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*By*

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*Summary.*—An analysis of aircraft response and loading conditions in symmetrical manoeuvres is presented with a particular recognition of the designer's needs.

The analysis is based on the theory of aircraft response to elevator induced longitudinal manoeuvres. Basic response functions have been derived for the chosen, exponential type of elevator motion, and from these, general expressions have been obtained for various derived response quantities, such as tailplane loads, elevator hinge moments, normal accelerations at the tail, etc.

A computational method which reduces the calculations to a routine is given in Appendix B. The method allows the evaluation of the complete time histories of response quantities or, alternatively their significant maxima.

The simplifying assumptions underlying the analysis are critically reviewed and possible limitations of the method are discussed.

1. *Introduction.*—For some time now it has been felt that new trends in aircraft design, manifesting themselves by new aerodynamic shapes, the attainment of still higher Mach numbers and higher performances in general have altered the characteristics of aircraft behaviour in flight, and in particular the aircraft response to various disturbances. This development is associated with changes in the numerical values of the different parameters affecting the aircraft behaviour. Some of these parameters whose effect could in the past be either disregarded or assumed to be the same for all aircraft types have now become of primary importance, whereas others have lost much of their original significance. Parameters which have become more important are (*a*) low air densities associated with high altitude (*b*) high static margins of swept-winged aircraft at high subsonic Mach numbers and (*c*) low aerodynamic damping and high wing loading, whereas the coefficient $C_{m0}$ (less tail) is now often less significant because of its small magnitude.

In this connection it is worth mentioning the significance of the non-dimensional coefficient $\omega$ which can be considered as the generalized static stability coefficient or as the main aerodynamic 'spring constant' (*cf.* sections A.1 and B.3). For the aircraft of five or ten years ago its numerical value fluctuated between 0 and 10; today values of 50 are not uncommon, 100 quite feasible, and 200 and more on their way. With a low $\omega$ value the aircraft, regarded as a dynamic system, is either overdamped or only slightly underdamped, resulting in an aperiodic or nearly aperiodic response to, say, an instantaneous or impulsive disturbance. A high $\omega$ value renders the aircraft considerably underdamped with all the dynamic consequences.

All these new trends have a direct bearing on the loading conditions of aircraft in longitudinal pilot-induced manoeuvres, which form the subject of this report. In view of the advantages of the present rational approach it was decided to investigate what could be done to simplify the calculations whilst preserving strictly the realistic character of the fundamental conditions.

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In the present analysis which relates to elevator-induced manoeuvres, several parameters which, as mentioned above, have been hitherto neglected, have been taken into account; others, *e.g.*, those connected with unsteady flow phenomena and with elastic structural deformations are still neglected.

A workable method is derived for obtaining time histories of the various response quantities and for the calculation of the maxima of those quantities reached in the pull-out manoeuvre considered, the severity of the manoeuvre being covered by means of suitable constants. It is hoped that the method will be of assistance to the aircraft designer.

This paper embraces various aspects of the same problem, and the following short review of the contents is made for the benefit of those readers who might be interested mainly in some particular aspect of the subject.

Sections 1 to 4 and section 13 contain *remarks of general interest.*

Sections 5 to 9 are devoted to the presentation and discussion of the *physical aspects* of a pull-out manoeuvre and of the various response quantities.

The *mathematical analysis* is presented in Appendix A.

Section 12 contains approximate formulae for a quick, though only *rough, estimation* of the maxima of the tailplane load.

In Appendix B the full *computational method* is presented; it should be used in conjunction with the explanatory remarks of section 10.

Remarks on the choice of the mean *rate of elevator movement* are given in section 11. These may be of interest to readers who want to use either the full method of Appendix B or the approximate formulae of section 12.

2. *Earlier Work.*—Historically the first rational method for calculating dynamic tailplane loads was presented in this country in 1921 by J. Case and S. B. Gates[1]. The method was based on rigorous solutions of the linearized equations of motion of the aircraft, and it was checked against solutions obtained by step-by-step integration of the equations of motion; the method could be applied to any particular case. One of the simplifying assumptions was to take the static stability of the aircraft to be zero, as it was then found that this parameter affected the tailplane loads only slightly. In 1928 Bolas and Allward suggested another method[2] based on the step-by-step solution of the equations of motion allowing the static stability of the aircraft to be taken into account.

In 1941 S. B. Gates[6] proposed some very simple formulae for the tail load estimation, and indicated the necessity for further refinement. As far as the writer is aware that was the first suggestion in the past for a generalized treatment of this problem for the designer's use.

Since 1938 several other methods [3, 7, 8, 9, 10, 11, 12, 18, 25] based on the aircraft response to a specified elevator motion have been published. An ' inverse method ', that is one where the time history of the normal acceleration is postulated, has been suggested in this country by Howard and Owen[4]. A similar, though in some respects more elaborate method, appeared in America[15] in 1951, the form of the time history of the normal acceleration being identical with that assumed by Howard and Owen. Other authors [5, 16, 17] have also been attracted by the idea of the ' inverse method '.

It is not claimed that this short review embraces all the existing methods for calculating tail loads in longitudinal manoeuvres. It is also outside the scope of this paper to compare the merits and limitations of the various methods. The mere number of these investigations indicates the growing interest and the importance of the problem, and perhaps also the difficulties encountered in deriving a satisfactory method.

The present method in a less elaborate form has been used for particular problems[14], and in its present form is now incorporated in Vol. 2 of Air Publication 970[19].

3. *Method of Approach.*—The basic problem is solved analytically (*cf.* Appendix A); the response of an aircraft to the specified type of elevator motion is obtained in general terms, and then expressions are derived for the various response quantities such as the tailplane load, the elevator hinge moment, the angular velocity and acceleration in pitch, together with general conditions for maxima of these quantities. Thus all these quantities are given in terms of the dimensions and inertias of the aircraft, its aerodynamic derivatives, the speed and height of flight, the maximum normal acceleration reached in the manoeuvre and the mean rate of elevator movement. The analysis covers only incremental values of the various quantities as affected by the manoeuvre itself; any response quantities shown in the main text or appendices represent incremental values only.

The notation adopted is, as far as possible, that commonly used in the analysis of the static and dynamic stability of aircraft; it is based on the 'Nomenclature' of Bryant and Gates[20] with some additional notation introduced by Dr. Neumark[21]. The notation used is given and explained in the List of Symbols and in section B.3 of Appendix B.

The presentation of the analytical work in terms of non-dimensional quantities allows the comparison between the results obtained for different aircraft types to be made in the most consistent way. The non-dimensional forms are in general also best suited for the derivation of workable computational schemes. A computational method for the calculation of normal accelerations, tailplane loads and other response quantities of interest to the designer is given in Appendix B. It permits the required computations to be performed merely as a routine.

The available test results, particularly those reported by Matheny[13], have been used to assess the reliability of the analytical treatment and of the simplifying assumptions. These assumptions are:

(*a*) The forward speed of the aircraft is taken to be constant throughout the manoeuvre

(*b*) The component of the aircraft weight along the normal to the flight path is taken to be constant

(*c*) The lift contribution due to the elevator deflection is neglected*

(*d*) The tailplane pitching moment about its own quarter-chord point due to the elevator deflection is neglected*

(*e*) The effects of the structural dynamic response of the aircraft is disregarded

(*f*) The disturbed motion of the aircraft conforms to the quasi-steady aerodynamic treatment

(*g*) The aerodynamic derivatives are constant.

Any possible limitations to the present method resulting from those assumptions are discussed in the next section.

Other flight test results reported by Matheny[23] and by H. H. Brown[24] have been used for the estimation of the maximum rates of elevator movement likely to be used by pilots in different types of aircraft. The derivation of a general expression for this rate is discussed in section 11. This is the quantity which—together with the maximum normal acceleration to be reached in the manoeuvre—has to be specified to make the general results of the present analysis applicable to practical cases.

4. *Simplifying Assumptions.*—The analytical treatment of the problem expounded in Appendix A is based on certain simplifying assumptions. Without these assumptions any generalization of the problem would be impossible. However, they may impose certain limitations on the final results obtained, and these will now be discussed.

The assumption of *small disturbances* makes it possible to linearize the equations of motion of the aircraft so that they can be integrated. Now the disturbances considered (elevator deflections

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* Appendices C and D indicate how these effects can be taken into account.

responsible for the manoeuvre) are far from small, and this might be expected to be responsible for considerable errors. However, the comparison of results obtained from linearized equations of motion with those obtained from step-by-step integrations, and also with flight test results[13] indicates that the errors involved are small and can be disregarded especially for manoeuvres of short duration which are the primary concern of this analysis.

The assumption of a *constant forward speed* of the aircraft throughout the manoeuvre normally introduces only small errors as can be seen from Matheny's work[13]. In exceptional cases, particularly at high subsonic speeds, they may become appreciable since a small decrease in forward speed and a corresponding decrease in Mach number during a manoeuvre may significantly affect the numerical values of the various aerodynamic derivatives as the manoeuvre develops. Particularly important here are changes in the $\partial C_m/\partial \alpha$ derivative. Thus the condition of *constant derivatives* postulated by the linearized equations of motion is not fulfilled in these rare cases, and the exact response of the aircraft can be obtained only through a step-by-step integration which is a rather laborious procedure. For design purposes it is suggested that two manoeuvres treated in accordance with the present method, one at the highest value of $\partial C_m/\partial \alpha$, and the other at its lowest value, would cover the true manoeuvre giving possibly somewhat conservative tailplane loads. Cases where the aeroplane becomes statically unstable are excluded from these considerations.

The linearized form of the equations of motion of the aircraft postulates also that the derivatives should be constant with varying angle of incidence and elevator angle. Very often they are constant or very nearly so, but if they are not it is usually sufficient to take their mean values over the estimated ranges of the appropriate angles.

The aircraft is assumed to be a *rigid body*. In many cases such an assumption should be quite satisfactory, but there may be instances where the elastic properties of the aircraft structure affect significantly the response in the manoeuvre, the important modes being wing and tailplane in torsion, and the rear fuselage in bending. These effects can be dealt with partly by estimating the aerodynamic derivatives as modified by structural elastic properties, and disregarding any dynamic effects. For this purpose the only additional quantities required are numerical values of the appropriate structural stiffnesses. The derivatives so modified should then be used. In most cases this procedure will be adequate. If, however, the natural frequencies of any of the elastic modes are comparatively low, and the initial rate of elevator movement high, the dynamic response of the structure may appreciably modify the basic aircraft response or any of the derived response quantities; *e.g.*, the maximum download at the tailplane occurring in the early stages of the manoeuvre might be considerably affected. The present approach does not cover such instances; they must be worked out on their own merits.

The contribution to the total *lift due to the elevator deflection* has been neglected in the first equation of motion, that of normal forces acting on the aeroplane. Errors due to this neglect are usually small on tailed aircraft. The effect becomes more pronounced for aeroplanes with short tail arms and very high static margins when errors may be as high as 10 per cent. In such cases the procedure of Appendix D may be applied.

The *unsteady flow phenomena* have been disregarded except for the delay in the downwash appearing at the tail. This has been taken into account in the usual way by the inclusion of the fourth term in the second equation of motion, equation (2), or the first term in equation (4). Further work is needed to investigate under which flight conditions these phenomena significantly affect the response of aircraft, and to assess the probable errors resulting from the application of the present method.

5. *Qualitative Remarks on the Manoeuvre.*—Before proceeding to a detailed discussion of the problem, some physical aspects of a longitudinal manoeuvre will first be considered. It is assumed that the manoeuvre itself, produced by a specified elevator movement, is not affected by the aircraft attitude during the steady flight immediately preceding it; this is true for a very rapid pull-out manoeuvre and particularly for its initial stages of a relatively short duration in which

all the response quantities reach their absolute maxima. Some numerical calculations have shown that even with a vertical dive as the initial flight condition the errors resulting from this assumption are not excessive. If the initial attitude corresponds to straight and level flight these errors are negligibly small.

A rapid pull-out manoeuvre is produced by an elevator deflected quickly upwards, held in this position for a short time, and then reversed. During such a manoeuvre the motion of the aircraft is continuously disturbed, and there are continuous changes in the angle of incidence $\alpha$, and in the angular pitching velocity $q$. These two quantities will be called basic response quantities. The derived response quantities, for instance the tailplane load or the elevator hinge moment, are linear combinations of the basic response quantities or their derivatives with respect to time.

When considering the dynamic response of an aircraft, the aircraft itself may be regarded as a dynamic system of two degrees of freedom, possessing its inertias, aerodynamic spring constants and aerodynamic damping coefficients. In modern aircraft these parameters are usually such that the system is less than critically damped, and the response is oscillatory in character showing appreciable overshoots in the various response quantities. Because of this the semi-empirical treatments of the past—invariably based on steady state considerations—should be abandoned.

The time histories of different response quantities in a rapid pull-out manoeuvre have been calculated for an example representative of modern aircraft, and are shown in Fig. 2 (a) to (h). At any time the numerical incremental values of the basic response quantities $\alpha$ and $q$ and their derivatives with respect to time are, in general, different from zero. There is a certain relationship between the angle of incidence $\alpha(=\hat{w})$, its derivative $d\alpha/dt$ and the angular velocity in pitch $q$ (proportional to $\hat{q}$); this relationship is given by equation (3). It follows that $d\alpha/dt$ is not equal to $q$ unless $\alpha$ is zero. These two quantities may even be of opposite signs. The incremental angle of incidence $\alpha$, the coefficient of normal acceleration $n$ and the velocity component $w$ of the centre of gravity of the aircraft are all proportional to one another, and thus their time histories are similar. The coefficient of normal acceleration increases right from the beginning of the manoeuvre and reaches its first maximum after a short time. If the initial rate of elevator movement is high enough, and the elevator is not reversed this maximum will be the absolute maximum: any subsequent maxima including the asymptotic value will be less. The difference between this first maximum and the asymptotic value will be termed ' the dynamic overshoot '.

Changes in any other response quantity with time are of a similar character: decaying oscillation —with the possible exceptions manifest during the very early stages of the manoeuvre. With the elevator deflected and held all those quantities tend to their respective asymptotic values which for all practical purposes are normally reached after a time of two or three seconds, and then the attitude of the aircraft corresponds to a steady circling in a vertical plane provided the simplifying assumptions still apply. The assumption of the component of the aircraft weight along the normal to the flight path being constant specially applies when the aircraft is flying along the near-horizontal portions of the circle.

The net aerodynamic load* at the tailplane is given by

$$P = \tfrac{1}{2}\rho V^2 S'(a_1 \alpha'_{\text{eff}} + a_2 \eta)$$

where $\alpha'_{\text{eff}}$ is the resultant effective angle of incidence at the tail. The two contributions to the net load, $P_w$ and $P_\eta$ corresponding to the two terms in this expression are numerically of opposite signs; in pull-out manoeuvres $P_w$ is positive (an upload).

In accordance with equation (13) of Appendix A the component $P_w$ itself is a linear combination of three quantities $\alpha$, $q$, and $d\alpha/dt$, as follows

$$P_w = \tfrac{1}{2}\rho V^2 S' a_1 \left[\left(1 - \frac{d\varepsilon}{d\alpha}\right)\alpha + \frac{l}{V} q + \frac{l}{V}\frac{d\varepsilon}{d\alpha}\frac{d\alpha}{dt}\right].$$

* The expression ' net load ' is used here to describe the total incremental aerodynamic load at the tailplane due to the manoeuvre alone.

Thus the net tail load has the following four components:

$\alpha$ component, responsible for the static stability of aircraft, may be termed *the restoring load.*

$q$ component is connected with the increase in the angle of incidence $\alpha'$ at the tail due to the angular velocity of the aircraft in pitch; it usually provides the main contribution to the damping of the short-period mode.

$d\alpha/dt$ component providing another contribution to the damping originates from the fact that any changes in the downwash at the wing appear at the tail only after a time $l/V$. Under steady flight conditions this component becomes zero.

$\eta$ component represents the disturbing force, and is solely responsible for the manoeuvre; as such it can rightly be called the *manoeuvring load*. In general this load is variable during a single manoeuvre.

These four components correspond to the second, third, fourth and fifth terms of the second equation of motion, equation (2) respectively.

As shown in Appendix A, section A.4.5 the net tail load may also be expressed thus:

$$P = A\left(B\hat{w} + C\frac{d\hat{w}}{d\tau} + a_2\eta\right)$$

where $\hat{w}$ is an alternative symbol for the incremental angle of incidence $\alpha$, and $\tau$ stands for the non-dimensional time, the unit of time being $\hat{t} = W/g\rho SV$. Thus $\tau = t/\hat{t}$.

The contribution to the net tail load due to the aircraft response alone is then

$$P_w = A(B\hat{w} + C\ d\hat{w}/d\tau),$$

and that due to the elevator movement is $P_\eta = Aa_2\eta$. Changes in the two contributions and also in their sum, the net load, can be studied in Fig. 2 (f), (g), and (h) (where all the loads are given per unit maximum coefficient of normal acceleration $n_m$).

Changes in $P_w$ are rather similar to those in $\alpha$ or $n$ but, owing to the $C\ d\hat{w}/d\tau$ term, $P_w$ reaches its first maximum sooner than $n$. $P_\eta$ is proportional to the elevator deflection throughout the manoeuvre.

Referring again to Fig. 2 (h) it can be seen that the first minimum of the net tail load $P_1$ (maximum download) occurs in the early stages of the manoeuvre when the elevator is not yet fully deflected, and there has already been some response. The first maximum upload $P_2$ occurs at a time when the two contributions to the net load, $P_w$ and $P_\eta$, have both practically reached their maxima. This maximum net load usually corresponds to the critical condition for the torque at the tailplane combined with a relatively high upload.

Should the elevator be reversed at about the time of $P_2$ a further increase in the upload, usually accompanied with a decrease in the tailplane torque, would result.

In a hypothetical manoeuvre where the elevator is deflected instantaneously—the case represented by dotted lines in Fig. 2—the maximum download $P_0$ occurs right at the beginning of the manoeuvre, its numerical value being greater than $P_1$, the maximum download in the case of a gradual elevator application. However, the first maximum upload $P_2$ differs little in the two cases.

In the empirical formulae of the past the tail load has been taken as the sum of two quantities: the 'balancing load' and the 'manoeuvring load', and the underlying physical picture of the manoeuvre has been extremely simplified. It seems as well at this point to review briefly that simplified picture and the terms usually associated with it in order to compare them with those offered by the present approach.

Within the empirical approach the aircraft is considered to be flying level when the angle of incidence is suddenly increased to produce the maximum specified acceleration at its centre of gravity; the angular acceleration is zero at the instant this normal acceleration is experienced, the balancing tail load being that calculated accordingly. Further it is assumed that any such manoeuvre is produced by an instantaneous application of a manoeuvring load at the tail calculated from a prescribed formula. The net tailplane load at the beginning of the manoeuvre is then the sum of the balancing load and that manoeuvring load, and it has been customary to take the chordwise centre of pressure position of the net load anywhere between the leading edge and the half-chord point of the mean tailplane chord without any reference to the angle of incidence or the elevator angle necessary to produce the corresponding chordwise load distribution.

In spite of the fact that such an empirical approach proved to be extremely useful in the past, the exact interpretation of the manoeuvre conceived in this way was difficult if not impossible. The sudden change in the angle of incidence accompanied by the sudden increase in the normal acceleration from $1g$ to, say, $ng$ would imply an eventual flight in a gravitational field of intensity $ng$. The aircraft does not rotate about its lateral axis even if ' manoeuvring ' load is defined through the specified angular acceleration about this axis. Thus the tailplane loading does not contain any damping contributions. Except for the sign of the manoeuvring load there is no true relationship between the numerical value of that load, as obtained from the various empirical formulae, and the maximum normal acceleration expected to be reached in a real manoeuvre.

6. *Elevator Movement.*—For the mathematical treatment of the problem an exponential function of time, equation (5) has been chosen to represent in a general form the time history of the elevator movement in pull-out manoeuvres. When choosing this function the following considerations have been taken into account.

There is practically an unlimited number of possible types of time histories of the elevator movement in longitudinal manoeuvres. However, the available records of such time histories indicate that for the most rapid manoeuvres the elevator movements may be reduced to a few, possibly only two types, namely:

(*a*) the elevator deflected quickly by some angle, held in this position for a short period, and then reversed

(*b*) the elevator deflected quickly and immediately reversed possibly well beyond its initial steady-state position.

An inspection of Fig. 1 shows that both these types of elevator motion can be represented by the exponential function chosen. General solutions for any such motion are given in Appendix A so that complete time histories of any of the desired response quantities may be obtained.

Of the two above elevator movements the first (*a*) is considered as possibly more appropriate for design purposes where the basic specified quantity is the maximum normal acceleration (say $n_1 g$) to be reached in the manoeuvre and never exceeded. It is suggested that a pilot is unlikely to apply the second type of movement (*b*) in manoeuvres involving the maximum normal acceleration specified, and if he attempted to do so he would often exceed that acceleration by a considerable amount. This view seems to be well substantiated by some flight test results reported by H. H. Brown[24] where for most of the type (*b*) manoeuvres the prescribed maximum incremental value of normal acceleration was exceeded by anything up to 50 per cent, and on one occasion by as much as 75 per cent.

The exponential function chosen, equation (5) is also rather ' flexible ' in that by varying the three parameters $k$, $f$, and $\tau_3$ a large variety of elevator motions can be obtained. This property may also prove useful when checking the results calculated by the present method against those from suitable flight tests; it is expected that for rapid manoeuvres it will usually be possible to approximate the recorded elevator motion by this exponential function.

At variance with the fairly popular assumption of elevator movement time histories represented by a few straight line segments, the function chosen is a continuous one, and thus the mathematical search for maxima of the various response quantities is more straightforward. In addition to the elevator motion given by equation (5) an alternative motion has been considered and suggested in connection with the method given in Vol. 2 of A.P. 970[19]. The proposal consists in using the same expression, equation (6) for both the initial and the reversed elevator movement (*cf.* section 9).

7. *Elevator Moved Instantaneously and then Held.*—The hypothetical manoeuvre produced by an instantaneous elevator deflection, being as it were a limiting manoeuvre, may be considered as a sort of useful datum for the discussion of other longitudinal manoeuvres. Formulae covering this case are given in Appendix A, sections A.3(*a*) and A.5, and Appendix B, section B.8. A numerical example is shown by dotted lines in Fig. 2 (a) to (h) where all the response quantities are plotted against the generalized time (*see* section A.1); with the exception of coefficients of normal acceleration they are given in terms of maximum incremental $g$ reached in the manoeuvre.

