_{i1}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

For a wing performing *rotary oscillations* we have an additional load contribution L_{iq}, which arises from the incidence term w_i in (38). For a rotary oscillation we have according to (7b) $w_r = -\,\vartheta^* =$ const, and

$$w_i = w_r\,\frac{x - x_0}{\bar{c}} \equiv -\,\frac{q(x - x_0)}{V} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

agrees with the incidence as it occurs on a wing in quasi-steady pitch. The corresponding load distribution is

$$L_{iq}(x, y) = -4\, w_r \frac{\partial}{\partial x}\left(\frac{x - x_0}{\bar{c}} \sqrt{\{s^2(x) - y^2\}}\right) \quad \ldots \quad (59)$$

as can be verified by means of equation (IV, 19) and (IV, 20) of Appendix IV. Thus we have for a rotary oscillation

$$L_i \equiv L_{i\vartheta} = L_{iw} + L_{iq} . \quad \ldots \quad (60)$$

4.3. *Forces and Moments.*—After the solutions L_r and L_i of equation (38) have been given in equation (52) and equations (57) and (60) of the previous sections, we can now determine the actual loads l_r and l_i by means of equations (28). The term, which has to be added to the imaginary part of the load, agrees with L_{i0} in equation (55). The result can be written in the following form, if $i\omega l_{iw}$ is the imaginary part of the load for a heaving oscillation ($w_r = -w^*/V$) and $i\omega l_{i\vartheta}$ is the imaginary part for an angular oscillation ($w_r = -\vartheta^*$) :

$$\left.\begin{aligned} l_r &= L_r \\ l_{iw} &= 2L_{i0} + L_{i1} \\ l_{i\vartheta} &= l_{iw} + L_{iq} \equiv l_{iw} + l_{iq} \end{aligned}\right\} \quad \ldots \quad (61)$$

Using the results (52), (55), (56) and (59) of the preceding sections, we obtain ($K = \tan \Lambda_t$) :

$$\left.\begin{aligned} l_r &= -4w_r \frac{s}{K\sqrt{(s^2 - y^2)}} , \\ l_{iw} &= -4w_r \frac{\sqrt{(s^2 - y^2)}}{\bar{c}} - 2\frac{w_r}{K^2}\frac{\partial}{\partial x}\left\{\frac{x}{\bar{c}}\sqrt{(s^2 - y^2)} \log \frac{\gamma\omega x}{2\bar{c}K^2}\right\} \\ l_{iq} &\equiv L_{iq} = -4w_r \frac{\partial}{\partial x}\left(\frac{x - x_0}{\bar{c}}\sqrt{(s^2 - y^2)}\right) \end{aligned}\right\} \quad \ldots \quad (62)$$

For the calculation of the forces and pitching moments the spanwise integrals of these loads are required. We obtain, after performing the differentiations with respect to x, for these integrals :

$$\left.\begin{aligned} \int_{-s}^{s} l_r \frac{dy}{\bar{c}} &= -\frac{4\pi w_r x}{K^2\bar{c}} \\ \int_{-s}^{s} l_{iw} \frac{dy}{\bar{c}} &= -\frac{2\pi w_r x^2}{K^2\bar{c}^2}\left[1 + \frac{1}{2K^2x^2}\frac{\partial}{\partial x}\left(x^3 \log \frac{\omega\gamma x}{8K^2\bar{c}}\right)\right] \\ \int_{-s}^{s} l_{iq} \frac{dy}{\bar{c}} &= -\frac{2\pi w_r x^2}{K^2\bar{c}^2}\left(3 - 2\frac{x_0}{x}\right) \end{aligned}\right\} \quad \ldots \quad (63)$$

We define the longitudinal stability derivatives in the usual way (*see* Ref. 2) by writing the z-component of the complex force due to a heaving oscillation $\bar{w} = -w_r V$ as

$$-\tfrac{1}{2}\rho V^2 \iint (l_r + i\omega l_{iw}) e^{int}\, dx\, dy = wZ_w + \dot{w}Z_{\dot{w}}$$

$$= \bar{w}e^{int}(Z_w + in\, Z_{\dot{w}}) = -\rho V^2 w_r S e^{int}(z_w + i\omega z_{\dot{w}})$$

and the complex pitching moment in a similar way.

Splitting this into real and imaginary parts leads to the following list of stability derivatives for heaving oscillations ($w_r = w^*/V$, $w_i = 0$) :

$$\left.\begin{aligned} 2w_r z_w &= \iint_S l_r \frac{dx\,dy}{S}\,, \qquad 2w_r z_{\dot{w}} = \iint_S l_{iw} \frac{dx\,dy}{S} \\ 2w_r m_w &= \iint_S l_r \frac{x - x_0}{\bar{c}} \frac{dx\,dy}{S}\,, \qquad 2w_r m_{\dot{w}} = \iint_S l_{iw} \frac{x - x_0}{\bar{c}} \frac{dxdy}{S} \end{aligned}\right\} \quad \ldots \quad (64)$$

For rotary oscillations ($w_r = -\vartheta^*$) we write accordingly for the force :

$$-\tfrac{1}{2}\rho V^2 \iint (l_r + i\omega l_{i\vartheta}) e^{int}\, dx\, dy = \vartheta Z_\vartheta + \dot{\vartheta} Z_{\dot{\vartheta}}$$

$$= \vartheta^* e^{int}(Z_\vartheta + inZ_{\dot{\vartheta}}) = -\rho V^2 S w_r e^{int}(z_\vartheta + i\omega z_{\dot{\vartheta}})$$

and a corresponding expression for the moment. Splitting this expression into real and imaginary parts leads to a list of derivatives similar to equation (64). By comparison we find that the 'real parts' of the derivatives agree for both kinds of oscillation :

$$z_w = z_\vartheta \qquad m_w = m_\vartheta \quad \ldots \quad (65)$$

and that the 'imaginary parts' satisfy the relations (compare equations (60) and (61)) :

$$z_{\dot{\vartheta}} = z_{\dot{w}} + z_q, \qquad m_{\dot{\vartheta}} = m_{\dot{w}} + m_q \quad \ldots \quad (66)$$

where z_q and m_q are defined as the derivatives which occur in the quasi-steady theory of a wing in pitch ; namely

$$2w_r z_q = \iint l_{iq} \frac{dx\,dy}{S}\,, \qquad 2w_r m_q = \iint l_{iq} \frac{x - x_0}{\bar{c}} \frac{dx\,dy}{S}\,. \quad \ldots \quad (67)$$

The equations (65) and (66) hold for any wing plan-form, as was demonstrated by Neumark and Thorpe[2].

We perform the integrations required in (64) and (67) using equation (63). The results are given in the first column of Table 1. Here $h = x_0/\bar{c}$ denotes the axis position in terms of the mean chord measured from the apex (h positive if pointing backwards).

4.4. *Discussion of Results*.—The results of the preceding calculations, which are compiled in the first column of Table 1, show that the relations

$$z_w = z_\vartheta\,, \qquad m_w = m_\vartheta \quad \ldots \quad (65)$$

hold also for sonic speeds ($M \sim 1$), if only small values of the reduced frequency ω are considered. The stability derivatives, which correspond to quantities in phase with the original oscillation of the wing, can be calculated from steady-flow theory (*see* Ref. 29). The same applies for the derivatives z_q and m_q. The latter remains negative for all axis-positions.

The other four derivatives which occur in a short period longitudinal oscillation with two degrees of freedom, $z_{\dot{w}}$, $m_{\dot{w}}$, $z_{\dot{\vartheta}}$, $m_{\dot{\vartheta}}$ are related through the following equations :

$$z_{\dot{\vartheta}} = z_q + z_{\dot{w}}, \qquad m_{\dot{\vartheta}} = m_q + m_{\dot{w}}\,. \quad \ldots \quad (66)$$

All these derivatives depend on the reduced frequency ω. In Fig. 3 the function :

$$z_{\dot{w}}/(4\pi \cot \Lambda_l) = -\frac{1}{6}\left[1 + \frac{3}{2}\cot^2 \Lambda_l . \log\left(\frac{\omega\gamma}{4} \cot^2 \Lambda_l\right)\right] \quad \ldots \quad (68)$$

is plotted against $\cot \Lambda_l$ for various values of ω. ($-z_{\dot{w}}$) is positive for small aspect ratios (large sweep angles) but assumes big negative values for larger aspect ratios. These values become even bigger for very small frequencies and decrease with increasing ω. In Fig. 4 the function

$$\frac{m_{\dot{w}}}{4\pi \cot \Lambda_l} = \frac{z_{\dot{w}}}{2\pi \cot \Lambda_l} . \frac{1}{2}\left(\frac{3}{2} - h\right) - \frac{1}{32}\cot^2 \Lambda_l \quad \ldots \quad (69)$$

is plotted against the axis position $h = x_0/\bar{c}$ for various values of the leading-edge sweep angle Λ_l and three values of ω. For small values of the aspect ratio $A = 4 \cot \Lambda_l$, $m_{\dot{w}}$ is negative as required for a damped oscillation (*see* below). But for a leading-edge sweep $\Lambda_l = 60$ deg, $(-m_{\dot{w}})$ becomes zero or negative for an axis position $h = 4/3$ at the aerodynamic centre and definitely negative for positions forward of the aerodynamic centre. Conditions can be improved slightly by increasing the reduced frequency ω.

The present calculations cover only frequencies up to $\omega = 0\cdot 1$, since it was felt that for bigger values of ω certain terms, which have been omitted in this paper, may become important. It may well be that the present analysis applies also for bigger ω's, since calculations in the supersonic and subsonic range by W. P. Jones have shown that the higher frequency terms do not affect the results very much. But in the sonic range this point requires further investigation. As can be seen from Figs. 3 and 4 the frequency ω has a bigger influence on the results for a smaller sweep angle and its influence decreases with decreasing aspect ratio.

