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Summary.

The problem of calculating the response of an aircraft in rolling manoeuvres when the mass distribution of the aircraft is such that the inertia terms in the equations of motion effect a cross-coupling of the usual lateral and longitudinal motions is considered. Solutions are outlined to two formulations of this problem: (1) Response to a given applied aileron and (2) Response corresponding to a specified time history of rate of roll. Detailed calculations are made only for the first of these, and the results compare favourably with digital-computer solutions.

Possible simplifications to the method of calculation are discussed.

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1. *Introduction.*

The trend towards long slender fuselages more evenly loaded than hitherto and often combined with considerable reduction of wing span has led to a new class of dynamic problems involving cross-coupling of the lateral and longitudinal motions. One of these is the complex cross-coupled motion associated with rapid rolling manoeuvres of some present day aeroplanes. This problem is the subject considered in the present paper.

The dynamics of aeroplane motion including cross-coupling effects have been the subject of much investigation recently, but most of the studies have emphasised the dynamics of specific aircraft, and have been conducted using either an analogue or digital computer for solving the equations of motion[2, 3, 4, 6, 11]. In contrast little has been done of a general analytic nature[5, 7]. However, as far back as 1948, W. H. Phillips[1] gave a simplified analysis of the stability of the coupled longitudinal and lateral motion following a disturbance from steady rolling flight. Neglecting damping and gravity terms Phillips arrives at two simplified criteria for stability. These can be written:

$$\frac{0 \cdot 196}{\left(\frac{ps}{V_e}\right)\left(\frac{k_B}{s}\right)} \sqrt{\left\{\frac{-\frac{\partial C_m}{\partial \alpha}\sigma\bar{c}}{W/S}\right\}} > 1,$$

and

$$\frac{0 \cdot 196}{\left(\frac{ps}{V_e}\right)\left(\frac{k_B}{s}\right)} \sqrt{\left\{\frac{\frac{\partial C_n}{\partial \beta}\sigma b}{(W/S)(1-I_x/I_y)}\right\}} > 1, \qquad (1)$$

and show why certain design trends should aggravate the problem of cross-coupled motions, for we see that there are four features tending to push the aircraft towards instability:

Increase of k_B (the radius of gyration, i.e. $mk_B^2 = I_y$) and the usually associated increase of $(1-I_x/I_y)$ due to redistribution of the mass of aircraft; increased wing loading, W/S; increased operational height, that is, reduced values of σ.

Furthermore, whilst the value of $-\partial C_m/\partial\alpha$ tends to be high for many supersonic aircraft, the values of $\dfrac{\partial C_n}{\partial\beta}$ have been tending to get smaller and decrease with increasing Mach number.

Interesting and instructive though this simplified analysis may be, it is not a sufficient basis for design of trouble-free aircraft since, away from the divergent or near divergent response, there can occur convergent responses having undesirable characteristics. It therefore becomes necessary to study the behaviour of an aircraft during practical rolling manoeuvres. As mentioned earlier, there have been a number of investigations relating to specific aircraft in which the computers have been employed to obtain numerical solutions. There is clearly a need for an extension of the analysis to cover either the response of the aircraft to a prescribed aileron input or that corresponding to a given rate-of-roll time history. The first attempt to do anything of this nature was made by Pinsker[3] who, on the same basis as Phillips, considered the response of an aircraft in the case when the rate of roll is represented by a square wave function. The present investigation is concerned with the more general problem. It was considered unwise to start with the drastic simplifications of the sort underlying these analyses. This naturally means that the resulting algebra is very complex, but the authors consider that adequate working approximations should be sought only when it has been demonstrated that the basic approach gives answers in agreement with the direct solution of the equations of motion using a digital or analogue computer.

Before proceeding to the description of the approximate method of dealing with the five degree of freedom equations, we shall make some general observations concerning the interplay of the inertia cross-coupling terms and the aerodynamic terms.

2. *Choice of Axis System and some General Observations.*

In discussing the dynamics of an aircraft it becomes necessary to define one or more systems of axes. The choice usually lies between the two systems of body axes usually referred to as the wind-body and the principal inertia axes systems. Each of these has particular advantages. The first, being defined by the steady state of flight condition, is best suited to the discussion of the stability of the aircraft and often facilitates physical description of factors involved in the motion. The other being of fixed orientation relative to the aircraft for all flight conditions, is preferred for problems of control systems using sensing instruments within the aircraft. It also has the mathematical advantage that it avoids the added complication of the product-of-inertia terms in the analysis.

The equations of motion of a rigid aircraft referred to *any* system of axes fixed in the body are of the form:

$$m(\dot{U}-rV+qW) = X \tag{2}$$

$$m(\dot{V}-pW+rU) = Y \tag{3}$$

$$m(\dot{W}-qU+pV) = Z \tag{4}$$

$$I_x\dot{p} - (I_y-I_z)qr - I_{yz}(q^2-r^2) - I_{zx}(\dot{r}+pq) - I_{xy}(\dot{q}-rp) = L \tag{5}$$

$$I_y\dot{q} - (I_z-I_x)rp - I_{zx}(r^2-p^2) - I_{xy}(\dot{p}+qr) - I_{yz}(\dot{r}-pq) = M \tag{6}$$

$$I_z\dot{r} - (I_x-I_y)pq - I_{xy}(p^2-q^2) - I_{yz}(\dot{q}+rp) - I_{zx}(\dot{p}-qr) = N. \tag{7}$$

For the rapid manoeuvre we are considering, and for usual aeroplane layouts (symmetrical with respect to the xz-plane) we may assume that the forward speed is constant, and $I_{xy} = I_{yz} \approx 0$. Our equations can then be written:

$$\frac{d\hat{V}}{dt} = \hat{W}p - r + Y/mV_e \tag{8}$$

$$\frac{d\hat{W}}{dt} = -\hat{V}p + q + Z/mV_e \tag{9}$$

$$\dot{p} = -\delta_x qr - e_x(\dot{r}+pq) + L/I_x \tag{10}$$

$$\dot{q} = -\delta_y pr - e_y(r^2-p^2) + M/I_y \tag{11}$$

$$\dot{r} = -\delta_z pq - e_z(\dot{p}-qr) + N/I_z . \tag{12}$$

If the principal inertia axes system is used, the terms in e_x, e_y, e_z disappear simplifying the equations as mentioned earlier. We shall now consider the nature of the second-order terms which are usually omitted, but which are of considerable significance in the present problem. The nature of the effect of the first terms on the right-hand side of the force equations can be appreciated most readily if we consider the limiting case of no aerodynamic forces and suppressed yawing ($r = 0$) and pitching motion ($q = 0$). Under these circumstances, we should have a cyclic interchange of the relative wind direction in the longitudinal and lateral plane with the aircraft rolling about its minimal inertia axis. This being so this effect is more readily appreciated in the principal inertia axes system. Suppose the x-axis of this system has an initial trimmed angle of incidence ϵ_0, then we would expect the motion referred to this axis system to begin with a decrease in angle of incidence (cf. Figs. 11, 13, 14).