From Fig. 2 (b) it is seen that the incremental normal acceleration increases right from the beginning of the manoeuvre, reaches its first maximum $n_m g$, and then through decaying oscillations tends to an asymptotic value $n_a g$. For a given elevator deflection $\eta_0$ the coefficient $n_a$ may be expressed thus

$$n_a = -\,\delta\eta_0 D\frac{K_a}{J^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (cf.\ \text{equation } (23))$$

or in terms of dimensional quantities

$$n_a = -\,\frac{\frac{1}{2}\rho V^2 S' l a_2 \eta_0}{W c H_m}$$

which is of course what one would obtain from static considerations of the equilibrium of pitching moments in a steady circling case. Strictly speaking the asymptotic value $n_a$ is reached after an infinite time but for all practical purposes it can be assumed that it is reached within a few seconds from the beginning of the manoeuvre. It should be noted that the same value of $n_a$ is obtained also in other manoeuvres where the initial elevator movement is not instantaneous but gradual, and that in general it does not depend on the time history of that movement, but on the final elevator angle alone.

From the last equation it can be seen that the elevator angle $\eta_0$ required to reach an incremental value of the asymptotic normal acceleration $n_a g$ increases directly with the manoeuvre margin $H_m$. The angle $\eta_0$ increases also with the aircraft weight $W$ and decreases with increasing air density $\rho$ at a constant true air speed $V$. However, when considering these changes it should be borne in mind that the manoeuvre margin itself is a function of $W$ and $\rho$; with the simplifying assumptions made in the present treatment $H_m$ may be written thus:

$$H_m = -\,\frac{1}{a}\frac{\partial C_m}{\partial \alpha} - \frac{g\rho S l^2}{W c}\, m_q\,.$$

The incremental value of the angle of incidence $\alpha_a$ is given as

$$\alpha_a = \frac{1}{D}\, n_a \ (see\ \text{equation } (15))$$

and the asymptotic value of the angular velocity in pitch $q_a$ in radians per second

$$q_a = \frac{a}{2\hat{t}D}\, n_a$$

or in terms of dimensional quantities

$$q_a = \frac{g}{V} n_a .$$

The first maximum normal acceleration $n_m g$ occurs at a time corresponding to $J\tau = \pi$ (*cf.* Fig. 2 (b)). With the usual values of significant parameters this time may be of the order of one or two seconds, and more, but in extreme cases it may be well below $\frac{1}{2}$ second.

The peak value $n_m g$ shows a dynamic overshoot. Within the short-period mode considered, the aircraft represents a dynamic system with one degree of freedom and thus obeys the same laws as a single mass-spring-damping mechanical system. In the non-dimensional notation used the basic parameters are:

$\sqrt{(R^2 + J^2)}$ natural undamped frequency

$J$ natural damped frequency

$R$ damping coefficient.

For an instantaneous disturbance (step function) the dynamic overshoot depends solely on the ratio $R/J$. At $R/J = 0$ (zero damping) the overshoot is 100 per cent of the steady asymptotic value. With $R$ and $J$ values of modern aircraft it may reach 50 per cent or even more. The conditions may be studied from Fig. 11 where $K_a$ and $K_\pi$ plotted against $R/J$ represent response factors for the asymptotic values of normal acceleration (or angle of incidence $\alpha$) and its first maximum respectively.

When the two parameters $R$ and $J$ are written in terms of dimensional quantities thus

$$J = \sqrt{\left\{\frac{Wc}{2g\rho S k_B^2} aH_m - R^2\right\}}$$

$$R = \tfrac{1}{2}\left[-\frac{l^2}{k_B^2}(m_q)_{\text{less tail}} + \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{1}{2}\frac{S'l^2}{Sk_B^2} a_1 + \tfrac{1}{2}a\right]$$

then in conjunction with Fig. 11 the effect of changes in any of the basic quantities can be discussed, especially if numerical values of the remaining quantities are known.

In general it can be expected that with an increase in the manoeuvre margin $H_m$ the overshoot is increased, but it is decreased if the tail arm $l$ is increased.

The time history of the net tail load is shown in Fig. 2 (h) by the dotted line. It starts with a value $P_0$ which is entirely due to the instantaneous deflection of the elevator through an angle $\eta_0$. Obviously

$$P_0 = Aa_2\eta_0 = \tfrac{1}{2}\rho V^2 S' a_2 \eta_0 .$$

In a pull-out manoeuvre $\eta_0$ represents an upward movement of the elevator and thus with the accepted sign convention it is negative, and so is also $P_0$ representing a download. As the manoeuvre develops the negative value of the tail load decreases, then changes its sign and reaches its maximum positive value $P_2$ slightly before the maximum acceleration $n_m g$ is reached. All the changes in the net load are due to the response of the aircraft. In accordance with equation (18) the general expression for the tail load can be written thus:

$$P = A\left(B\hat{w} + C\frac{d\hat{w}}{d\tau} + a_2\eta\right).$$

The last term in the bracket corresponds to the elevator contribution $P_\eta$ to the net load, which is $P_0$ in this case and remains constant throughout the manoeuvre. The other two terms taken together represent the contribution due to the aircraft response $P_w$ which numerically is of opposite sign to the $\eta$ contribution. It should be remembered that $\hat{w}$ in the above expression is equal to the angle of incidence $\alpha$ and is proportional to the coefficient of normal acceleration $n$. Usually the coefficient $C$ is much smaller than $B$, and therefore the changes in the $P_w$ contribution to net tail load are rather similar to the changes in the normal acceleration (*see* Fig. 2 (b) and (g)).

The peak value $P_2$ is usually an upload and is combined with a high torque on the tailplane. A pure torque is experienced earlier in the manoeuvre when the net load becomes zero, *i.e.*, when the two contributions $P_w$ and $P_\eta$ are numerically equal and opposite. The numerical value of this torque is obtained by multiplying $P_0$ by the distance between the two centre of pressure positions: one, the chordwise c.p. of the load due to the angle of incidence alone, and the other due to the elevator deflection when $\alpha = 0$. Since this statement applies to incremental loads only it needs to be suitably modified when the initial loading conditions preceding the manoeuvre are taken into account.

In cases when the static stability of the aircraft less tail is positive, *i.e.*, when $(\partial C_m/\partial\alpha)_{\text{less tail}}$ is negative, the maximum upload $P_2$ becomes effectively a down load being then the absolute minimum down load in the manoeuvre.

The asymptotic value of the net load at the tail may be expressed thus :

$$P_a = -\,\delta\eta_0 \frac{A}{J^2} BK_a + Aa_2\eta_0$$

where the two contributions are clearly separated. In terms of dimensional quantities $P_a$ becomes

$$P_a = \tfrac{1}{2}\rho V^2 S'\left[-\frac{S'l}{Sc}\frac{1}{aH_m}\left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right)a_1 + 1\right]a_2\eta_0$$

where

$$\mu = \frac{W}{g\rho Sl}.$$

The effective angle of incidence at the tail is

$$\alpha'_{\text{eff}} = -\frac{S'l}{Sc}\frac{1}{aH_m}\left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right)a_2\eta_0 .$$

The rotational character of the motion under steady-state conditions is recognizable in these expressions by the appearance of the manoeuvre margin $H_m$ and of the term $a/2\mu$ in the round bracket.

The last expression for $P_a$ permits a discussion on the effects of changing some of the basic parameters. In a pull-out manoeuvre $P_a$ decreases with increasing manoeuvre margin $H_m$ the decrease being much more rapid than in the case of the normal acceleration $n_a$. The difference between the absolute maximum $P_2$ and the asymptotic value $P_a$ may be regarded as the dynamic overshoot, although in relation to the net loads, this concept is of little practical use. In addition to normal accelerations and tailplane loads, Fig. 2 (d) and (e) show also the time histories of the angular velocity in pitch $q$ and of the normal acceleration at the tail $n_t$. Owing to the assumption of an instantaneous elevator application, $n_t$ has a finite value right at the beginning of the manoeuvre.

8. *Elevator Moved Gradually and then Held.*—The type of elevator movement chosen is given by the equation $\eta = \eta_0(1 - e^{-k\tau})$ which for all practical purposes may be considered as representing an elevator gradually deflected at first, and then held at $\eta_0$. If the mean rate of

its movement (*see* section A.2) is relatively high then the behaviour of the various response quantities is, on the whole, similar to that described in the preceding section. In the early stages of the manoeuvre there may be major changes in those response quantities which depend directly on the time history of the elevator movement. The asymptotic values of all the quantities, however, remain unchanged. Full lines in Fig. 2 apply to the manoeuvre as obtained with gradual elevator deflection.

The changes in normal acceleration are similar to those observed with an instantaneous elevator application; the corresponding full curve is now slightly displaced from the dotted curve; the first maximum $n_m$ attains a somewhat lower value, and occurs later than $J\tau = \pi$. After that, through decaying oscillations, the coefficient $n$ approaches its asymptotic value. The deviations from the instantaneous case become the more pronounced the lower the mean rate of elevator movement; the primary purpose of the present analysis is to find the critical loading conditions for an aircraft in longitudinal manoeuvres; these are obtained with the highest rates of elevator movement which a pilot is able or likely to apply.

For the general treatment this rate, expressed in degrees per second, does not represent the parameter which really matters, that means one whose significance could readily be appreciated with reference to the response properties of various aircraft. The non-dimensional generalized rate $k$ as defined by equation (7) is here the more suitable parameter. By itself or in combination with the two main factors $R$ and $J$ it affects in the most direct way the behaviour of the aircraft, and particularly the numerical values of the maxima of response quantities. On one occasion it has been found that rates of about 10 deg/sec and 100 deg/sec meant for two different types of aircraft the same ' quickness ' of the elevator movement in that the dynamic character of the ensuing manoeuvre was in both cases identical.

With a decrease in the mean elevator rate (or $k$ value) the coefficient of the first maximum acceleration $n_m$ decreases, and the time of its occurrence is delayed. At a certain relatively low rate the peak value $n_m$ becomes equal to the asymptotic value $n_a$. At such low rates $n_m$ has also ceased to be the absolute maximum; instead, one of the subsequent maxima shows the greatest overshoot, but this overshoot is now negligibly small.

When the elevator rate is further decreased down to a value corresponding to $k = R$ a kind of limiting case is obtained; no maxima of $n$ can now be observed; the time $J\tau_m$ of the would-be first maximum is delayed to $2\pi$ and the corresponding $[J\tau,\ n]$ curve shows there an inflection point with a horizontal tangent; further similar inflection points occur at $J\tau = 4\pi,\ 6\pi$, etc. This type of response is shown in Fig. 12. The normal acceleration increases monotonically throughout the manoeuvre and approaches its asymptotic value $n_a$; the response becomes quasi-aperiodic in its character. The very low elevator rate considered ($k = R$) has no practical significance from the standpoint of rapid pull-out manoeuvres but it is of interest as a limiting case.

As compared with the instantaneous case of section 7 the time history of the net tailplane load is very different during the early stages of the manoeuvre. The load starts from zero and soon reaches a maximum download $P_1$ (strictly speaking a minimum, *see* Fig. 2 (h)). This maximum occurs when the elevator has not yet been fully deflected, and the relieving contribution of the aircraft response (or of the effective angle of incidence at the tail) is about to override the increase in the $\eta$-contribution. The numerical value of $P_1$ is always less than the instantaneous value $P_0$ of the preceding section. Thus the ratio $P_1/P_0$ is less than one, and for a given aircraft and specified flight conditions it depends solely on the mean elevator rate $d\eta/dt$ or on the value of the coefficient $k$. The numerical example of Fig. 2 for the gradual elevator application has been calculated for a rate given by the condition $k = 4J + R$ (*cf.* section 11); the corresponding $d\eta/dt$ is of the order of 90 deg/sec. In order to ascertain the effects of changes in the mean elevator rate values of $P_1$ have been calculated together with the times of occurrence $J\tau$ for a few elevator rates. The ringed points in Fig. 2 (h) show the positions of the corresponding peaks. It should be noted that in each of the calculated cases the value of the maximum elevator angle $\eta_0$ was adjusted so that the *first* maximum normal acceleration should be equal to the specified value of $6 \cdot 5g$. The effects of changing the elevator rate can best be studied from Fig. 3 where $P_1$, the time $J\tau$, and $\eta_0$ are plotted against the rate in degrees per second. The maximum elevator angle $\eta_0$ varies very little within the range of high elevator rates; at lower rates $\eta_0$ becomes greater, but below a certain rate it loses its practical significance since—as has been seen—either the asymptotic value of normal acceleration or one of the later maxima represents the absolute maximum. The time of occurrence of the maximum down load $J\tau_1$ decreases with increasing the rate, and becomes zero for the instantaneous elevator application. The net download $P_1$ increases when the rate is increased, the changes being more rapid at lower values of the rates; thus for the numerical example considered a change in the rates say, from 20 deg/sec to 40 deg/sec increases $P_1$ by 37 per cent, but a change from 120 deg/sec to 140 deg/sec would raise $P_1$ by only 4 per cent. Calculations made on several types of aircraft show that with practical mean elevator rates the ratio $P_1/P_0$ often assumes values of the order of $0 \cdot 6$ to $0 \cdot 7$ but anything from $0 \cdot 5$ to $0 \cdot 8$ may be expected. In exceptional cases $P_1/P_0$ values even outside this range may be obtained, say, if the aerodynamic damping of the aircraft (given by the coefficient $R$) is unusually low or if the elevator power control system unduly restricts the maximum elevator rate. Changes similar to those shown in Fig. 3 can be found for any aircraft and any condition of flight though the numerical implications of this figure apply only to the particular example considered. The maximum value of the net upload at the tail $P_2$ is little affected by the initial elevator rate; with decreasing rate it decreases slightly and is somewhat delayed, but it always occurs before the peak maximum acceleration $n_m g$. At high elevator rates $P_2$ is reached when, for all practical purposes, the elevator has already been deflected to its maximum angle $\eta_0$. The general remarks on $P_2$ made in the preceding section apply here as well. The asymptotic value of the net tail load $P_a$ is the same as in the case of instantaneous elevator movement and thus remarks on $P_a$ made in section 7 again apply in this case.

Consider now the changes in the angular velocity in pitch $q$ which occur in a pull-out manoeuvre. As soon as the elevator starts to deflect the aircraft begins to rotate about its lateral axis at a varying rate $q$ with respect to a system of axes fixed in space. In general $q$ differs from the rate $d\alpha/dt$ at which the angle of incidence changes with time; in fact $q$ is a function of both the rate $d\alpha/dt$ and the angle $\alpha$ itself. The time history of $q$ shown in Fig. 2 (e) is fairly typical; $q$ increases from zero to reach its first absolute maximum at about $J\tau = \pi/2$, then decreases considerably to a minimum which is still positive (nose-up), and finally through decaying oscillations approaches its asymptotic value. As can be seen from the graph the angular velocities obtained with instantaneous and high-rate gradual elevator movements differ relatively little from each other. It is of interest to note the rather large dynamic overshoot of the first maximum of this response quantity. Changes in the angular acceleration in pitch, $dq/dt$ can be estimated from Fig. 2 (c). The first maximum depends largely on the initial mean rate of elevator deflection, whereas the subsequent minimum is only slightly affected by it both being very nearly proportional to the maximum normal acceleration reached in the manoeuvre. This is probably the main reason for the considerable scatter of recorded values of the first maximum angular acceleration per $g$ found in abrupt pull-out manoeuvres by Matheny[23].

The normal acceleration at any point of the aircraft is a sum of the normal acceleration $ng$ at the centre of gravity and the acceleration due to the angular acceleration in pitch $dq/dt$. The latter component varies linearly with the distance from the centre of gravity. Fig. 2 (d) shows a typical time history of the normal acceleration at the tail, $n_t$. The first negative maximum occurs in the early stages of the manoeuvre; its numerical value for a fighter aircraft may be of the order of one or two $g$. The times of occurrence of this maximum and of the maximum down load $P_1$ do not coincide in general. However, when stressing the tail and rear fuselage for $P_1$ it may be found that in certain cases the simultaneous relieving inertia forces due to the negative normal acceleration are quite substantial.

The subsequent positive maximum of $n_t$ appearing in the vicinity of $J\tau = \pi$ is affected by $dq/dt$ to a small though not negligible extent. This maximum is always bigger than $n_m$. For the steady circling case the asymptotic value of the normal acceleration at the tail is of course, equal to that at the centre of gravity of the aircraft since $dq/dt$ is then zero.

9. *Elevator Reversed.*—A pull-out manoeuvre as discussed in the preceding section will be completed by the application of the elevator in the opposite direction in order to bring the aircraft back to a 1$g$ steady level flight condition. To cover this case, the full expression for the elevator deflection, equation (5) may now be used. Time histories of the various response quantities can be obtained by substituting the expressions of section A.3 ($c$) into the appropriate formulae of section A.4. This procedure can be adopted in all cases where the knowledge of aircraft response in such manoeuvres is required. In particular when comparing appropriate flight test results with the results calculated in accordance with the present analytical method the recorded elevator movement must be approximated by equation (5) as closely as possible.

In order to calculate for design purposes the response and maxima of the various response quantities in a full pull-out manoeuvre the elevator movement, equation (5) could be used. In this case an additional assumption must be made, as to how far the elevator would be reversed. However, an alternative approach will be made here. For the purposes of A.P. 970[19] it has already been suggested that the type of reversed elevator movement should be the same as that used for the initiation of the manoeuvre, *i.e.*, that given by equation (6) but with the reversed sign The additional assumptions are:

(i) Before the reversed elevator movement is applied the aircraft is in a steady circling attitude in a vertical plane, the steady incremental normal acceleration being now equal to the maximum permissible $n_m g$.

The elevator angle required to maintain this steady attitude is somewhat greater than the angle $\eta_0$ of the first stage of the manoeuvre as discussed in section 8

(ii) The mean rate of the reversed elevator movement is equal to the mean rate of the initial movement taken with the opposite sign

(iii) The maximum reversed elevator angle is such as to reduce the steady circling incremental value of $n_m g$ value down to zero the latter being the absolute minimum during this stage of the manoeuvre. It follows that this maximum reversed angle is equal to the elevator angle $\eta_0$ used for the initiation of the manoeuvre, taken with the opposite sign.

This type of reversed elevator motion has been taken as a basis for the computational scheme given in Appendix B, section B.5; it reduces considerably the computational work required.

Of the various maxima of response quantities occurring during the second stage of the manoeuvre there are usually only two which are of interest from the design standpoint. These are: a further maximum upload at the tailplane $P_3$, and a maximum normal acceleration at the tail both appearing at roughly the same time. The occurrence of the latter maximum is due solely to the changes in the $dq/dt$ contribution; its numerical value for fighter aircraft may be larger than $n_m g$ by one or two $g$.

10. *Computational Method.*—Appendix A shows the analytical treatment of the problem, and the derivation of formulae for the different response quantities. It contains all the information required to calculate time histories and maxima of these quantities, allowing a certain latitude in making assumptions as to the manoeuvre itself. Appendix B is based on the results of Appendix A, and gives a method for computing the various response quantities particularly those required for design purposes. No direct reference to the derivation of formulae is made, and the numerical computations may be performed even with little understanding of the physics of the problem. When interpreting the numerical results obtained it should be borne in mind that

(i) All the response quantities are related to the coefficient of maximum incremental normal acceleration $n_m$ reached in the manoeuvre with the exception of the coefficients of normal acceleration themselves. However, for any new value of $n_m$, the quantities $\eta_0$, $k$, and $K_m$ should be calculated afresh as indicated in section B.5.1.0

(ii) The maximum elevator deflection $\eta_0$ and the response quantities considered represent incremental values in the manoeuvre. Total values at any time during the manoeuvre are obtained by adding the incremental ones to those present in the steady level flight conditions preceding the manoeuvre.

The manoeuvre underlying the computational method of Appendix B may be considered as consisting of two stages, namely :

*First stage, the pull-out proper.*—The initial attitude of the aeroplane corresponds to steady level flight conditions at the selected true air speed $V$. The elevator is moved gradually in accordance with the general formula.

$$\eta = \eta_0(1 - e^{-k\tau}) = \eta_0\left(1 - e^{-\frac{k}{J}J\tau}\right)$$

where $\tau$ is the aerodynamic time (*cf.* section A.1), and $k$ a coefficient related to the mean rate of the elevator movement. The maximum elevator angle $\eta_0$ is chosen in such a way that the maximum incremental normal acceleration reached in the manoeuvre is $n_m g$, and the corresponding total acceleration $(n_m + 1)g$. (In practice this latter normally corresponds to the boundary of the appropriate flight envelope at the chosen speed $V$). After a short time asymptotic conditions will be reached with a normal acceleration $n_a g$ usually somewhat less than $n_m g$. In order to reach a steady incremental normal acceleration of $n_m g$, which is the initial condition for the second stage of the manoeuvre, the elevator deflection will usually have to be increased to $\eta_a$. This additional small deflection is assumed to be made slowly in comparison with the initial rather rapid movement.

*Second stage, the return to level flight.*—This stage of the manoeuvre is initiated when the aircraft is flying along the near-horizontal portion of a circle in the vertical plane at a total steady normal acceleration of $(n_m + 1)g$. The elevator is moved according to the same general formula as in the first stage, but in the reverse direction, the time being reckoned anew from zero. The mean rate of elevator movement is assumed to be the same as before, and thus the numerical value of the coefficient $k$ remains unchanged. The maximum incremental normal acceleration reached in this stage is $- n_m g$ reducing the total acceleration to $1g$. It follows that the maximum reversed elevator angle equals $\eta_0$. These two stages of a pull-out manoeuvre are conceived here as forming one single manoeuvre but they can be treated separately as it is immaterial for the second stage how its initial steady flight conditions have been reached. If instantaneous elevator movements are assumed for the two stages, the resulting manoeuvre, which may be taken as a limiting manoeuvre, is the most rapid of all the manoeuvres of the type considered, and maxima of the various response quantities, particularly those appearing in the early stages of the manoeuvre normally reach their highest values. Thus if this limiting manoeuvre is assumed for design purposes conservative loading conditions are usually obtained. The computational scheme, however, becomes much simpler; it is given in section B.8 of Appendix B. The method of Appendix B allows the calculation of the time histories of some or all of the response quantities, or else the calculation of the various maxima of those quantities only, without obtaining the full responses. The last procedure is the shorter ; it is advisable, however, to have the full response curves as this provides a better insight into the manoeuvre and may also serve as an additional check on the computed values.