Figs. 5 and 6 show plottings of the derivatives $z_{\dot{\vartheta}}$ and $m_{\dot{\vartheta}}$ for a Delta wing at sonic speeds for various axis positions $h = x_0/\bar{c}$. Since z_q and m_q are independent of the frequency, the influence of ω on $z_{\dot{\vartheta}}$ and $m_{\dot{\vartheta}}$ is similar to its influence on $z_{\dot{w}}$ and $m_{\dot{w}}$ as described above.

The damping of the short-period oscillation depends on the derivatives $-z_w$ and $-m_{\dot{\vartheta}}$ (compare equation (74) in section 4.5). The first quantity $-z_w$ is always damping, whereas $-m_{\dot{\vartheta}}$ may become zero or negative and thus cancel the favourable influence of $(-z_w)$. In order to give an idea of this unfavourable influence of $-m_{\dot{\vartheta}}$ on the damping, Fig. 7 shows the curves $-m_{\dot{\vartheta}} = 0$ as functions of the aspect ratio A and the axis position h for various values of the reduced frequency ω. For all points below these curves, where $-m_{\dot{\vartheta}}$ is positive, we may expect stable or damped oscillations. For an axis position behind $\frac{3}{4}$ of the root chord, all oscillations becomes damped, but this c.g. position can hardly be realised in an actual aircraft. For centre of pressure positions forward from the $\frac{3}{4}$ root-chord point, the aspect ratio for which the oscillations remain damped, is restricted to fairly small values. This applies even for axis positions 2 or 3 mean chords ahead of the apex. Thus an efficient tail with a sweep of 45 deg would have to work on a sufficiently long arm (about 3 or 4 tail mean chords) according to this calculation which neglects all effects of the downwash, induced by the wing, on the tailplane and also the effects of a vertical shift of the tailplane position.

It is not easy to discuss the present results for $M = 1$ in relation to subsonic results for a Delta wing since they are not available yet up to sufficiently high Mach numbers. But supersonic results have been calculated before. They are based on the assumption that ω is small compared to $(M^2 - 1)/M^2$ (*see* Appendix III and Refs. 17, 18, 19, 23). Column 2 in Table 1 shows these results for the longitudinal stability derivatives of a Delta wing. As can be seen the sonic results can be obtained from the supersonic results by taking the limit $M \rightarrow 1$, except for the term:

$$\frac{M^2(1 - H)}{M^2 - 1}$$

which occurs in the ' dotted ' derivatives and tends to

$$\cot^2 \Lambda_l \log\left(\frac{4 \tan \Lambda_l}{\sqrt{(M^2 - 1)}}\right) \rightarrow + \infty .$$

In our sonic solution we have a corresponding term

$$-\frac{1}{2} \cot^2 \Lambda_l \log \frac{\omega\gamma \cot^2 \Lambda_l}{4}$$

which also tends to $+\infty$, if ω tends to zero. This is quite satisfactory from the mathematical point of view, since in the supersonic case ω was assumed to be small compared to $M^2 - 1$, which means that ω must tend to zero as M approaches 1, whereas in this paper ω was retained for the sonic calculation.

On our present knowledge it is not easy to close the gap between the two theories. In Fig. 8 an attempt has been made to interpolate between $M = 1$ and the supersonic region. The broken curves which connect the values for $M = 1$ of the present theory with the supersonic results depend on the frequency ω, but with increasing Mach number the results become more and more independent of ω, provided ω is small so that one single (asymptotic) curve can be used. Obviously, these interpolated curves which require confirmation by an actual calculation, have to be used with some caution. Nevertheless it seems to be evident that the 'dotted' stability derivatives for a Delta wing of moderate sweep undergo (for a fixed ω) rapid variations with M near the speed of sound. This variation means an appreciable loss in the damping of the short period longitudinal oscillation. For $\Lambda_l \geqslant 60$ deg this loss of damping does not occur. If the interpolated curves in Fig. 8 are trustworthy, it appears that the supersonic results compiled in Ref. 23, hold for Mach numbers down to about $M^2 = 1 \cdot 2$ as far as small frequencies are concerned.

For wings of a very small aspect ratio, the term proportional to $\cot^3 \Lambda_l$ can be neglected in the results of Table 1. We obtain agreement with the results of Garrick (Appendix B of Ref. 14), which are valid for any Mach number at a very small aspect ratio.

4.5. *Application to a Particular Aircraft.*—In order to illustrate the application of these results to actual aircraft, an example is given here, to indicate broadly what happens. According to the elementary theory[2, 23] of the short-period oscillations the reduced frequency ω is given by

$$\omega = \sqrt{\left\{\frac{-z_w}{\mu i_B}\left(\frac{m_w}{z_w} - \frac{m_q}{\mu}\right)\right\}} \qquad (70)$$

with

$$\mu = \frac{W}{g\rho \bar{c} S}, \quad i_B = \frac{k_B^2}{\bar{c}^2} \qquad (71)$$

(W = weight of the aircraft, k_B = radius of gyration, μ = density ratio, i_B = moment of inertia ratio). We introduce the restoring margin H_n, which is according to Table 1 (for a Delta wing inside the Mach cone from the apex) :

$$H_n = \frac{m_w}{z_w} = \frac{4}{3} - h \qquad (72)$$

and have

$$\omega = \sqrt{\left\{\frac{\pi \cot \Lambda_l}{E\mu i_B}\left[H_n + \frac{\pi \cot \Lambda_l}{E\mu}\left(\left(H_n + H - \frac{2}{3}\right)^2 + \left(\frac{4}{3} - H\right)\left(H - \frac{1}{3}\right)\right)\right]\right\}} \qquad (73)$$

where E and H depend on $a = \cot \Lambda_l \sqrt{(M^2 - 1)}$ (*see* Ref. 23). The damping per period is given by $e^{-B\pi/\mu\omega}$ where the reduced frequency ω of the (undamped) oscillation is introduced according to (73) and B is

$$B = -z_w - \frac{m_{\dot{\vartheta}}}{i_B} . \qquad (74)$$

In order to have some idea about the magnitude of these quantities, an example was calculated. The inertia ratio was chosen as $i_B = 0 \cdot 25$ and the density ratio as $\mu = 200$. Fig. 9 shows the reduced frequency ω and also the damping per period $\exp(-B\pi/(\mu\omega))$ for a Delta wing with 60-deg leading-edge sweep at sonic speeds ($i_B = 0 \cdot 25$, $\mu = 200$) for various values of the static margin $H_n = 4/3 - h$. This particular aircraft is dynamically stable for $H_n > 0$, but $h = 4/3 - H_n$ should be less than $1 \cdot 00$ to obtain static stability at low speeds (subsonic a.c. at $h = 1$).

In order to show the variation of the damping with M this function has been plotted for the same aircraft ($i_B = 0 \cdot 25$, $\mu = 200$, $\Lambda_l = 60$ deg) against M for a static margin $H_n = 0 \cdot 333$, $0 \cdot 1$ and $0 \cdot 033$. For $M = 1$ the results of this paper are used and for $M^2 \geqslant 1 \cdot 2$ the results of

Ref. 23 (based on the assumption $\omega << M^2 - 1$). The intermediate values are interpolated and have to be used with some caution. This particular aircraft ($i_B = 0 \cdot 25$, $\mu = 200$, $\Lambda_l = 60$ deg) is stable for all Mach numbers and all static margins H_n given in Fig. 9. Results for a wing of 45-deg sweep ($i_B = 0 \cdot 25$, $\mu = 200$) are also shown. The stability of this aircraft would be fairly poor for a Mach number range between $M^2 = 1$ and $M^2 = 1 \cdot 245$, if $h = 1$.

Before any general conclusions can be drawn from these calculations the effects of a change in the plan-form of the wing (taper) will have to be investigated.

5. *Conclusions.*—A new approach to the theory of a lifting surface performing harmonic oscillations (heaving and pitching oscillations) is suggested, which covers also the sonic range. It is shown, that for small reduced frequencies the problem can be reduced to the solution of an integral equation which has the same form as the integral equation for the lifting surface in steady flow (*see* section 2.5 and Appendices II and III). This equation has been solved for sonic flow ($M = 1$) and all 8 stability derivatives for a short-period longitudinal oscillation (with two degrees of freedom) have been calculated for a Delta wing (Table 1). The results can be found in Figs. 3 to 7. They show a marked decrease in the damping of the short period longitudinal oscillation at sonic speeds for wings of moderate leading-edge sweep. This oscillation becomes undamped for all axis positions in front of the aerodynamic centre unless the aspect ratio $A = 4 \cot \Lambda_l$ is sufficiently small. A leading-edge sweep of about 60 deg is required to obtain damping for a tailless aircraft with a positive static margin. The shape and position of a tail must be carefully chosen if it is to improve damping. The damping improves slightly with increasing frequency.

Although these calculations are based on potential theory, which neglects the effects of thickness, friction and shock waves, they probably show the main features of the actual flow. It seems unlikely that the bad results of the potential theory can be off-set to a predominating extent by the influence of shock waves or friction. The results will depend a great deal on the plan-form of the wing, and further calculations are required to show this influence, *e.g.*, the effect of taper. Such plan-forms, for which even this theory shows rapid variations of the stability in the sonic range, should be abandoned in favour of other plan-forms with better characteristics. These will have to be tested as to their behaviour in a real flow. Thus it is hoped that the linearized theory outlined in this report, will provide a useful tool in the investigation of transonic stability problems.

