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One-Dimensional Treatment of Non-Uniform Flow

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Summary.—For one-dimensional flow of a perfect gas, conditions at a station of a duct are defined by any four independent properties. Standard methods exist for the calculation of any other desired property from the given four independent properties.

The object of this paper is to illustrate the errors likely to arise when the simple one-dimensional flow methods are applied to a circular section duct in which a boundary layer exists. Graphical results are presented, for the case of a one-seventh power law boundary-layer velocity profile, showing the ratio of the true mean values calculated with allowance for the boundary-layer, to quantities derived from the simple one-dimensional calculation. Various boundary-layer thicknesses and a range of Mach numbers are dealt with.

Specifically three examples are worked out in detail, with different selections of the four independent variables, the selections being chosen to cover problems of common interest. The results of the first two examples might be applied, for instance, to the problem of the performance or design of a duct discharging adiathermally to atmosphere, from a reservoir with known stagnation conditions. The errors are usually small. Thus calculations by simple one-dimensional theory differ by less than about $2\frac{1}{2}$ per cent up to a 'one-dimensional' Mach number $M = 1$, and 5 per cent up to $M = 2$, from the values obtained by assuming a boundary-layer thickness at exit of 10 per cent of the duct radius. For other boundary-layer thicknesses the errors are roughly in proportion. The results of the third example indicate the errors likely to arise in the analysis of other quantities at a station, from measurements of mass flow, area, total temperature and static pressure. Here the accuracy of the one-dimensional method is within 2 per cent up to a free-stream Mach number $M' = 2$ for any boundary-layer thickness. Total pressure is an exception, the error in this case approaching 10 per cent at $M' = 2$.

General equations are presented for use in cases not covered by these examples. They are analogous to the one-dimensional equations, and give ratios of mean flow quantities to their sonic values, as functions of Mach number and correction factors, graphically presented, which depend on the velocity distribution. As a further illustration of possible application of the theory, the correction factors may be used for the calculation of momentum flux or kinetic energy flux from the mean velocity and mean density.

Introduction.—The analysis of many flow problems assuming the flow to be one-dimensional, is a useful approximation. At first sight this is surprising, since the flow model which forms the basis of the one-dimensional method differs profoundly from the actual physical situation. For instance the one-dimensional method ignores the existence of a non-uniform and possibly changing velocity profile, and it assumes happenings at the walls of the duct, *e.g.*, friction, to be felt instantaneously over the whole cross-section. However the one-dimensional method is so to be preferred for its simplicity, that it is important, as indeed in all approximate theories, to have some knowledge of its accuracy.

One aspect of the general problem of the assessment of the accuracy of the one-dimensional method is considered here. The cue is taken from the statement that 'suitable mean values' of the parameters of flow should be used in the one-dimensional equations. To investigate the significance of this, the one-dimensional uniform distributions of flow parameters have been replaced by non-uniform distributions, and equations developed which are analogous to the one-dimensional equations, but which are in terms of 'suitable mean values'. Results calculated using this refinement, have been compared with results obtained using one-dimensional formulae. In this way, some idea of the percentage inaccuracy resulting from the use of one-dimensional formulae for flow calculations should have been obtained.

The discussion has been restricted to problems which may be solved without reference to the one-dimensional friction factor. For simplicity the fluid has been assumed to be a perfect gas with constant specific heats.

LIST OF SYMBOLS

A	Cross-sectional area of duct
c_p	Specific heat at constant pressure
F	Impulse
h	Specific enthalpy
i	0, 1, 2, 3
j	0, 1, 2, 3 . . .
m	Mass flow rate
M	Mach number
n	Expansion index in equation (A.26)
p	Pressure
T	Temperature
q	Velocity
Q	Heat received by fluid across walls of duct per unit mass of flow
r	Radial distance from centre-line of duct
R	Gas constant
R_1	Radius of duct
s	Specific entropy
W	Work delivered by fluid across walls of duct per unit mass of flow
x	Co-ordinate measured along duct
X	Force in direction of flow
Y	Defined by equation (A.29)
Z	Defined by equation (A.35)
γ	Ratio of specific heats
δ	Boundary-layer thickness
ξ_i, η	Parameters defined by equations (A.5) and (A.6)
μ	Coefficient of dynamic viscosity
Φ	Dissipation function
ρ	Density
Λ	Availability

Suffix $_a$ refers to atmospheric conditions

Suffix $_0$ refers to stagnation conditions

Suffix $_s$ refers to sonic conditions

Prime indicates free-stream conditions

Bars indicate mean values

Equations and Assumptions of One-dimensional Theory.—Consider the steady continuous one-dimensional flow of the fluid which initially occupies the portion ABCD of the duct of Fig. 1, between any two sections 1 and 2, drawn at right-angles to the straight centre-line. Let the fluid receive heat Q and deliver work W across the walls AB and CD of the duct per unit mass of

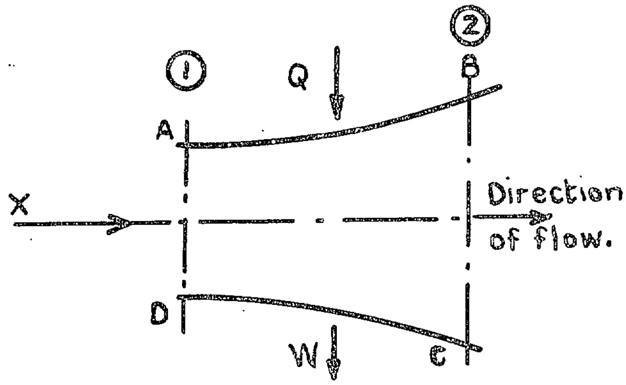

FIG. 1.

flow, and let a force X in the direction of flow be exerted on the fluid by the walls of the duct and by any objects within ABCD. Then in the absence of gravity or any other body force, application of the law of conservation of matter, Newton's second law, and the first law of thermodynamics yields:

Continuity Equation:

$$\rho_1 A_1 q_1 = \rho_2 A_2 q_2 \qquad (1)$$

Momentum Equation:

$$X = F_2 - F_1 \qquad (2)$$

Energy Equation:

$$Q - W = c_p T_{02} - c_p T_{01} \qquad (3)$$

where

$$m = \rho A q = \text{mass flux} \qquad (4)$$

$$\frac{F}{A} = p + \rho q^2 = \text{impulse/unit area} \qquad (5)$$

$$T_0 = T + \frac{q^2}{2c_p} = \text{total temperature} \qquad (6)$$

The assumptions inherent in these equations are that at sections 1 and 2:

(*a*) axial heat conduction is negligible

(*b*) only normal stresses are exerted on the fluid under consideration by surrounding fluid or moving surfaces

(*c*) velocity components at right-angles to the centre-line are negligible

(*d*) the normal stress $\frac{4}{3}\mu\frac{\partial^2 q}{\partial x^2}$ is negligible

(*e*) the difference in kinetic energy of turbulent motion between sections 1 and 2 is negligible

(*f*) all parameters of flow are uniformly distributed across the duct.

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It is desirable to stress the fact that these assumptions apply only to conditions at the places where fluid enters or leaves the region ABCD, *i.e.*, at sections 1 and 2. It is to be noted that any violation of the assumptions within ABCD may be tolerated. Thus viscous dissipation within ABCD, for example, represents a purely internal effect and does not constitute a flow of energy, momentum or mass to or from the fluid under consideration. Also, in crossing the boundaries DA and BC, the fluid may experience shear stresses at stationary surfaces, such as would be found at the walls or in a porous boundary, because the fluid actually in contact is then at rest relative to them, and hence no work is done.

It is conceivable that the conditions of assumptions (*a*), (*c*), (*d*) and (*e*) will be fulfilled in most cases of flow in ducts, except in the vicinity of shock waves, or for ducts of rapidly changing area. However, the presence of a boundary layer at the walls of the duct will always cause the flow to depart from the conditions of assumptions (*b*) and (*f*). The former is tantamount to assuming the dissipation function Φ to be negligible over sections 1 and 2.