The further point we note from an examination of the force equations is that, in the absence of aerodynamic terms, we would require a rate of yaw which varies as $\epsilon_0 p$, to make $d\hat{V}/dt$, and hence $\hat{V}$ or β, very small, while the rate of pitch, q, has to remain zero.

In the first of the moment equations we see that the inertia cross-coupling terms are unlikely to be large being in the principal inertia axes system of order qr. Of the aerodynamic terms we may expect that the term due to the rolling moment induced by sideslip ($L_v\hat{V}$) cannot generally be neglected. We would expect some deviation from the simple one degree of freedom solution. The direction of the sideslip development, as we have seen above, depends primarily on the inclination of the principal inertia axis. Initial sideslip is positive when the roll rate and the initial angle of incidence (not small) of the principal inertia axis are of the same sign, and generally negative when the signs are opposite. For small incidence, matters depend more critically on detailed aerodynamics.

In the equation for the acceleration in pitch we have two inertial terms which may become important; these are $\delta_y pr$ which is of order pr, and this term has become known as the *gyroscopic term* (cf. action occurring during precession of a conventional gyroscope) and the term $e_y(r^2-p^2)$, which is of order $\hat{W}_0 p^2$ or $\epsilon_0 p^2$. When the roll rate is large and of the same sign as the rate of yaw, then the first of these terms causes an upward pitch acceleration of appreciable magnitude, and can lead if opposed by only a small amount of aerodynamic restoring moment to a tendency to diverge in pitch. The second term is such that it will pitch the nose down if the forward principal inertia axis lies below the line of flight, and *vice versa*. It does not appear if we refer the motion to the principal inertia axes but we have to bear in mind that the rate of yaw differs by $\hat{W}_0 p$ in the two systems of axes. Its effect is therefore included in a modified gyroscopic term.

In the last of our equations of motion we have, similarly, two inertia terms whose order is pq and $-\hat{W}_0\dot{p}$ respectively. The first of these plays a similar role in the yawing equation to that of pr in the pitch equation.

The action of the second term is again determined by the inclination of the forward principal inertia axis relative to the initial flight path (or x-wind-body axis). Its action is obscured by the use of principal inertia axes since the term then disappears. We recall, however, that the yaw acceleration differs by $\hat{W}_0\dot{p}$ in the two systems of axes, and the sideslip response is the same in both systems of axes as it should be since the $\hat{W}_0 p$ component of the rate of yaw which no longer exists in principal axes is compensated by just this difference in the $\hat{W}p$ term of the sideslip acceleration equation.

Of the aerodynamic terms the most important is usually the $N_v\hat{V}$ term (yawing moment due to sideslip) which will tend to reduce the sideslip response. The action of the damping in yaw term, $N_r r$, is different in different flight conditions. It restricts the development of rate of yaw, r, and this is undesirable in the case of large positive $\hat{W}_0$ for a positive rate of roll. Unaugmented, the contribution of this term is probably not large, but the effect is of significance in considering aeroplanes with yaw autostabilisation. The yawing moment produced by deflection of the aileron has an effect on the motion whose significance depends on the sign of yawing moment due to aileron, and the magnitude and sign of $\hat{W}_0$ or ϵ_0.

The above discussion is clearly of restricted usefulness only, since the interaction of the various factors is intrinsically simultaneous, and this cannot be allowed for in the above description. It does nevertheless outline the nature of the equations of motion to which we are seeking a solution.

3. *Approximate Solution of the Equations of Motion.*

3.1. *Equations of Motion.*

We now rewrite the equations of motion by introducing a set of non-dimensional quantities formed by dividing forces and moments by $\rho V_e^2 S$ and $\rho V_e^2 Ss$ respectively, time by $\hat{t} = m/\rho V_e S$, mass by the mass (m) of the aeroplane and introduce the semi-span, s, as a characteristic length so that moments of inertia are divided by ms^2. This is the usual system of units used in the uncoupled lateral motion, and so all lateral derivatives retain their usual form. The longitudinal moment derivatives are, however, modified thus*:

$$\frac{M_w}{\rho V_e Ss} = \frac{\bar{c}}{s} m_w ,$$

$$\frac{M_q}{\rho V_e Ss^2} = \left(\frac{\bar{c}}{s}\right)^2 m_q ,$$

$$\frac{M_{\dot{w}}}{\rho Ss^2} = \left(\frac{\bar{c}}{s}\right)^2 m_{\dot{w}} .$$

* It may be worth noting that although the derivatives themselves are modified the corresponding concise quantity has exactly the same value as it would have had if the more usual characteristic length had been used in the definition of μ, i_B and the derivative.

The equations of motion can now be written in the form:

$$\left.\begin{aligned}
D\hat{p} + \delta_x \hat{q}\hat{r} &= \frac{\mu l_\xi}{i_A}\xi + \frac{\mu l_v}{i_A}\hat{v} + \frac{l_p}{i_A}\hat{p} + \frac{l_r}{i_A}\hat{r} \\
D\hat{q} + \delta_y \hat{p}\hat{r} &= \frac{\mu \frac{\bar{c}}{s} m_w}{i_B}\hat{w} + \frac{\left(\frac{\bar{c}}{s}\right)^2 m_q}{i_B}\hat{q} + \frac{\left(\frac{\bar{c}}{s}\right)^2 m_{\dot{w}}}{i_B} D\hat{w} \\
D\hat{r} + \delta_z \hat{p}\hat{q} &= \frac{\mu n_\xi}{i_C}\xi + \frac{\mu n_v}{i_C}\hat{v} + \frac{n_p}{i_C}\hat{p} + \frac{n_r}{i_C}\hat{r} \\
D\hat{v} - (\hat{w} + \hat{W}_0)\hat{p} + \hat{r} &= y_v \hat{v} + \frac{C_{Le}}{2}\cos\theta \sin\phi \\
D\hat{w} - \hat{q} + \hat{v}\hat{p} &= z_w \hat{w} + \frac{C_{Le}}{2}(\cos\theta\cos\phi - \cos\theta_0)\,.
\end{aligned}\right\} \qquad (13)$$

3.2. *Approximations Made in Dealing with Inertia Product Terms.*

To proceed we write:

$$\hat{p} = \hat{p}_0(t) + \hat{p}', \quad D\hat{p} = D\hat{p}_0 + D\hat{p}',$$

where $\hat{p}_0(t)$ is an approximation to $\hat{p}$ such that we may further approximate as follows:

$$\hat{p}\hat{r} = (\hat{p}_0 + \hat{p}')\hat{r} \approx \hat{p}_0\hat{r},$$

$$\hat{p}\hat{q} = (\hat{p}_0 + \hat{p}')\hat{q} \approx \hat{p}_0\hat{q},$$

$$\hat{v}\hat{p} = (\hat{p}_0 + \hat{p}')\hat{v} \approx \hat{p}_0\hat{v},$$

and $(\hat{W}_0 + \hat{w})\hat{p} \approx \hat{W}_0\hat{p} + \hat{w}\hat{p}_0 \approx \hat{W}_0\hat{p}_0 + \hat{w}\hat{p}_0 + \hat{W}_0\hat{p}'$, $\hat{p}'$, $\hat{q}$, $\hat{r}$, $\hat{w}$, $\hat{v}$, θ and θ_0 being assumed small of first order. The term $\delta_x\hat{q}\hat{r}$ is accordingly neglected.