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APPENDIX I

The function $L(x, y)$

We assume a periodical oscillation and the velocity potential ϕ in the form

$$\phi = \bar{\phi}(x, y, z) . e^{int} . \qquad \text{(I, 1)}$$

Then $\bar{\phi}$ satisfies Bernoulli's equation (6) in the form :

$$V\left(\frac{in}{V} + \frac{\partial}{\partial x}\right)\bar{\phi} = -(\bar{I} - \bar{I}_\infty) . \qquad \text{(I, 2)}$$

According to (15) the discontinuity $\Delta\bar{I}$ along the wing surface produces a load $\bar{l}$ and a discontinuity $\Delta\bar{\phi}$ of the velocity potential

$$\Delta\phi = \Delta\bar{\phi} . e^{int} \equiv 2e^{int} V . \phi^*(x, y, +0) \qquad \text{(I, 3)}$$

and we have from (I, 2) ($\phi^* = 0$ along leading edge) :

$$4\left(\frac{in}{V} + \frac{\partial}{\partial x}\right)\phi^* = 2\frac{\Delta\bar{I}}{V^2} = \bar{l}(x, y) . \qquad \text{(I, 4)}$$

We insert this in (23) and obtain after an integration by parts

$$\left.\begin{aligned} l(x, y) &= 4\left[\frac{\partial\phi^*}{\partial x} + \frac{in}{V}\phi^*\right] \\ L(x, y) &= 4\left[\frac{\partial\phi^*}{\partial x} + \frac{in}{V}\frac{M}{1+M}\phi^*\right] \end{aligned}\right\} \quad \ldots \quad \text{(I, 5)}$$

APPENDIX II

Small Frequency Oscillations in Subsonic Flow

We consider the case of small frequencies ω only. Then we may simplify (25) as

$$\left.\begin{aligned} L_{tr}(y) &= \frac{\omega^2}{1+M}\int_{x_l}^{x_t}\left[L_i(\xi, y) - \frac{M}{1+M}\frac{x_t-\xi}{\bar{c}}L_r(\xi, y)\right]\frac{d\xi}{\bar{c}} = 0 \\ L_{ti}(y) &= -\frac{1}{1+M}\int_{x_l}^{x_t}L_r(\xi, y)\frac{d\xi}{\bar{c}} \end{aligned}\right\} \quad \ldots \quad \text{(II, 1)}$$

With the same approximation we have for $x = X$:

$$\left.\begin{aligned} (\cos H)_{x=X} &= \cos H_2 - H_1 \sin H_2 \simeq \cos H_2 \\ (\sin H)_{x=X} &= H_1 \cos H_2 + \sin H_2 \end{aligned}\right\} \quad \ldots \quad \text{(II, 2)}$$

For a wing of finite aspect ratio H_2 is finite, except in the limiting case $M \to 1$, where the integration has to be extended over the forward part S^* of the wing, where $x' \leqslant X$ and H_2 tends to infinity as x' tends to X. Now we have shown in section 3.2 that even in this limiting case the following approximation is valid for small frequencies :

$$\left.\begin{aligned} (\cos H)_{x=X} &= 1 \\ (\sin H)_{x=X} &= H_1 + \sin H_2 \end{aligned}\right\} \quad \ldots \quad \text{(II, 3)}$$

and therefore it applies also in the case $M \leqslant 1$. When introducing these simplifications in (26) we obtain the system of equations (31), which is valid for small frequencies ($\varepsilon = +1$). Finally we may go back to the original load distribution $\bar{l} = l_r + i\omega l_i$ by means of equation (28).

The solution l_r will be independent of ω, if ω is small enough. This is not true for l_i and L_i at least not for Mach numbers M near 1. In this case $F(x, y)$ in (31c) depends on ω and so does the function l_i and the stability derivatives derived from l_i. If M is small enough, so that

$$\frac{1}{\omega}\sin\frac{\omega M\sigma^2}{\bar{c}(x-x'+r)} = -\frac{1}{\omega}\sin\frac{\omega M(x-x'-r)}{(1-M^2)\bar{c}}$$

can be replaced by $-\dfrac{M}{1-M^2}\dfrac{x-x'-r}{\bar{c}}$, the load l_i becomes independent of ω.

In this particular case where $\omega << |1 - M^2|$ we may simplify the equations (31) by introducing the functions

$$\left.\begin{aligned} \frac{\bar{w}_1}{V}(x, y) &= \frac{\bar{w}}{V} \exp\left(-\frac{i\omega M^2 x}{(1 - M^2)\bar{c}}\right) \\ \phi_1^*(x', y') &= \phi^* \exp\left(-\frac{i\omega M^2 x'}{(1 - M^2)\bar{c}}\right) \end{aligned}\right\} \quad \ldots \quad \text{(II, 4)}$$

where $\phi^*(x', y')$ as defined in (I, 3) describes the discontinuity in the velocity potential between both faces of the wing. Then we have according to (I, 5)

$$L(x', y') \exp\left(-\frac{i\omega M^2 x'}{(1 - M^2)\bar{c}}\right) = 4\left(\frac{\partial \phi_1^*}{\partial x'} + \frac{i\omega M}{1 - M^2} \frac{\phi_1^*}{\bar{c}}\right). \quad \ldots \quad \text{(II, 5)}$$

When taking real and imaginary part and inserting this in (26) we obtain, using (I, 5) for L_t, a simpler system of two equations for the real and imaginary part of ϕ_1^*. If we prefer to retain the complex notation, we introduce the transformation (II, 4) in equation (24) and obtain :

$$\frac{\bar{w}_1}{V}(X, y) = \frac{1}{2\pi} \iint_S \left[\frac{\partial \phi_1^*}{\partial x'} + \frac{i\omega M}{1 - M^2} \frac{\phi_1^*}{\bar{c}}\right] \left[\exp\left(-\frac{i\varepsilon\omega M r}{(1 - M^2)\bar{c}}\right)\left(\varepsilon + \frac{x - x'}{r}\right)\right]_{\substack{x=X \\ z=0}} \frac{dx'\,dy'}{(y - y')^2}$$

$$- \frac{i\omega}{2\pi(1 + M)} \int_{-b/2}^{b/2} \frac{\phi_1^*(x_t, y')}{\bar{c}} \int_{-\infty}^{X} \left[\exp(-iH^*)\left(\varepsilon + \frac{x - x'}{r}\right)\right]_{\substack{x'=x_t \\ z=0}} dx \frac{dy'}{(y - y')^2} \qquad \text{(II, 6)}$$

with

$$H^* = H - \frac{i\omega M^2 x'}{(1 - M^2)\bar{c}}.$$

When retaining only terms of the first order in $\omega/|1 - M^2|$ the exponential in the wake term may be replaced by 1 and the integrand in the first term of (II, 6) may be written as

$$\left[\frac{\partial \phi_1^*}{\partial x'} + \frac{i\omega M}{1 - M^2}\left(\frac{\phi_1^*}{\bar{c}} - \varepsilon \frac{r}{\bar{c}} \frac{\partial \phi_1^*}{\partial x'}\right)\right]\left(\varepsilon + \frac{x - x'}{r}\right)$$

$$= \frac{\partial \phi_1^*}{\partial x'}\left(\varepsilon + \frac{x - x'}{r}\right) - \frac{i\omega M}{(1 - M^2)\bar{c}} \frac{\partial}{\partial x'} [\phi_1^*(r + \varepsilon(x - x'))].$$

Thus we obtain (since $\phi_1^* = 0$ along leading edge) :

$$\frac{w_1}{V}(x, y) = \frac{1}{2\pi} \iint_S \frac{\partial \phi_1^*}{\partial x'}\left(\varepsilon + \frac{x - x'}{r}\right)_{z=0} \frac{dx'\,dy'}{(y - y')^2}$$

$$- \frac{i\omega}{2\pi(1 - M^2)} \int_{-b/2}^{b/2} \frac{\phi_1^*(x_t, y')}{\bar{c}} \left(\varepsilon(x - x') + r\right)_{\substack{x'=x_t \\ z=0}} \frac{dy'}{(y - y')^2}. \quad \ldots \quad \text{(II, 7)}$$

When separating this equation into real and imaginary parts, we obtain two equations for ϕ_{1r}^* and ϕ_{1i}^*, each of which is of the same form as the corresponding equation for steady flow, the incidence term for ϕ_{1i}^* being modified by an 'induced incidence', which depends on ϕ_{1i}^*.

This system of equations for ϕ_{1r}^* and ϕ_{1i}^* is equivalent to the method suggested by W. P. Jones in a number of papers (*see* Ref. 7, 8). It was also suggested by H. Multhopp in an unpublished paper, derived there without reference to the complete wave equation. Multhopp started instead with the transformation (II, 4) and after neglecting terms of higher order in $\omega/(1 - M^2)$, solved the ordinary Laplace equation for ϕ^*, in terms of an integral equation.

It may be pointed out that the simplifications, which lead from (26) to the system (31a) to (31b), are not justifiable for a wing of infinite aspect ratio. In order to show this, we consider an incompressible flow ($M = 0$). In the two-dimensional case, where L and w is independent of y, the spanwise integration in (26) can easily be performed. We obtain from (26)

$$\frac{w}{V}(X, 0) = \frac{1}{4\pi}\int_{x_l}^{x_t}\frac{L(x')\,dx'}{x' - X} + \frac{L(x_t)}{4\pi}\int_{-\infty}^{X} e^{i\omega\frac{X-x}{\bar{c}}}\frac{dx}{x_t - x} \qquad \text{(II, 8)}$$

where

$$L(x_t) = -i\omega\int_{x_l}^{x_t} L(\xi)\,\frac{d\xi}{\bar{c}}. \qquad \text{(II, 9)}$$

The second integral in (II, 8) (which represents the wake influence) does not converge, if the exponental function is expanded in powers of ω, as it was done for a wing of finite aspect ratio. This integral can be written as

$$\int_{-\infty}^{X} e^{i\omega\frac{X-x}{\bar{c}}}\frac{dx}{x_t - x} = e^{i\omega\frac{X-x_t}{\bar{c}}}\int_{U}^{\infty} e^{iu}\frac{du}{u}$$

with

$$U = \omega\frac{x_t - X}{\bar{c}}.$$

We introduce the integral-cosine and integral-sine, as defined in (IV, 5) and (IV, 6) and have

$$e^{i\omega\frac{X-x_t}{\bar{c}}}\left[-\mathrm{Ci}(U) + i\left(\frac{\pi}{2} - \mathrm{Si}(U)\right)\right]$$

$$\simeq \left(1 + i\omega\frac{X - x_t}{\bar{c}}\right)\left[-\log\frac{\gamma\omega(x_t - X)}{\bar{c}} + i\left(\frac{\pi}{2} - \frac{\omega(x_t - X)}{\bar{c}}\right)\right]$$

($\gamma = \log C = 1{\cdot}781072$, C = Eulerian constant). Thus we obtain a term $\log\omega$, which occurs in the imaginary part L_i and l_i as can be seen by splitting (II, 8) in its real and imaginary part. The 'dotted' stability derivatives depend on $\log\omega$.