Equations of Flow in Terms of Mean Values.—It is proposed to examine the particular assumption of one-dimensional theory, that all parameters of flow are uniformly distributed across the duct at the entry and exit sections of the portion of the duct under consideration. To do this, equations have been developed which take account of the non-uniform distributions due to the presence of the boundary layer. Their derivation is given in Appendix I. Assumptions (*a*) to (*e*) of the one-dimensional theory have been retained, but the last assumption has been replaced by the more realistic assumption that only static pressure and total temperature are uniform across the sections 1 and 2 of the duct*. In particular the velocity is allowed to be non-uniformly distributed, and this entails non-uniform distributions of density, temperature, total pressure, etc.

The equations are:

Continuity Equation:

$$\bar{\rho}_1 A_1 \bar{q}_1 = \bar{\rho}_2 A_2 \bar{q}_2 \qquad (7)$$

Momentum Equation:

$$X = \bar{F}_2 - \bar{F}_1 \qquad (8)$$

Energy Equation:

$$Q - W = c_p T_{02} - c_p T_{01} \qquad (9)$$

where

$$m = \bar{\rho} A \bar{q} \qquad (10)$$

$$\frac{F}{A} = p + \xi_2 \bar{\rho} \bar{q}^2 \qquad (11)$$

$$T_0 = \eta \bar{T} + \frac{\xi_3 \bar{q}^2}{2c_p} . \qquad (12)$$

In these equations mean density $\bar{\rho}$ and mean velocity $\bar{q}$ are such that

$\bar{\rho}\bar{q}$ is the mass flux per unit area,

$\xi_2 \bar{\rho} \bar{q}^2$ is the momentum flux per unit area,

and $\xi_3 \bar{\rho} \bar{q}^3$ is the kinetic energy flux per unit area.

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* It may be shown from the equations of motion that the static pressure across a turbulent boundary layer is approximately uniform, and Young[2] states that 'the few available experimental data confirm that the powerful mechanism of energy interchange of eddying motion tends to produce a nearly uniform distribution of energy across the boundary layer when the surface is insulated'. Within the laminar sublayer, however, total temperature will only be strictly uniform for flow with the Prandtl number unity and the surface insulated.

Mean temperature $\bar{T}$ has been defined in such a way that

$$\frac{p}{\bar{\rho}} = R\bar{T} \qquad (13)$$

and then $\bar{T}$ is the mean mixing temperature

and $\eta\bar{q}$ is the volume flux per unit area.

The quantities ξ_2, ξ_3 and η, of which general definitions are given in Appendix I, may be seen to be correction factors, which take account of the fact that differently weighted means must be taken when determining mass flux, momentum flux, etc. ξ_2, ξ_3 and η become unity if the flow is uniformly distributed, and in general their values depend on the velocity distribution. Explicit formulae for ξ_2, ξ_3 and η for the particular case of flow in a circular pipe with the velocity distribution:

$$\left.\begin{aligned} \frac{q}{q'} &= \left(\frac{y}{\delta}\right)^{1/7} && 0 \leqslant y \leqslant \delta \\ \frac{q}{q'} &= 1 && \delta \leqslant y \leqslant R \end{aligned}\right\} \qquad (14)$$

are derived in Appendix I and plotted against δ/R_1 for various values of free-stream Mach number M' in Fig. 2. For moderate Reynolds numbers this velocity distribution approximates to that of flow in a pipe of radius R_1, with a turbulent boundary layer of thickness δ, q' being the free-stream velocity and y the distance from the wall.

In developing the theory it has been found convenient to define a quantity

$$\bar{M}^2 = \frac{\xi_3\bar{q}^2}{\gamma R\eta\bar{T}}. \qquad (15)$$

$\bar{M}$ is referred to as 'mean Mach number'; it might be interpreted physically by regarding it, in the light of elementary kinetic theory, as proportional to the square root of the ratio of directed to random kinetic energy of the molecules.

Mean total pressure $\bar{p}_0$ has been defined as the total pressure of the corresponding uniform stream which possesses the same value of the availability flux as the non-uniform stream. This definition associates changes of total pressure with losses, since the availability of a system is defined as the maximum value of the useful work that can be obtained from the system-atmosphere combination for all possible changes of state of the system[3].

With this definition it is shown in Appendix II that

$$\log \bar{p}_0 = \frac{1}{m}\int_A \rho q \log p_0 \, dA \,. \qquad (16)$$

General Formulae for Mean Flow Quantities.—It is possible to express all mean quantities in terms of mass flow rate, area, total temperature and static pressure, which have unequivocal values at any section. It proves more convenient however, to express all quantities, including static pressure, in terms of mass flow rate, area, total temperature and mean Mach number, which serves as a parameter. The derivations are given in Appendix I, and the equations, with the exception of that for mean total pressure, are compared with the corresponding one-dimensional equations below. It may be observed that if the flow is uniformly distributed, the mean value equations become identical with the one-dimensional equations.

Mean Value Equations

$$\bar{p} = \sqrt{\left(\frac{R}{\gamma}\right)} \frac{m\sqrt{T_0}}{A} \frac{\sqrt{\xi_3}}{\eta\sqrt{\left\{\bar{M}^2\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\right\}}} \quad \ldots \quad (17)$$

$$\bar{T} = T_0 \frac{1}{\eta\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)} \quad \ldots \quad (18)$$

$$\bar{\rho} = \frac{1}{\sqrt{(\gamma R)}} \frac{m}{A\sqrt{T_0}} \sqrt{\xi_3} \sqrt{\left\{\frac{1 + \frac{\gamma - 1}{2}\bar{M}^2}{\bar{M}^2}\right\}} \quad \ldots \quad (19)$$

$$\bar{q} = \sqrt{(\gamma R)} \frac{\sqrt{T_0}}{\sqrt{\xi_3}} \sqrt{\left\{\frac{\bar{M}^2}{1 + \frac{\gamma - 1}{2}\bar{M}^2}\right\}} \quad \ldots \quad (20)$$

$$\bar{F} = \sqrt{\left(\frac{R}{\gamma}\right)} m\sqrt{T_0} \frac{\sqrt{\xi_3}\left\{1 + \left(\frac{\xi_2\eta}{\xi_3}\right)\bar{M}^2\right\}}{\eta\sqrt{\left\{\bar{M}^2\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\right\}}} \quad \ldots \quad (21)$$

One-dimensional Equations

$$p = \sqrt{\left(\frac{R}{\gamma}\right)} \frac{m\sqrt{T_0}}{A} \frac{1}{\bar{M}^2\left(1 + \frac{\gamma - 1}{2}M^2\right)} \quad \ldots \quad (22)$$

$$T = T_0 \frac{1}{1 + \frac{\gamma - 1}{2}M^2} \quad \ldots \quad (23)$$

$$\rho = \frac{1}{\sqrt{(\gamma R)}} \frac{m}{A\sqrt{T_0}} \sqrt{\left\{\frac{1 + \frac{\gamma - 1}{2}M^2}{M^2}\right\}} \quad \ldots \quad (24)$$

$$q = \sqrt{(\gamma R)} \sqrt{T_0} \sqrt{\left\{\frac{M^2}{1 + \frac{\gamma - 1}{2}M^2}\right\}} \quad \ldots \quad (25)$$

$$F = \sqrt{\left(\frac{R}{\gamma}\right)} m\sqrt{T_0} \frac{1 + \gamma M^2}{\sqrt{\left\{M^2\left(1 + \frac{\gamma - 1}{2}M^2\right)\right\}}} \quad \ldots \quad (26)$$

Flow in Entry Length.—Most instances of compressible flow in ducts fall into the category of entry length problems, since, particularly in supersonic flow, ducts will rarely be long enough for pipe flow to be established. Indeed for flow in parallel pipes, the choking length may be shorter than the entry length. The discussion that follows has therefore been restricted to entry length flows, the assumption being made that the flow consists of two parts, namely the boundary layer in which all viscous action is confined, and a free stream in which, in the absence of shock waves, the flow is homentropic. Flow with heat exchange has not been considered, because the assumption of uniform total temperature then ceases to be valid.