Substitution of these approximations in the equations of motion, (13), do not greatly simplify matters unless we can make $\hat{p}_0(t) = \hat{p}_0 =$ constant.* The next step is to assume that over certain intervals of time we may approximate in this manner. It is, therefore, seen that the method of calculation we shall now develop can be described as a step-by-step integration of the equations involving only few steps and with the integration formula within each step being an analytic solution of approximations to the equations of motion.

---

* It may be mentioned that high rates of roll, and hence cross-coupling effects, may follow rudder application for aircraft having large 'Dutch-roll' ratios. In such a case, an alternative (and in some ways a more desirable) course would be to insert the linearised solution for the product terms. The solution may then be sought as a perturbation of the linear solution. This would result in linear equations with time-dependent coefficients. To reduce the problem to the same extent as done herein would require substitution for product terms only and treating these as inputs into the system of equations. We did not pursue this line of approach any further as it was considered that it would not be so accurate where large values of p are involved whilst at the same time not offering much simplification.

We are thus led to consider equations of the form:

$$\left.\begin{aligned}
D\hat{p} &= \frac{\mu l_\xi}{i_A}\,\xi + \frac{\mu l_v}{i_A}\,\hat{v} + \frac{l_p}{i_A}\,\hat{p} + \frac{l_r}{i_A}\,\hat{r} \\
D\hat{q} + \delta_y \hat{p}_0 \hat{r} &= \frac{\mu m_w}{i_B}\left(\frac{\bar{c}}{s}\right)\hat{w} + \frac{m_q}{i_B}\left(\frac{\bar{c}}{s}\right)^2 \hat{q} + \frac{m_{\dot{w}}}{i_B}\left(\frac{\bar{c}}{s}\right)^2 D\hat{w} \\
D\hat{r} + \delta_z \hat{p}_0 \hat{q} &= \frac{\mu n_\xi}{i_C}\,\xi + \frac{\mu n_v}{i_C}\,\hat{v} + \frac{n_p}{i_C}\,\hat{p} + \frac{n_r}{i_C}\,\hat{r} \\
D\hat{v} - \hat{p}_0\hat{w} - \hat{W}_0\hat{p} + \hat{r} &= y_v\hat{v} + \frac{C_{Le}}{2}\sin\phi \\
D\hat{w} - \hat{q} + \hat{p}_0\hat{v} &= z_w\hat{w} - \frac{C_{Le}}{2}(1-\cos\phi)\,.
\end{aligned}\right\} \tag{14}$$

3.3. *Treatment of the Gravity Terms.*

Apart from the gravity terms the equations (14) are in a linearised form admitting of standard solution. The effect of these gravity terms has been found to be small in such investigations as have been made, but these have usually involved only low values of C_{Le} and their significance will increase for larger values of C_{Le}. An assessment of the importance of these terms was made when the aeroplane was assumed to perform a constant rate of roll manoeuvre as in Phillips' analysis. The details are given in Appendix II. This analysis shows that the effect is small provided the rate of roll is in the range where inertia cross-coupling effects are appreciable, and C_{Le} is small to moderate in value, *see* Figs. 3 to 10. It is, however, unnecessary to neglect the gravity terms completely to render our problem manageable and within the assumptions underlying equations (14) we may approximate by writing:

$$\begin{aligned}
\phi &= \phi_i + \hat{p}_0\tau + \varphi\,, \\
\sin\phi &= \sin(\phi_i + \hat{p}_0\tau + \varphi) \\
&\approx \sin\phi_i\cos\hat{p}_0\tau + \cos\phi_i\sin\hat{p}_0\tau
\end{aligned}$$

and

$$\cos\phi \approx \cos\phi_i\cos\hat{p}_0\tau - \sin\phi_i\sin\hat{p}_0\tau\,,$$

where φ is a perturbation angle of bank.

This implies the neglect of terms of order $C_{Le}\varphi$, which is consistent with the neglect of terms such as $\hat{p}'\hat{w}$, $\hat{p}'\hat{r}$ etc.

3.4. *Solutions of the Final Approximate Form of Equations.*

We are thus led to consider our equations of motion in the form:

$$\left.\begin{array}{lllll}
(D+\nu_l)\hat{p} & & -\nu_{lr}\hat{r} & +\omega_l\hat{v} & = -\delta_{l\xi}\xi \\
 & (D+\nu)\hat{q} & +\delta_y\hat{p}_0\hat{r} + (\omega+\chi D)\hat{w} & & = 0 \\
\nu_{np}\hat{p} & +\delta_z\hat{p}_0\hat{q} & +(D+\nu_n)\hat{r} & -\omega_n\hat{v} & = -\delta_{n\xi}\xi \\
-\hat{W}_0\hat{p} & & +\hat{r} & -\hat{p}_0\hat{w} + (D+\bar{y}_v)\hat{v} & = \dfrac{C_{Le}}{2}(\sin\phi_i\cos\hat{p}_0\tau + \cos\phi_i\sin\hat{p}_0\tau) \\
 & -\hat{q} & +(D-z_w)\hat{w} & +\hat{p}_0\hat{v} & = \dfrac{C_{Le}}{2}(\cos\phi_i\cos\hat{p}_0\tau - \sin\phi_i\sin\hat{p}_0\tau - 1)\,.
\end{array}\right\} \tag{15}$$