APPENDIX III

Supersonic Flow

The procedure which leads from the general equation (26) to the system (31) for small values of the frequency, can be justified in the same way as this was done for subsonic speeds in Appendix II. Thus the system (31) holds also for $M > 1$, provided that the integration is extended over the part S^* of the wing area S, which is inside the forward Mach cone from the point $(X, y, 0)$. The terms with $\varepsilon = +1$ and $\varepsilon = -1$ must be added.

Thus we find again that the 'induced' incidence $F(x, y)$ depends on ω so that L_i and l_i and finally the 'dotted' stability derivatives depend on ω, if M is near to 1.

If M is large enough so that $\omega << M^2 - 1$, then the transformation (II, 4) can also be introduced and the system (31) simplifies. The resultant equations can formally be obtained from (II, 7), if the terms for $\varepsilon = +1$ and $\varepsilon = -1$ are added and the integration is extended over S^*.

APPENDIX IV

Some Mathematical Aids

Appendix IV, I. Some integrals.—In this appendix we shall first derive some useful integrals, which can be used later on, to estimate the order of magnitude of certain terms neglected in the main part of the paper. We consider (for $m = 0, 1\ 2 \ldots$) the integral

$$A_m \equiv \int_{x_0}^{x} \sin \frac{a}{x - x'} \cdot (x - x')^{m-1}\, dx' = a^m \int_{u_0}^{\infty} \frac{\sin u\, du}{u^{m-1}} \quad \ldots \quad (\text{IV, } 1)$$

where $a = n\sigma^2/2V$ is independent of x' and

$$u = a/(x - x') \qquad u_0 = a/(x - x_0) \quad \ldots \quad (\text{IV, } 2)$$

and the integral

$$B_m \equiv \int_{x_0}^{x} \cos \left(\frac{a}{x - x'}\right) \cdot \ (x - x')^{m-1}\, dx' = a^m \int_{u_0}^{\infty} \frac{\cos u\, du}{u^{m+1}} . \quad \ldots \quad (\text{IV, } 3)$$

Integrating by parts we have for $m > 0$

$$\left.\begin{aligned} \int_{u_0}^{\infty} \frac{\sin u\, du}{u^{m+1}} &= \frac{\sin u_0}{m u_0^{\ m}} + \int_{u_0}^{\infty} \frac{\cos u\, du}{m u^m} \\ \int_{u_0}^{\infty} \frac{\cos u\, du}{u^{m+1}} &= \frac{\cos u_0}{m u_0^{\ m}} - \int_{u_0}^{\infty} \frac{\sin u\, du}{m u^m} \end{aligned}\right\} \quad \ldots \quad (\text{IV, } 4)$$

so that all these integrals can be reduced to integrals with an exponent 1 in the denominator. For these we find

$$A_0 = \int_{u_0}^{\infty} \frac{\sin u\, du}{u} = \frac{\pi}{2} - \text{Si}\,(u_0) = \frac{\pi}{2} - \int_0^{u_0} \frac{\sin u\, du}{u} \quad \ldots \quad (\text{IV, } 5)$$

$$B_0 = \int_{u_0}^{\infty} \frac{\cos u\, du}{u} = -\ \text{Ci}\,(u_0) = -\ C - \log u_0 + \int_0^{u_0} \frac{1 - \cos u}{u}\, du \quad \ldots \quad (\text{IV, } 6)$$

(C = Eulerian constant ; $\gamma = e^C = 1{\cdot}781072$). Si and Ci denote the integral-sine and integral-cosine respectively (compare Ref. 38).

From (IV, 4) we find for $m \geqslant 1$:

$$\left.\begin{aligned} A_m &= \frac{a}{m}\left[\frac{\sin u_0}{u_0}\,(x \ - x_0)^{m-1} + B_{m-1}\right] \\ B_m &= \frac{1}{m}\left[\cos u_0 .\ (x - x_0)^m - a A_{m-1}\right] \end{aligned}\right\} \quad \ldots \quad (\text{IV, } 7)$$

and thus in the limiting case $a \to 0$, which implies $u_0 \to 0$:

$$\left.\begin{aligned} &\lim_{a\to 0} B_m = \frac{(x - x_0)^m}{m} \ (m = 1, 2, \ldots) \\ &\lim_{a\to 0} \frac{A_m}{a} = \frac{(x - x_0)^{m-1}}{m}\left(1 + \frac{1}{m-1}\right) = \frac{(x - x_0)}{m-1}\ (m = 2, 3 \ldots) \end{aligned}\right\} \quad \ldots \quad (\text{IV, } 8)$$

$$\left.\begin{aligned}&\lim_{a\to 0}\frac{A_1}{a}=1+B_0,\ \lim_{a\to 0}A_0=\frac{\pi}{2}\\&\lim_{a\to 0}B_0=-\lim_{u_0\to 0}\log(\gamma u_0)\end{aligned}\right\} \qquad \text{(IV, 9)}$$

Finally we find for $m \geqslant 1$:

$$C_m \equiv \int_{x_0}^{x}\left(1-\cos\frac{a}{x-x'}\right)(x-x')^{m-1}dx' = \left(1-\cos\frac{a}{x-x_0}\right)\frac{(x-x_0)^m}{m}$$

$$+\frac{a}{m}\int_{x_0}^{x}\sin\frac{a}{x-x'}\,.\,(x-x')^{m-2}dx' = \frac{a^2}{m}\left[\frac{1-\cos u_0}{u_0{}^2}(x-x_0)^{m-2}+\frac{A^{m-1}}{a}\right] \qquad \text{(IV, 10)}$$

and in the limit $a \to 0$:

$$\left.\begin{aligned}&\lim_{a\to 0}\frac{C_m}{a^2}=\frac{(x-x_0)^{m-2}}{2(m-2)}\ (m=3,4\ldots)\\&\lim_{a\to 0}\frac{C_2}{a^2}=\frac{1}{2}\left(\frac{3}{2}+B_0\right)\\&\lim_{a\to 0}\frac{C_1}{a}=\frac{\pi}{2}\end{aligned}\right\} \qquad \text{(IV, 11)}$$

Appendix IV, 2. *Magnitude of some integrals.*—The relations, obtained in Appendix IV, 1, enable us to show the order of magnitude of some integrals, which have been neglected in the treatment of the integral equations (36) and (38). In (36) the cosine-function was replaced by 1, which is permissible if the integral :

$$\frac{1}{4\pi}\iint_{S^*}L_r(x',y')\left(1-\cos\frac{n\sigma^2}{2V(x-x')}\right)\frac{dx'\,dy'}{\sigma^2} \qquad \text{(IV, 12)}$$

becomes sufficiently small for $\omega \to 0$. When integrating with respect to x', we choose a point $x' = x_0$ near enough to x, so that $L_r(x', y')$ is approximately equal to $L_r(x, y')$. On the other hand the interval $x - x_0$ must be chosen big enough, so that for $x' < x_0$ the expression $n\sigma^2/2V(x - x') < n\sigma^2/2V(x - x_0)$ is bounded, so that it tends to zero for $\omega \to 0$ and the integral between $x' = x_l$ and $x' = x_0$ is finite and vanishes as ω^2. The second part of the integral (IV, 12), between $x' = x_0$ and $x' = x$ is then (for every value of y') proportional to the integral C_1 in (IV, 10) and according to (IV, 11) proportional to a or to $\omega = n\bar{c}/V$, q.e.d. Thus the simplification of (36) and (37) which leads to (38) and (39) is justified for small ω's. This argument does not apply to wings of infinite aspect ratio since then the spanwise integration produces an infinite contribution so that this term cannot be neglected.

A similar argument can be applied to show that the integral :

$$\frac{1}{4\pi}\iint_{S^*}\left[L_r(x',y')-L_r(x,y')\right]\left[\frac{\sigma^2}{2\bar{c}(x-x')}-\frac{V}{\bar{c}n}\sin\frac{n\sigma^2}{2V(x-x')}\right]\frac{dx'\,dy'}{\sigma^2} \qquad \text{(IV,13)}$$

which is the difference of equations (43a) and (46), is also proportional to ω, and can therefore be neglected for small ω. To prove this, we divide again the interval $x_l < x' < x$ into two parts. For the first part $x_l < x' < x_0$ the distance $x - x' > x - x_0$ remains positive and the sine function can be expanded, so that this part of the integral becomes proportional to ω^2.

The second part of the integral for $x_0 < x' < x$ is of the form (compare (IV, 1)):

$$\iint \text{const.}\left(1 - \frac{x - x'}{a} \sin \frac{a}{x - x'}\right)\frac{dx'\,dy'}{2\bar{c}} = \text{const.} \int \left(x - x_0 - \frac{A_2}{a}\right)\frac{dy'}{2\bar{c}}$$

which tends to zero for $a \to 0$ according to (IV, 8).