Under these conditions any flow is particularized by constant values of mass flow rate m, total temperature T_0 and free-stream total pressure p_0'. It is shown in Appendix I that the stream area A is related to these quantities by the expression

$$A = \sqrt{\left(\frac{R}{\gamma}\right)} \frac{m\sqrt{T_0}}{p_0'} \frac{1}{Y_1} \sqrt{\left[\frac{1 + \frac{\gamma - 1}{2} \frac{\xi_2\eta}{\xi_3} \frac{\bar{M}^2}{Y_2}}{\frac{\xi_2\eta}{\xi_3} \frac{\bar{M}^2}{Y_2}}\right]} \quad \ldots \quad (27)$$

with

$$Y_i = \frac{\bar{\rho}}{\rho'} \left(\frac{\bar{q}}{q'}\right)^i \xi_i \,, \qquad i = 1, 2 \,. \quad \ldots \quad (28)$$

It is convenient, as in one-dimensional theory[1] to refer all flow quantities to their value at the sonic condition, defined as that at which $\bar{M} = 1$ and $\xi_2 = \xi_3 = \eta = 1$, *i.e.*, the flow is uniformly distributed. The sonic values, designated by suffix $_s$, serve merely as reference values. It is irrelevant whether or not the flow ever attains the sonic condition.

Equation (27) together with equations (17) to (21) now give:

General Value

$$A = \sqrt{\left(\frac{R}{\gamma}\right)} \frac{m\sqrt{T_0}}{p_0'} \frac{1}{Y_1} \sqrt{\left[\frac{\left(1 + \frac{\gamma - 1}{2} \frac{\xi_2\eta}{\xi_3} \frac{\bar{M}^2}{Y_2}\right)^{\frac{\gamma+1}{\gamma-1}}}{\frac{\xi_2\eta}{\xi_3} \frac{\bar{M}^2}{Y_2}}\right]} \quad \ldots \quad (29)$$

$$p = p_0' \sqrt{\left\{\frac{Y_0/\eta}{\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)\left(1 + \frac{\gamma - 1}{2} \frac{\xi_2\eta}{\xi_3} \frac{\bar{M}^2}{Y_2}\right)^{\frac{\gamma+1}{\gamma-1}}}\right\}} \quad \ldots \quad (30)$$

$$T = T_0 \frac{1}{\eta\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)} \quad \ldots \quad (31)$$

$$\bar{\rho} = \frac{p_0'}{RT_0} \sqrt{\left[\frac{Y_0\eta\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)}{\left(1 + \frac{\gamma - 1}{2} \frac{\xi_2\eta}{\xi_3} \frac{\bar{M}}{Y_2}\right)^{\frac{\gamma+1}{\gamma-1}}}\right]} \quad \ldots \quad (32)$$

$$\bar{q} = \sqrt{(\gamma R)} \frac{\sqrt{T_0}}{\sqrt{\xi_3}} \sqrt{\left\{\frac{\bar{M}^2}{1 + \frac{\gamma - 1}{2} \bar{M}^2}\right\}} \quad \ldots \quad (33)$$

$$\bar{F} = \sqrt{\left(\frac{R}{\gamma}\right)} m\sqrt{T_0} \frac{\sqrt{\xi_3}}{\eta} \frac{\left\{1 + \gamma\left(\frac{\xi_2\eta}{\xi_3}\right)\bar{M}^2\right\}}{\sqrt{\left\{\bar{M}^2\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)\right\}}} \quad \ldots \quad (34)$$

Ratio

$$\frac{A}{A_s} = \frac{1}{Y_1} \sqrt{\left[\frac{\left(1 + \frac{\gamma - 1}{2} \frac{\xi_2}{\xi_3} \frac{\bar{M}^2}{Y_2}\right)^{\frac{\gamma+1}{\gamma-1}}}{\frac{\xi_2 \eta}{\xi_3} \frac{\bar{M}^2}{Y_2} \left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}} \right]} \quad \ldots \quad (35)$$

$$\frac{p}{p_s} = \sqrt{\left[\frac{\frac{Y_0}{\eta}\left(\frac{\gamma + 1}{2}\right)^{\frac{2\gamma}{\gamma-1}}}{\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)\left(1 + \frac{\gamma - 1}{2} \frac{\xi_2 \eta}{\xi_3} \frac{\bar{M}^2}{Y_2}\right)^{\frac{\gamma+1}{\gamma-1}}} \right]} \quad \ldots \quad (36)$$

$$\frac{\bar{T}}{T_s} = \frac{\frac{\gamma + 1}{2}}{\eta\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)} \quad \ldots \quad (37)$$

$$\frac{\bar{\rho}}{\rho_s} = \sqrt{\left[\frac{Y_0 \eta \left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)\left(\frac{\gamma + 1}{2}\right)^{\frac{2}{\gamma-1}}}{\left(1 + \frac{\gamma - 1}{2} \frac{\xi_2 \eta}{\xi_3} \frac{\bar{M}^2}{Y_2}\right)^{\frac{\gamma+1}{\gamma-1}}} \right]} \quad \ldots \quad (38)$$

$$\frac{\bar{q}}{q_s} = \frac{1}{\sqrt{\xi_3}} \sqrt{\left[\frac{\frac{\gamma + 1}{2} \bar{M}^2}{1 + \frac{\gamma - 1}{2} \bar{M}^2} \right]} \quad \ldots \quad (39)$$

$$\frac{\bar{F}}{F_s} = \frac{\sqrt{\xi_3}}{\eta} \sqrt{\left[\frac{\frac{\gamma + 1}{2}}{\bar{M}^2\left(1 + \frac{\gamma - 1}{2} \bar{M}^2\right)} \right]} \frac{1 + \gamma\left(\frac{\xi_2 \eta}{\xi_3}\right)\bar{M}^2}{1 + \gamma} \quad \ldots \quad (40)$$

Sonic Value

$$A_s = \sqrt{\left(\frac{R}{\gamma}\right)} \frac{m\sqrt{T_0}}{p_0'} \sqrt{\left\{\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma+1}{\gamma-1}}\right\}} \quad \ldots \quad (41)$$

$$p_s = \frac{p_0'}{\left(\frac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \quad \ldots \quad (42)$$

$$T_s = \frac{T_0}{\left(\frac{\gamma + 1}{2}\right)} \quad \ldots \quad (43)$$

$$\rho_s = \frac{p_0'}{RT_0} \frac{1}{\left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}}} \quad \ldots \quad (44)$$

Sonic Value—continued

$$q_s = \frac{\sqrt{(\gamma R T_0)}}{\sqrt{\left\{\frac{\gamma+1}{2}\right\}}} \qquad (45)$$

$$F_s = \sqrt{\left(\frac{R}{\gamma}\right)} \sqrt{T_0} \frac{\gamma+1}{\sqrt{\left\{\frac{\gamma+1}{2}\right\}}} \qquad (46)$$

One-dimensional Formula

$$\frac{A}{A_s} = \sqrt{\left[\frac{1}{M^2}\left\{\frac{1+\frac{\gamma-1}{2}M^2}{\frac{\gamma+1}{2}}\right\}^{\frac{\gamma+1}{\gamma-1}}\right]} \qquad (47)$$

$$\frac{p}{p_s} = \left\{\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right\}^{\frac{\gamma}{\gamma-1}} \qquad (48)$$

$$\frac{T}{T_s} = \frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2} \qquad (49)$$

$$\frac{\rho}{\rho_s} = \left\{\frac{\frac{\gamma+1}{2}}{1+\frac{\gamma-1}{2}M^2}\right\}^{\frac{1}{\gamma-1}} \qquad (50)$$

$$\frac{q}{q_s} = \sqrt{\left[\frac{\frac{\gamma+1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\right]} \qquad (51)$$

$$\frac{F}{F_s} = \sqrt{\left[\frac{\frac{\gamma+1}{2}}{M^2\left(1+\frac{\gamma-1}{2}M^2\right)}\right]} \frac{1+\gamma M^2}{1+\gamma} . \qquad (52)$$

Application to Typical Flow Calculations.—The above equations enable such calculations to be carried out as are normally made with one-dimensional formulae, but with regard to the existence of non-uniform velocity profiles at sections 1 and 2 of the duct under consideration, and hence enable the accuracy of the one-dimensional results to be assessed.