In operational form, including terms representing initial values of $\hat{p}$, $\hat{q}$, $\hat{r}$, $\hat{v}$, $\hat{w}$, these equations become[8]:

$$
\left.\begin{aligned}
&(D+\nu_l)\hat{p} - \nu_{lr}\hat{r} + \omega_l\hat{v} = -\delta_{l\xi}\bar{\xi} + \hat{p}_i D \\
&(D+\nu)\hat{q} + \delta_y\hat{p}_0\hat{r} + (\omega+\chi D)\hat{w} = \hat{q}_i D + \chi\hat{w}_i D \\
&\nu_{np}\hat{p} + \delta_z\hat{p}_0\hat{q} + (D+\nu_n)\hat{r} - \omega_n\hat{v} = -\delta_{n\xi}\bar{\xi} + \hat{r}_i D \\
&-\hat{W}_0\hat{p} + \hat{r} - \hat{p}_0\hat{w} + (D+\bar{y}_v)\hat{v} = \hat{v}_i D + \frac{C_{Le}}{2}\frac{(\sin\phi_i D^2 + \cos\phi_i\hat{p}_0 D)}{(D^2+\hat{p}_0^2)} \\
&-\hat{q} + (D-z_w)\hat{w} + \hat{p}_0\hat{v} = \hat{w}_i D + \frac{C_{Le}}{2}\frac{(\cos\phi_i D^2 - \sin\phi_i\hat{p}_0 D)}{(D^2+\hat{p}_0^2)} - \\
&\qquad - \frac{C_{Le}}{2}.
\end{aligned}\right\} \quad (16)
$$

The solution of these equations is now reasonably straightforward but the detailed algebra is extremely lengthy and is accordingly omitted almost completely.

For the case of constant aileron angle, $\xi = \xi_0 = \bar{\xi}$ (Heaviside operational equivalent) the operational solution of the above equations has the form:

$$
\left.\begin{aligned}
\hat{p} &= \frac{H_{01}D^7 + H_{11}D^6 + H_{21}D^5 + H_{31}D^4 + H_{41}D^3 + H_{51}D^2 + H_{61}D + H_{71}}{(D^2+\hat{p}_0^2)(G_0D^5+G_1D^4+G_2D^3+G_3D^2+G_4D+G_5)}, \\
\hat{q} &= \frac{H_{02}D^7 + H_{12}D^6 + H_{22}D^5 + H_{32}D^4 + H_{42}D^3 + H_{52}D^2 + H_{62}D + H_{72}}{(D^2+\hat{p}_0^2)(G_0D^5+G_1D^4+G_2D^3+G_3D^2+G_4D+G_5)},
\end{aligned}\right\} \quad (17)
$$

with similar expressions for $\hat{r}$, $\hat{w}$, $\hat{v}$. The second number of the suffix of the H's denotes the variable in question according to the scheme 1, 2, 3, 4, 5 correspond to p, q, r, $\hat{w}$, $\hat{v}$. (Formulae for G's and H's are given in Appendix I.) The second factor in the denominator can be written:

$$
\left\{\begin{vmatrix} D+\bar{y}_0 & 0 & 1 \\ \omega_l & D+\nu_l & -\nu_{lr} \\ -\omega_n & \nu_{np} & D+\nu_n \end{vmatrix} + \hat{W}_0\begin{vmatrix} \omega_l & -\nu_{lr} \\ -\omega_n & D+\nu_n \end{vmatrix}\right\}\begin{vmatrix} D+\nu & \omega+\chi D \\ -1 & D-z_w \end{vmatrix} +
$$

$$
+ \hat{p}_0^2[(D+\nu)(D+\nu_n)(D+\nu_l) - \delta_y\delta_z(D-z_w)(D+\nu_l)(D+\bar{y}_0) - \delta_z(\omega+\chi D)(D+\nu_l) + \delta_y\omega_n(D+\nu_l) -
$$

$$
- \delta_y\delta_z\hat{p}_0^2(D+\nu_l) + \nu_{np}\nu_{lr}(D+\nu) + \delta_y\nu_{np}\omega_l + \hat{W}_0 D\{\delta_z\nu_{lr}(\omega+\chi D) - \delta_y\delta_z\omega_l(D-z_w)\}].
$$

The first term represents the product of the two uncoupled motions and the second term represents the coupling effect and as such disappears when $\hat{p}_0$ is zero.

To proceed we have to split the right hand side of equation (17) into its partial fractions. It is seen, cf. Appendix III, that the polynomial

$$G_0\lambda^5 + G_1\lambda^4 + G_2\lambda^3 + G_3\lambda^2 + G_4\lambda + G_5$$

in a typical case factorises into

$$(\lambda^2+a_1\lambda+b_1)(\lambda^2+a_2\lambda+b_2)(\lambda+b_3)$$

or

$$(\lambda^2+a_1\lambda+b_1)(\lambda+b_3)(\lambda+b_4)(\lambda+b_5).$$

Transforming we thus have solutions of the form:

$$A_{0n} + A_{1n}e^{-b_3\tau} + e^{-r_2\tau}(A_{2n}\cos s_2\tau + A_{3n}\sin s_2\tau) + e^{-r_1\tau}(A_{4n}\cos s_1\tau + A_{5n}\sin s_1\tau)$$

or

$$B_{0n} + B_{1n}e^{-b_3\tau} + B_{2n}e^{-b_4\tau} + B_{3n}e^{-b_5\tau} + e^{-r_1\tau}(B_{4n}\cos s_1\tau + B_{5n}\sin s_1\tau).$$

The coefficients A and B can be evaluated directly from equations (17) by well known methods, *see*, for example, Refs. 8 and 9. Usually b_1 is large (indicating a fast mode) and in Appendix III use is made of this to develop approximate expressions for the coefficients A_4, B_4, A_5, B_5 and the factors of the above polynomial.

3.5. *Discussion of Choice of the Value of $\hat{p}_0$.*

To apply the method outlined above to the calculation of the response of the aeroplane to a given aileron input we need to specify the value or values of $\hat{p}_0$ to be used. We recall that during the initial stages (and often beyond this) the response in roll is not expected to differ appreciably from that given by the simple single degree of freedom calculation. For the two aileron inputs considered herein this is illustrated in Figs. 1 and 2. It was anticipated, and the examples calculated confirmed, that it would be unnecessary to commence with a very low value of $\hat{p}_0$, since the error incurred would be small for small initial values of $\hat{w}$, $\hat{v}$, q and r. If the simple roll calculation indicates a relatively slow growth of p, it seems reasonable to select for initial value of p_0 a value of the order of $\frac{1}{2}p_{\max}$. In such cases as the single degree of freedom calculation indicates a rapid growth of rate of roll as, for example, for aileron input of the multi-square wave type, Fig. 2, a higher value can be used without much apparent loss of accuracy (*see* Figs. 13, 14, 15, 16 and 17).

The choice of the subsequent steps in $\hat{p}_0$ is determined by the trends indicated by the single degree of freedom calculation suitably modified, if necessary, by any marked departure of the more exact time history of p from the simple roll result. Thus, although the response in α (or $\hat{w}$) and β (or $\hat{v}$) may be of most immediate interest, it is advisable, in dealing with a general aileron-angle input, to compute the rate of roll as well.