11. *The Rate of Elevator Movement.*—In order to determine exactly a pull-out manoeuvre of the type considered numerical values must be known for all the parameters relating to the aircraft and its flight conditions, and two other basic quantities must also be specified, namely the maximum normal acceleration to be reached in the manoeuvre and the maximum value of the mean rate of the elevator movement $[d\eta/dt]$. The proper choice of these two quantities in particular cases is outside the scope of the present paper. However, suggestions have already been made[19] for the estimation of the mean elevator rates. They have been expressed in terms of certain quantities, and it seems suitable to show here how they have been arrived at.

If the elevator control circuit imposes no limits to the elevator rates the pilot can apply, then the proper choice of the maximum rate becomes a difficult task. The statistical analysis of suitable data may give hints on how to make such a choice in particular cases. Flight test data of some 250 pull-out manoeuvres made on twenty different aircraft types have been collected by Matheny[23] and out of these nearly 40 of the most rapid manoeuvres have been selected for analysis of the trends in the variation of the elevator rate.

Inspection of the data indicated that the size of the elevator circuit could have only a secondary effect on the rate. Several attempts were made to correlate the elevator rate with various aircraft data, aerodynamic data or their combinations but in each case considerable scatter was observed. It was, however, realized that there existed a certain relationship between the elevator rate, maximum normal acceleration and the time at which that maximum occurred, in other words that the rate is affected by the way the aircraft responds to imposed disturbances. It seemed plausible that pilots tend to apply higher elevator rates in aircraft responding quickly to disturbances, and *vice versa*.

In order to see whether any such trend existed, the available dimensional data were interpreted in terms of the generalized non-dimensional quantities of Appendix A. These considerations led to the belief that the expression $(k - R)/J$ might be a constant for the most rapid manoeuvres. This is borne out by test data, the scatter being quite moderate. It has been suggested for design purposes[19] that the value of $(k - R)/J$ should be 4, or the generalized elevator rate $k = 4J + R$. The whole proposal must be treated as tentative, and further tests are needed to get more information.

12. *Approximations*.—Probably the most important quantity from the designer's standpoint is the net load at the tailplane, and especially its maxima occurring during the manoeuvre. During the early design stages rough estimates of these maxima may be needed, and the following formulae can be used for this purpose. These formulae are empirical in character and are based on the results obtained through the application of the method given in Appendix B to several types of aircraft.

The formulae for the first *maximum incremental download* $P_1$ are

maximum net load $\quad P_1 = P_0\sqrt{\left\{\dfrac{1}{1 + 10R/k}\right\}}$

elevator contribution $\quad P_{\eta 1} = P_0\sqrt{\left\{\dfrac{1}{1 + 3 \cdot 3R/k}\right\}}$

and thus the relief due to the aircraft response is

$$P_{w1} = P_0\left[\sqrt{\left\{\frac{1}{1 + 10R/k}\right\}} - \sqrt{\left\{\frac{1}{1 + 3 \cdot 3R/k}\right\}}\right].$$

Now $P_0$ represents the download due to the instantaneous elevator deflection in the limiting manoeuvre of section B.8. It is given by the formula

$$P_0 = -\frac{F}{K_\pi}\frac{a_2}{\delta}J^2 n_m$$

or if expressed as far as possible in terms of the basic dimensional quantities

$$P_0 = -W\frac{c}{l}H_m\frac{1}{1 + e^{-\pi R/J}}n_m$$

where $H_m$ is the manoeuvre margin at constant speed $V$.

The ratios $P_1/P_0$ and $P_{\eta 1}/P_0$ corresponding to the above empirical formulae are plotted in Fig. 4 against $k/R$. The ringed points show the true values calculated for the four different elevator rates of the numerical example discussed in section 8 (*see* also Figs. 2 (h) and 3).

The application of the simplified formulae for the estimation of the ratios $P_1/P_0$ and $P_{\eta 1}/P_0$ may usually involve errors of the order of $\pm$ 10 per cent or more.

The first *maximum incremental upload* $P_2$ can be roughly estimated from the formula

$$P_2 = \frac{WS'}{Sa}\left(1 - \frac{d\varepsilon}{d\alpha} + \frac{g\rho Sl}{2W}\, a\right)a_1 n_m + P_0$$

where the first term represents the contribution due to the effective angle of incidence at the tail, and the second the contribution due to the elevator deflection.

Similarly the approximate formula for the *second maximum incremental upload* $P_3$ is

$$P_3 = \left[\frac{WS'}{Sa}\left(1 - \frac{d\varepsilon}{d\alpha} + \frac{g\rho Sl}{2W}\, a\right)a_1 - \frac{Wc}{l}\, H_m\right]n_m - P_1\,.$$

The elevator can be considered to be returned to its initial position.

13. *Concluding Remarks.*—The method presented for the estimation of aircraft behaviour in symmetric pull-out manoeuvres and for the calculation of various response quantities, including the tailplane loads, and of their maxima has been derived in terms of the significant geometric and aerodynamic parameters. Thus reliable results will be obtained if exact numerical values of the aerodynamic derivatives in question are used. Any errors in their estimation affect the accuracy of final results. In this respect the manoeuvre margin $H_m$ is the most important of all the derivatives. Within the simplifying assumptions made for the present method all the aerodynamic derivatives involved are the same as those required for the calculation of quantities in steady flight conditions and also for the estimation of the longitudinal dynamic stability characteristics. It is thus essential that these derivatives be obtained from the best available data.

The analytical treatment presented is based on several simplifying assumptions. The available experimental evidence proves them to be quite adequate. It is, however, not unlikely that under special circumstances the errors involved may become considerable. It is, therefore, desirable that further theoretical and experimental work should be undertaken to obtain the necessary insight into those additional problems, of which the dynamic structural response and the unsteady flow effects are perhaps the two most important.

The present work applies to tailed aircraft only. A similar approach could also be used for the investigation of loading conditions of tailless and delta aircraft in longitudinal manoeuvres. Although the disturbance and the significant response quantities are rather different, parts of the present analysis could be used directly.

## LIST OF SYMBOLS

| | | |
|---|---|---|
| $A$ | | Coefficient in equation (18); also coefficient in equation (40) |
| $A_1$ | | Coefficient in equation (63) |
| $a$ | = | $\partial C_L/\partial \alpha$ for the whole aircraft |
| $a_1$ | = | $\partial C_L'/\partial \alpha'$ |
| $a_2$ | = | $\partial C_L'/\partial \eta$ including the effects of tabs if used |
| $B$ | | Coefficient in equation (18) |
| $b_1$ | = | $\partial C_h/\partial \alpha'$ |
| $b_2$ | = | $\partial C_h/\partial \eta$ including the effects of tabs if used |
| $C$ | | Coefficient in equation (18) |
| $C_h$ | | Hinge-moment coefficient of elevator |
| $C_m$ | | Pitching-moment coefficient of aeroplane about its c.g. |
| $C_L$ | | Lift coefficient of aeroplane |
| $C_L'$ | | Lift coefficient of tailplane |
| $c$ | | Standard mean chord of wing |
| $c'$ | | Standard mean chord of tailplane |
| $D$ | | Coefficient in equation (15) |
| $E_f$ | | Special function (*see* equations (9)) |
| $E_k$ | = | $\bar{E}$, special function (*see* equations (9)) |
| $F$ | | Coefficient in equation (27) |
| $f$ | | Generalized mean rate or reversed elevator movement (non-dimensional), *cf.* section A.2 |
| $g$ | | Acceleration due to gravity |
| $H$ | | Special function, (*see* equations (9)) |
| $H_m$ | | Manoeuvre margin, stick fixed (*see* section 7) |
| $H_n$ | | Restoring margin, stick fixed ($= \partial C_m/\partial C_L$) |
| $I$ | | Part of the real stability root (*see* equation (8a)) |
| $i$ | = | $\sqrt{-1}$ |
| $J$ | | Non-dimensional frequency of pitching oscillation |
| $K$ | | Special function (*see* equations (9)) |
| $K_f$ | | Special function (*see* equations (9)) |
| $K_k$ | = | $\bar{K}$, special function (*see* equations (9)) |
| $K_m$ | | Value of function $K$ at the time of $n_m$ |
| $K_\pi$ | | Value of function $K$ at $J\tau = \pi$ |
| $k$ | | Generalized mean rate of elevator movement (non-dimensional) (*see* section A.2, equation (7)) |
| $k_B$ | | Radius of gyration of aeroplane about the lateral axis |



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| | |
|---|---|
| $L$ | Special function (*see* equations (9)) |
| $l$ | Distance from the c.g. of the aeroplane to the quarter-chord point of the tailplane |
| $M$ | Coefficient in the equation of section B.6.4 |
| $m_q$ | Damping derivative in pitch $\left[= \dfrac{c}{2l} \dfrac{\partial C_m}{\partial\left(\dfrac{ql}{V}\right)}\right]$ |
| $N$ | Coefficient in the equation of section B.6.4 |
| $n$ | Coefficient of incremental normal acceleration at c.g. of aeroplane |
| $\bar{n}$ | Coefficient of normal acceleration at the tailplane due to angular acceleration in pitch |
| $n_t$ | Coefficient of total normal acceleration at the tailplane |
| $n_m$ | Maximum value of $n$ in a manoeuvre, incremental value |
| $P$ | Incremental tailplane load |
| $P_0$ | $P$ due to elevator angle $\eta_0$ |
| $P_1$ | First maximum download, net incremental value |
| $P_2$ | First maximum upload, net incremental value |
| $Q$ | Coefficient in equation (46) |
| $q$ | Angular velocity of the aeroplane in pitch |
| $\hat{q}$ | Non-dimensional form of $q$ $(= \hat{t}q)$ |
| $R$ | Non-dimensional damping factor of pitching oscillation |
| $S$ | Wing area |
| $S'$ | Tailplane area |
| $T$ | Coefficient in equation (46) |
| $T_h$ | Coefficient in equation (55) |
| $t$ | Time in seconds |
| $\hat{t}$ | Unit of aerodynamic time (seconds), (*see* section B.3) |
| $U$ | Coefficient in equation (40) |
| $U_1$ | Coefficient in equation (63) |
| $V$ | True air speed of aeroplane |
| $W$ | Weight of aeroplane |
| $w$ | Incremental velocity component along $z$ axis |
| $\alpha$ | Wing incidence, incremental value |
| $\alpha'_{\mathrm{eff}}$ | Effective tail incidence, incremental value |
| $\Gamma$ | Coefficient (*see* equation (46)) |
| $\Delta$ | Coefficient (*see* equation (40)) |

## LIST OF SYMBOLS—*continued*

| | |
|---|---|
| $\Delta_1$ | Coefficient (*see* equation (63)) |
| $\delta$ | Elevator effectiveness |
| $\varepsilon$ | Downwash angle at the tail |
| $\eta$ | Elevator deflection responsible for the manoeuvre |
| $\eta_0$ | Maximum value of $\eta$ |
| $\lambda$ | Root of stability equation (*see* equations (8a) and (8b)) |
| $\mu$ | Relative density of aeroplane (*see* section B.3) |
| $\nu$ | Rotary damping coefficient (*see* section B.3) |
| $\rho$ | True air density |
| $\Sigma$ | Coefficient in equation (58) |
| $\tau$ | Non-dimensional time ($= t/\hat{t}$) |
| $\chi$ | Downwash damping coefficient (*see* section B.3) |
| $\omega$ | Static stability coefficient (*see* section B.3) |

*Suffixes*

| | |
|---|---|
| $_a$ | Associated with steady circling conditions |
| $_m$ | Associated with maximum incremental normal acceleration |
| $_w$ | Due to the effective tail incidence |
| $_\eta$ | Due to the elevator deflection |

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## TABLE 1

*Data for the Numerical Example*

As a numerical example a fighter type aircraft has been chosen flying at 600 ft/sec T.A.S., at a height of 30,000 ft. The aircraft performs a rapid pull-out manoeuvre in which it reaches a maximum incremental normal acceleration of $6 \cdot 5g$.

| | |
|---|---|
| $B = 1 \cdot 319$ | $C = 0 \cdot 0556$ |
| $D = 11 \cdot 68$ | $F = 732 \cdot 4$ |
| $\mu = 78$ | $\hat{t} = 2 \cdot 62$ sec. |
| $\omega = 43 \cdot 09$ | $\delta = 68 \cdot 66$ |
| $\nu = 2 \cdot 58$ | $\chi = 0 \cdot 7745$ |
| $J = 6 \cdot 41$ | $R/J = 0 \cdot 39$ |
| $R = 2 \cdot 5$ | |

$k = 4J + R = 28 \cdot 14$* corresponding to an elevator rate

$$\left[\frac{d\eta}{dt}\right] = -\ 91 \cdot 4 \text{ deg/sec.}$$

*Cf. section 11.

## APPENDIX A

### *Theoretical Analysis*

A.1. *Equations of Motion.*—The equations of motion of an aircraft in symmetric manoeuvres can be written thus:

$$\frac{W}{g}\left(\frac{dw}{dt} - qV\right) + \tfrac{1}{2}\rho V^2 S \frac{\partial C_L}{\partial \alpha}\alpha = 0 \quad \ldots \quad (1)$$

$$\frac{W}{g}k_B{}^2\frac{dq}{dt} - \tfrac{1}{2}\rho V^2 Sc\frac{\partial C_m}{\partial \alpha}\alpha - \tfrac{1}{2}\rho V^2 Sc\frac{\partial C_m}{\partial q}q + \tfrac{1}{2}\rho V^2 S' l\frac{\partial C_L{}'}{\partial \alpha'}\frac{l}{V}\frac{d\varepsilon}{d\alpha}\frac{d\alpha}{dt}$$

$$= -\tfrac{1}{2}\rho V^2 S' l\frac{\partial C_L{}'}{\partial \eta}\eta \quad \ldots \quad (2)$$

where $\alpha$ and $\eta$ stand for the incremental values of the angle of incidence and of the elevator deflection respectively. These equations correspond to a system of moving axes fixed in the aircraft. They have been established under the following assumptions:

(*a*) The forward speed of the aircraft and the overall Mach number are taken to be constant throughout the manoeuvre

(*b*) The component of the aircraft weight along the normal to the flight path is taken to be constant

(*c*) The tailplane lift contribution due to the elevator deflection has been neglected in equation (1)

(*d*) The tailplane pitching moment about its quarter-chord datum due to the elevator deflection has been neglected in equation (2)

(*e*) The dynamic response of the elastic aircraft structure is disregarded

(*f*) The disturbed motion of the aircraft conforms to the quasi-steady aerodynamic treatment

(*g*) The aerodymanic derivatives are constant within the ranges of the effective angles of incidence and of the elevator deflection.

The equations of motion in non-dimensional notation assume the following form

$$\frac{d\hat{w}}{d\tau} + \tfrac{1}{2}a\hat{w} - \hat{q} = 0 \quad \ldots \quad (3)$$

$$\chi\frac{d\hat{w}}{d\tau} + \omega\hat{w} + \frac{d\hat{q}}{d\tau} + \nu\hat{q} = -\delta\eta\,. \quad \ldots \quad (4)$$

The non-dimensional notation used is basically that established by Bryant and Gates[20] with modifications introduced by Neumark[21] and others. For the non-dimensional velocity components symbols $\hat{w}$ and $\hat{q}$ are used instead of $w/V$ and $\hat{t}q$. Also the letter $k$ is used for a parameter characterizing the elevator motion (*see* equation 5).

Owing to the non-dimensional notation adopted, the independent variable, time may be expressed either as

(i) the aerodynamic time $\tau$, the time unit being $\hat{t} = W/g\rho SV$ seconds

or

(ii) the generalized time $J\tau$ (an angular quantity). Its value $J\tau = 2\pi$ corresponds to the natural period of the damped short-period oscillation of the aircraft in pitch. For any value of $J\tau$ the corresponding dimensional time in seconds can be evaluated thus:

$$t = \frac{\hat{t}}{J}(J\tau) \text{ seconds}$$

where $J\tau$ should be taken in radians.

A.2. *The Disturbing Function.*—The disturbance is due to the pilot's action; the elevator motion has been assumed to be given by the following function of time

$$\eta = \eta_0[(1 - e^{-k\tau}) + (1 - e^{f\tau})e^{-f\tau_3}] . \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Under the condition that numerical values of $k, f$, and $\tau_3$ are large, it can be taken that of the two distinct portions of this expression the first, $\eta_0(1 - e^{-k\tau})$ represents the initial elevator movement, whereas the second, $\eta_0(1 - e^{f\tau})\, e^{-f\tau_3}$—the reversed movement. This can be readily checked by comparing the initial motions in Figs. 1 (a) and 1 (b), both obtained for the same value of $k$; for all practical purposes the reversed movement of Fig. 1 (b) does not affect the initial one.

At $\tau = 0$, $\eta = 0$.

At $\tau = \tau_3$ the elevator deflection becomes

$$\eta_3 = \eta_0(e^{-f\tau_3} - e^{-k\tau_3}) .$$

In most practical cases both $f\tau_3$ and $k\tau_3$ are numerically large and possibly do not differ much from each other. Thus $\eta_3$ is very nearly zero (*cf.* Fig. 1 (c)).

For the case of an elevator applied gradually and then held only the first portion of the right-hand side of equation (5) is to be retained, *i.e.*,

$$\eta = \eta_0(1 - e^{-k\tau}) ; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

then the rate at which the elevator is moved is given by

$$\frac{d\eta}{d\tau} = \eta_0 k\, e^{-k\tau} .$$

It is continuously variable with time, but a mean rate will be specified as half the $d\eta/d\tau$ value at $\tau = 0$ (*see* Fig. 1 (a)). Thus if $\eta_0$ is known, the coefficient $k$ can be expressed in terms of the mean elevator rate $[d\eta/d\tau]$

$$k = 2\frac{1}{\eta_0}\left[\frac{d\eta}{d\tau}\right]$$

or, if the mean rate is given in radians per second

$$k = 2\frac{\hat{t}}{\eta_0}\left[\frac{d\eta}{dt}\right] . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

A.3. *Complete Solutions.*—From the equations of motion, equations (3) and (4), the following characteristic stability equation is obtained

$$\lambda^2 + (\nu + \chi + \tfrac{1}{2}a)\lambda + \omega + \nu\tfrac{1}{2}a = 0 .$$

The roots of this equation may be either real or complex; they will be written thus:

real roots $\lambda_{1,2} = -R \pm I$ .. .. .. .. .. .. (8a)

complex roots $\lambda_{1,2} = -R \pm Ji$ .. .. .. .. .. .. (8b)

where $R = \tfrac{1}{2}(\nu + \chi + \tfrac{1}{2}a)$ , $I = \sqrt{\{R^2 - (\omega + \tfrac{1}{2}\nu a)\}}$

and $J = \sqrt{(\omega + \tfrac{1}{2}\nu a - R^2)}$ .

Complete solutions of the equations of motion can be expressed in terms of the following auxiliary functions of time

$$\left.\begin{aligned}
H &= e^{-\frac{R}{J}J\tau}\cos J\tau \\
L &= e^{-\frac{R}{J}J\tau}\sin J\tau \\
K &= \frac{1}{\left(\frac{R}{J}\right)^2+1}\left[1-e^{-\frac{R}{J}J\tau}\left(\cos J\tau+\frac{R}{J}\sin J\tau\right)\right] \\
K_k &= \frac{1}{\left(\frac{R-k}{J}\right)^2+1}\left[1-e^{-\frac{R-k}{J}J\tau}\left(\cos J\tau+\frac{R-k}{J}\sin J\tau\right)\right](=\bar{K}) \\
K_f &= \frac{1}{\left(\frac{R+f}{J}\right)^2+1}\left[1-e^{-\frac{R+f}{J}J\tau}\left(\cos J\tau+\frac{R+f}{J}\sin J\tau\right)\right] \\
E_k &= e^{-\frac{k}{J}J\tau}(=E) \\
E_f &= e^{\frac{f}{J}J\tau}
\end{aligned}\right\} \quad \ldots \quad (9)$$

Graphs of some of these functions are given in Figs. 5 to 8.

First derivatives of the auxiliary functions with respect to the generalized time are as follows:

$$\left.\begin{aligned}
&\frac{dH}{d(J\tau)} = -L-\frac{R}{J}H, \qquad \frac{dL}{d(J\tau)} = H-\frac{R}{J}L, \qquad \frac{dK}{d(J\tau)} = L, \\
&\frac{dK_k}{d(J\tau)} = \frac{L}{E_k}, \qquad \frac{dK_f}{d(J\tau)} = \frac{L}{E_f}, \\
&\frac{dE_k}{d(J\tau)} = -\frac{k}{J}E_k, \qquad \frac{dE_f}{d(J\tau)} = \frac{f}{J}E_f, \\
&\frac{d}{d(J\tau)}\left[E_kK_k\right] = -\frac{k}{J}E_kK_k+L, \\
&\frac{d}{d(J\tau)}\left[E_fK_f\right] = \frac{f}{J}E_fK_f+L.
\end{aligned}\right\} \quad \ldots \quad (9a)$$

With the initial conditions $\tau = \hat{w} = \hat{q} = 0$ complete solutions for $\hat{w}$, and their first two derivatives can now be written thus:

(*a*) *Elevator moved instantaneously and held*.

$$\eta = \eta_0$$

$$\frac{1}{\delta\eta_0}\hat{w} = -\frac{K}{J^2} \quad \ldots \quad (10a)$$

$$\frac{1}{\delta\eta_0}\frac{d\hat{w}}{d\tau} = -\frac{L}{J} \qquad \text{(10b)}$$

$$\frac{1}{\delta\eta_0}\frac{d^2\hat{w}}{d\tau^2} = -H + \frac{R}{J}L\,. \qquad \text{(10c)}$$

(*b*) *Elevator moved gradually and then held*

$$\eta = \eta_0(1 - e^{-k\tau})$$

$$\frac{1}{\delta\eta_0}\hat{w} = -\frac{1}{J^2}\Big[K - E_k K_k\Big] \qquad \text{(11a)}$$

$$\frac{1}{\delta\eta_0}\frac{d\hat{w}}{d\tau} = -\frac{k}{J^2}E_k K_k \qquad \text{(11b)}$$

$$\frac{1}{\delta\eta_0}\frac{d^2\hat{w}}{d\tau^2} = -\frac{k}{J}L + \frac{k^2}{J^2}E_k K_k\,. \qquad \text{(11c)}$$

(*c*) *Elevator moved gradually and then reversed*

$$\eta = \eta_0[(1 - e^{-k\tau}) + (1 - e^{f\tau})e^{-f\tau_3}]$$

$$\frac{1}{\delta\eta_0}\hat{w} = -\frac{1}{J^2}\Big[K - E_k K_k\Big] - \frac{1}{J^2}\Big[K - E_f K_f\Big]e^{-f\tau_3} \qquad \text{(12a)}$$

$$\frac{1}{\delta\eta_0}\frac{d\hat{w}}{d\tau} = -\frac{k}{J^2}E_k K_k + \frac{f}{J^2}E_f K_f\, e^{-f\tau_3} \qquad \text{(12b)}$$

$$\frac{1}{\delta\eta_0}\frac{d^2\hat{w}}{d\tau^2} = -\frac{k}{J}L + \frac{k^2}{J^2}E_k K_k + \Big[\frac{f}{J}L + \frac{f^2}{J^2}E_f K_f\Big]e^{-f\tau_3}\,. \qquad \text{(12c)}$$

The auxiliary functions, equations (9) apply to cases of complex roots of the stability equation. The following detailed analysis is also made with respect to such cases. The changes to be introduced in the various formulae to cope with real stability roots are given in section B.7 of Appendix B.