Appendix IV, 3. *Evaluation of some integrals.*—We consider an integral of the form

$$\frac{1}{4\pi} \iint_{S^*} \frac{\partial}{\partial x'}\left(f(x')\sqrt{(s'^2 - y'^2)}\right)\frac{dx'\,dy'}{(y - y')^2} = h(x, y) \qquad \text{(IV, 14)}$$

where $f(x')$ is bounded for $x_l < x' < x$ and $s' = s(x')$, $x = x_l(s')$ describes the leading edge of the wing. We perform the integration with respect to x' for constant y' between $s(x') = y'$ and $x' = x$:

$$h(x, y) = \frac{f(x)}{4\pi} \int_{-s(x)}^{s(x)} \frac{\sqrt{(s^2(x) - y'^2)}\,dy'}{(y - y')^2}.$$

Here the principal value of the integral must be taken[36]. Since:

$$\int \frac{\sqrt{(s^2 - y'^2)}\,dy'}{(y - y')^2} = - \frac{\sqrt{(s^2 - y'^2)}}{y' - y} - y \int \frac{dy'}{(y' - y)\sqrt{(s^2 - y'^2)}} - \sin^{-1} \frac{y'}{s} \qquad \text{(IV, 15)}$$

and

$$\int_{-s}^{s} \frac{dy'}{(y' - y)\sqrt{(s^2 - y'^2)}} \quad \left\{ \begin{array}{ll} = 0 & \text{for } y^2 < s^2 \\ = \dfrac{-\pi\,|y|}{y\sqrt{(y^2 - s^2)}} & \text{for } y^2 > s^2 \end{array} \right\} \qquad \text{(IV, 16)}$$

we have finally for $y^2 < s^2$, the only case we are interested in,

$$h(x, y) = - \tfrac{1}{4} f(x). \qquad \text{(IV, 17)}$$

Thus we find as a particular case ($f(x) = 1$):

$$\frac{1}{4\pi} \iint_{S^*} \frac{s}{\sqrt{(s^2 - y'^2)}} \frac{ds}{dx'} \frac{dx'\,dy'}{\sigma^2} = - \frac{1}{4}. \qquad \text{(IV, 18)}$$

At the same time we have proved that the integral equation:

$$\frac{1}{4\pi} \iint_{S^*} H(x', y) \frac{dx'\,dy'}{(y - y')^2} = h(x) \qquad \text{(IV, 19)}$$

where $h(x)$ is known, is solved by

$$H(x, y) = - \frac{\partial}{\partial x}\left(4h(x)\sqrt{(s^2 - y^2)}\right). \qquad \text{(IV, 20)}$$

It can be shown, that this solution is unique if the only solutions to be admitted must have a singularity of no higher order than $1/\sqrt{(s^2 - y^2)}$ along the edge $y = \pm s(x)$.

Appendix IV, 4. *Evaluation of an integral.*—The last term in equation (48) is proportional to the integral :

$$I(\eta) = \frac{1}{\pi}\int_{-1}^{+1} \log \frac{1}{4(\eta - \eta')^2} \frac{d\eta'}{\sqrt{(1 - \eta'^2)}}$$

where $\eta' < 1$. Now it can be shown that for $\eta^2 < 1$

$$\frac{dI(\eta)}{d\eta} = \frac{1}{\pi}\int_{-1}^{+1} \frac{2}{\eta - \eta'} \frac{d\eta'}{\sqrt{(1 - \eta'^2)}} = 0$$

so that

$$I(\eta) = I(0) = \frac{4}{\pi}\int_{0}^{1} \log \frac{1}{2\eta'} \frac{d\eta'}{\sqrt{(1 - \eta'^2)}}$$

$$= -\frac{4}{\pi}\left[\frac{\pi}{2}\log 2 + \int_{0}^{1} \frac{\log \eta' \, d\eta'}{\sqrt{(1 - \eta'^2)}}\right] = 0.$$

TABLE 1

	$M \sim 1,\ \omega = \dfrac{n\bar{c}}{V} << 1$	$M^2 > 1,\ \omega << M^2 - 1,\ a = \sqrt{(1-k^2)} = \sqrt{(M^2-1)}\cot \Lambda_l \leqslant 1$
$z_w = z_\theta$	$-\pi \cot \Lambda_l$	$-\dfrac{\pi}{E}\cot \Lambda_l$
$m_w = m_\theta$	$-\pi \cot \Lambda_l\left(\dfrac{4}{3} - h\right)$	$-\dfrac{\pi}{E}\cot \Lambda_l\left(\dfrac{4}{3} - h\right)$
$z_{\dot{w}}$	$-\dfrac{2\pi}{3}\cot \Lambda_l\left\{1 + \dfrac{3}{2}\cot^2 \Lambda_l . \log\left(\dfrac{\omega\gamma}{4}\cot^2 \Lambda_l\right)\right\}$	$-\dfrac{2\pi}{3E}\cot \Lambda_l\left\{1 - 3M^2\dfrac{1-H}{M^2-1}\right\}$
$m_{\dot{w}}$	$-\dfrac{2\pi}{3}\cot \Lambda_l\left(\dfrac{3}{2} - h\right)\left\{1 + \dfrac{3}{2}\cot^2 \Lambda_l . \log\left(\dfrac{\omega\gamma}{4}\cot^2 \Lambda_l\right)\right\}$ $-\dfrac{\pi}{8}\cot^3 \Lambda_l$	$-\dfrac{2\pi}{3E}\cot \Lambda_l\left(\dfrac{3}{2} - h\right)\left\{1 - 3M^2\dfrac{1-H}{M^2-1}\right\}$
$z_{\dot{\theta}}$	$-\pi\cot \Lambda_l\left(\dfrac{8}{3} - h\right) - \pi\cot^3 \Lambda_l . \log\left(\dfrac{\omega\gamma}{4}\cot^2 \Lambda_l\right)$	$-\dfrac{\pi}{E}\cot \Lambda_l\left(\dfrac{2}{3} + 2H - h\right) + \dfrac{2\pi}{E}\cot \Lambda_l\, M^2\dfrac{1-H}{M^2-1}$
$m_{\dot{\theta}}$	$-\pi\cot \Lambda_l(2-h)^2 - \pi\cot^3 \Lambda_l\left(\dfrac{3}{2} - h\right)\log\left(\dfrac{\omega\gamma}{4}\cot^2 \Lambda_l\right)$ $-\dfrac{\pi}{8}\cot^3 \Lambda_l$	$-\dfrac{\pi}{E}\cot \Lambda_l\,(1 + 3H - 2h(1+H) + h^2)$ $+\dfrac{2\pi}{E}\cot \Lambda_l\left(\dfrac{3}{2} - h\right)M^2\dfrac{1-H}{M^2-1}$
z_q	$-\pi\cot \Lambda_l\,(2-h)$	$-\dfrac{\pi}{E}\cot \Lambda_l\,(2H - h)$
m_q	$-\pi\cot \Lambda_l\left[\left(\dfrac{5}{3} - h\right)^2 + \dfrac{2}{9}\right]$	$-\dfrac{\pi}{E}\cot \Lambda_l\left[\left(\dfrac{2}{3} + H - h\right)^2 + \left(\dfrac{4}{3} - H\right)\left(H - \dfrac{1}{3}\right)\right]$
	$A = 4\cot \Lambda_l \qquad -\log\dfrac{\gamma}{4} = 0\cdot 80907$	$1 - H = \dfrac{a^2(K-E)}{(1-a^2)E + a^2(K-E)} \xrightarrow[M\to 1]{} (M^2-1)\cot^2 \Lambda_l \log\left(\dfrac{4\tan \Lambda_l}{\sqrt{(M^2-1)}}\right)$ $E = E(k) \xrightarrow[M\to 1]{} 1,\ K = K(k) \xrightarrow[M\to 1]{} \infty,\ H \xrightarrow[m\to 1]{} 1$

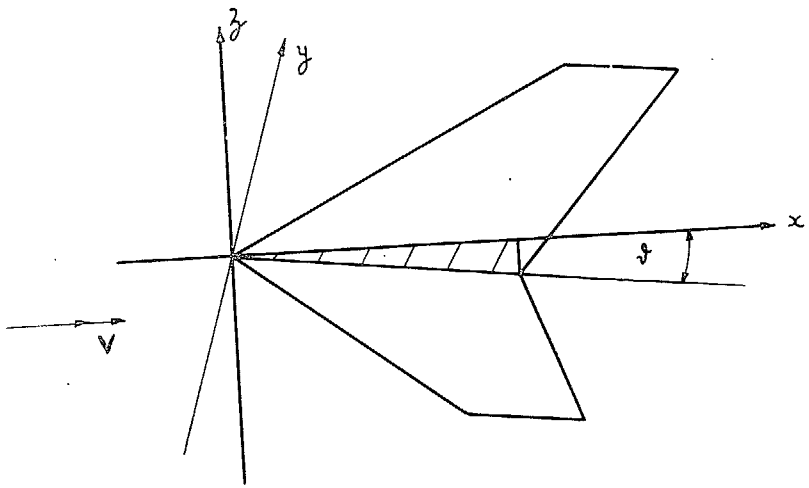

FIG. 1a. Co-ordinates used in the theory.

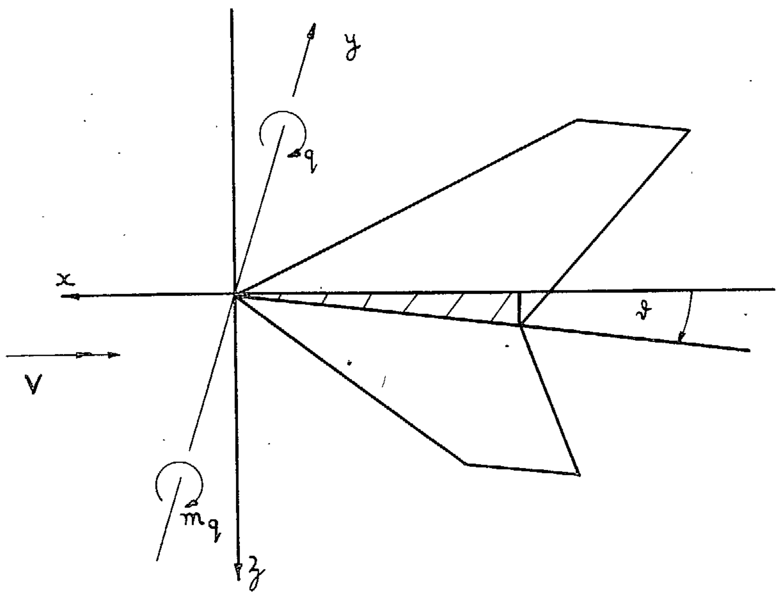

FIG. 1b. Stability axis (stability derivatives).