Having described the method of calculation in general terms, we now proceed to discuss the results obtained for an aircraft having the aerodynamic derivatives set out in Table 1.

4. *Numerical Examples.*

To check on the validity of the approximations made in the solution outlined above we shall compare the response as calculated by this method with that obtained by exact solution of the equations of motion as obtained by use of the DEUCE digital computer.

An aeroplane having the aerodynamic characteristics, geometry, and inertias shown in Table 1 is considered, as well as one which has its forward inertia axis inclined 5° below the flight path in steady flight instead of 5° above but which is otherwise identical. Flight at one speed and height only is considered, namely, $M = 0 \cdot 8$ and 40,000 ft. The responses to two types of aileron input have been calculated (*see* Figs. 1 and 2).

Fig. 11, which refers to a positive value of inclination of forward inertia axis, $\epsilon_0 = 5°$, shows the response in the rate of roll, (p), the incidence, (α), the sideslip angle, (β), the rate of pitch, (q), and the rate of yaw, (r), following a simple single wave input of aileron angle ($\xi = 8°$ for $0 < t \leqslant 1 \cdot 8$ sec, and zero thereafter).

The full-line curves are the exact solutions, and circled points are used to indicate the results obtained by using the approximate method of Section 3. Values of p_0 used at different stages of the calculations are indicated on the rate-of-roll curve. The agreement of the two solutions is good throughout.

The same calculation was repeated with $\epsilon_0 = -5°$, and to illustrate the effect of changes in p_0 step pattern a rather crude approximation is used, *see* Fig. 12. As might be expected the agreement with the exact solution is not so good but is still reasonable. Comparison of the response in this case with that illustrated by the preceding figure shows the importance of the parameter ϵ_0, as mentioned in Section 2 and as found by other investigators.

The simple aileron input used in the calculations just described leads to a somewhat unrealistic rolling manoeuvre. It may be argued that it is more desirable to specify the rate-of-roll time history but, as the analysis is developed for a solution following a specified aileron input, this is not immediately practicable. A more realistic rate-of-roll curve can, however, be obtained if the single degree of freedom calculation is used for determining the aileron input required to give a specified rate of roll, and the aileron input so obtained modified to suit the convenience of the five degree of freedom calculation. Such a process is described in Appendix V, and is the basis of the second type of aileron input (Fig. 2) considered. It should be noted that this aileron input depends on the two aircraft characteristics, damping in roll (l_p) and the moment of inertia in roll (i_A).

In the first of the calculations for the aileron input of the type illustrated in Fig. 2 the aileron angles were so arranged as to give a rate of roll corresponding to $\hat{p}$ value of $-6 \cdot 76$ approximately, so as to enable use to be made of data already computed for $\hat{p}_0 = -6 \cdot 76$. The value of $\hat{p}_0$ is kept constant at this level for the time interval corresponding to 0 to $0 \cdot 35$ in τ. After this time the motion is assumed virtually uncoupled, that is, $\hat{p}_0 = 0$. The method of Section 3 gives again results in good agreement with the exact solution (*see* Fig. 13). Also shown on the same figure is the response as computed using the simplified analysis of Appendix III. This is in sufficiently good agreement with the other solution as to provide acceptably good estimates of the maximum disturbance in α and β. The initial incidence of the forward principal inertia axis was taken as $+5°$.

To study the effect of a faster roll, the aileron displacements were increased to give a maximum negative rate of roll of about 136°/sec. Fig. 14 illustrates the effect of these changes on the response in the other variables. Again agreement of the approximate solutions with the exact values is fairly good, although naturally not so good as in Fig. 13 which refers to the somewhat slower roll.

Lastly, the calculations were repeated with the input as in the preceding example, but with the principal inertia axis inclined below the flight path initially by 5° ($\epsilon_0 = -5°$). The results of this set of calculations are shown in Fig. 15 and again, as in the previous examples with ϵ_0 negative, the rate of roll does not tend to decay after the final centralisation of the aileron. The time interval over which the response is known, in all the calculations already referred to, is insufficient to give a clear indication of the behaviour of the aircraft some time after the aileron had been centralised. The practical significance of the subsequent behaviour may be questioned on the grounds that the pilot would increase either the interval of time over which reversed aileron is applied, or the amount applied. These actions can, provided the rate of roll at the moment the aileron is finally centralised is small, result in the return to virtually uncoupled damped motion as shown in Figs. 16 and 17. However, as this is a matter of judgement, it is clear that the subsequent behaviour in the event of finishing with too rapid a rate of roll is of some importance. The response to the input of Fig. 15 over a prolonged time interval is shown in Fig. 18. It is interesting to compare the results of digital computer with the analysis of alternative steady states which emerge if gravity terms are neglected. The non-linear steady-state equations have solutions other than that giving zero values for all the variables, p, q, r, $\hat{w}$, and $\hat{v}$. These involve steady rotation about all three axes and constant incidence and sideslip. Details of the calculation of these steady states and their stability are given in Appendix IV.

For the aircraft characteristics assumed throughout the present investigation, the motion seems to be alternate oscillations about two values of the variables which are in fair agreement with steady-state values as given by the analysis of Appendix IV.

These steady states are in the present instance unstable with respect to small disturbances. Accordingly the completely linearised response would show a tendency to depart from the steady state. This is illustrated in Fig. 19. Here a point in the time history of Fig. 16 was chosen at which the main deviation from the steady-state conditions is in rate of roll, incidence and rate of yaw, and the linearised response to these initial disturbances calculated. At first the rate of roll varies more or less in accordance with the linearised response but after half an oscillation has elapsed it departs appreciably from the exact (digital-computer) solution. To what can we attribute this departure? It is either the effect of the gravity terms (which on the basis of Appendix II we would expect to be negligible when p is fairly large), or the effect of the inertia cross-coupling terms, that is, we are not permitted to write these as $\hat{p}_s \delta \hat{q}$ etc. as in Appendix IV but should treat them on the lines of the main text or Appendix III. To assess the first effect directly the digital-computer calculation has been repeated omitting gravity terms. For a comparison of the two sets of results *see* Fig. 20. It is clear that we may rule out the gravity terms as being the primary factor, and so we conclude that in the type of motion illustrated by Figs. 15 and 18 the lateral and longitudinal motions are both affected by the inertia terms in pq etc.

5. *Conclusions.*

In calculating the response to aileron application of an aircraft in which the yawing and pitching inertias are large compared with the rolling inertias it is necessary to include products of the rate of rotation about the roll axis and either of the other two axes (pq and pr terms).

Such calculations are normally performed using either analogue or digital computers. The method developed in the present paper or its simplified version (Appendix III) offers an alternative, with the added attraction of possibilities of further simplification. Comparison of the results of the method described herein with those given by the digital-computer calculations indicates a satisfactory accuracy.