A.4. *Derived Response Quantities.*—Values of various derived response quantities can be calculated with the use of complete solutions of the preceding section. General formulae for some of those quantities will now be derived.

A.4.1. *Angle of Incidence at the Wing.*—The linearized form of equations of motion requires the incremental angle of incidence at the wing to be identical with $\hat{w}$; thus

$$\alpha \equiv \hat{w}\,.$$

A.4.2. *Angle of Incidence at the Tail.*—Its effective incremental value is given by

$$\alpha'_{\text{eff}} = \Big(1 - \frac{d\varepsilon}{d\alpha}\Big)\alpha + \frac{lq}{V} + \frac{l}{V}\frac{d\varepsilon}{d\alpha}\frac{d\alpha}{dt}\,. \qquad \text{(13)}$$

The first term of the right-hand side of this equation represents the increase in the angle of incidence due to the incremental velocity of the aircraft in the $z$ direction together with the (static) downwash correction; the second term corresponds to the increment due to the angular velocity in pitch of the aircraft; the last term provides a correction required by the fact that in a general case the downwash effect is overestimated in the first term since any change in the downwash produced at the wing appears at the tail only after a time $l/V$.

By using equation (1) $q$ can be eliminated from the last expression which, after being reduced to a non-dimensional form, becomes

$$\alpha'_{\text{eff}} = \left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right)\hat{w} + \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{1}{\mu}\frac{d\hat{w}}{d\tau}\,. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

A.4.3. *Normal Acceleration at c.g.* of the aircraft (upwards) is given by $(qV - dw/dt)$; in accordance with equation (1) it is proportional to $\alpha$. Thus the coefficient of normal acceleration can be written as

$$n = D \,.\, \hat{w} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where

$$D = \frac{\frac{1}{2}\rho V^2}{W/S}\,a\,.$$

A.4.4. *Normal Acceleration at any Point* of the aircraft consists in general of two components; one is equal to the acceleration at the c.g., the other is due to the angular acceleration in pitch $dq/dt$. The value of the latter component at the tail is $(-\,l\,dq/dt)$. Differentiating equation (1) enables the corresponding coefficient of acceleration to be written in non-dimensional notation thus

$$\bar{n} = -\frac{D}{\mu}\left(\frac{2}{a}\frac{d^2\hat{w}}{d\tau^2} + \frac{d\hat{w}}{d\tau}\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

and the coefficient of total normal acceleration at the tail

$$n_t = n + \bar{n} = D\left[\hat{w} - \frac{1}{\mu}\left(\frac{2}{a}\frac{d^2\hat{w}}{d\tau^2} + \frac{d\hat{w}}{d\tau}\right)\right]. \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

A.4.5. *Tailplane Load.*—The general expression for the incremental tailplane load is

$$P = \tfrac{1}{2}\rho V^2 S'(a_1\alpha'_{\text{eff}} + a_2\eta)$$

and after substituting $\alpha'_{\text{eff}}$ from equation (14) it can be written thus

$$P = A\left(B\hat{w} + C\frac{d\hat{w}}{d\tau} + a_2\eta\right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

where

$$A = \tfrac{1}{2}\rho V^2 S'$$

$$B = \left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right) a_1$$

$$C = \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{a_1}{\mu}\,.$$

A.4.6. *Angular velocity in pitch* is obtained directly from the first equation of motion, equation (3) as

$$q = \frac{1}{\hat{t}}\left(\frac{d\hat{w}}{d\tau} + \frac{a}{2}\hat{w}\right). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

A.4.7. *Angular acceleration in pitch* is obtained with the use of equation (19):

$$\frac{dq}{dt} = \frac{1}{\hat{t}^2}\left(\frac{d^2\hat{w}}{d\tau^2} + \frac{a}{2}\frac{d\hat{w}}{d\tau}\right) \quad \ldots \quad (20)$$

A.4.8. *Elevator hinge moment coefficient* is given by the general formula

$$C_h = b_1\alpha'_{\text{eff}} + b_2\eta$$

and using equation (14) it can be expressed thus

$$C_h = \frac{b_1}{a_1}B\hat{w} + \frac{b_1}{a_1}C\frac{d\hat{w}}{d\tau} + b_2\eta \,. \quad \ldots \quad (21)$$

A.5. *Elevator Moved Instantaneously and Held.*—In this case only normal accelerations at c.g. and tailplane loads will be considered.

Time histories of the angle of incidence at the wing and of the *normal acceleration at the c.g.* may be obtained with the basic response, equation (10a) as follows:

angle of incidence $\alpha = -\ \delta\eta_0\dfrac{K}{J^2}$

coefficient of normal acceleration

$$n = -\ \delta\eta_0 D\frac{K}{J^2} \quad \ldots \quad (22)$$

thus either of them varies as function $K$. This function is shown for various $R/J$ values in Figs. 8 (a) and 8 (b).

In accordance with equation (10b) the condition for maxima of either $\alpha$ or $n$ is given by $L = 0$. Of the infinite number of roots of this equation the two $J\tau_m = \pi$ and $J\tau_a = \infty$ correspond to the first (absolute) maximum and to the asymptotic value respectively. Thus

first maximum $n_m = -\ \delta\eta_0 D\dfrac{K_\pi}{J^2}$ $\quad \ldots \quad$ (23)

asymptotic value $n_a = -\ \delta\eta_0 D\dfrac{K_a}{J^2}\,.$ $\quad \ldots \quad$ (24)

It follows that the elevator angle required to produce a specified incremental normal acceleration $n_m g$ is

$$\eta_0 = -\ \frac{J^2}{\delta DK_\pi}n_m \,. \quad \ldots \quad (25)$$

From equations (18), (10a) and (10b) the expression for the *tailplane load* can be written thus:

$$P = \delta\eta_0\frac{A}{J^2}\left(-\ BK - CJL + J^2\frac{a_2}{\delta}\right) \quad \ldots \quad (26)$$

or related to the coefficient of maximum normal acceleration equation (23)

$$\frac{P}{n_m} = \frac{F}{K_\pi}\left(BK + CJL - J^2\frac{a_2}{\delta}\right) \quad \ldots \quad (27)$$

where $F = \dfrac{WS'}{Sa}\,.$

At the beginning of the manoeuvre, when $K = L = 0$, equation (26) reduces to

$$P = P_0 = Aa_2\eta_0 . \quad \ldots \quad (28)$$

In a pull-out manoeuvre ($\eta_0$ negative) $P_0$ represents the maximum download at the tail; though strictly speaking it is not an algebraic minimum. If regarded as the $\eta$ contribution to the net tailplane load (*cf.* section 5) $P_0$ remains constant throughout the manoeuvre. The following formula gives $P_0$ related to the coefficient of maximum incremental normal acceleration, equation (23).

$$\frac{P_0}{n_m} = -F\frac{a_2}{\delta}\frac{J^2}{K_n} . \quad \ldots \quad (29)$$

With $\eta = \eta_0 = \text{const.}$ the condition for maximum upload in a pull-out manoeuvre is obtained from equation (18) as

$$B\frac{d\hat{w}}{d\tau} + C\frac{d^2\hat{w}}{d\tau^2} = 0$$

which is also the condition for the maximum effective angle of incidence at the tail. Using equations (10) this condition can be written thus

$$(CR - B)L = CJH$$

or explicitly

$$\tan J\tau = \frac{J}{-\left(\frac{B}{C} - R\right)} . \quad \ldots \quad (30)$$

For a negative $\eta_0$ the first positive root of this equation is associated with the first (absolute) maximum upload. This maximum can be obtained as required, either from equation (26) or equation (27) with $K$ and $L$ calculated for $J\tau_2$ from equations (9). $K$ and $L$ may also be estimated from Fig. 8 (a), 8 (b), and 6.

The asymptotic value of the tailplane load per incremental maximum $g$ is given by

$$\frac{P_a}{n_m} = \frac{F}{K_n}\left(BK_a - J^2\frac{a_2}{\delta}\right) \quad \ldots \quad (31)$$

or if related to $n_a$

$$\frac{P_a}{n_a} = \frac{F}{K_a}\left(BK_a - J^2\frac{a_2}{\delta}\right) . \quad \ldots \quad (32)$$

A.6. *Elevator Moved Gradually and then Held.*—A.6.1. *Normal Accelerations.*—With equations (15) and (11a) the coefficient of normal acceleration at the c.g. can be written thus

$$n = -\delta\eta_0 D\frac{1}{J^2}\left[K - E_k K_k\right] . \quad \ldots \quad (33)$$

The second term in the bracket depends on the generalised elevator rate $k$; for an instantaneous elevator movement $k = \infty$, and $E_k K_k = 0$ reducing this expression to the corresponding one of the preceding section, equation (22).

Now the condition for maxima of $n$ is $dn/d\tau = 0$ or explicitly

$$\cos J\tau + \frac{R - k}{J}\sin J\tau = e^{\frac{R-k}{J}J\tau} . \quad \ldots \quad (34)$$

In most practical cases the numerical value of $k$ is large so that $R < k$, and then there is an infinite number of roots of this equation each corresponding to a maximum or minimum of $n$. The first positive root $J\tau_m$ gives the time of occurrence of the first maximum $n_m$, which is in this context the absolute maximum. This root may be calculated from the above equation or may be read from the graph, Fig. 10. The numerical value $n_m$ is then obtained as

$$n_m = -\delta\eta_0 D \frac{K_m}{J^2} \quad \ldots \quad (35$$

where $K_m$ is the value of the function $K$ for $J\tau = J\tau_m$.

(For the rare cases when $k \leqslant R$ the response in $n$ does not show any distinct maxima with the exception of the aysmptotic value $n_a$ given by equation (24).)

The maximum elevator angle $\eta_0$ required to reach a given incremental acceleration $n_m g$ is obtained from equation (35) as

$$\eta_0 = -\frac{J^2}{\delta D K_m} n_m . \quad \ldots \quad (36)$$

The value $K_m$ depends on the coefficient $k$. Thus $\eta_0$ is also a function of $k$ or of the mean elevator rate $[d\eta/dt]$; *see* equation (7).

The mean rate may be specified in different ways. Two of them will be considered here.

(*a*) The mean elevator rate $[d\eta/dt]$ is specified by the time $t'$ in seconds after which the maximum angle $\eta_0$ would be reached at that rate; thus

$$\left[\frac{d\eta}{dt}\right] = \frac{\eta_0}{t'}$$

and equation (7) becomes $k = 2\hat{t}/t'$. With this value, $J\tau_m$ can be calculated from equation (34) and then $K_m$, and $\eta_0$ from equation (36).

(*b*) The mean rate $[d\eta/dt]$ is specified directly in degrees per second. In this case the elevator angle does not disappear from equation (7). There are therefore four simultaneous equations (7), (9), (34) and (36) to be solved for $k$, $K_m$, $J\tau_m$ and $\eta_0$. Because of the transcendental forms involved the elimination of variables is not practicable and the solution must be obtained by successive approximations. For this purpose the following scheme may be used:

(i) Assume $K_m = K_\pi$ (as the first approximation)

(ii) Find $\eta_0$ from equation (36)

(iii) Find $k$ from equation (7). It should be noted that $[d\eta/dt]$ is of the same sign as $\eta_0$. Thus the coefficient $k$ is always positive

(iv) Calculate $-(R-k)/J$

(v) Find the first positive root $J\tau_m$ of equation (34)

(vi) Find the new value of $K_m$ for $J\tau_m$ obtained under (v)

Repeat the procedure with the new $K_m$ value.

The process is a convergent one; in many cases already the first approximation will be found satisfactory.

Now, with $\eta_0$ known, the formula for finding the elevator angle at any time during the manoeuvre is given by

$$\eta = \eta_0(1 - E_k) = -\frac{J^2}{\delta D K_m} n_m(1 - E_k) \,. \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

By eliminating $\eta_0$ from equations (33) and (36) the coefficient of normal acceleration at the c.g. of the aircraft can be expressed in terms of its maximum value $n_m$ thus:

$$n = \frac{n_m}{K_m}(K - E_k K_k) \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

The formula for the coefficient of *normal (upward) acceleration* at the tail due to the angular velocity in pitch of the aircraft is obtained from equations (16), (11b) and (11c) as

$$\bar{n} = -\delta\eta_0 \frac{D}{J^2}\frac{k}{\mu a}\Big[(2k - a)E_k K_k - 2JL\Big]$$

or expressed in terms of $n_m$

$$\bar{n} = \frac{k}{\mu a}\frac{n_m}{K_m}\Big[(2k - a)E_k K_k - 2JL\Big] \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

The coefficient of the *total normal acceleration* at the tail, $n_t$ is equal to the algebraic sum of $n$ and $\bar{n}$, equations (38) and (39) respectively. The condition for its maxima is

$$A \cos J\tau + U \sin J\tau = e^{-\frac{k-R}{J}J\tau} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

where

$$A = 1 - \frac{2J}{\Delta}, \qquad U = \frac{2(R + k) - a}{\Delta} - \frac{k - R}{J},$$

and

$$\Delta = \frac{1}{J}\frac{k(2k - a) - \mu a}{\left(\frac{R - k}{J}\right)^2 + 1}.$$

Usually the second positive root of equation (40) gives the time of the absolute maximum of $n_t$.

After an infinite time the contribution $\bar{n}$ due to the angular acceleration becomes zero, and the asymptotic value of the coefficient of normal acceleration is given by

$$n_a = \frac{K_a}{K_m} n_m \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

and is of course the same for all points of the aircraft.

A.6.2. *Tailplane Loads.*—The form of equation (18) implies two components of the net tailplane load, one due to the effective angle of incidence at the tail $\alpha'_{\text{eff}}$, the other due to the elevator angle $\eta$. Using the formulae of section A.3 (*b*) these two components may be written thus:

tail load due to $\alpha'_{\text{eff}}$:

$$P_w = A\left(B\hat{w} + C\frac{d\hat{w}}{d\tau}\right) = -\delta\eta_0\frac{A}{J^2}\Big[BK + (Ck - B)E_k K_k\Big] \quad \ldots \quad (42)$$

tail load due to $\eta$:

$$P_\eta = Aa_2\eta = Aa_2\eta_0(1 - E_k) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

and when related to the maximum normal acceleration in the manoeuvre $(n_m g)$, equation (35), they become respectively:

$$\frac{P_w}{n_m} = \frac{F}{K_m}\Big[BK + (Ck - B)E_k K_k\Big] \qquad \qquad (44)$$

$$\frac{P}{n_m} = -\frac{F}{K_m}\frac{a_2}{\delta}J^2(1 - E_k)\,. \qquad \qquad (45)$$

The net load per $g$ is then

$$\frac{P}{n_m} = \frac{P_w}{n_m} + \frac{P_\eta}{n_m}\,.$$

The condition for its maxima is $d[P/n_m]/d\tau = 0$; using equations (11) and (9) it can be written thus

$$\cos J\tau + Q \sin J\tau = T\, \mathrm{e}^{-\frac{k-R}{J}J\tau} \qquad \qquad (46)$$

where $$Q = \frac{CJ}{\Gamma} - \frac{k-R}{J}\,, \qquad T = 1 + \frac{J^2}{\Gamma}\frac{a_2}{\delta}\,,$$

and $$\Gamma = \frac{Ck - B}{\left(\frac{k-R}{J}\right)^2 + 1}\,.$$

In a pull-out manoeuvre $[\eta_0 < 0,\ n_m > 0]$ the first and second roots of this equation are associated with the maximum download and the maximum upload respectively. For the usual high values of the coefficient $k$ these two maxima represent absolute maxima.

After an infinite time the asymptotic values of the two components of the tail load $P_w$ and $P_\eta$, equations (42) and (43) respectively become

$$P_{wa} = -\,\delta\eta_0\frac{A}{J^2}BK_a$$

and $$P_{\eta a} = Aa_2\eta_0$$

and again, when related to the maximum normal acceleration $n_m g$,

$$\frac{P_{wa}}{n_m} = FB\frac{K_a}{K_m} \qquad \qquad (47)$$

$$\frac{P_{\eta a}}{n_m} = \frac{F}{K_m}\frac{a_2}{\delta}J^2\,. \qquad \qquad (48)$$

The same components when related to the asymptotic value of the normal acceleration $n_a = -\,\delta\eta_0\,.\,DK_m/J^2$ reduce to

$$\frac{P_{wa}}{n_a} = FB \qquad \qquad (49)$$

$$\frac{P_{\eta a}}{n_a} = \frac{F}{K_a}\frac{a_2}{\delta}J^2\,. \qquad \qquad (50)$$

A.6.3. *Elevator Hinge-moment Coefficient.*—The general formula for the hinge-moment coefficient of the elevator, equation (21), is similar to that for the net tailplane load, equation (18); thus the various expressions for the two components of $C_h$, the condition for maxima, etc., are derived in a similar way; they are as follows:

$$C_h \text{ due to } \alpha'_{\text{eff}}: \quad C_{hw} = -\,\delta\eta_0 \frac{1}{J^2}\Big[BK + (Ck - B)E_k K_k\Big]\frac{b_1}{a_1} \qquad (51)$$

$$C_h \text{ due to } \eta: \quad C_{h\eta} = b_2\eta_0(1 - E_k)\,. \qquad (52)$$

The same related to $n_m$ :

$$\frac{C_{hw}}{n_m} = \frac{1}{DK_m}\Big[BK + (Ck - B)E_k K_k\Big]\frac{b_1}{a_1} \qquad (53)$$

$$\frac{C_{h\eta}}{n_m} = -\frac{b_2}{\delta}\frac{J^2}{DK_m}(1 - E_k)\,. \qquad (54)$$

The net value of $C_h$ per $g$

$$\frac{C_h}{n_m} = \frac{C_{hw}}{n_m} + \frac{C_{h\eta}}{n_m}\,.$$

The condition for maxima of the net hinge moment is given by

$$\cos J\tau + Q\sin J\tau = T_h\, e^{-\frac{k-R}{J}J\tau} \qquad (55)$$

where

$$T_h = 1 + \frac{J^2}{\Gamma}\frac{a_1}{\delta}\frac{b_2}{b_1}\,.$$

Asymptotic values of $C_{hw}$ and $C_{h\eta}$ become respectively:

$$(C_{hw})_a = -\,\delta\eta_0\frac{1}{J^2}BK_a\frac{b_1}{a_1}\,, \quad (C_{h\eta})_a = b_2\eta_0\,.$$

The same related to $n_m$ become

$$\frac{(C_{hw})_a}{n_m} = \frac{b_1}{a_1}\frac{B}{D}\frac{K_a}{K_m}\,, \qquad \frac{(C_{h\eta})_a}{n_m} = -\frac{b_2}{\delta}\frac{J^2}{DK_m}\,,$$

but related to the asymptotic value $n_a$

$$\frac{(C_{hw})_a}{n_a} = \frac{b_1}{a_1}\frac{B}{D}\,, \qquad \frac{(C_{h\eta})_a}{n_a} = -\frac{b_2}{\delta}\frac{J^2}{DK_a}\,.$$

A.6.4. *Angular Velocity in Pitch.*—In accordance with equation (19) and (11) the angular velocity of the aircraft $q$ in radians per second is given by the following expression

$$q = -\,\delta\eta_0\frac{1}{2\hat{t}J^2}\Big[aK + (\tfrac{1}{2}k - a)E_k K_k\Big] \qquad (56)$$

and related to the coefficient of maximum normal acceleration $n_m$

$$\frac{q}{n_m} = \frac{1}{2\hat{t}DK_m}\left[aK + (\tfrac{1}{2}k - a)E_kK_k\right] \quad \ldots \quad (57)$$

Maxima of $q$ occur when

$$\cos J\tau + \Sigma \sin J\tau = e^{-\frac{k-R}{J}J\tau} \quad \ldots \quad (58)$$

where $$\Sigma = \left[\left(\frac{k-R}{J}\right)^2 + 1\right]\frac{J}{k-\frac{1}{2}a} - \frac{k-R}{J}.$$

The asymptotic value of $q$ becomes

$$q_a = -\,\delta\eta_0 \frac{aK_a}{2\hat{t}J^2}$$

and in terms of $n_m$ and $n_a$ respectively

$$\frac{q_a}{n_m} = \frac{a}{2\hat{t}D}\frac{K_a}{K_m} \quad \ldots \quad (59)$$

$$\frac{q_a}{n_a} = \frac{a}{2\hat{t}D}. \quad \ldots \quad (60)$$

A.6.5. *Angular Acceleration in Pitch.*—From equations (20), (11), and (35) the following expressions for this quantity are obtained

$$\frac{dq}{dt} = \delta\eta_0 \frac{k}{\hat{t}^2J^2}\left[\left(k - \frac{a}{2}\right)E_kK_k - JL\right] \quad \ldots \quad (61)$$

$$\frac{1}{n_m}\frac{dq}{dt} = -\frac{k}{\hat{t}^2D}\frac{1}{K_m}\left[\left(k - \frac{a}{2}\right)E_kK_k - JL\right]. \quad \ldots \quad (62)$$

The condition for maxima is

$$\frac{d^3\hat{w}}{d\tau^3} + \frac{a}{2}\frac{d^2\hat{w}}{d\tau^2} = 0$$

or explicitly

$$A_1 \cos J\tau + U_1 \sin J\tau = e^{-\frac{k-R}{J}J\tau} \quad \ldots \quad (63)$$

where $$A_1 = 1 - \frac{2J}{\Delta_1}, \qquad U_1 = \frac{2(k+R) - a}{\Delta_1} - \frac{k-R}{J},$$

and $$\Delta_1 = \frac{1}{J}\frac{k(2k-a)}{\left(\frac{k-R}{J}\right)^2 + 1}.$$

A.7. *Elevator Moved Gradually and then Reversed.*—This case is characterized by the elevator deflection given by equation (5); the basic response is shown in section A.3 (*c*). The significance of this case can be recognized when interpreting flight test results of rapid pull-out manoeuvres. However, it has not been used in the derivation of the computational method of Appendix B.