For simplicity all flows discussed will be supposed to originate from a region where conditions are uniform, which will be taken as section 1. This could be either a stagnation region of infinite area, or a region where the boundary-layer thickness is zero.

For illustration, three commonly arising types of problem will be discussed:

(i) Analysis of the performance of a duct of given design: the stagnation conditions are given at 1, the geometry (area) of the duct is known, and one other boundary condition (*e.g.*, pressure) is given at section 2. It is required to calculate all other flow quantities at section 2

(ii) Design of duct for given performance: the stagnation conditions are given at 1, the duct area is to be determined from data given at 2 (*e.g.*, rate of mass flow and pressure)

(iii) Analysis of flow at a section: values of mass flow rate, total temperature, area and static pressure are known at a section from (easily performed) measurement; all the other quantities are to be deduced at that section.

The numerical examples which follow are for flow in a duct of circular section of radius R_1, boundary-layer thickness δ, with the velocity distribution given by equation (14).

Example 1.—Consider a problem of type (i) above, where the static pressure is given at 2, *e.g.*, a duct of given exit area discharges fluid from a large reservoir to atmosphere, under homentropic free-stream conditions. It is required to calculate the rate of mass flow, and the stream properties at the exit, section 2.

The one-dimensional procedure in this instance is as follows:—values of p_0', T_0, p_2 and A_2 are given. Appropriate use is made of equations (41) to (52). p_s is first calculated from p_0'. Hence $p_2/p_s \to M_2 \to A_2/A_s \to A_s \to m$ knowing p_0' and T_0. Also $M_2 \to T_2/T_s$, ρ_2/ρ_s etc., $\to T_2$, ρ_2 etc., since the sonic values are known from T_0 and p_0'.

The generalized procedure is the same, except that equations (35)–(46) are used, together with the values of Y_i, ξ_i and η appropriate to the Mach number and boundary-layer thickness at 2.

Fig. 3 shows the results of such a calculation. The ratios of the one-dimensional results to the results accounting for the non-uniform velocity distribution, are plotted against δ/R_1 for different exit pressures. These pressures are designated by the one-dimensional values of the exit Mach numbers M. The $M = 0$ curves are for the limiting case of incompressible flow (any exit pressure), obtained separately by simple calculation.

Table 1 gives actual figures for use if the curves of Fig. 3 are not sufficiently accurate.

TABLE 1

M	0			1			2		
δ/R_1	0·2	0·6	1·0	0·2	0·6	1·0	0·2	0·6	1·0
$m/\bar{m}$	1·050	1·144	1·224	1·063	1·188	1·295	1·098	1·304	1·496
$p_0/\bar{p}_0$	1·000	1·000	1·000	1·045	1·136	1·215	1·154	1·543	1·976
$T/\eta\bar{T}$	1·000	1·000	1·000	0·987	0·964	0·945	0·956	0·878	0·819
$T/\bar{T}$	1·000	1·000	1·000	0·984	0·959	0·941	0·948	0·864	0·806
$q/\bar{q}$	1·050	1·144	1·224	1·047	1·139	1·219	1·040	1·126	1·206
$q/\eta\bar{q}$	1·050	1·144	1·224	1·050	1·144	1·224	1·050	1·144	1·225
$q/\xi_2\bar{q}$	1·040	1·093	1·200	1·038	1·118	1·195	1·033	1·108	1·185
$q/\sqrt{\xi_3}\,.\,\bar{q}$	1·036	1·115	1·190	1·035	1·111	1·186	1·030	1·101	1·176
$\rho/\bar{\rho}$	1·000	1·000	1·000	1·016	1·043	1·063	1·055	1·158	1·240
$F/\bar{F}$	1·000	1·000	1·000	1·058	1·168	1·260	1·111	1·353	1·586

All values are unity at $\delta/R_1 = 0$.

Example 2.—Consider a problem of type (ii) above, *e.g.*, calculate the exit area of a nozzle discharging fluid from a large reservoir, given the mass flow rate and the exit pressure. The procedure is similar to that for example 1, and the results are given in Fig. 4.

Example 3.—Consider a problem of type (iii) above, where it is required to deduce values of other flow quantities from measurements of m, T_0, A and p. The simplest procedure here is to calculate $\bar{M}$ from equation (17) for the appropriate boundary-layer thickness at the section, and hence the values of T/T_0, etc., from equations (18) to (21). These values may then be compared with the values of T/T_0, etc., obtained one-dimensionally using equations (22) to (26). The results are plotted in Fig. 5. In this case the ratio of the one-dimensional values to the mean values are plotted against δ/R_1 at constant values of free-stream Mach number M', a procedure which simplified the computations.

Discussion of Results of Flow Calculations.—The curves in Figs. 3, 4 and 5, which present ratios of the one-dimensional solutions to the mean-value solutions, indicate directly the errors inherent in the one-dimensional method due to its neglect of the existence of the boundary layer. In most cases the error tends to increase with Mach number, and also with boundary-layer thickness.

It may be seen that for examples 1 and 2, which concerned ducts working between known upstream stagnation conditions and downstream static pressure, the one-dimensional method gives results accurate to about $2\frac{1}{2}$ per cent up to $M = 1$ and 5 per cent up to $M = 2$, so long as the boundary-layer thickness does not exceed 10 per cent of the radius. For thicker boundary layers the errors increase nearly in proportion.

For example 3, namely the analysis of mean quantities at a section from measurements of m, T_0, A and p, the accuracy of the one-dimensional method is within 2 per cent up to $M' = 2$ for any boundary-layer thickness. (Note that the vertical scale in Fig. 5 is ten times that of Figs. 3 and 4.) Total pressure is an exception, the error in this case approaching 10 per cent at $M' = 2$. This means that a pitot traverse cannot be dispensed with if mean total pressure, or losses, are accurately required.

Conclusions.—The review of the basic assumptions of the one-dimensional method has stressed that for many applications one-dimensional conditions need only be met at the places where fluid enters or leaves the portion of the duct under consideration.

A generalization of the one-dimensional method has been developed, and used to assess the influence of the boundary layer on its accuracy. The generalization takes account of the non-uniform distributions of flow parameters due to the presence of the boundary layer. Equations, analogous to the one-dimensional equations, have been presented which give the ratios of mean flow quantities to their sonic values, as functions of Mach number and correction factors which depend on the velocity distribution.

Calculations were made using these equations, of flow in a duct of circular cross-section, under the assumption of homentropic free-stream conditions, and a turbulent boundary layer having a one-seventh power law velocity distribution. Results were compared with results of one-dimensional calculations, for typical flow problems.

In the case of ducts working between known upstream stagnation conditions and downstream static pressure, the one-dimensional method gave results accurate to about $2\frac{1}{2}$ per cent up to $M = 1$ and 5 per cent up to $M = 2$, so long as the boundary layer thickness did not exceed 10 per cent of the radius. For thicker boundary layers the errors increased roughly in proportion.