The next step is to simplify as much as possible without undue loss of accuracy. It is considered that many of the aerodynamic terms, retained here for completeness and to remove all possibility of doubt in the assessment of accuracy, may be neglected. For example we may be justified in retaining only terms in m_w and n_v with 'effective'* m_q and n_r, together with l_v.

Although the main purpose of the present investigation is the proving of the accuracy of the proposed method, we find, in agreement with other investigators, that the inclination of the forward principal inertia axis to the flight path in the initial equilibrium condition is an important parameter (Refs. 2, 3, 6, 11). The effects of changes in the aerodynamics have not been considered numerically but it is clear that the emphasis lies in the derivatives n_v and m_w, and it is this which makes it reasonable to anticipate the possibility of further simplification.

On the basis of the calculations for an aircraft rolling continuously at a constant rate, comparisons of digital-computer solutions with and without gravity terms, as well as comparison of the results of the simplified calculation of Appendix III with other results, we conclude that, provided the rate of roll is not small, the gravity terms may be neglected.

---

* 'Effective' is used in here in the sense that the values are adjusted to give correct damping of the uncoupled motions, e.g.

$$\text{Effective } m_q = m_q + m_{\dot{w}}$$

Aerodynamic derivatives have been treated as constant throughout the present paper. The most serious omissions in this way are any non-linear properties of m_w and n_v, with dependence of l_v, n_ξ, and n_p on incidence being of secondary importance, but still having some significance in certain cases.

The method described can be adapted to deal with the rolling pull-out manoeuvre, and this is particularly straightforward if the elevator is applied and centralised sufficiently ahead of the aileron for the response to it to be calculated as an uncoupled motion.

For some applications, for example in checking structural integrity, the analysis of Appendix VI, namely, the calculation of the response in incidence and sideslip when a prescribed rate-of-roll time history is achieved, will be a more appropriate approach than the direct problem considered elsewhere.

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LIST OF SYMBOLS

b		Aircraft span
C_L		Lift coefficient $= \dfrac{\text{Lift}}{\frac{1}{2}\rho V^2 S}$
C_m		Pitching-moment coefficient $= \dfrac{M}{\frac{1}{2}\rho V^2 S\bar{c}}$
$\bar{c}$		Wing mean chord $= \dfrac{S}{b}$
D		Total differential operator with respect to time
e_x	=	$-I_{zx}/I_x$ ratio of product of inertia to moment of inertia about each axis
e_y	=	$-I_{zx}/I_y$
e_z	=	$-I_{zx}/I_z$
G, H		Coefficients in operational solutions (*see* Section 3.4 and Appendix I)
I_x		Moment of inertia about x-axis
I_y		Moment of inertia about y-axis
I_z		Moment of inertia about z-axis
I_{yz}		Product of inertia with respect to x-axis
I_{zx}		Product of inertia with respect to y-axis
I_{xy}		Product of inertia with respect to z-axis
i_A	=	I_x/ms^2
i_B	=	I_y/ms^2
i_C	=	I_z/ms^2
L		Rolling moment about x-axis
l_p, l_r, l_v, l_ξ		Rolling-moment derivatives with respect to the principal inertia axes (*see* Royal Aeronautical Society Data Sheets)
M		Pitching moment about y-axis
m_q, m_w, $m_{\dot{w}}$		Pitching-moment derivatives with respect to the principal inertia axes (*see* Royal Aeronautical Society Data Sheets)

$$\left.\begin{aligned} \left(\frac{\bar{c}}{s}\right)^2 m_q &= \frac{M_q}{\rho V_e S s^2} \\ \frac{\bar{c}}{s} m_w &= \frac{M_w}{\rho V_e S s} \\ \left(\frac{\bar{c}}{s}\right)^2 m_{\dot{w}} &= \frac{M_{\dot{w}}}{\rho S s^2} \end{aligned}\right\} \text{modified longitudinal moment derivatives}$$

N Yawing moment about z-axis

n_p, n_r, n_v, n_ξ Yawing moment derivatives with respect to the principal inertia axes (*see* Royal Aeronautical Society Data Sheets)

p Rate of roll about x-axis

$\hat{p}$ $=$ $p\hat{t}$, angular velocity in roll (non-dimensional form)

$\hat{p}_0$ An assumed angular velocity in roll (non-dimensional form) used in the analysis of Section 3.2

$\hat{p}'$ Perturbation rate of roll (non-dimensional form) $= \hat{p} - \hat{p}_0$

q Rate of pitch about y-axis

$\hat{q}$ $=$ $q\hat{t}$, angular velocity in pitch (non-dimensional form)

r Rate of yaw about z-axis

$\hat{r}$ $=$ $r\hat{t}$, angular velocity in yaw (non-dimensional form)

S Wing area

s $=$ $\frac{b}{2}$, wing semi-span

t Time

$\hat{t}$ $=$ $\frac{m}{\rho V_e S}$

U Velocity component along x-axis

V Velocity component along y-axis

V_e Resultant steady-state velocity

W Velocity component along z-axis

v, w Small disturbance values of V and W

$\hat{v}$ $=$ $v/V_e = \beta$, sideslip angle

$\hat{w}$ $=$ $w/V_e = \alpha - \epsilon_0$, change in angle of incidence

X		Force component (including gravity terms where applicable) along x-axis
Y		Force component (including gravity terms where applicable) along y-axis
Z		Force component (including gravity terms where applicable) along z-axis
y_v		Y-force derivative $(\bar{y}_v = -y_v)$ (*see* Royal Aeronautical Society Data Sheets)
z_w		Z-force derivative (*see* Royal Aeronautical Society Data Sheets)
α		Angle of incidence of principal inertia axis
β		Sideslip angle
δ_x	=	$\dfrac{I_z - I_y}{I_x}$
δ_y	=	$\dfrac{I_x - I_z}{I_y}$
δ_z	=	$\dfrac{I_y - I_x}{I_z}$
$\delta_{l\xi}$	=	$-\dfrac{\mu l_\xi}{i_A}$, concise aileron effectiveness derivative
$\delta_{n\xi}$	=	$-\dfrac{\mu n_\xi}{i_C}$, concise yawing-moment due to aileron derivative
ϵ_0		$(= \hat{W}_0)$ Initial angle of incidence of principal inertia axis
θ		Angle between x-axis and horizontal
μ	=	$\dfrac{m}{\rho S s} = \dfrac{V_e \hat{t}}{s}$, relative density of aircraft referred to semi-span
ν	=	$-\dfrac{\left(\dfrac{\bar{c}}{s}\right)^2 m_q}{i_B}$, concise derivative for rotary damping in pitch
ν_l	=	$-\dfrac{l_p}{i_A}$, concise damping-in-roll derivative
ν_{lr}	=	$\dfrac{l_r}{i_A}$, concise rolling-moment derivative due to yaw
ν_{np}	=	$-\dfrac{n_p}{i_C}$, concise yawing-moment derivative due to roll
ν_n	=	$-\dfrac{n_r}{i_C}$, concise damping-in-yaw derivative
ξ		Aileron angle
τ	=	$\dfrac{t}{\hat{t}}$, time parameter (*see* Section 3.1)