The various response quantities can be obtained in exactly the same way as in section A.6, that means by substituting the response formulae of section A.3 (*c*) into the required expressions for the derived response quantities of section A.4.

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## APPENDIX B

### *Computational Method*

B.1. *Introductory Remarks.*—The computational scheme for the calculation of time histories of various response quantities and their maxima presented in this Appendix has been arranged in such a way that no reference need be made to the derivation of the formulae given in Appendix A. However, for the physical interpretation of the manoeuvre and of the computed responses the main text of the report, and particularly section 10 should be consulted.

The various formulae in the following sections are expressed in terms of certain portmanteau coefficients to be calculated from the basic data, and also in terms of quantities $E$, $H$, $K$ and $L$ which are functions of $J\tau$. Now $J\tau$, which may be treated throughout as a single symbol, represents a measure of time having, however, the character of an angle; thus it can be expressed in degrees or radians. Changes in any of the response quantities may be plotted against $J\tau$ or if required against time $t$ in seconds. For this purpose the $J\tau$ values taken in radians will have to be multiplied by $\hat{t}/J$ to obtain $t$ in seconds; thus

$$t = \frac{\hat{t}}{J}\,(J\tau) \text{ seconds.}$$

$J\tau$ should also be taken in radians in all cases where it appears in an exponent of exponential functions.

If only maxima of response quantities are required the time histories of those quantities need not be computed. In this case $J\tau$ values or times of occurrence should be calculated from the conditions for maxima, and then substituted in the appropriate expressions.

All the quantities and responses can be calculated from the given formulae and equations, but the use of graphs and graphical methods as indicated may considerably reduce the labour of computation.

B.2. *Numerical Data Required.*—

| | | |
|---|---|---|
| $a$ | = | $\partial C_L/\partial\alpha$ for the whole aircraft |
| $a_1$ | = | $\partial C_L'/\partial\alpha'$ |
| $a_2$ | = | $\partial C_L'/\partial\eta$ including the effects of tabs if used |
| $b_1$ | = | $\partial C_h/\partial\alpha'$ |
| $b_2$ | = | $\partial C_h/\partial\eta$ including the effects of tabs if used |
| $c$ | | Standard mean chord of wing in feet |

| | |
|---|---|
| $g$ | Acceleration due to gravity ($= 32{\cdot}2$ ft/sec$^2$) |
| $k_B$ | Radius of gyration of aeroplane about the lateral axis (ft) |
| $l$ | Distance from aeroplane c.g. to mean quarter-chord point of the tailplane (ft) |
| $m_{q\,\text{l.t.}}$ | Damping derivative in pitch of the aircraft without tailplane, if different from zero |
| $n_{\text{boundary}}$ | Coefficient of maximum normal acceleration to be reached in the manoeuvre |
| $S$ | Wing area (ft$^2$) |
| $S'$ | Tailplane area (ft$^2$) |
| $V$ | True air speed of aeroplane (ft/sec) |
| $W$ | Weight of aeroplane (lb) |
| $\rho$ | True air density (slug/ft$^3$) |
| $\left(\frac{\partial C_m}{\partial \alpha}\right)_{\text{less tail}}$ | Static stability derivative for the aircraft without tailplane |
| $\frac{d\varepsilon}{d\alpha}$ | Downwash derivative at the tail |
| $\left[\frac{d\eta}{dt}\right]$ | Mean elevator rate in degrees per second (in some cases the coefficient $k$ may be specified instead of the mean elevator rate). |

B.3. *Basic Formulae.*—

$$\mu = \frac{W}{g\rho S l} \qquad \hat{t} = \mu \frac{l}{V}$$

$$B = \left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right)a_1 \qquad C = \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{a_1}{\mu}$$

$$D = \tfrac{1}{2}\rho V^2 \frac{a}{W/S} \qquad F = \frac{WS'}{Sa}$$

$$\frac{\partial C_m}{\partial \alpha} = \left(\frac{\partial C_m}{\partial \alpha}\right)_{\text{less tail}} - \frac{S'l}{Sc}\left(1 - \frac{d\varepsilon}{d\alpha}\right)a_1 \qquad \omega = -\frac{Wc}{2g\rho S k_B^2}\frac{\partial C_m}{\partial \alpha}$$

$$\delta = \frac{Wc}{2g\rho S k_B^2}\frac{S'l}{Sc}a_2 \qquad \nu = \nu_{\text{tail}} + \nu_{\text{less tail}}$$

$$\nu_{\text{tail}} = \tfrac{1}{2}\frac{S'l^2}{S k_B^2}a_1 \qquad \nu_{\text{less tail}} = -\frac{l^2}{k_B^2}(m_q)_{\text{less tail}}$$

$$\chi = \frac{d\varepsilon}{d\alpha}\nu_{\text{tail}} \qquad R^2 + J^2 = \omega + \tfrac{1}{2}a\nu = \frac{Wc}{2g\rho S k_B^2}aH_m$$

$$R = \tfrac{1}{2}(\nu + \chi + \tfrac{1}{2}a) \qquad J = \sqrt{(\omega + \tfrac{1}{2}a\nu - R^2)}$$

$$n_m = n_{\text{boundary}} - 1$$

*Note.*—For the coefficient $k$ see section B.5.1.0.

$$\Gamma = \frac{Ck - B}{\left(\frac{k-R}{J}\right)^2 + 1} \qquad Q = \frac{CJ}{\Gamma} - \frac{k-R}{J}$$

$$T = 1 + \frac{J^2}{\Gamma}\frac{a_2}{\delta} \qquad T_h = 1 + \frac{J^2}{\Gamma}\frac{a_1}{\delta}\frac{b_2}{b_1}$$

$$\Delta = \frac{1}{J}\frac{k(2k-a) - \mu a}{\left(\frac{k-R}{J}\right)^2 + 1} \qquad \Delta_1 = \frac{1}{J}\frac{k(2k-a)}{\left(\frac{k-R}{J}\right)^2 + 1}$$

$$A = 1 - \frac{2J}{\Delta} \qquad A_1 = 1 - \frac{2J}{\Delta_1} \qquad \Sigma = \left[\left(\frac{k-R}{J}\right)^2 + 1\right]\frac{J}{k - \frac{a}{2}} - \frac{k-R}{J}$$

$$U = \frac{2(k+R) - a}{\Delta} - \frac{k-R}{J} \qquad U_1 = \frac{2(k+R) - a}{\Delta_1} - \frac{k-R}{J}$$

$$K_\pi = \frac{1 + e^{-\pi R/J}}{\left(\frac{R}{J}\right)^2 + 1} \quad (\textit{cf.}\ \text{Fig. } 11) \qquad K_a = \frac{1}{\left(\frac{R}{J}\right)^2 + 1} \quad (\textit{cf.}\ \text{Fig. } 11)\,.$$

B.4. *Special Functions.*—

$$H = e^{-\frac{R}{J}J\tau}\cos J\tau \quad (\textit{cf.}\ \text{Fig. } 5)$$

$$L = e^{-\frac{R}{J}J\tau}\sin J\tau \quad (\textit{cf.}\ \text{Fig. } 6)$$

$$E = e^{-\frac{k}{J}J\tau} \quad (\textit{cf.}\ \text{Fig. } 7)$$

$$K = \frac{1}{\left(\frac{R}{J}\right)^2 + 1}\left[1 - e^{-\frac{R}{J}J\tau}\left(\cos J\tau + \frac{R}{J}\sin J\tau\right)\right] \quad (\textit{cf.}\ \text{Fig. } 8)$$

$$\bar{K} = \frac{1}{\left(\frac{k-R}{J}\right)^2 + 1}\left[1 - e^{\frac{k-R}{J}J\tau}\left(\cos J\tau - \frac{k-R}{J}\sin J\tau\right)\right].$$

B.5. *Elevator Moved Gradually.*—B.5.1. *Formulae for the First Stage of the Manoeuvre* (*cf.* *section* 10).—B.5.1.0. *Evaluation of* $\eta_0$, $k$, $J\tau_m$, $K_m$.—

(i) With the $R/J$ value appropriate to the case considered assume as the first approximation $K_m = K_\pi$ (*see* section B.3 or Fig. 11)

(ii) Find the maximum elevator deflection $\eta_0$

$$\eta_0 = -\frac{J^2}{\delta D}\frac{n_m}{K_m}\ \text{radians}$$

(iii) Find $k$ from

$$k = 2\hat{t}\left[\frac{d\eta}{dt}\right]\frac{1}{\eta_0}$$

where $\left[\frac{d\eta}{dt}\right]$ is the specified mean elevator rate being of the same sign as $\eta_0$ Thus the coefficient $k$ is always positive.

*Note.*—If $\left[\frac{d\eta}{dt}\right]$ is expressed in degrees per second then $\eta_0$ should be taken in degrees.

(iv) Find $\frac{k - R}{J}$

(v) Find the first positive root $J\tau_m$ of the equation $\bar{K} = 0$ (*see* section B.4 or Fig. 10).

(vi) Find the new value of $K_m$ for $J\tau_m$ obtained under (v) (*see* section B.4 or Fig. 8). Repeat the procedure with the new $K_m$ value.

The process is a convergent one; usually only two or three repetitions are required to obtain satisfactory approximation.

*Note.*—The whole calculation need not be repeated if either $\frac{k - R}{J}$ as found under (iv) is 20 or more, or the ratio $R/J$ is $1 \cdot 0$ or more. In such cases the elevator deflection $\eta_0$ should be calculated with $K_m = K_\pi$.

B.5.1.1. *Tailplane loads in pounds per g.*—The contribution due to the effective angle of incidence at the tail $\alpha'_{\text{eff}}$:

$$\frac{P_w}{n_m} = \frac{F}{K_m}\left[BK + \Gamma\left(E - H + \frac{k - R}{J}L\right)\right].$$

The contribution due to the elevator angle $\eta$:

$$\frac{P_\eta}{n_m} = -\frac{F}{K_m}\frac{a_2}{\delta}J^2(1 - E).$$

The net tailplane load per $g$:

$$\frac{P}{n_m} = \frac{P_w}{n_m} + \frac{P_\eta}{n_m}.$$

Time histories of the net tail load and of its two components can be obtained by calculating the numerical values of these three expressions for several values of $J\tau$ between 0 and 240 deg. It is within this range that the significant maxima occur. A similar remark applies to response quantities of the following sections.

The condition for tail load maxima is given by the equation

$$\cos J\tau + Q \sin J\tau = T\,\mathrm{e}^{-\frac{k-R}{J}J\tau}.$$

The first two positive roots of this equation $J\tau_1$ and $J\tau_2$ correspond with the first maximum download and the subsequent maximum upload respectively. The roots can be found from this equation by trial and error or graphically as indicated in section B.6.4.

B.5.1.2. *Angle of incidence at the wing α in radians per g.*—

$$\frac{\alpha}{n_m} = \frac{1}{DK_m}\left[K - \frac{1}{\left(\frac{k-R}{J}\right)^2 + 1}\left(E - H + \frac{k-R}{J} L\right)\right].$$

At $J\tau_m$ (time of maximum acceleration) this becomes

$$\frac{\alpha_m}{n_m} = \frac{1}{D}.$$

B.5.1.3. *Effective angle of incidence $\alpha'_{\text{eff}}$ at the tail in radians per g.*—

$$\frac{\alpha'_{\text{eff}}}{n_m} = \frac{1}{DK_m a_1}\left[BK + \Gamma\left(E - H + \frac{k-R}{J} L\right)\right].$$

At $J\tau_m$ (time of maximum acceleration) this becomes

$$\frac{(\alpha'_{\text{eff}})_m}{n_m} = \frac{B}{Da_1}.$$

B.5.1.4. *Elevator angle η in radians per g.*—

$$\frac{\eta}{n_m} = -\frac{J^2}{\delta DK_m}(1 - E).$$

B.5.1.5. *Coefficient of normal acceleration n at the c.g. of the aeroplane.*—

$$n = \frac{n_m}{K_m}\left[K - \frac{1}{\left(\frac{k-R}{J}\right)^2 + 1}\left(E - H + \frac{k-R}{J} L\right)\right].$$

Its maximum value $n_m$ occurs at $J\tau = J\tau_m$ (*cf.* section B.5.1.0. (v)).

B.5.1.6. *Coefficient of normal acceleration $\bar{n}$ at the tail due to the angular acceleration in pitch only.*—

$$\bar{n} = \frac{kn_m}{\mu a K_m}\left[\frac{2k - a}{\left(\frac{k-R}{J}\right)^2 + 1}\left(E - H + \frac{k-R}{J} L\right) - 2JL\right].$$

Condition for maxima:

$$A_1 \cos J\tau + U_1 \sin J\tau = e^{-\frac{k-R}{J}J\tau}.$$

B.5.1.7. *Total coefficient of normal acceleration $n_t$ at the tail.*—

$$n_t = n + \bar{n}.$$

Condition for maxima:

$$A \cos J\tau + U \sin J\tau = e^{-\frac{k-R}{J}J\tau}.$$

The maximum upward normal acceleration at the tail in the first stage of the manoeuvre is associated with the second positive root of this equation.

B.5.1.8. *Angular velocity in pitch q in radians per second per g.*—

$$\frac{q}{n_m} = \frac{1}{2\hat{t}DK_m}\left[\frac{2k-a}{\left(\frac{k-R}{J}\right)^2+1}\left(E-H+\frac{k-R}{J}L\right)+aK\right].$$

Condition for maximum:

$$\cos J\tau + \Sigma \sin J\tau = e^{-\frac{k-R}{J}J\tau}.$$

The first positive root has to be considered.

B.5.1.9. *Elevator hinge-moment coefficient per g.*—The contribution due to the effective angle of incidence at the tail $\alpha'_{\text{eff}}$:

$$\frac{C_{hw}}{n_m} = \frac{b_1}{a_1}\frac{1}{DK_m}\left[BK + \Gamma\left(E-H+\frac{k-R}{J}L\right)\right].$$

The contribution due to the elevator angle $\eta$:

$$\frac{C_{h\eta}}{n_m} = -\frac{b_2}{\delta}\frac{J^2}{DK_m}(1-E).$$

The total hinge-moment coefficient per $g$:

$$\frac{C_h}{n_m} = \frac{C_{hw}}{n_m} + \frac{C_{h\eta}}{n_m}.$$

Condition for maxima:

$$\cos J\tau + Q \sin J\tau = T_h\, e^{-\frac{k-R}{J}J\tau}.$$

The first two positive roots should be considered.

B.5.2. *Formulae for the Initial Condition of the Second Stage (Steady Circling at $n_m g$) cf. Section 10.*—B.5.2.1. *Tailplane load in pounds per g.*—The contribution due to the effective angle of incidence $\alpha'_{\text{eff}}$:

$$\frac{P_{wa}}{n_m} = FB.$$

The contribution due to the elevator angle $\eta$

$$\frac{P_{\eta a}}{n_m} = -\frac{Fa_2J^2}{\delta K_a}.$$

The net tail load per $g$

$$\frac{P_a}{n_m} = \frac{P_{wa}}{n_m} + \frac{P_{\eta a}}{n_m} \quad \left[\text{or, roughly, } \frac{Wc}{l}\left(\frac{\partial C_m}{\partial C_L}\right)_{\text{less tail}}\right].$$

B.5.2.2. *Angle of incidence at the wing $\alpha_a$ in radians per g.*—

$$\frac{\alpha_a}{n_m} = \frac{1}{D}.$$

B.5.2.3. *Effective angle of incidence at the tail* $(\alpha'_{\text{eff}})_a$ *in radians per g.*—

$$\frac{(\alpha'_{\text{eff}})_a}{n_m} = \frac{B}{Da_1}\,.$$

B.5.2.4. *Elevator deflection* $\eta_a$ *per g in radians.*—

$$\frac{\eta_a}{n_m} = -\frac{J^2}{\delta D K_a}\,.$$

B.5.2.5. *Coefficient of normal acceleration at the c.g. of the aeroplane.*—

$$n_a = n_m\,.$$

B.5.2.6. *Coefficient of normal acceleration* $\bar{n}_a$ *at the tail due to the angular acceleration in pitch only.*—

$$\bar{n}_a = 0\,.$$

B.5.2.7. *Total coefficient or normal acceleration* $n_t$ *at the tail.*—

$$n_t = n_m\,.$$

B.5.2.8. *Angular velocity in pitch* $q_a$ *in radians per second per g.*—

$$\frac{q_a}{n_m} = \frac{a}{2\bar{t}D} \qquad \left[\text{or } \frac{g}{V}\right].$$

B.5.2.9. *Elevator hinge-moment coefficient per g.*—The contribution due to the effective angle of incidence

$$\frac{(C_{hw})_a}{n_m} = \frac{b_1}{a_1}\frac{B}{D}\,.$$

The contribution due to the elevator angle

$$\frac{(C_{h\eta})_a}{n_m} = -\frac{b_2}{\delta}\frac{J^2}{DK_a}\,.$$

The total hinge-moment coefficient

$$\frac{(C_h)_a}{n_m} = \frac{(C_{hw})_a}{n_m} + \frac{(C_{h\eta})_a}{n_m}\,.$$

B.5.3. *Formulae for the Second Stage, cf. Section* 10.—The various quantities occurring in this stage of the manoeuvre, $P_w$, $P_\eta$, $P$, $\alpha$, etc., can be obtained by adding values found in the first stage (taken with opposite signs) to the corresponding values for the steady circling case given in section B.5.2.

For instance the tail load contribution due to $\alpha'_{\text{eff}}$ becomes

$$\frac{P_w}{n_m} = -\frac{F}{K_m}\left[BK + \Gamma\left(E - H + \frac{k-R}{J}L\right)\right] + FB$$

and the contribution due to $\eta$

$$\frac{P_\eta}{n_m} = \frac{F}{K_m}\frac{a_2}{\delta}J^2(1-E) - \frac{Fa_2J^2}{\delta K_a}\,.$$

The general conditions for maxima of the various quantities remain the same as in the first stage of the manoeuvre, and also the times of occurrence ($J\tau$ values) are the same, time being reckoned anew from zero at the beginning of this second stage.

In particular the tailplane load at $J\tau_1$ represents now another maximum upload with the elevator partly reversed. The other maximum at $J\tau_2$ being small may be disregarded.

B.6. *Remarks on the Use of Graphs.*—

B.6.1. The formulae of sections B.3 and B.4 suffice to calculate the response quantities for any values of the parameters involved. The computational work may greatly be reduced, however, by making use of the graphs, Figs. 5 to 11, where various functions are plotted against $J\tau$ in degrees. If it becomes necessary, however, to compute exponential functions the values of $J\tau$ should of course be expressed in radians.

B.6.2. Graphs of the four functions $H$, $L$, $E$ and $K$ are given in Figs. 5 to 8 covering the practical ranges of the ratio $R/J$ and of the $J\tau$ values. $K_a$ values are also shown in Fig. 8, and $K_\pi$ values given in Fig. 11 correspond to the maxima at $J\tau = 180$ deg. $H$, $L$ and $K$ functions could not be obtained from the graphs at low $J\tau$ values; it is suggested that in many cases values of these functions may not be needed in this range of $J\tau$. But if required reference to the original equations of section B.4 should be made.

B.6.3. The root $J\tau_m$ of section B.5.1.0 (v) can be obtained directly from the graph, Fig. 10.

B.6.4. Conditions for maxima of the net tail load, of the coefficients of normal acceleration $n_t$ and $\bar{n}$, of the angular velocity in pitch, and of the elevator hinge-moment coefficient are given in sections B.5.1.1, B.5.1.6, B.5.1.7, B.5.1.8 and B.5.1.9 as transcendental equations of the general form

$$M \cos J\tau + N \sin J\tau = e^{-\frac{k-R}{J} J\tau}.$$

The first two positive roots of such an equation can be found graphically with the use of the graphs, Figs. 9 (a) and 9 (b) thus:

Draw a circle passing through the origin, the co-ordinates of the centre of the circle being $\frac{1}{2}M$ along the horizontal axis and $\frac{1}{2}N$ along the vertical axis. Draw straight lines from the origin to the two inter-sections of this circle with the appropriate spiral curve $(k - R)/J$ and produce these straight lines to the peripheral scale, which gives the required values of $J\tau$ in degrees.

The centres of the circles corresponding to the response quantities considered are determined as follows:

| | abscissa | ordinate | $N/M$ |
|---|---|---|---|
| Net tail load .. .. .. .. .. | $\frac{1}{2}T$ | $\frac{1}{2}Q/T$ | $Q$ |
| Coefficient of normal acceleration $n_t$ .. .. | $\frac{1}{2}A$ | $\frac{1}{2}U$ | $U/A$ |
| Coefficient of normal acceleration $\bar{n}$ .. .. | $\frac{1}{2}A_1$ | $\frac{1}{2}U_1$ | $U_1/A_1$ |
| Angular velocity $q$ .. .. .. .. | $\frac{1}{2}$ | $\frac{1}{2}\Sigma$ | $\Sigma$ |
| Hinge-moment coefficient .. .. .. | $\frac{1}{2}T_h$ | $\frac{1}{2}Q/T_h$ | $Q$ |

*Note.*—(*a*) In cases when the second point of intersection is close to the origin the corresponding root may be calculated from the formula

$$J\tau = \frac{\pi}{2} + \tan^{-1}\frac{N}{M} \text{ radians}$$

(*b*) If the numerical value of the first root is small, say below 5 deg, then it is advisable to check this root against the original transcendental equation, and to correct it if necessary.