FIGS. 1a and 1b. Notations.

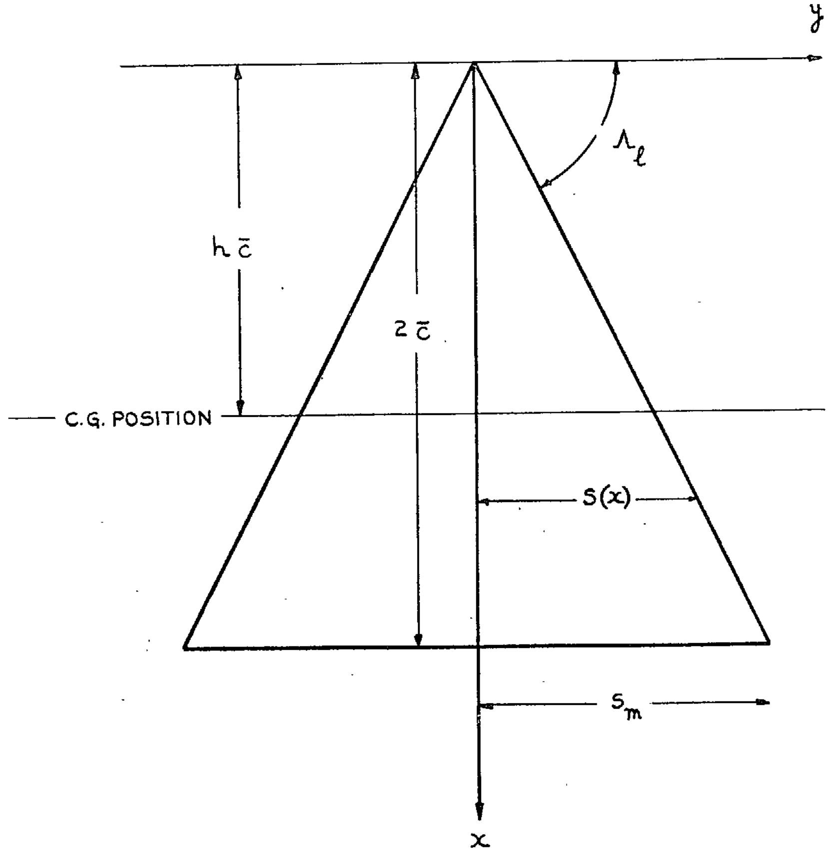

FIG. 2. The delta wing.

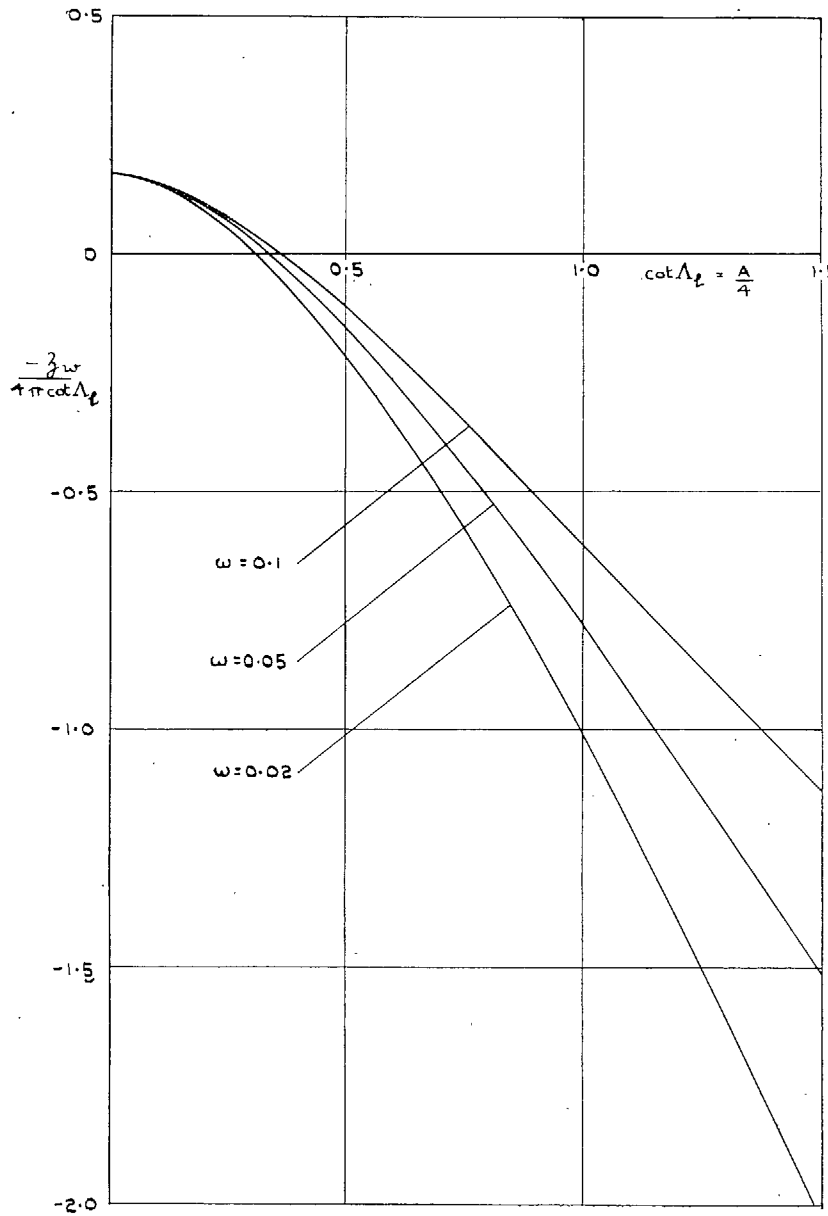

FIG. 3. The derivative $-z_{\dot w}$ for a delta wing for various frequencies ω and aspect ratios $A = 4 \cot \Lambda_l$ at sonic speed ($M \sim 1$).

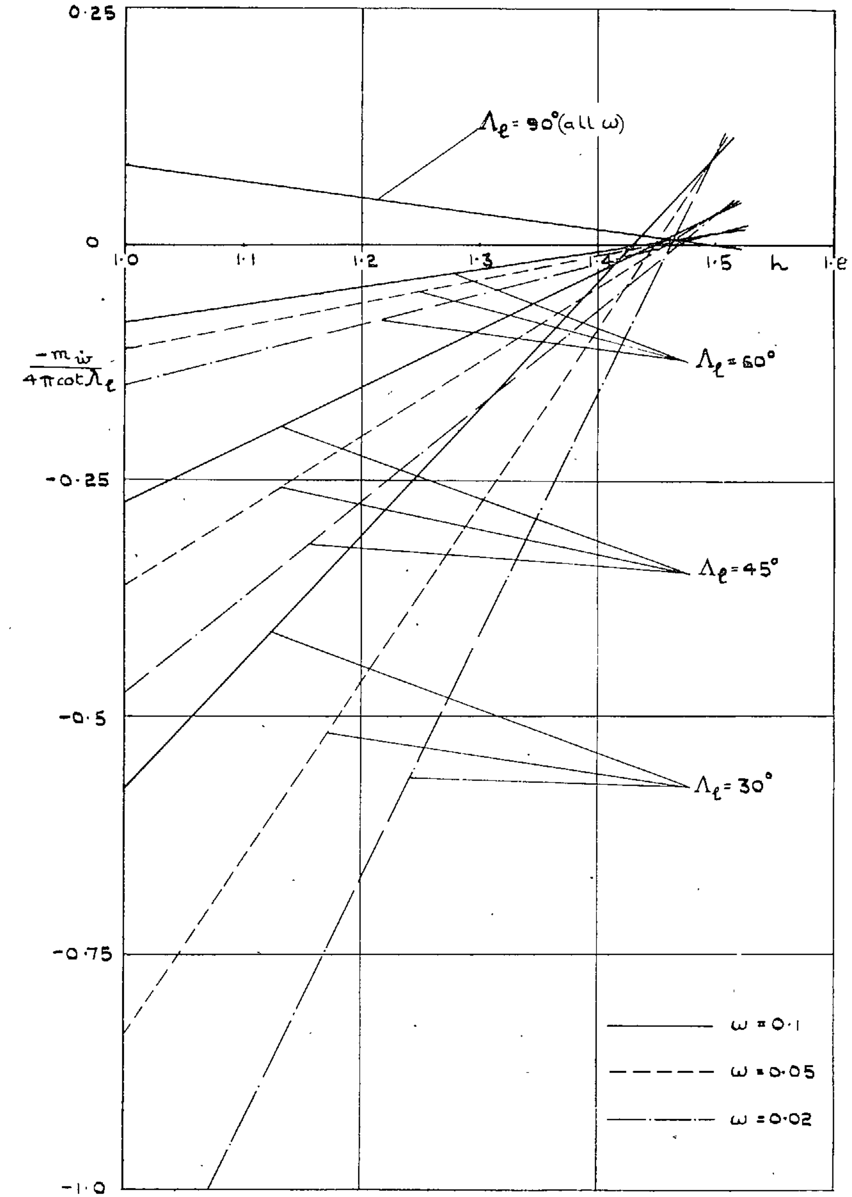

FIG. 4. The derivative $-m_{\dot w}$ for a delta wing at sonic speed against the axis position h for various reduced frequencies ω and various aspect ratios $A = 4 \cot \Lambda_l$.