ϕ Angle of bank

φ Perturbation angle of bank $(= \phi - \phi_i - \hat{p}_0 \tau)$

$\chi \quad = \quad -\dfrac{\left(\dfrac{\bar{c}}{s}\right)^2 m_{\dot{w}}}{i_B}$

$\omega \quad = \quad -\dfrac{\mu \dfrac{\bar{c}}{s} m_w}{i_B}$, concise restoring-moment derivative in pitch

$\omega_n \quad = \quad \dfrac{\mu n_v}{i_C}$, concise weathercock stability derivative

$\omega_l \quad = \quad -\dfrac{\mu l_v}{i_A}$, concise rolling-moment derivative due to sideslip

Suffices

i Denotes the initial value of the quantity. (When $\hat{p}_0$ changes value the initial conditions are defined by the end conditions of preceding time interval)

e Denotes initial steady-state value of a variable

s Denotes steady-state values of a variable

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APPENDIX I

Coefficients of Numerators and Denominators in the Operational Solutions of the Approximate Equations of Motion

Denominator.

The form of the denominator is:

$$G_0D^5 + G_1D^4 + G_2D^3 + G_3D^2 + G_4D + G_5$$

where

$$G_0 = 1$$

$$G_1 = k_{11} + k_{10}$$

$$G_2 = k_{21} + k_{20} + k_{11}k_{10} + \hat{p}_0{}^2(1-\delta_y\delta_z) + \hat{W}_0\omega_l$$

$$G_3 = k_{30} + k_{10}k_{21} + k_{11}k_{20} + \hat{p}_0{}^2\{-\delta_y\delta_z(\nu_l - z_w + \bar{y}_v) - \delta_z\chi + \nu + \nu_n + \nu_l\} + \\ + \hat{W}_0\omega_l(\nu_n + k_{11}) - \hat{W}_0\nu_{lr}\omega_n$$

$$G_4 = k_{20}k_{21} + k_{11}k_{30} + \hat{p}_0{}^2\{-\delta_y\delta_z(\bar{y}_v\nu_l - \bar{y}_v z_w - z_w\nu_l + \hat{p}_0{}^2) + \delta_y\omega_n - \delta_z(\nu_l\chi + \omega) + \\ + \nu_n(\nu + \nu_l) + \nu\nu_l + \nu_{np}\nu_{lr}\} + \hat{W}_0\{\omega_l(k_{21} + \nu_n k_{11} - \delta_y\delta_z\hat{p}_0{}^2) + \nu_{lr}(\delta_z\hat{p}_0{}^2\chi - \omega_n k_{11})\}$$

$$G_5 = k_{30}k_{21} + \hat{p}_0{}^2\{-\delta_y\delta_z(\hat{p}_0{}^2 - \bar{y}_v z_w)\nu_l + \delta_y(\omega_n\nu_l + \nu_{np}\omega_l) - \delta_z\nu_l\omega + \nu(\nu_n\nu_l + \nu_{np}\nu_{lr})\} + \\ + \hat{W}_0\{\omega_l(\nu_n k_{21} + \delta_y\delta_z\hat{p}_0{}^2 z_w) + \nu_{lr}(\delta_z\hat{p}_0{}^2\omega - \omega_n k_{21})\}$$

and

$$k_{10} = \bar{y}_v + \nu_n + \nu_l$$

$$k_{20} = \bar{y}_v(\nu_l + \nu_n) + \nu_n\nu_l + \nu_{lr}\nu_{np} + \omega_n$$

$$k_{30} = \bar{y}_v(\nu_l\nu_n + \nu_{lr}\nu_{np}) + \omega_l\nu_{np} + \omega_n\nu_l$$

$$k_{11} = \nu + \chi - z_w$$

$$k_{21} = \omega - \nu z_w.$$

Numerators.

The numerators are of the form:

$$\frac{1}{D^2 + \hat{p}_0{}^2}[H_{0n}D^7 + H_{1n}D^6 + H_{2n}D^5 + H_{3n}D^4 + H_{4n}D^3 + H_{5n}D^2 + H_{6n}D + H_{7n}].$$

Numerator for $\hat{p}$.

The numerator coefficients are given below:

$$H_{01} = \hat{p}_i = \hat{p}_i' + \hat{p}_0$$

$$H_{11} = \hat{p}_i'\lambda_{11} + P - \hat{v}_i\omega_l + \hat{r}_i\nu_{lr} + \hat{p}_0G_1$$

$$H_{21} = \hat{p}_i'\lambda_{21} + P\lambda_{11} - \hat{v}_i\psi_{11} + \hat{r}_i\gamma_{11} + \nu_{lr}Q - \omega_l S + \hat{w}_i\delta_{11} - \hat{p}_0(\delta_z\nu_{lr}R - G_2) + \hat{p}_0{}^2(\hat{p}_i' + G_0)$$

$$H_{31} = \hat{p}_i'\lambda_{31} + P\lambda_{21} - \hat{v}_i\psi_{2i} + \hat{r}_i\gamma_{21} - S\psi_{11} + \hat{w}_i\delta_{21} - \omega_l T + \gamma_{11}Q + \epsilon_{11}R + \\ + \frac{C_{Lc}}{2}(\cos\phi_i - 1)\delta_{11} + \hat{p}_0G_3 + \hat{p}_0{}^2(P + \hat{p}_i'\lambda_{11} + \nu_{lr}\hat{r}_i + G_1)$$

Footnote.—It will be noticed that in the numerator functions given herein the incremental rate of roll, $\hat{p}'$, was used as one of the basic variables. The use of the total rate of roll, p, would have yielded slightly simpler expressions (cf. Appendix III), but as the complicated analysis including gravity is considered to be only a stage towards a more simplified solution it was not considered worthwhile attempting the slight improvement that results from rearranging terms.