When used to deduce mean values of flow quantities at a section from measurements of mass flow rate, area, total temperature and static pressure, the one-dimensional method was accurate to within 2 per cent up to $M' = 2$ for any boundary-layer thickness, for all quantities except total pressure, for which the errors approached 10 per cent.

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APPENDIX I

Derivation of Mean Value Equations

Reference is made to Fig. 1. With non-uniform distributions of velocity across AD and BC, and the assumptions given after equation (6) the equations of continuity, momentum and energy, (1), (2) and (3) may be written for the single streamtube of area δA normal to the flow, and summed over the whole cross-section to give

$$\sum_1 \rho q \, \delta A = m = \sum_2 \rho q \, \delta A \qquad \text{(A.1)}$$

$$X = \sum_2 (p + \rho q^2) \, \delta A - \sum_1 (p + \rho q^2) \, \delta A \qquad \text{(A.2)}$$

$$m(Q - W) = \sum_2 \rho q (c_p T + \tfrac{1}{2} q^2) \, \delta A - \sum_1 \rho q (c_p T + \tfrac{1}{2} q^2) \, \delta A \qquad \text{(A.3)}$$

Following Le Fevre[4] let

$$\Sigma \rho q^i \delta A = \bar{\rho} \bar{q^i} A \qquad i = 0, 1, 2, 3 \qquad \text{(A.4)}$$

$$\Sigma q \, \delta A = \eta \bar{q} A \qquad \text{(A.5)}$$

$$\bar{q^i} = \xi_i \bar{q}^i \qquad \text{(A.6)}$$

$$p/\bar{\rho} = R\bar{T} \qquad \text{(A.7)}$$

(A.4) and (A.6) give

$$\Sigma \rho q \, \delta A = \bar{\rho} \bar{q} A \qquad \text{(A.8)}$$

$$\Sigma \rho q^2 \, \delta A = \bar{\rho} \xi_2 \bar{q}^2 A \qquad \text{(A.9)}$$

$$\Sigma \rho q^3 \, \delta A = \bar{\rho} \xi_3 \bar{q}^3 A \qquad \text{(A.10)}$$

Hence equation (A.1) may be written

$$\bar{\rho}_1 A_1 \bar{q}_1 = m = \bar{\rho}_2 A_2 \bar{q}_2 \qquad \text{(A.11)}$$

Since p = constant across any section, (A.2) becomes, using (A.9)

$$X = (Ap + \bar{\rho} A \xi_2 \bar{q}^2)_2 - (Ap + \bar{\rho} A \xi_2 \bar{q}^2)_1 \quad \text{.. (A.12)}$$

or

$$X = \bar{F}_2 - \bar{F}_1 \quad \text{.. (A.13)}$$

where

$$F = Ap + \bar{\rho} A \xi_2 \bar{q}^2 \quad \text{.. (A.14)}$$

= mean impulse

From (A.3) using $p/\rho = RT$ and equations (A.5) and (A.10)

$$m(Q - W) = \sum_2 \left(\frac{c_p}{R} pq + \tfrac{1}{2}\rho q^3\right)\delta A - \sum_1 \left(\frac{c_p}{R} pq + \tfrac{1}{2}\rho q^3\right)\delta A$$

$$= \left(\frac{c_p}{R} p\eta A\bar{q} + \tfrac{1}{2}\bar{\rho} A \xi_3 \bar{q}^3\right)_2 - \left(\frac{c_p}{R} p\eta A\bar{q} + \tfrac{1}{2}\bar{\rho} A \xi_3 \bar{q}^3\right)_1 .$$

Dividing by $m = \bar{\rho}_1 A_1 \bar{q}_1 = \bar{\rho}_2 A_2 \bar{q}_2$.

$$Q - W = \left(\frac{c_p}{R}\frac{p}{\bar{\rho}}\eta + \tfrac{1}{2}\xi_3\bar{q}^2\right)_2 - \left(\frac{c_p}{R}\frac{p}{\bar{\rho}}\eta + \tfrac{1}{2}\xi_3\bar{q}^2\right)_1 .$$

Using (A.7)

$$Q - W = (c_p\eta\bar{T} + \tfrac{1}{2}\xi_3\bar{q}^2)_2 - (c_p\eta\bar{T} + \tfrac{1}{2}\xi_3\bar{q}^2)_1 \quad \text{.. (A.15)}$$

or

$$Q - W = c_p T_{02} - c_p T_{01} \quad \text{.. (A.16)}$$

where

$$T_0 = \eta\bar{T} + \frac{\xi_3\bar{q}^2}{2c_p} \quad \text{.. (A.17)}$$

T_0 = total temperature, assumed uniform across any section.

Equations (A.11), (A.12) and (A.15) are respectively the equations of continuity, momentum and energy for flow in a duct with non-uniform distributions of flow parameters, in terms of the mean values of density, velocity and temperature, and the parameters ξ_2, ξ_3 and η defined by equations (A.4) to (A.7).

Derivation of General Formulae for Mean Flow Quantities.—The mean flow quantities may be expressed in terms of mass flow rate, area and total temperature as follows.

From equation (A.17), division by $\eta\bar{T}$ gives

$$1 + \tfrac{1}{2}\frac{\xi_3\bar{q}^2}{c_p\eta\bar{T}} = \frac{T_0}{\eta\bar{T}} . \quad \text{.. (A.18)}$$

This suggests defining a mean Mach number by the relation

$$\bar{M}^2 = \frac{\xi_3\bar{q}^2}{\gamma R\eta\bar{T}} . \quad \text{.. (A.19)}$$

With this definition (A.18) becomes

$$1 + \frac{\gamma - 1}{2}\bar{M}^2 = \frac{T_0}{\eta\bar{T}}$$

$$\bar{T} = \frac{T_0}{\eta\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)} . \quad \text{.. (A.20)}$$

Elimination of $\eta\bar{T}$ from (A.19) and (A.20) gives

$$\bar{q} = \sqrt{(\gamma R)}\,\frac{\sqrt{T_0}}{\sqrt{\xi_3}}\sqrt{\left\{\frac{\bar{M}^2}{1 + \frac{\gamma - 1}{2}\bar{M}^2}\right\}} \quad \ldots \quad (A.21)$$

From the equation of state (A.7) and continuity (A.11)

$$p = \frac{mR\bar{T}}{A\bar{q}} \quad \ldots \quad (A.22)$$

and elimination of $\bar{q}$ and $\bar{T}$ from (A.20), (A.21) and (A.22) gives

$$p = \sqrt{\left(\frac{R}{\gamma}\right)}\,\frac{m\sqrt{T_0}}{A}\,\frac{\sqrt{\xi_3}}{\eta\sqrt{\left\{\bar{M}^2\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\right\}}} \quad \ldots \quad (A.23)$$

From $m = \bar{\rho}A\bar{q}$ and (A.21)

$$\bar{\rho} = \frac{1}{\sqrt{(\gamma R)}}\,\frac{m}{A\sqrt{T_0}}\,\sqrt{\xi_3}\sqrt{\left\{\frac{1 + \frac{\gamma - 1}{2}\bar{M}^2}{\bar{M}^2}\right\}} \quad \ldots \quad (A.24)$$

From (A.14), (A.21), (A.23) and (A.24)

$$\bar{F} = \sqrt{\left(\frac{R}{\gamma}\right)}\,m\sqrt{T_0}\,\frac{\sqrt{\xi_3}}{\eta}\,\frac{\left\{1 + \gamma\left(\frac{\xi_2\eta}{\xi_3}\right)\bar{M}^2\right\}}{\sqrt{\left\{\bar{M}^2\left(1 + \frac{\gamma - 1}{2}\bar{M}^2\right)\right\}}} \quad \ldots \quad (A.25)$$

These equations give the mean values of temperature, velocity, pressure, density and impulse in terms of mass flow rate, area, total temperature and mean Mach number.