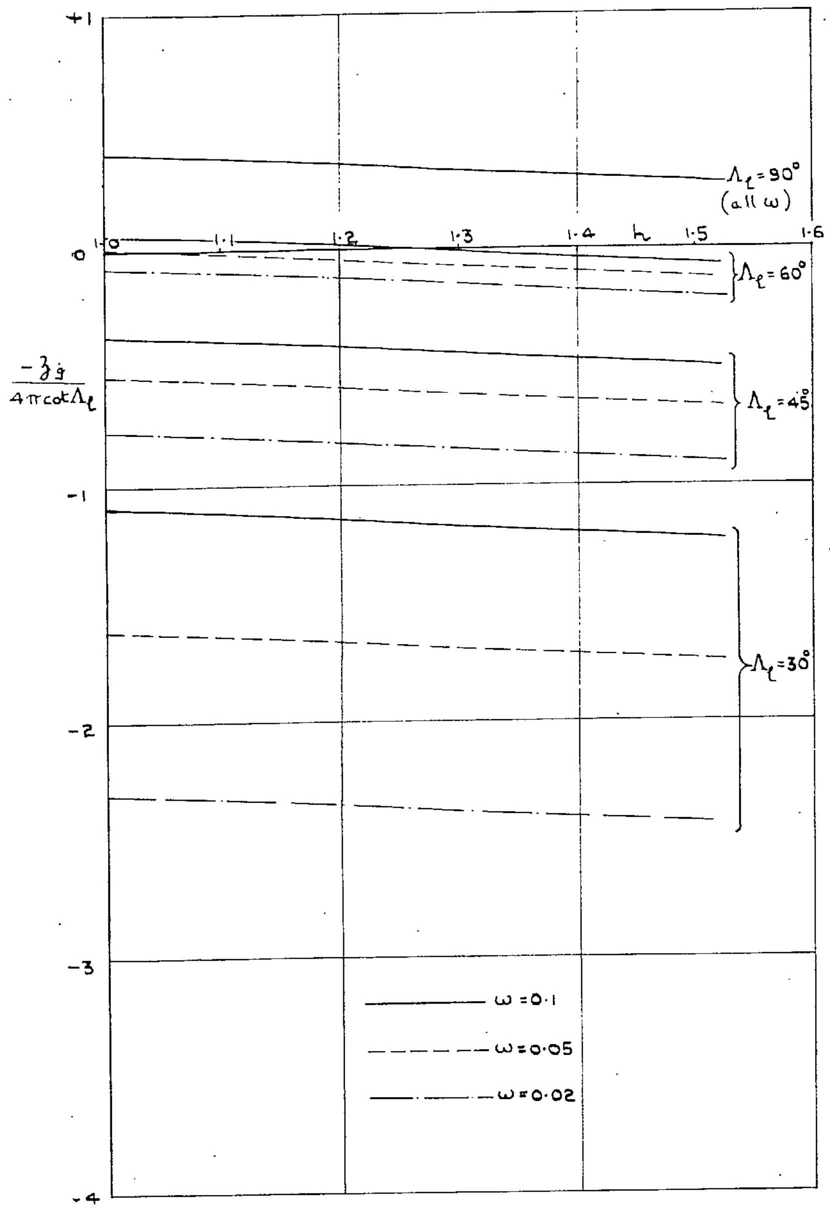

FIG. 5. The derivative $-z_{\dot{\vartheta}}$ for a delta wing at sonic speed against the axis position h for various frequencies ω and aspect ratios $A = 4 \cot \Lambda_l$.

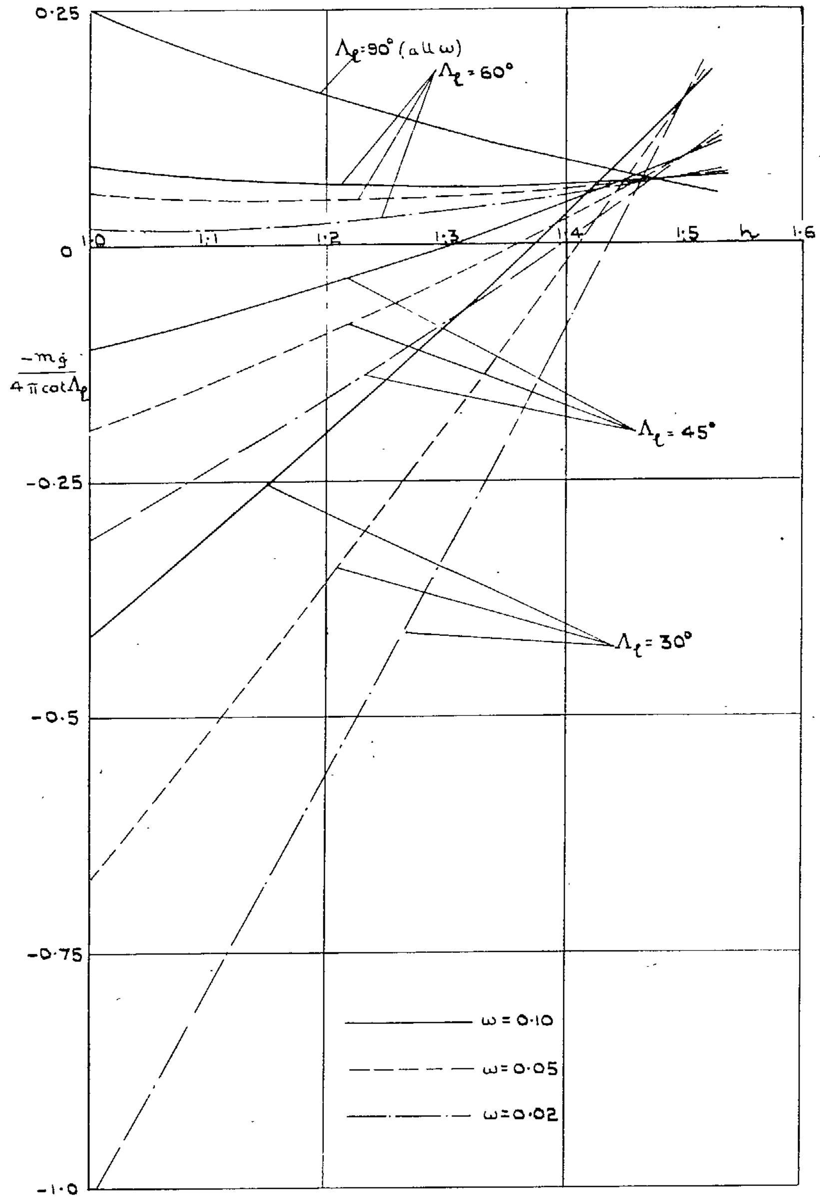

FIG. 6. The derivative $-m_{\dot{\vartheta}}$ for a delta wing at sonic speed against the axis position h for various frequencies ω and aspect ratios $A = 4 \cot \Lambda_l$.

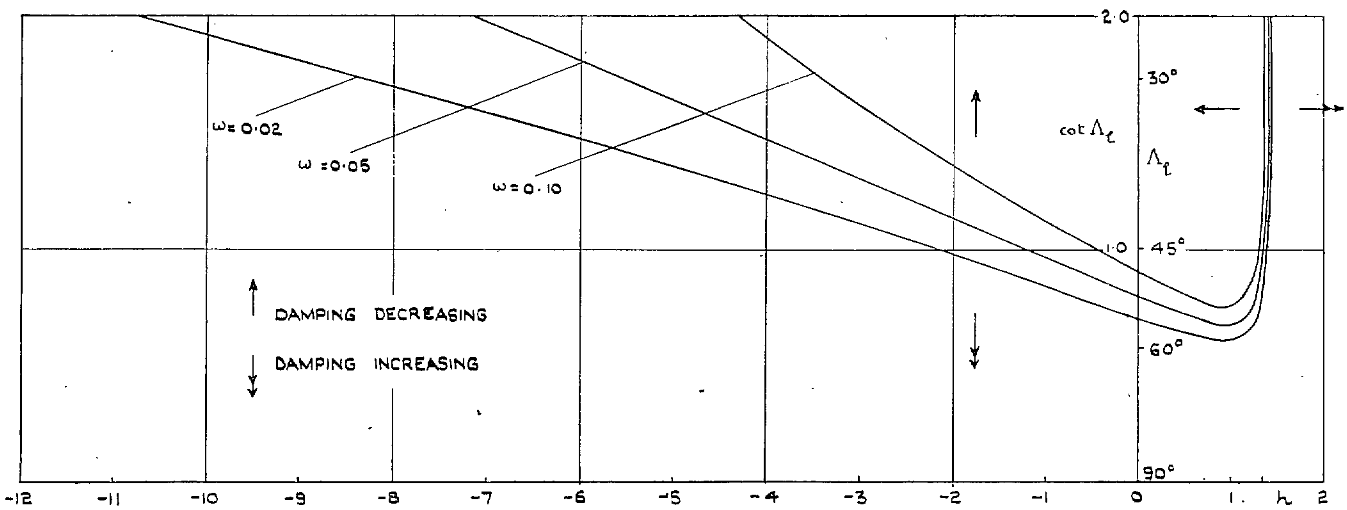

FIG. 7. Curves $-m_{\dot{\theta}} = 0$ for a delta wing at sonic speeds ($-m_{\dot{\theta}}$ is positive and contributes to the damping for all points below and to the right of the curves).