B.7. *Response Quantities when J is Imaginary.*—B.7.1. All the formulae given in the preceding section can be used also in cases when $J$ becomes imaginary, say $J = iI$, where

$$I = \sqrt{\{R^2 - (\omega + \tfrac{1}{2}a\nu)\}}\,.$$

The direct use of those formulae would necessitate dealing with imaginary quantities. This inconvenience may be obviated by the following changes introduced in the basic formulae, special functions and general formulae:

(i) $J$ replaced by $I$, but

(ii) expressions $\left[\left(\frac{R}{J}\right)^2 + 1\right]$ and $\left[\left(\frac{k-R}{J}\right)^2 + 1\right]$ replaced by

$$\left[\left(\frac{R}{I}\right)^2 - 1\right] \text{ and } \left[\left(\frac{k-R}{I}\right)^2 - 1\right] \text{ respectively,}$$

(iii) circular functions replaced by the corresponding hyperbolic ones.

Thus for instance the function $K$ (section B.4) becomes

$$K = \frac{1}{\left(\frac{R}{I}\right)^2 - 1}\left[1 - \mathrm{e}^{-\frac{R}{I}I\tau}\left(\cosh I\tau + \frac{R}{I}\sinh I\tau\right)\right],$$

and the conditions for tail load maxima (section B.5.1.1)

$$\cosh I\tau + Q\sinh I\tau = T\,\mathrm{e}^{-\frac{k-R}{I}I\tau}$$

$Q$ and $T$ (section B.3) being also altered as required.

B.7.2. In cases of imaginary $J$ values the maximum normal acceleration is reached after an infinite time or, for all practical purposes, a few seconds after the beginning of the manoeuvre. The procedure of section B.5.1.0 does not apply here; instead the maximum elevator angle $\eta_0$ can be found from

$$\eta_0 = -\frac{I^2}{\delta D}\frac{n_m}{K_a} \text{ radians}$$

where

$$K_a = \frac{1}{\left(\frac{R}{I}\right)^2 - 1},$$

and the coefficient $k = 2\hat{t}\left[\frac{d\eta}{dt}\right]\frac{1}{\eta_0}$.

B.7.3. The equations giving conditions for maxima of the tailplane load, and of the elevator hinge-moment coefficient given in sections B.5.1.1. and B.5.1.9 could be used here for finding only the first maximum provided they are altered in accordance with section B.7.1.

The calculations of the first maxima per unit incremental normal acceleration may be considerably simplified by using the following approximate formulae:

$$\text{first maximum down load} = -\frac{Fa_2I^2}{\delta K_a}$$

$$\text{first maximum hinge-moment coefficient} = -\frac{b_2I^2}{\delta DK_a}$$

which give slightly conservative values.

Second maxima of these quantities and maxima of all other quantities occurring after an infinite time from the beginning of the manoeuvre may be calculated from the formulae for the steady circling case given in section B.5.2.

Thus for instance:

maximum up-load at the tail with the elevator fully deflected $= FB - \dfrac{Fa_2 I^2}{\delta K_a}$,

maximum up-load at the tail with the elevator reversed (third stage) $= FB$.

B.8. *Simplified Procedure for Use with Instantaneous Elevator Deflection.*—The following procedure and formulae apply when the elevator is moved instantaneously through the angle $\eta_0$ at the beginning of the first stage of the manoeuvre, and moved instantaneously through $\eta_0$ in the reverse direction at the beginning of the second stage.

The basic formulae from $\Gamma$ to $U_1$ inclusive in section B.3 are not required, nor are the functions $E$ and $K$ from section B.4. Other basic formulae and special functions are retained unchanged.

B.8.1. *Response and maxima during the first stage of the manoeuvre.*—The following table contains formulae for tail loads, etc., which correspond with the general formulae of sections B.5.1.1 to B.5.1.9 as shown by the first column of the table.

| Section | Formula | Condition for maxima* |
|---|---|---|
| B.5.1.1 | $\dfrac{P_w}{n_m} = \dfrac{F}{K_\pi}(BK + CJL)$ | |
| B.5.1.1 | $\dfrac{P_\eta}{n_m} = -\dfrac{F}{K_\pi}\dfrac{a_2}{\delta}J^2 = \text{const.}$ | |
| B.5.1.1 | $\dfrac{P}{n_m} = \dfrac{P_w}{n_m} + \dfrac{P_\eta}{n_m}$ | $\tan J\tau = \dfrac{-J}{B/C - R}$ |
| B.5.1.2 | $\dfrac{\alpha}{n_m} = \dfrac{1}{DK_\pi}K$ | $J\tau = \pi$, (1st max.) |
| B.5.1.3 | $\dfrac{\alpha'_{\text{eff}}}{n_m} = \dfrac{1}{DK_\pi a_1}(BK + CJL)$ | |
| B.5.1.4 | $\dfrac{\eta_0}{n_m} = -\dfrac{J^2}{\delta DK_\pi} = \text{const.}$ | |
| B.5.1.5 | $n = \dfrac{n_m}{K_\pi}K$ | $J\tau = \pi$, (1st max.) |
| B.5.1.6 | $\bar{n} = \dfrac{Jn_m}{\mu a K_\pi}[(2R - a)L - 2JH]$ | $\tan J\tau = \dfrac{-J(4R - a)}{2(J^2 - R^2) + aR}$ |
| B.5.1.7 | $n_t = n + \bar{n}$ | $\tan J\tau = \dfrac{-J(4R - a)}{2(J^2 - R^2) - a(\mu - R)}$ |
| B.5.1.8 | $\dfrac{q}{n_m} = \dfrac{1}{2\hat{t}DK_\pi}(2JL + aK)$ | $\tan J\tau = \dfrac{2J}{2R - a}$ |
| B.5.1.9 | $\dfrac{C_{hw}}{n_m} = \dfrac{b_1}{a_1}\dfrac{1}{DK_\pi}(BK + CJL)$ | |
| B.5.1.9 | $\dfrac{C_{h\eta}}{n_m} = -\dfrac{b_2}{\delta}\dfrac{J^2}{DK_\pi} = \text{const.}$ | |
| B.5.1.9 | $\dfrac{C_h}{n_m} = \dfrac{C_{hw}}{n_m} + \dfrac{C_{h\eta}}{n_m}$ | $\tan J\tau = \dfrac{-J}{B/C - R}$ |

* *See* also the remark at the end of this section.

*The complete response* in any of the quantities $P$, $n$, $q$, etc., is obtained thus:

(i) calculate the numerical values of the coefficients required (*see* section B.3) and find the appropriate value of $K_\pi$ (section B.3 or Fig. 11)

(ii) find numerical values of as many of the functions $H$, $L$ and $K$ as are required, for several values of $J\tau$ from section B.4 or Figs. 5 to 8, and for each $J\tau$ calculate the value of the quantity considered from the formula in the above table

(iii) plot the quantity considered against $J\tau$, or $\tau$, or $t$ in seconds as required, remembering that if $J\tau$ is in radians,

$$t = \frac{\hat{t}}{J} \cdot J\tau \text{ seconds} .$$

*The first or second maximum* of the quantity considered may be found without calculating the complete response thus:

(i) find the first root $J\tau_1$, or, where applicable, the second root $J\tau_2$ of the condition given in the table

(ii) find numerical values of $H$, $L$ and $K$ as required for $J\tau_1$, or $J\tau_2$ from section B.4 or Figs. 5, 6 and 8

(iii) by substituting these values of $H$, $L$ and $K$ into the formula for the quantity considered obtain the first or second maximum of this quantity.

The value of any other response quantity at the time of the maxima considered can be calculated from the appropriate formula using the same values of $H$, $L$ and $K$.

In addition to the above maxima both $P/n_m$ and $C_h/n_m$ assume extreme values at the beginning of the manoeuvre, namely

$$\frac{P_0}{n_m} = \frac{P_\eta}{n_m} \quad \text{and} \quad \frac{C_{h0}}{n_m} = \frac{C_{h\eta}}{n_m} .$$

B.8.2. *Steady Circling at $n_m g$.*—The formulae of section B.5.2 remain unchanged.

B.8.3. *Response and Maxima During the Second Stage.*—As in section B.5.3 the values of the various quantities are found by adding the values for the first stage with signs changed to the values of the steady circling condition. Time is reckoned anew from zero at the beginning of the second stage.

B.8.4. *Response Quantities when $J$ is Imaginary.*—The general provisions of section B.7 should be applied.

B.9. *Application of the Method to All-moving Tailplanes.*—The method given in the preceding sections in a form suitable for its direct application to aeroplanes with conventional tailplane arrangements can also be applied to aeroplanes with all-moving tailplanes provided the following points are observed:

(i) The derivative $a_2$ becomes numerically equal to $a_1$

(ii) The hinge-moment derivative $b_2$ becomes numerically equal to $b_1$, and $b_1$ itself should be evaluated as the pitching-moment derivative of the tailplane with respect to the hinge axis of the tailplane

(iii) The symbol $\eta$ stands for the variable tailplane setting as affected by control stick movements

(iv) The sum $(\alpha'_{\text{eff}} + \eta)$ gives the total incremental angle of incidence at the tailplane.

Thus in order to modify the expressions in sections B.3 and B.5 for use with aeroplanes with all-moving tailplanes, it is only necessary to replace $a_2$ and $b_2$ by $a_1$ and $b_1$ respectively.

The hinge-moment coefficient and the tailplane load now vary in the same way, each being proportional to $(\alpha'_{\text{eff}} + \eta)$.

# APPENDIX C

*Inclusion of the Tailplane Pitching Moment due to $\eta$*

Both the analysis presented in Appendix A and the computational method of Appendix B are based among others on the assumption that the tailplane pitching moment due to the elevator motion can be neglected. In a case of an unusually short tail arm and large tailplane chord it may, however, be required to take this effect into account. This can be done in the following manner.

The additional term representing the tailplane pitching moment due to $\eta$ is to be included in the second equation of motion, equation (2) of Appendix A so that the right-hand side of that equation becomes

$$-\tfrac{1}{2}\rho V^2 S' l \frac{\partial C'_L}{\partial \eta}\eta + \tfrac{1}{2}\rho V^2 S' c' \frac{\partial C'_m}{\partial \eta}\eta$$

and in terms of the non-dimensional notation of equation (4)

$$-\delta\eta - \delta'\eta \quad \text{or} \quad -\delta_t\eta$$

where the total elevator effectiveness $\delta_t = \delta + \delta'$,

$$\delta = \frac{Wc}{2g\rho S k_B^2}\frac{S' l}{Sc} a_2 \text{ (as before), and}$$

$$\delta' = -\frac{Wc}{2g\rho S k_B^2}\frac{S' c'}{Sc}\frac{\partial C'_m}{\partial \eta}.$$

Let $\bar{x}_\eta$ be the distance from the tailplane quarter-chord point to the c.p. position of the tailplane lift due to $\eta$, then

$$\frac{\partial C'_m}{\partial \eta} = -\frac{\bar{x}_\eta}{c'} a_2 .$$

Now the quantity $\delta'$ may be written thus:

$$\delta' = \frac{Wc}{2g\rho S k_B^2}\frac{S'\bar{x}_\eta}{Sc} a_2$$

and therefore

$$\delta_t = \delta + \delta' = \frac{Wc}{2g\rho S k_B^2}\frac{S'(l + \bar{x}_\eta)}{Sc} a_2 .$$

Thus in order to include the effect of the tailplane pitching moment due to the elevator movement one needs only to increase the tail arm $l$ appearing in the formula for $\delta$ by the length $\bar{x}_\eta$.

---

## APPENDIX D

*Inclusion of the Tailplane Lift due to $\eta$*

The effect of the tailplane lift due to the elevator deflection in a pull-out manoeuvre has been neglected in the preceding appendices. In cases where this effect is estimated to be significant and is to be included the following procedure may be adopted.

The first equation of motion, that of the equilibrium of normal forces, equation (1) of Appendix A must now comprise the term representing the lift due to $\eta$ which will appear on the right-hand side of that equation as

$$-\tfrac{1}{2}\rho V^2 S' \frac{\partial C_L'}{\partial \eta}\eta\,.$$

The corresponding term in the non-dimensional form of equation (3) is $z_\eta \eta$ where

$$z_\eta = -\tfrac{1}{2}\frac{\partial C_L}{\partial \eta} = -\tfrac{1}{2}\frac{S'}{S}a_2\,.$$

Thus disturbance terms due to the variable elevator angle $\eta$ appear now in both equations of motion, the second remaining unaltered.

With the initial conditions $\tau = \hat{w} = \hat{q} = 0$, and the same disturbance as in section A.3 (*b*), i.e.,

$$\eta = \eta_0(1 - e^{-k\tau})$$

the complete solution may be written in terms of the auxiliary functions of section A.3 thus:

$$\frac{1}{\eta_0}\hat{w} = -\frac{1}{J^2}\Big[(\delta - \nu z_\eta)(K - E_k K_k) - k z_\eta E_k K_k\Big]$$

$$\frac{1}{\eta_0}\frac{d\hat{w}}{d\tau} = -\frac{k}{J^2}\Big[\{\delta + (k - \nu)z_\eta\}E_k K_k - J z_\eta L\Big]$$

$$\frac{1}{\eta_0}\frac{d^2\hat{w}}{d\tau^2} = -\frac{k}{J}\Big[\delta + (R + k - \nu)z_\eta\Big]L + k z_\eta H + \frac{k^2}{J^2}\Big[\delta + (k - \nu)z_\eta\Big]E_k K_k\,.$$

Now with these solutions any derived response quantity can be obtained, and then its time history computed. However, the general formulae of section A.4 do not apply here. Instead the following should be used.

(i) Angle of incidence at the wing

$$\alpha \equiv \hat{w}$$

(ii) Angle of incidence at the tail

$$\alpha'_{\text{eff}} = \left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right)\hat{w} + \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{1}{\mu}\frac{d\hat{w}}{d\tau} - \frac{z_\eta}{\mu}\eta$$

(iii) Coefficient of normal acceleration at c.g.

$$n = D\left(\hat{w} + \frac{S'a_2}{Sa}\eta\right)$$

(iv) Coefficient of total normal acceleration at the tail

$$n_t = D\left[\hat{w} + \frac{S'a_2}{Sa}\eta - \frac{1}{\mu}\left(\frac{2}{a}\frac{d^2\hat{w}}{d\tau^2} + \frac{d\hat{w}}{d\tau} + \frac{S'a_2}{Sa}\frac{d\eta}{d\tau}\right)\right]$$

(v) Tailplane load

$$P = A\left[B\hat{w} + C\frac{d\hat{w}}{d\tau} + \left(a_2 - \frac{a_1}{\mu}z_\eta\right)\eta\right]$$

(vi) Angular velocity in pitch

$$q = \frac{1}{\hat{t}}\left(\frac{d\hat{w}}{d\tau} + \tfrac{1}{2}a\hat{w} - z_\eta\eta\right)$$

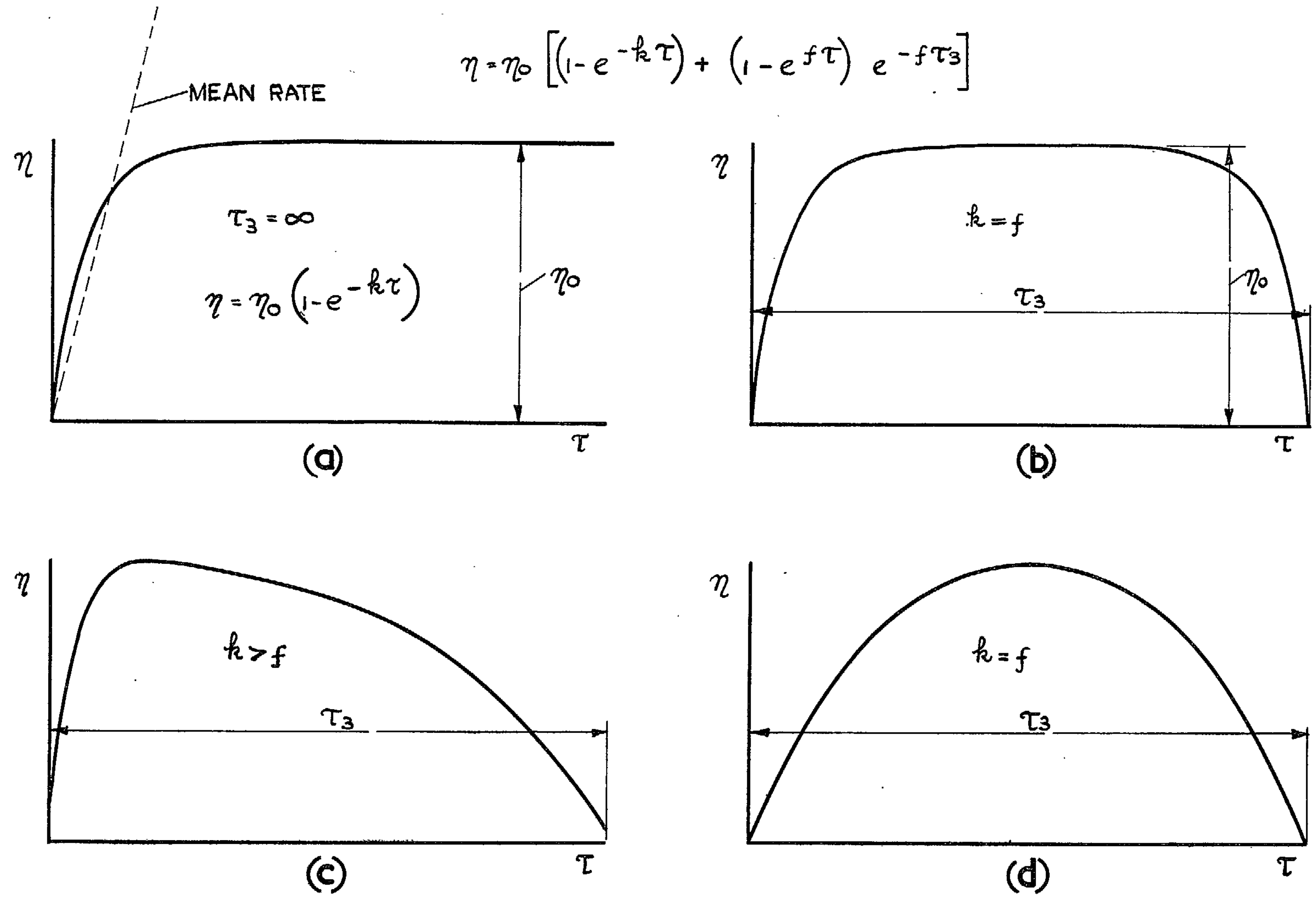
(vii) Angular acceleration in pitch

$$\frac{dq}{dt} = \frac{1}{\hat{t}^2}\left(\frac{d^2\hat{w}}{d\tau^2} + \tfrac{1}{2}a\frac{d\hat{w}}{d\tau} - z_\eta\frac{d\eta}{d\tau}\right)$$

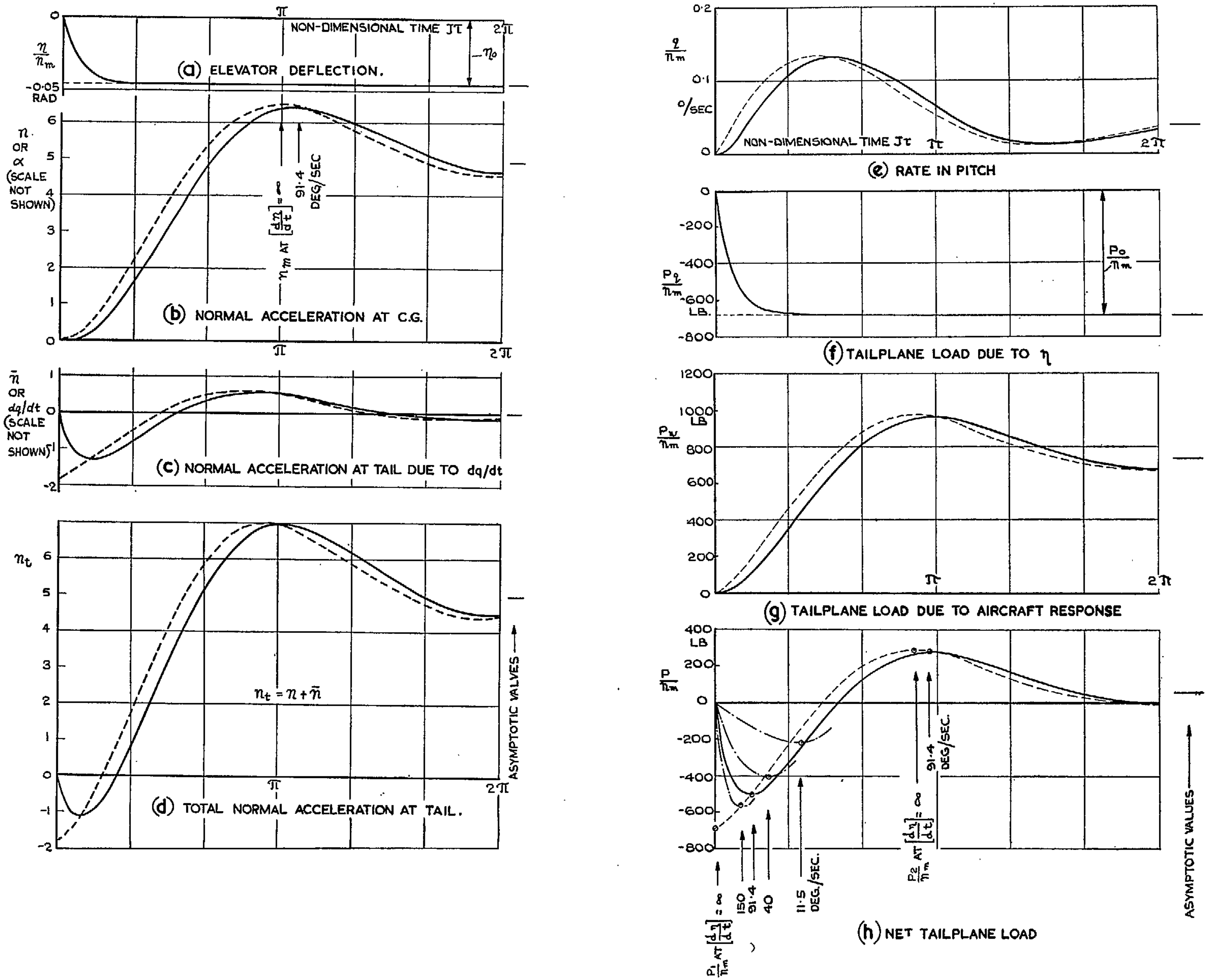
(viii) Elevator hinge-moment coefficient

$$C_h = \frac{b_1}{a_1}B\hat{w} + \frac{b_1}{a_1}C\frac{d\hat{w}}{d\tau} + \left(b_2 - \frac{b_1}{\mu}z_\eta\right)\eta\,.$$