Expressions for η and ξ_i.—Formal expressions will now be developed for η and ξ_i using the assumed velocity distribution (*see* equation (14))

$$\left.\begin{aligned} \frac{q}{q'} &= \left(\frac{y}{\delta}\right)^{1/n} \quad 0 \leqslant y \leqslant \delta, \text{ i.e., in the boundary layer} \\ \frac{q}{q'} &= 1 \quad \delta \leqslant y \leqslant R_1, \text{ i.e., in the free stream} \end{aligned}\right\} \quad \ldots \quad (A.26)$$

In each case the independent variable, upon which the values of ξ_i and η ultimately depend, will be n, the expansion index in equations (A.26), δ/R_1, the proportional boundary-layer thickness and M', the free-stream Mach number.

From the definition of η, equation (A.5)

$$\eta\bar{q}A = \Sigma q\,\delta A\,.$$

Applying this to a circular duct of radius R_1 and with the velocity distribution given by equations (A.26)

$$\eta\bar{q}A = q'\int_0^{R_1}\frac{q}{q'}\,2\pi r\,dr$$

$$\eta\bar{q}\pi R_1^2 = q'\left[\int_0^{\delta}\left(\frac{y}{\delta}\right)^{1/n}2\pi(R_1 - y)\,dy + \int_{\delta}^{R_1}2\pi(R_1 - y)\,dy\right]$$

$$\eta = 2\frac{q'}{\bar{q}}\left(\frac{\delta}{R_1}\right)^2\left[\int_0^1\left(\frac{y}{\delta}\right)^{1/n}\left(\frac{R_1}{\delta} - \frac{y}{\delta}\right)d\left(\frac{y}{\delta}\right) + \int_1^{R_1/\delta}\left(\frac{R_1}{\delta} - \frac{y}{\delta}\right)d\left(\frac{y}{\delta}\right)\right].$$

Integrating

$$\eta = \frac{q'}{\bar{q}}\left[\frac{1}{2n+1}\left(\frac{\delta}{R}\right)^2 - \frac{2}{n+1}\left(\frac{\delta}{R_1}\right) + 1\right]. \quad \ldots \quad (A.27)$$

From the definition of ξ_i equation (A.6)

$$\xi_i = \frac{\overline{q^i}}{\bar{q}^i}.$$

Using (A.4)

$$\xi_i = \frac{\Sigma\rho q^i\,\delta A}{(\Sigma\rho q\,\delta A)^i}\frac{(\bar{\rho}A)^i}{(\bar{\rho}A)}$$

$$\xi_i = \frac{\Sigma\rho q^i\,\delta A}{(\Sigma\rho q\,\delta A)^i}(\bar{\rho}A)^{i-1}$$

$$\xi_i = \frac{\left(\frac{\rho}{\rho'}\right)\left(\frac{q}{q'}\right)^i\frac{\delta A}{A}}{\left\{\Sigma\left(\frac{\rho}{\rho'}\right)\left(\frac{q}{q'}\right)\frac{\delta A}{A}\right\}^i}\left\{\Sigma\left(\frac{\rho}{\rho'}\right)\frac{\delta A}{A}\right\}^{i-1}. \quad \ldots \quad (A.28)$$

Define

$$Y_i = \Sigma\left(\frac{\rho}{\rho'}\right)\left(\frac{q}{q'}\right)^i\frac{\delta A}{A}. \quad \ldots \quad (A.29)$$

Since p and T_0 are constant across any section

$$\frac{\rho}{\rho'} = \frac{T'}{T} \quad \ldots \quad (A.30)$$

and

$$\left(1+\frac{\gamma-1}{2}M'^2\right)T' = \left(1+\frac{\gamma-1}{2}M^2\right)T. \quad \ldots \quad (A.31)$$

Therefore

$$\left(1+\frac{\gamma-1}{2}M'^2\right)T' = \left\{1+\frac{\gamma-1}{2}M'^2\left(\frac{q}{q'}\right)^2\frac{T'}{T}\right\}T.$$

Therefore

$$\frac{T'}{T} = \frac{1}{1+\frac{\gamma-1}{2}M'^2\left\{1-\left(\frac{q}{q'}\right)^2\right\}} = \frac{\rho}{\rho'}. \quad \ldots \quad (A.32)$$

(A.32) in (A.29) yields

$$Y_i = 2\left(\frac{\delta}{R_1}\right)^2\left[\int_0^1\frac{\left(\frac{y}{\delta}\right)^{i/n}\left(\frac{R_1}{\delta}-\frac{y}{\delta}\right)d\left(\frac{y}{\delta}\right)}{1+\frac{\gamma-1}{2}M'^2\left\{1-\left(\frac{y}{\delta}\right)^{2/n}\right\}} + \int_1^{R_1/\delta}\left(\frac{R}{\delta}-\frac{y}{\delta}\right)d\left(\frac{y}{\delta}\right)\right] \quad (A.33)$$

$$Y_i = Z_i - \left(\frac{\delta}{R_1}\right)Z_{i+n} + \left(1-\frac{\delta}{R_1}\right)^2 \quad \ldots \quad (A.34)$$

where

$$Z_i = 2\left(\frac{\delta}{R_1}\right)\int_0^1\frac{\left(\frac{y}{\delta}\right)^{i/n}d\left(\frac{y}{\delta}\right)}{1+\frac{\gamma-1}{2}M'^2\left\{1-\left(\frac{y}{\delta}\right)^{2/n}\right\}}. \quad \ldots \quad (A.35)$$

Performing the integration in equation (A.35) after expanding the integrand by the Binomial Theorem

$$Z_i = \frac{2\left(\frac{\delta}{R_1}\right)^2 \sum_{j=0}^{j=\infty} \left(\frac{\frac{\gamma-1}{2}M'^2}{1+\frac{\gamma-1}{2}M'^2}\right)^2 \frac{n}{2j+i+n}}{1+\frac{\gamma-1}{2}M'^2} \quad \ldots \quad \text{(A.36)}$$

Now from (A.4) with $i = 1$ and $i = 0$ respectively

$$\bar{q} = \frac{\Sigma \rho q \, \delta A}{\bar{\rho} A}, \qquad \bar{\rho} = \frac{\Sigma \rho \, \delta A}{A}$$

$$\bar{q} = \frac{\Sigma \rho q \, \delta A}{\Sigma \rho \, \delta A}$$

$$= q' \frac{\Sigma \frac{\rho}{\rho'} \frac{q}{q'} \, \delta A}{\Sigma \frac{\rho}{\rho'} \, \delta A}.$$

From (A.29) $$\frac{\bar{q}}{q'} = \frac{Y_1}{Y_0} \quad \ldots \quad \text{(A.37)}$$

and substituting in (A.27)

$$\eta = \frac{Y_0}{Y_1}\left[\frac{1}{2n+1}\left(\frac{\delta}{R_1}\right)^2 - \frac{2}{n+1}\left(\frac{\delta}{R_1}\right) + 1\right]. \quad \ldots \quad \text{(A.38)}$$

From (A.28) and (A.29)

$$\xi_i = \left(\frac{Y_0}{Y_1}\right)^i \left(\frac{Y_i}{Y_0}\right) \quad \ldots \quad \text{(A.39)}$$

with Y_i given by equations (A.34) and (A.36).

Using equations (A.38) and (A.39) ξ_i and η may be calculated for any values of free-stream Mach number M', relative boundary-layer thickness δ/R_1 and the parameter n.