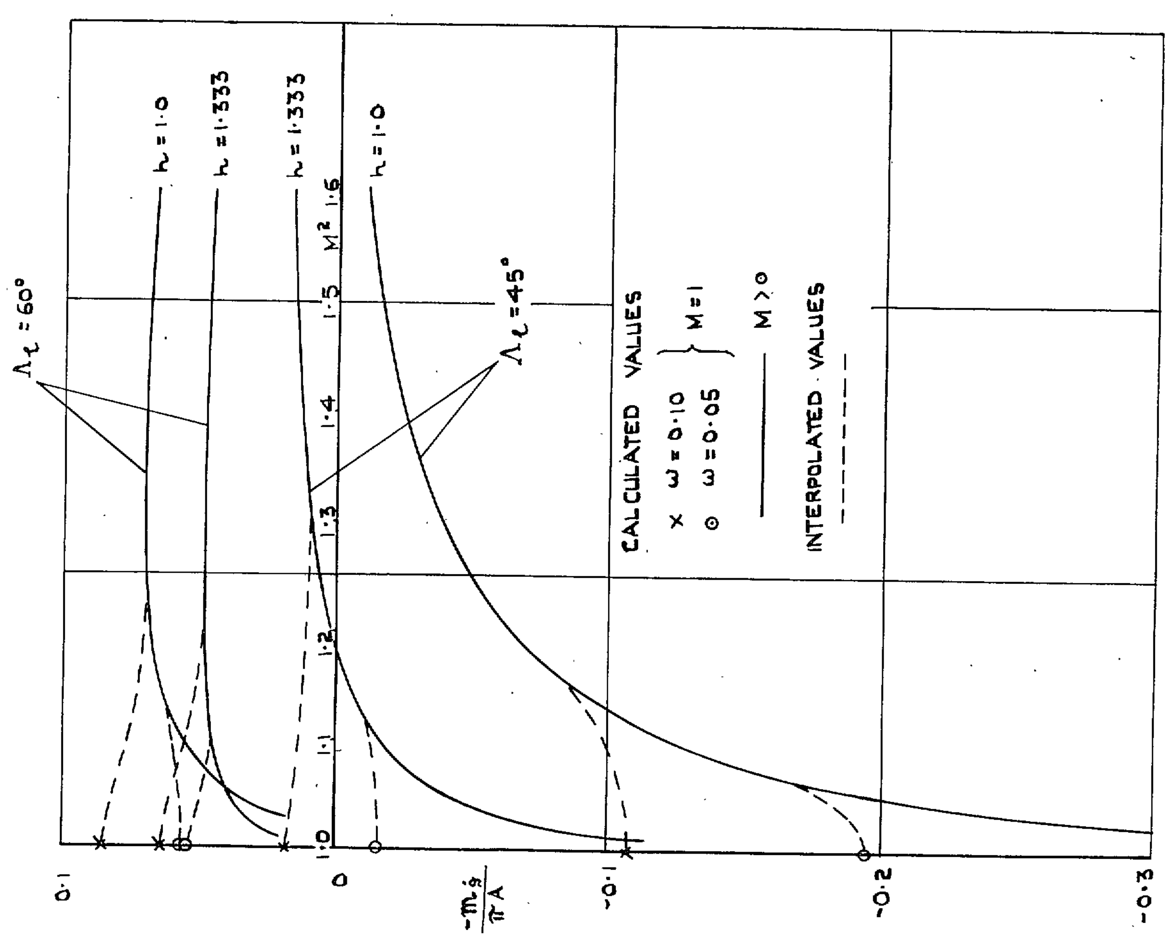

FIG. 8. The derivative $-m_{\dot{\theta}}/\pi A$ for a delta wing of aspect ratio $A = 4$ and $A = 2 \cdot 309$ for two axis positions $h = 1 \cdot 0$ and $h = 1 \cdot 333$ at sonic and supersonic speeds.

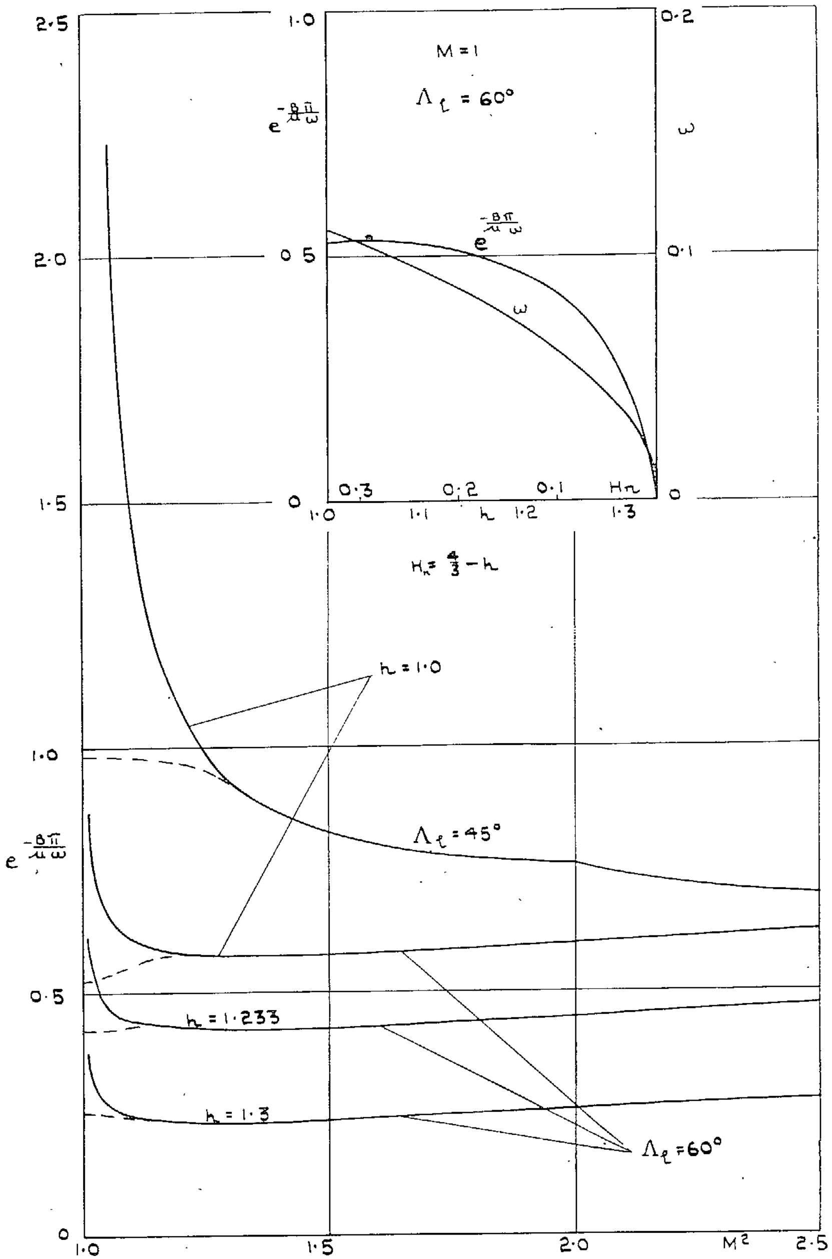

FIG. 9. Damping per period of the short-period oscillation in a characteristic case ($i_B = 0 \cdot 25$, $\mu = 200$).

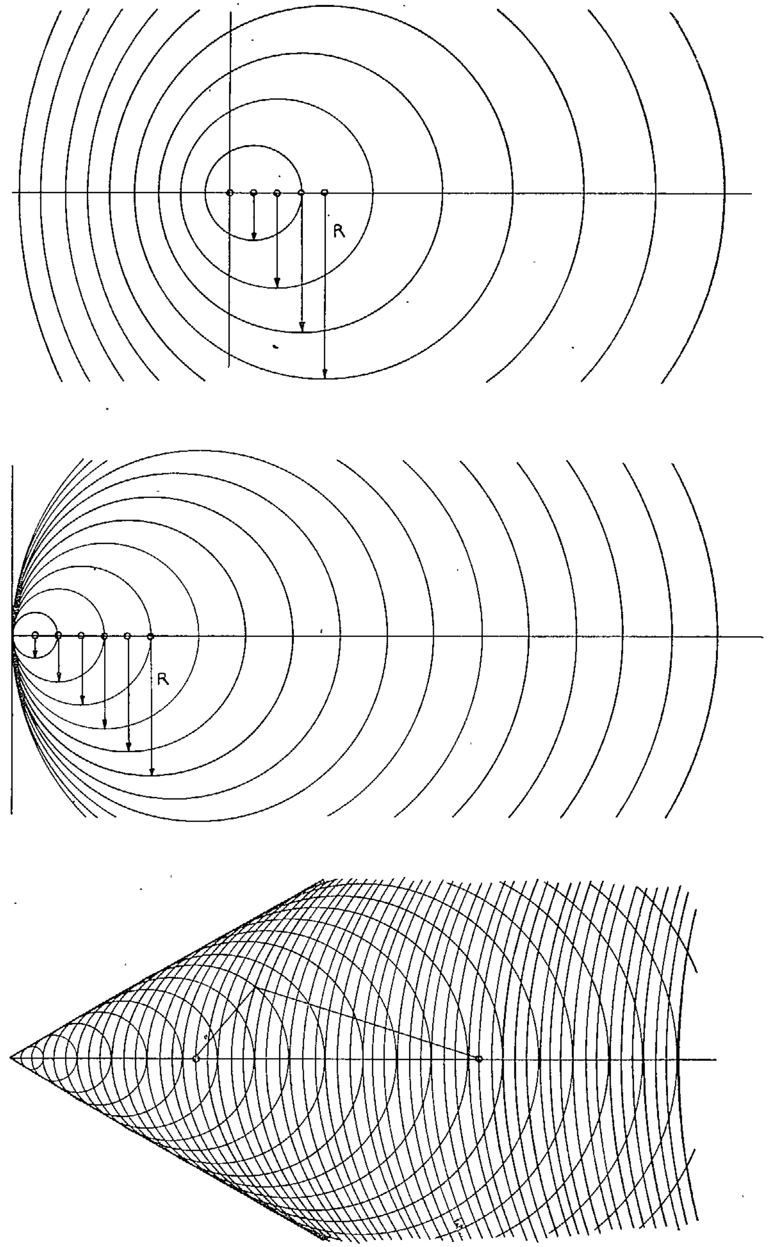

FIG. 10. Expansion of a disturbance wave in a subsonic, sonic, and supersonic flow.

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