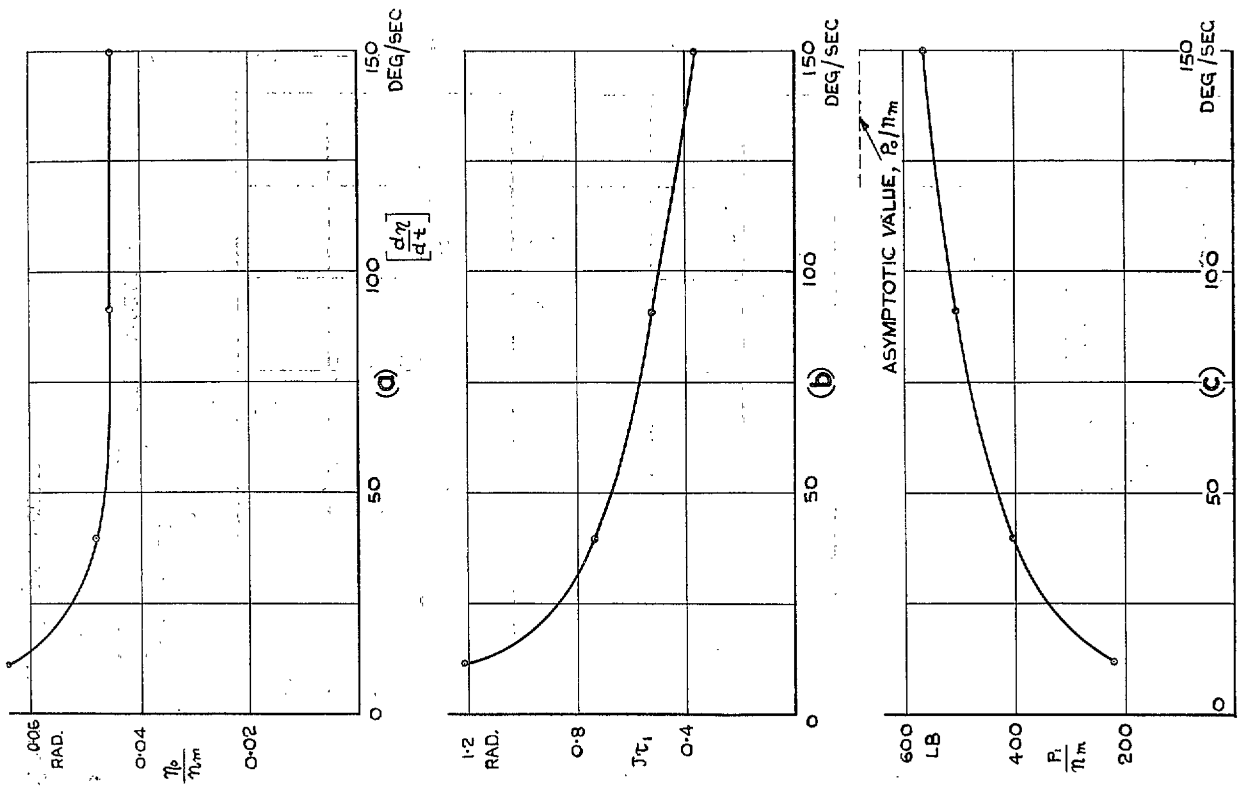
In order to obtain the significant maxima of these response quantities it is best to calculate their time histories over periods covering the maxima rather than to search for mathematical maxima, a procedure which might be here too cumbersome.




FIGS. 1 (a) to 1 (d). Types of elevator deflection curves.




FIGS. 2 (a) to 2 (h). Numerical example. Response curves.




FIGS. 3 (a) to 3 (c). Numerical example. Effect of varying elevator rate.

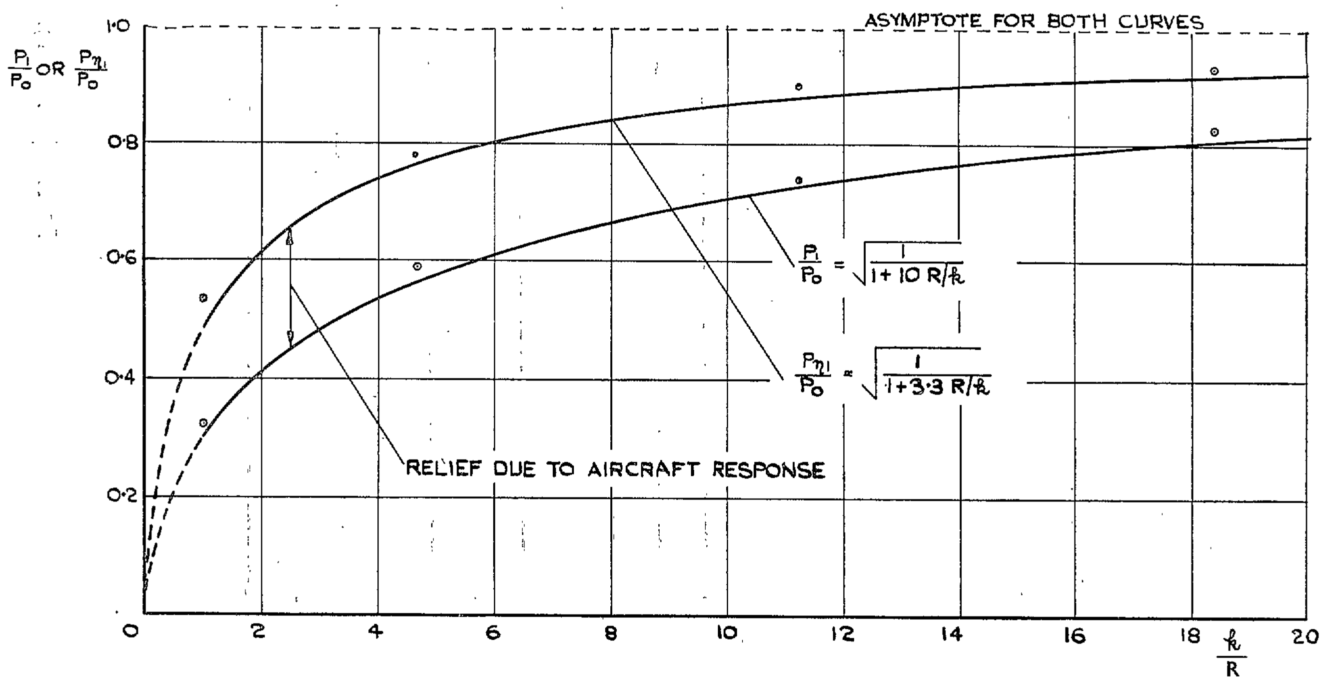



FIG. 4. Graph of empirical formulae.

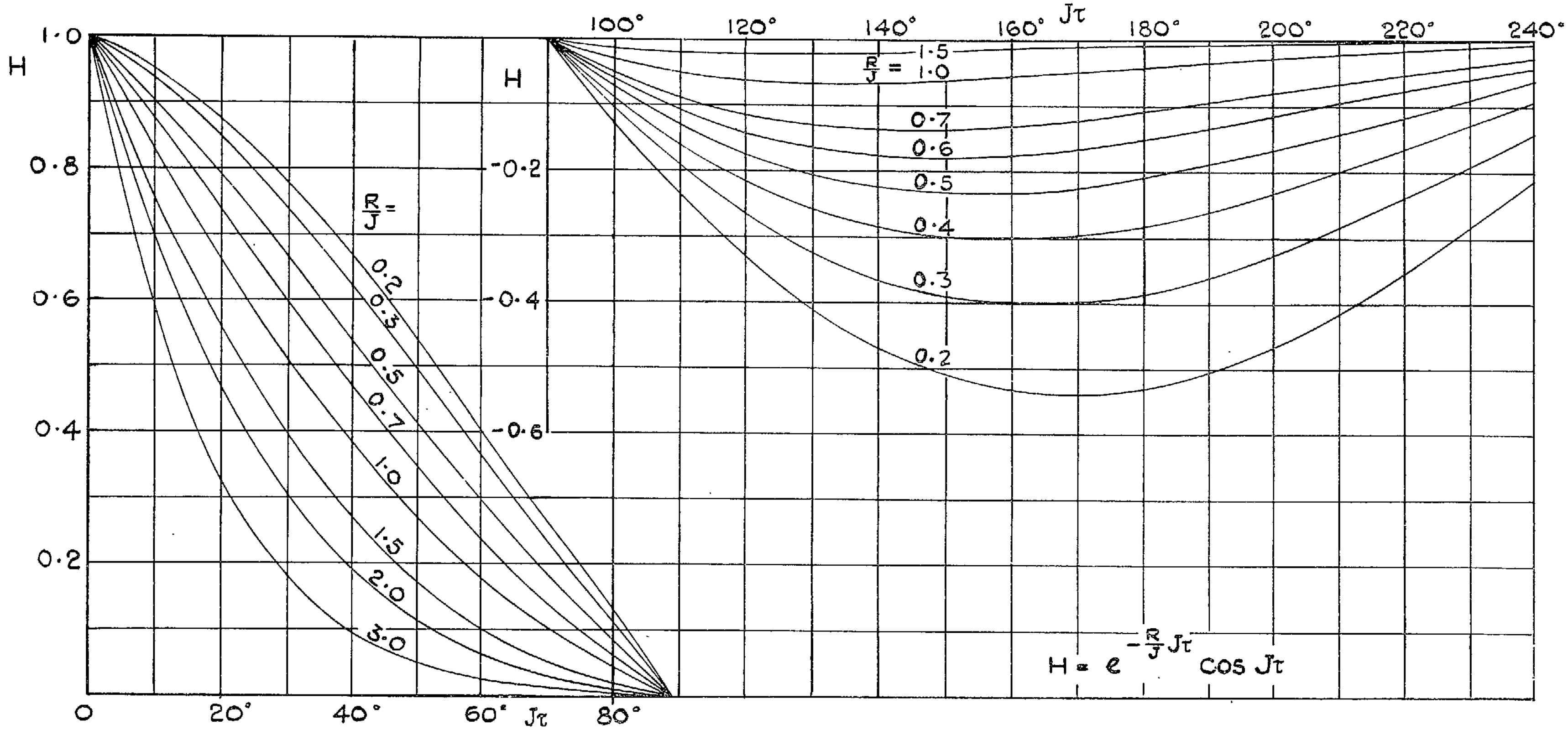



FIG. 5. Function $H$.

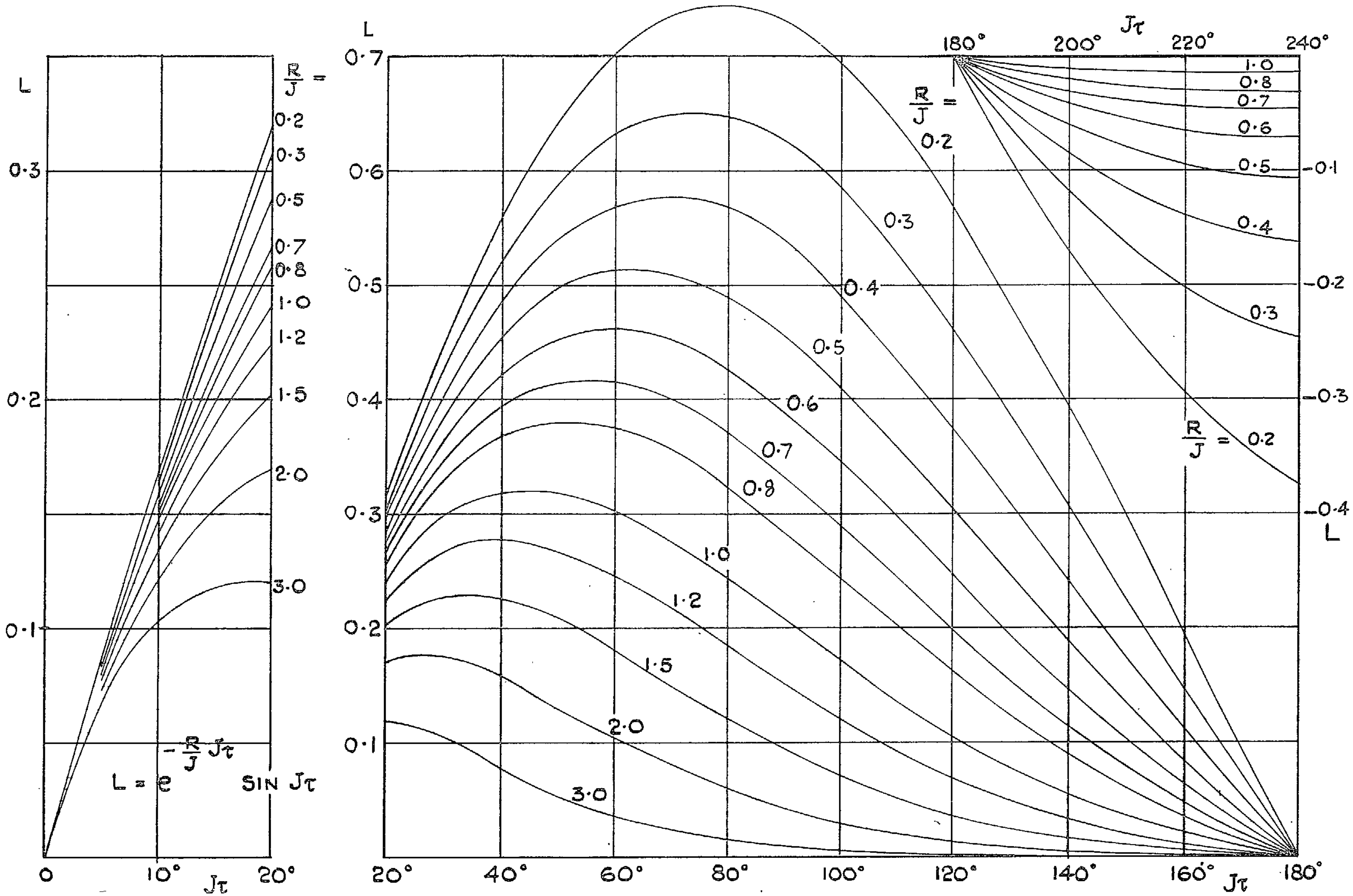



FIG. 6. Function $L$.

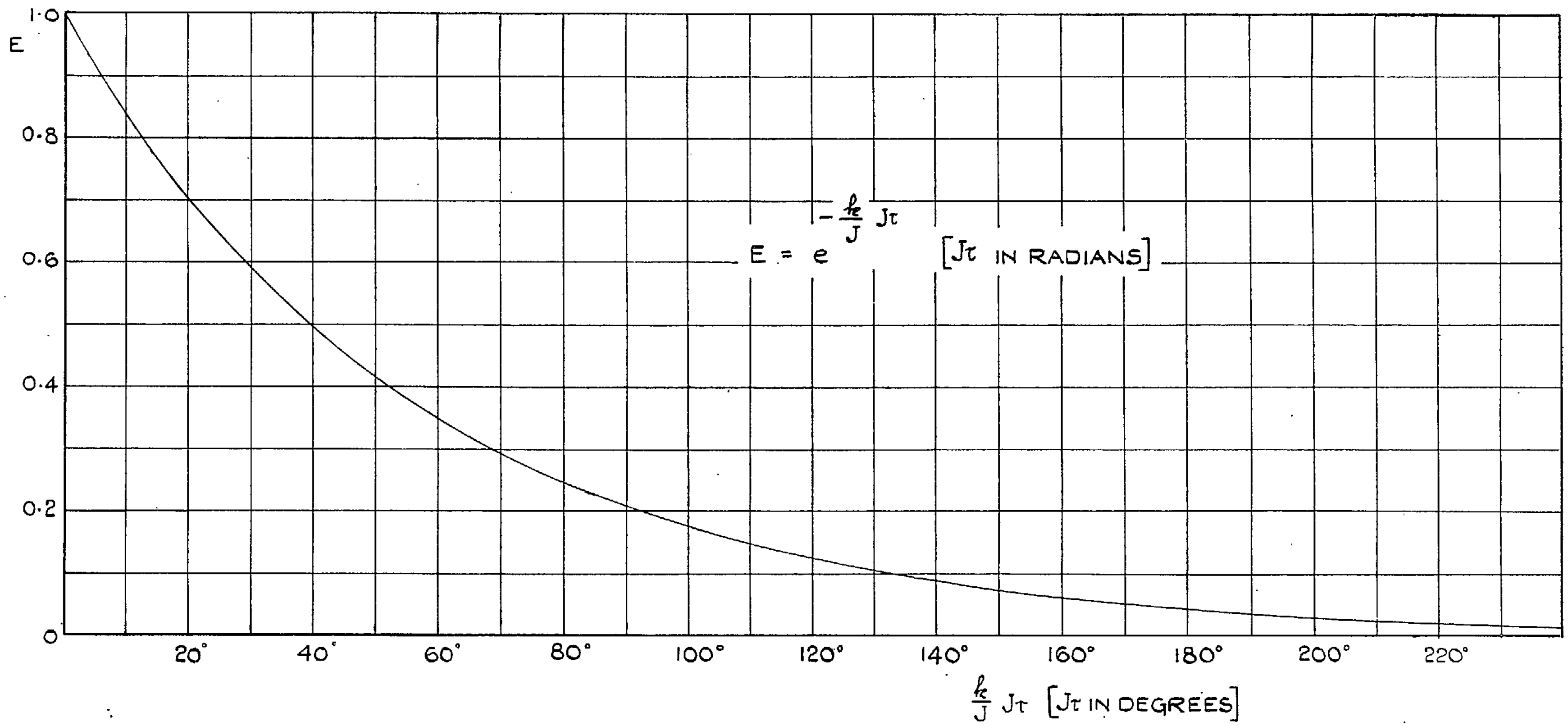



FIG. 7. Function $E$.

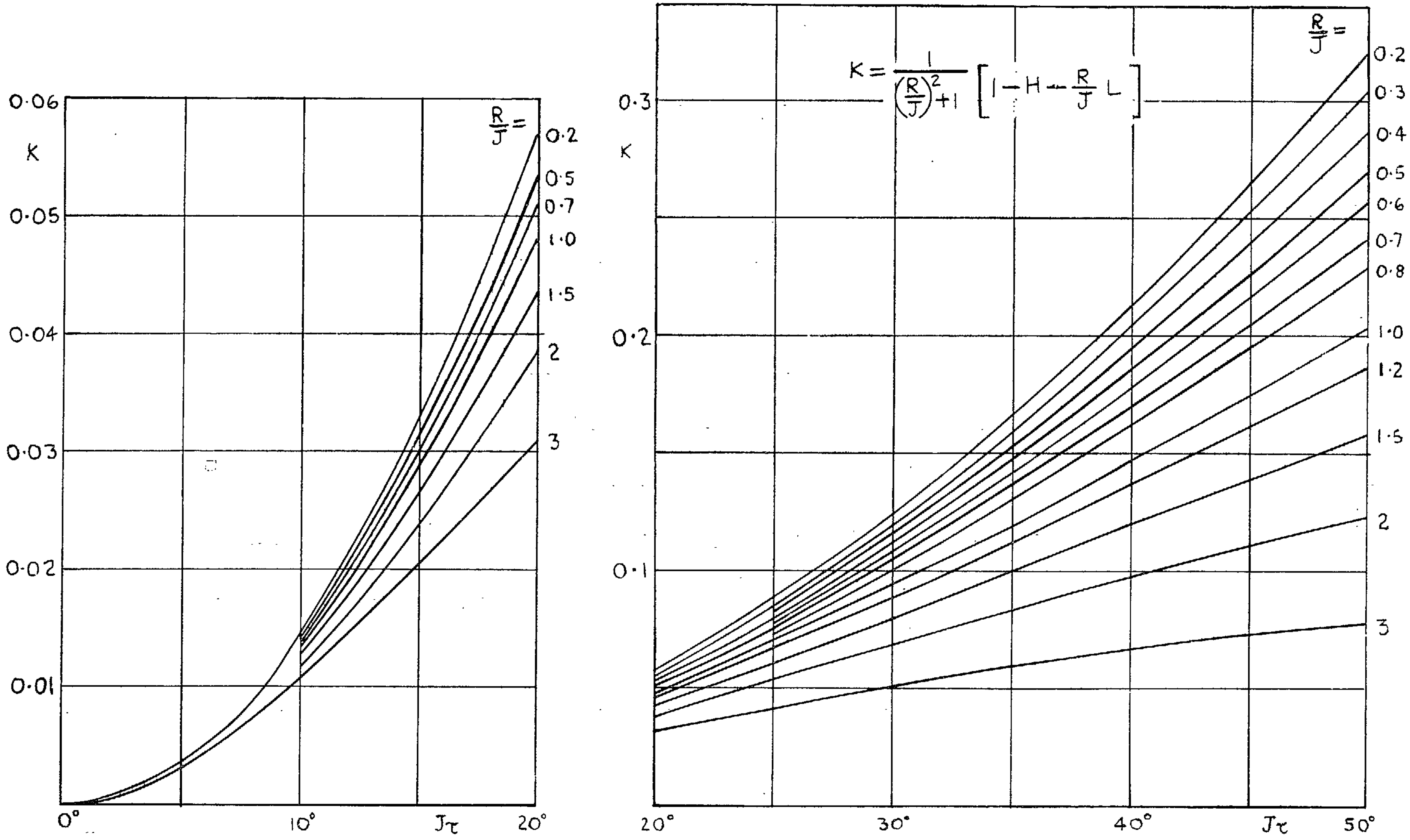



FIG. 8 (a). Function $K$.