Relation Between Free-stream and Mean Mach Numbers.—It is desirable to relate the free-stream Mach number M', in terms of which the expressions for ξ_i and η are given, to the mean Mach number $\bar{M}$, in terms of which the formula for mean flow quantities are given

$$\bar{M}^2 = \frac{\xi_3 \bar{q}^2}{\gamma R \eta \bar{T}}$$

and $$M'^2 = \frac{q'^2}{\gamma R T'}$$

$$\frac{\bar{M}^2}{M'^2} = \frac{\xi_3}{\eta}\left(\frac{\bar{q}}{q'}\right)^2 \frac{\bar{\rho}}{\rho'} \quad \ldots \quad \text{(A.40)}$$

as $$\frac{\bar{\rho}}{\rho'} = \frac{T'}{\bar{T}}$$

since $p =$ constant.

Now from (A.4) and (A.29) with

$$i = 0 , \qquad \frac{\bar{\rho}}{\rho'} = Y_0 . \qquad \text{.. .. ..} \quad \text{(A.41)}$$

Using (A.37) $\quad \dfrac{\bar{M}^2}{M'^2} = \dfrac{\xi_3}{\eta}\left(\dfrac{Y_1}{Y_0}\right)^2 Y_0 .$

Using (A.39) $\quad \dfrac{\bar{M}^2}{M'^2} = \dfrac{\xi_3}{\xi_2\eta} Y_2 . \qquad \text{.. .. ..} \quad \text{(A.42)}$

Expression for $\sqrt{\left(\dfrac{R}{\gamma}\right)}\dfrac{m\sqrt{T}}{Ap_0'}$.

By continuity

$$\frac{m}{A} = \bar{\rho}\bar{q}$$

$$= \frac{1}{Y_1}\rho' q'$$

from (A.37) and (A.41).

$$\frac{m}{A} = \frac{1}{Y_1}\frac{p_0'}{\sqrt{T_0}}\sqrt{\left(\frac{\gamma}{R}\right)} M'\left(1 + \frac{\gamma - 1}{2} M'^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \qquad \text{.. .. ..} \quad \text{(A.43)}$$

by standard one-dimensional formula, applicable in the free stream. Substituting for M' from (A.42)

$$\sqrt{\left(\frac{R}{\gamma}\right)}\frac{m\sqrt{T_0}}{Ap_0'} = \frac{1}{Y_1}\sqrt{\left\{\frac{\xi_2\eta}{\xi_3}\frac{\bar{M}^2}{Y_2}\left(1 + \frac{\gamma - 1}{2}\frac{\xi_2\eta}{\xi_3}\frac{\bar{M}^2}{Y_2}\right)^{-\frac{\gamma+1}{\gamma-1}}\right\}} . \qquad \text{.. .. ..} \quad \text{(A.44)}$$

APPENDIX II

Interpretation of Mean Total Pressure

Definition.—The mean total pressure $\bar{p}_0$ of a non-uniform stream having uniform total temperature is defined as the total pressure of that uniform stream having the same total temperature and mass flux as the non-uniform stream, and possessing the same value of availability flux.

Consider two thermodynamic systems consisting of the mass m of fluid that flows past a given section in unit time, of the uniform and non-uniform streams respectively. Then for atmospheric conditions designated by suffix $_a$, the availability of the uniform flow system is[3]

$$\Lambda = m(h_0 - T_a s) - m(h_a - T_a s_a) \qquad \text{.. .. ..} \quad \text{(A.45)}$$

whilst that of the non-uniform flow system is

$$\bar{\Lambda} = \int_A \rho q(h_0 - T_a s)\, dA - m(h_a - T_a s_a)$$

$$= mh_0 - \int_A \rho q T_a s\, dA - m(h_a - T_a s_a) \qquad \text{.. .. .. ..} \quad \text{(A.46)}$$

since h_0, the stagnation enthalpy is uniform.

Equating A and $\bar{A}$, and writing $s = R \log_e \frac{(T_0)^{\frac{\gamma}{\gamma-1}}}{\bar{p}_0}$ in (A.45) and $s = R \log_e \frac{(T_0)^{\frac{\gamma}{\gamma-1}}}{p_0}$ in (A.46) gives

$$m \log_e \bar{p}_0 = \int_A \rho q \log_e p_0 \, dA \quad \text{.. .. .. (A.47)}$$

or

$$\log_e \bar{p}_0 = \frac{1}{m} \int_A \rho q \log_e p_0 \, dA \,. \quad \text{.. .. .. (A.48)}$$

For the velocity distribution (A.26), the following expression may be derived for $\log_e(\bar{p}_0/p_0')$ using methods similar to those of Appendix I:

$$\log_e \frac{\bar{p}_0}{p_0'} = \left[\frac{R_1}{\delta} \sum_{j=1}^{j=\infty} \frac{n}{2j+1+n} a_j \left\{ \frac{\frac{\gamma-1}{2} M'^2}{1 + \frac{\gamma-1}{2} M'^2} \right\}^j \right.$$

$$\left. - \sum_{j=1}^{j=\infty} \frac{n}{2j+1+2n} a_j \left\{ \frac{\frac{\gamma-1}{2} M'^2}{1 + \frac{\gamma-1}{2} M'^2} \right\}^j \right] \frac{\frac{2\gamma}{\gamma-1} \left(\frac{\delta}{R_1} \right)^2}{Y_1 \left(1 + \frac{\gamma-1}{2} M'^2 \right)}$$

$$- \frac{\gamma}{\gamma-1} \frac{Y_1 - \left(1 - \frac{\delta}{R_1} \right)^2}{Y_1} \log_e \left(1 + \frac{\gamma-1}{2} M'^2 \right) \quad \text{.. .. .. (A.49)}$$

where

$$a_j = \sum_{j=1}^{j=j} \frac{1}{j} \,. \quad \text{.. .. .. (A.50)}$$

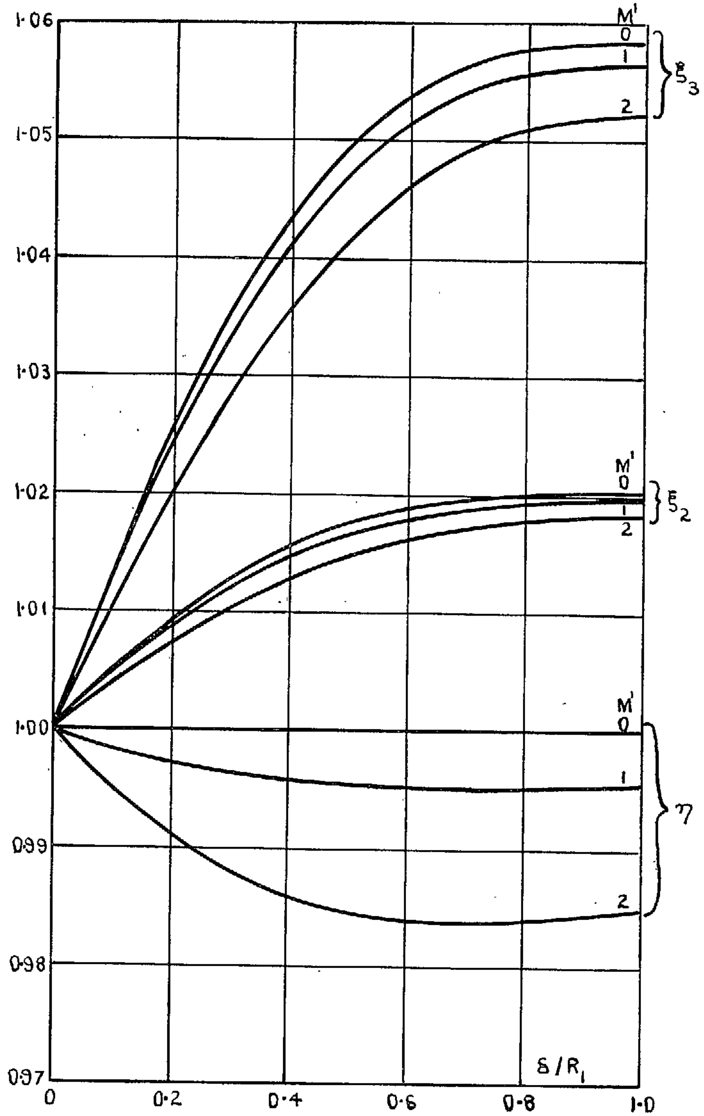

FIG. 2. ξ_2, ξ_3, and η vs. δ/R_1 for various values of M'.