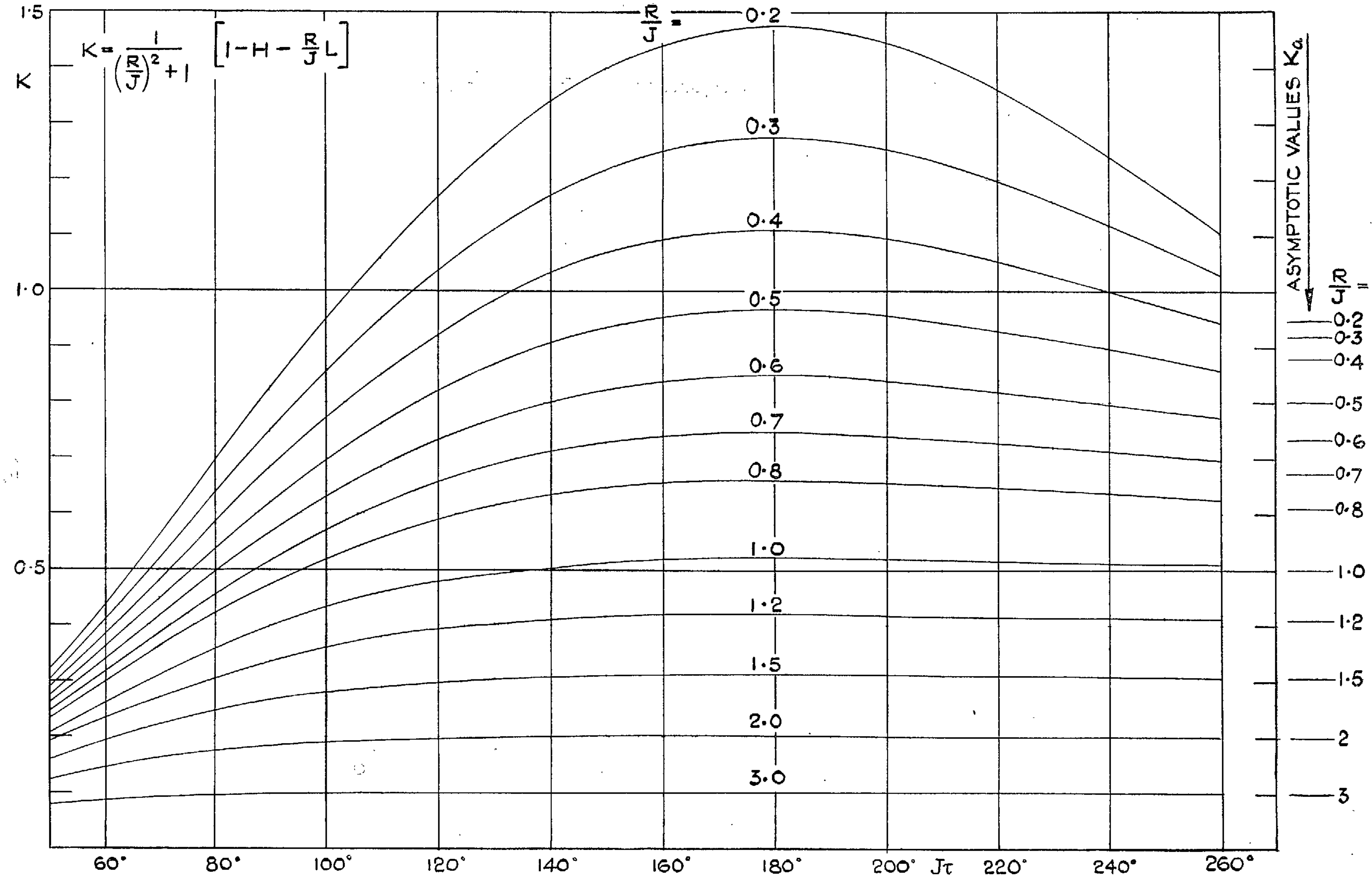



FIG. 8 (b). Function $K$.

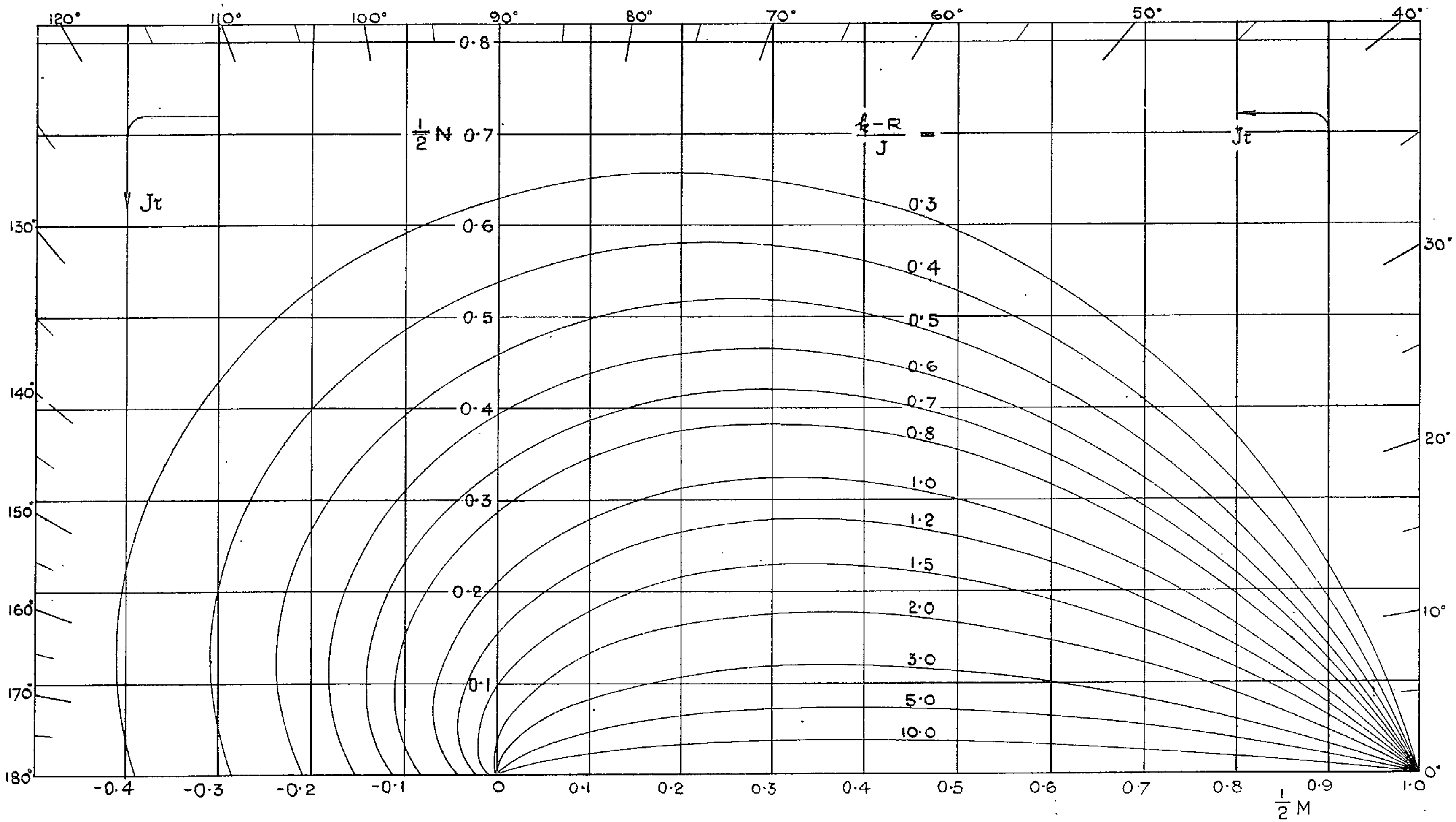



FIG. 9 (a). Graph for the solution of transcendental equations.

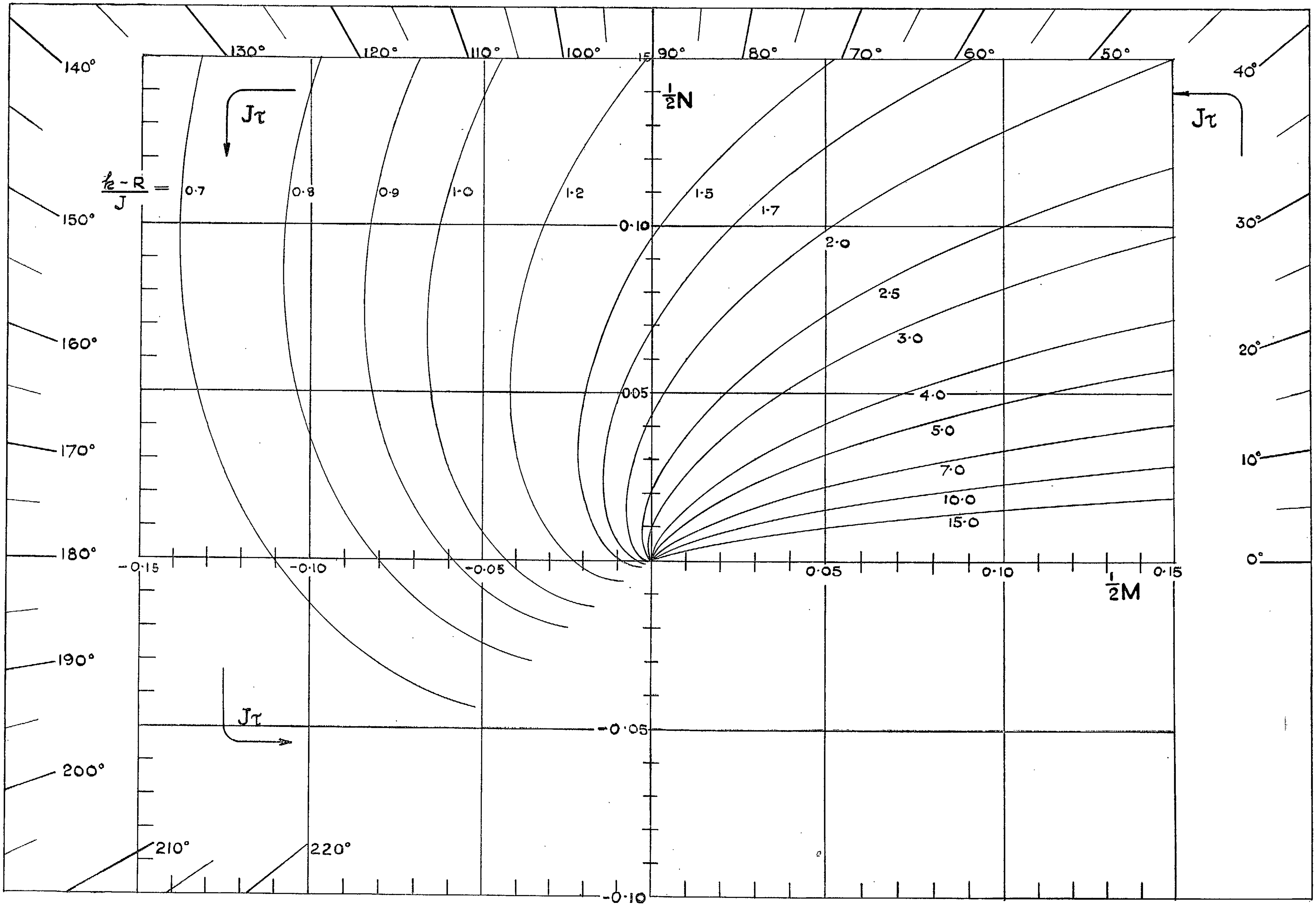



FIG. 9 (b). Graph for the solution of transcendental equations.

(4651)

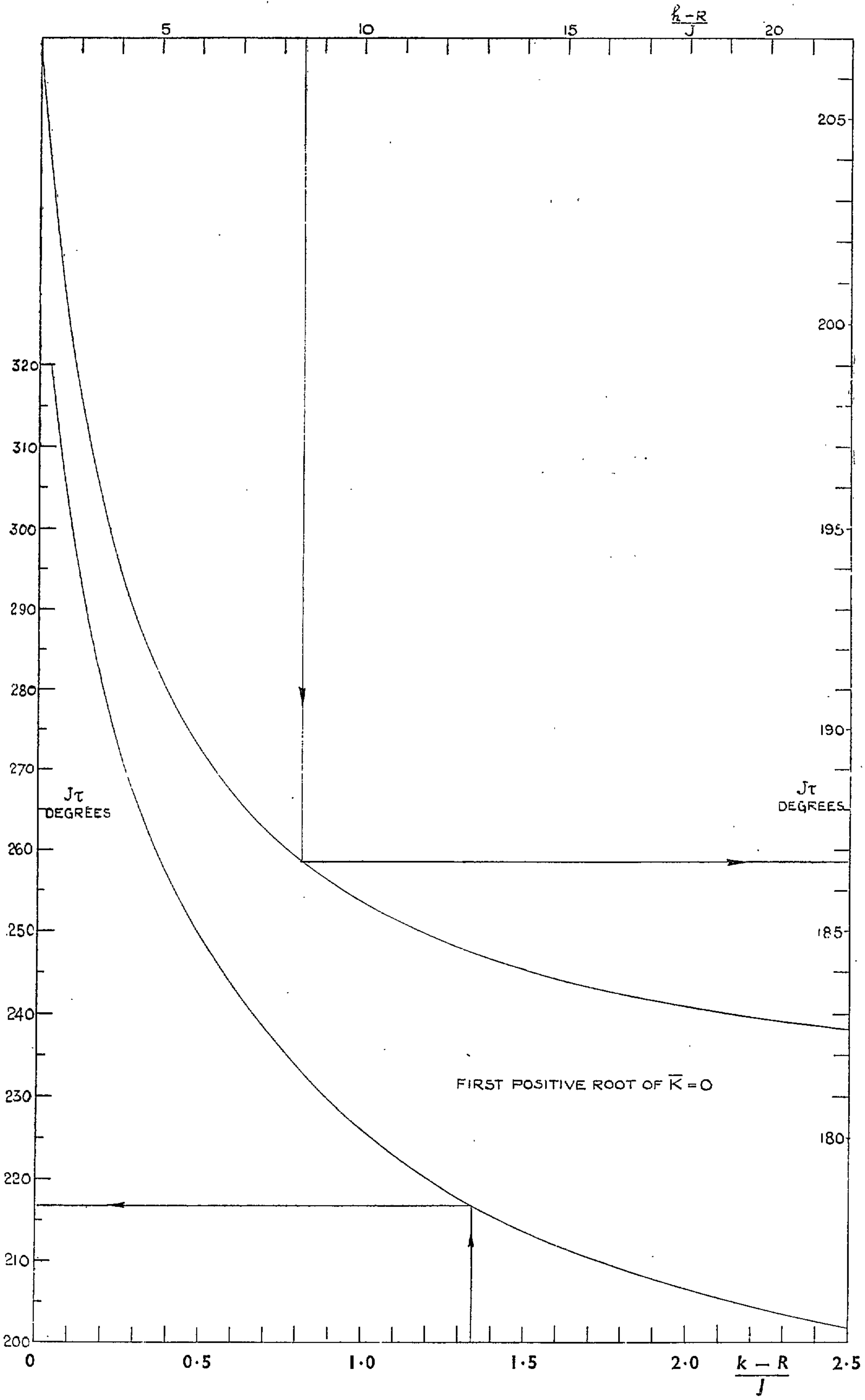



FIG. 10. First positive root of $\overline{K} = 0$.

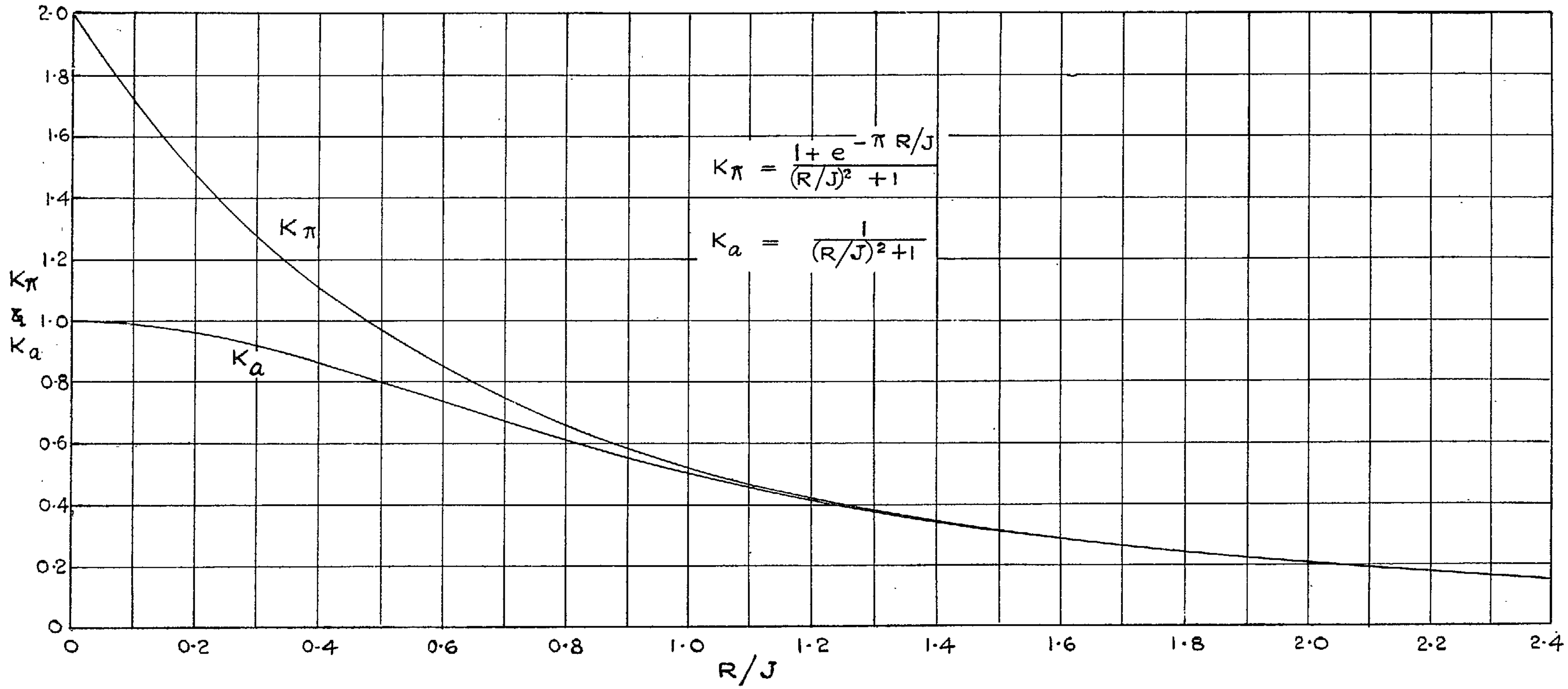



FIG. 11. Values of $K_\pi$ and $K_a$.

(4651) Wt. 20/9036 K.7 10/56 Hw.



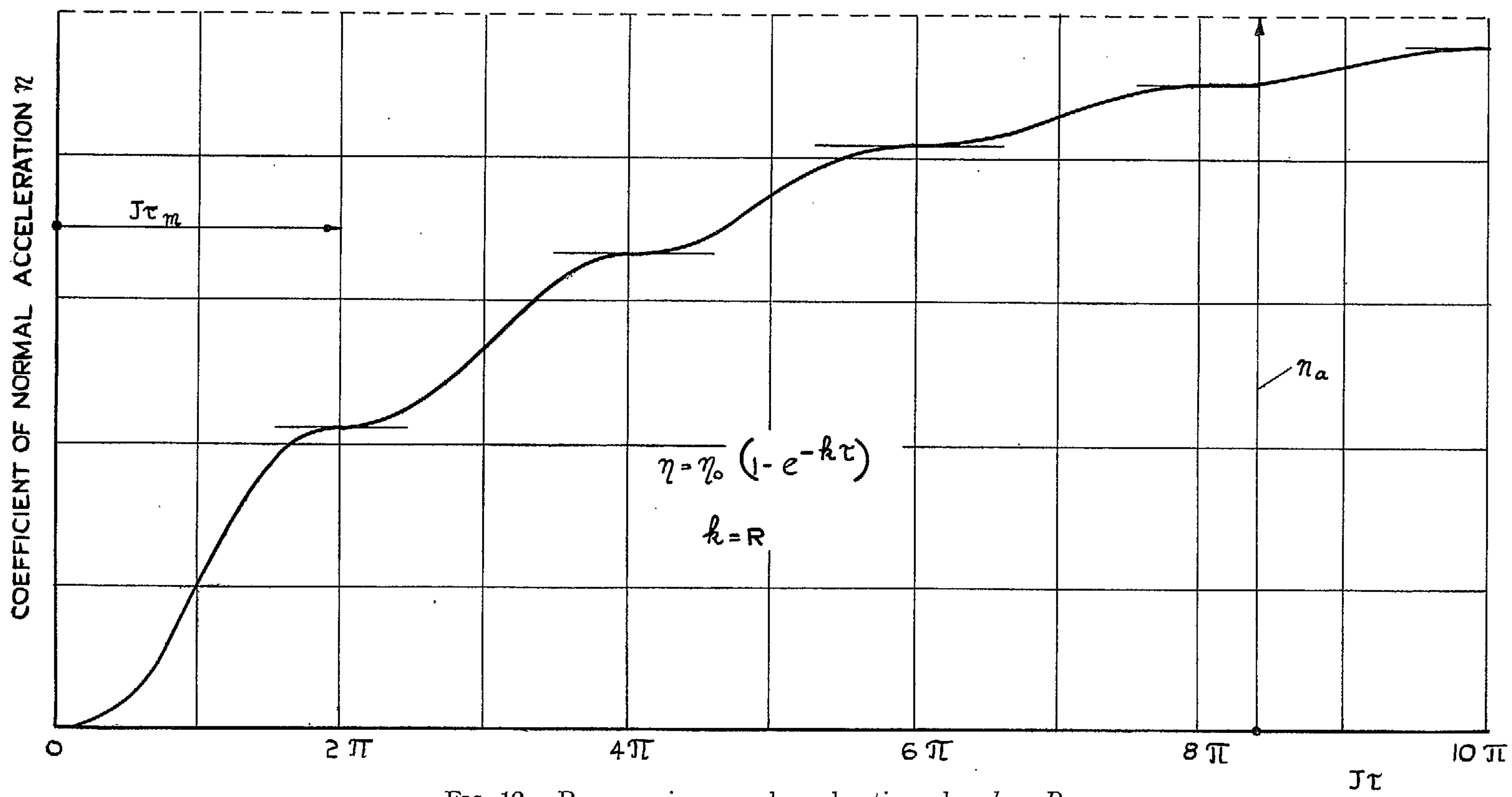



FIG. 12. Response in normal acceleration when $k = R$.

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