ξ_2, ξ_3 and η are correction factors defined by equations (A.4) to (A.6) (*see* also after equation (12)). Values plotted here are for a duct of radius R_1 having a turbulent boundary layer of thickness δ with a 1/7th power law velocity profile, and a uniform free stream of Mach number M'.

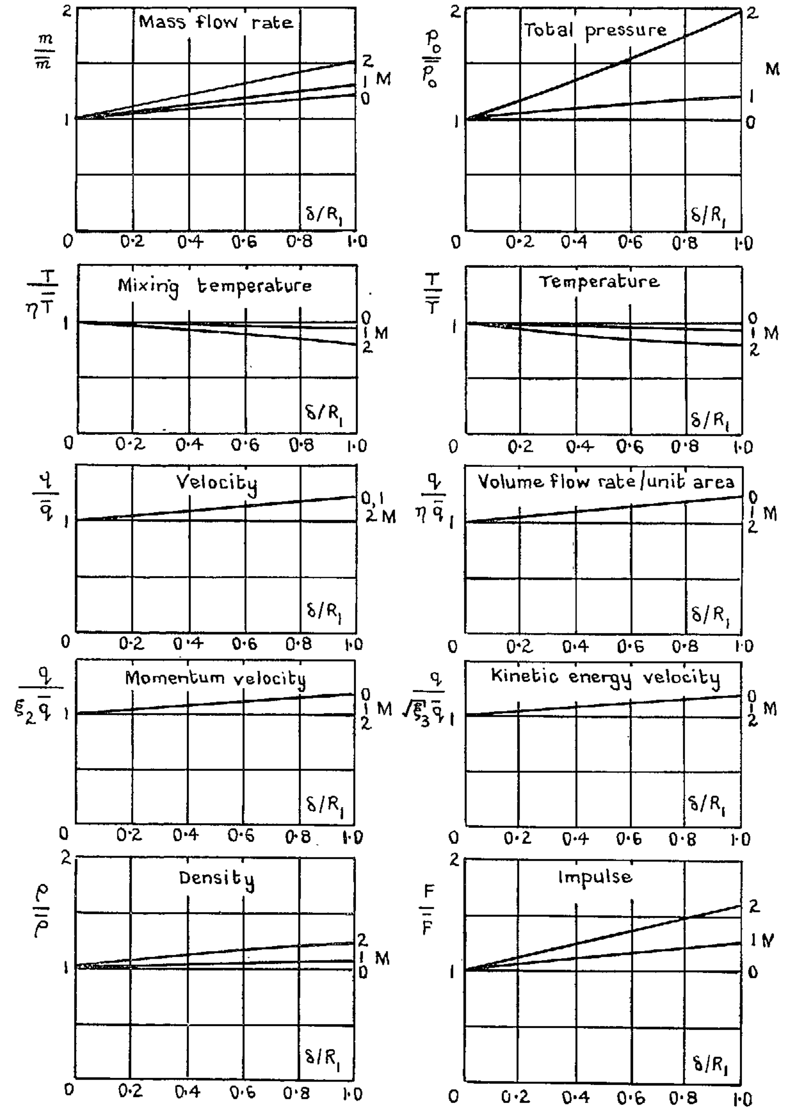

FIG. 3. Results of Example 1.

Curves give ratio of one-dimensional values to mean values at exit of a duct of given exit radius R_1, working between known upstream stagnation conditions and exit pressure, for various exit boundary-layer thicknesses δ, and Mach numbers M (*see* example 1). Homentropic free stream; turbulent boundary layer with 1/7th power law velocity distribution.

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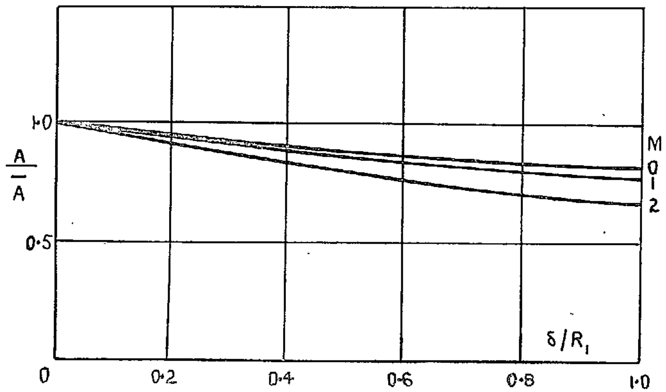

FIG. 4. Result of example 2.

Curves give ratio of exit area calculated one-dimensionally to true exit area, for given mass flow rate through a duct of exit radius R_1, working between known upstream stagnation conditions and exit pressure, for various exit boundary-layer thicknesses δ, and Mach number M (*see* example 1). Homentropic free stream ; turbulent boundary layer with 1/7th power law velocity distribution.

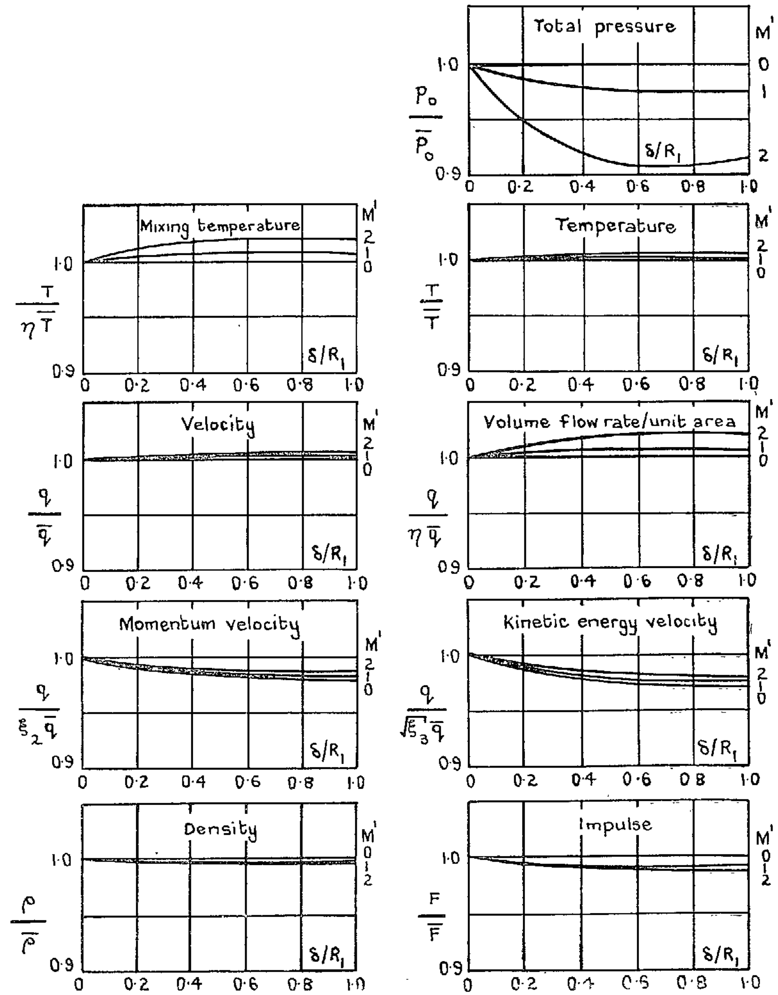

FIG. 5. Result of example 3.

Curves give ratio of one-dimensional values to mean values of flow parameters, deduced from given values of mass flow, area, total temperature and static pressure at a section. Duct of radius R_1 ; turbulent boundary of thickness δ with 1/7th power law velocity profile ; uniform free stream of Mach number M'.

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