$$
\begin{aligned}
H_{41} = {} & \hat{p}_i'\lambda_{41} + P\lambda_{31} - \hat{v}_i\psi_{31} + \hat{r}_i\gamma_{31} - S\psi_{21} + \hat{w}_i\delta_{31} - T\psi_{11} + \gamma_{21}Q + U\delta_{11} + \epsilon_{21}R + \\
& + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{21} + \hat{p}_0 G_4 + \hat{p}_0^{\,2}(\hat{p}_i'\lambda_{21} + P\lambda_{11} + \hat{r}_i\gamma_{11} + \nu_{lr}Q + G_2) - \\
& - \hat{p}_0^{\,3}(\omega_l \hat{W}_0 + \delta_z \nu_{lr} R) \\
H_{51} = {} & P\lambda_{41} - S\psi_{31} - T\psi_{21} + Q\gamma_{31} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{31} + U\delta_{21} + \hat{p}_0 G_5 + \\
& + \hat{p}_0^{\,2}\left(\hat{p}_i'\lambda_{31} + P\lambda_{21} + \hat{r}_i\gamma_{21} + Q\gamma_{11} + R\epsilon_{11} - \frac{C_{Le}}{2}\delta_{11} + G_3\right) - \hat{p}_0^{\,3}\hat{W}_0\psi_{11} \\
H_{61} = {} & U\delta_{31} - T\psi_{31} + \hat{p}_0^{\,2}\left(\hat{p}_i'\lambda_{41} + P\lambda_{31} + \hat{r}_i\gamma_{31} + Q\gamma_{21} + R\epsilon_{21} - \frac{C_{Le}}{2}\delta_{21} + G_4\right) - \hat{p}_0^{\,3}\hat{W}_0\psi_{21} \\
H_{71} = {} & \hat{p}_0^{\,2}\left[P\lambda_{41} + Q\gamma_{31} - \frac{C_{Le}}{2}\delta_{31} + G_5 - \hat{p}_0\hat{W}_0\psi_{31}\right]
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_{11} &= \omega_l(k_{11} + \nu_n) - \omega_n\nu_{lr} \\
\psi_{21} &= \omega_l(k_{21} + \nu_n k_{11}) - \omega_n\nu_{lr}k_{11} + \hat{p}_0^{\,2}(-\delta_y\delta_z\omega_l + \nu_{lr}\delta_z\chi) \\
\psi_{31} &= (\omega_l\nu_n - \omega_n\nu_{lr})k_{21} + \hat{p}_0^{\,2}\delta_z(\delta_y\omega_l z_w + \nu_{lr}\omega) \\
\lambda_{11} &= k_{11} + \nu_n + \bar{y}_v \\
\lambda_{21} &= k_{21} + k_{11}(\nu_n + \bar{y}_v) + \omega_n + \nu_n + \bar{y}_v + (1 - \delta_y\delta_z)\hat{p}_0^{\,2} \\
\lambda_{31} &= k_{21}(\nu_n + \bar{y}_v) + k_{11}(\omega_n + \nu_n + \bar{y}_v) + \hat{p}_0^{\,2}\{-\delta_y\delta_z(\bar{y}_v - z_w) - \delta_z\chi + \nu_n + \nu\} \\
\lambda_{41} &= k_{21}(\omega_n + \nu_n + \bar{y}_v) + \hat{p}_0^{\,2}\{-\delta_y(\delta_z\hat{p}_0^{\,2} - \delta_z z_w\bar{y}_v - \omega_n) - \delta_z\omega + \nu\nu_n\} \\
\gamma_{11} &= \nu_{lr}(k_{11} + \bar{y}_v) + \omega_l \\
\gamma_{21} &= \nu_{lr}(k_{21} + k_{11}\bar{y}_v + \hat{p}_0^{\,2}) + k_{11}\omega_l \\
\gamma_{31} &= \nu_{lr}(\bar{y}_v k_{21} + \hat{p}_0^{\,2}\nu) + \delta_y\omega_l\hat{p}_0^{\,2} + k_{21}\omega_l \\
\delta_{11} &= \hat{p}_0(\nu_{lr}\delta_z\chi - \omega_l) \\
\delta_{21} &= \hat{p}_0\{\omega_l(\delta_z\chi - \nu - \nu_n) + \nu_{lr}(\omega_n + \delta_z\chi\bar{y}_v + \delta_z\omega)\} \\
\delta_{31} &= \hat{p}_0\{\omega_l(\delta_z\omega + \delta_y\delta_z\hat{p}_0^{\,2} - \nu_n\nu) + \nu_{lr}(\omega_n\nu + \delta_z\bar{y}_v\omega)\} \\
\epsilon_{11} &= \hat{p}_0\{\nu_{lr}(\delta_z z_w - \delta_z\bar{y}_v) - \omega_l(1 + \delta_z)\} \\
\epsilon_{21} &= \hat{p}_0\{\nu_{lr}(\omega_n - \delta_z\hat{p}_0^{\,2} + \delta_z z_w\bar{y}_v) + \omega_l(\delta_z z_w - \nu_n)\}
\end{aligned}
$$

and also

$$
\begin{aligned}
P &= -\delta_{l\xi}\bar{\xi} - \nu_l\hat{p}_0 \\
Q &= -\delta_{n\xi}\bar{\xi} - \nu_{np}\hat{p}_0 \\
R &= \hat{q}_i + \chi\hat{w}_i \\
S &= \hat{W}_0\hat{p}_0 + \frac{C_{Le}}{2}\sin\phi_i \\
T &= \hat{v}_i\hat{p}_0^{\,2} + \frac{C_{Le}}{2}\hat{p}_0\cos\phi_i \\
U &= \hat{w}_i\hat{p}_0^{\,2} - \frac{C_{Le}}{2}\hat{p}_0\sin\phi_i,
\end{aligned}
$$

throughout this Appendix.

Numerator for $\hat{q}$.

$$H_{02} = \hat{q}_i$$

$$H_{12} = R\epsilon_{12} + \hat{w}_i\delta_{12} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\chi + \hat{p}_0(\hat{v}_i\chi - \delta_y\hat{r}_i)$$

$$H_{22} = R\epsilon_{22} + \hat{w}_i\delta_{22} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{12} - U\chi + \hat{v}_i\psi_{12} + \hat{r}_i\gamma_{12} + \lambda_{12}\hat{p}_i' + R\hat{p}_0^2 + \\ + \hat{p}_0(S\chi - \delta_y Q)$$

$$H_{32} = R\epsilon_{32} + \hat{w}_i\delta_{32} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{22} + U\delta_{12} + \hat{v}_i\psi_{22} + \hat{r}_i\gamma_{22} + \lambda_{22}\hat{p}_i' + S\psi_{12} + \\ + T\hat{p}_0\chi + Q\gamma_{12} + P\lambda_{12} + \hat{p}_0^2\left(R\epsilon_{12} + \chi\frac{C_{Le}}{2} - \delta_y\hat{p}_0\hat{r}_i\right)$$

$$H_{42} = R\epsilon_{42} + \hat{w}_i\delta_{42} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{32} + U\delta_{22} + \hat{v}_i\psi_{32} + S\psi_{22} + T\psi_{12} + \hat{r}_i\gamma_{32} + \\ + Q\gamma_{22} + \hat{p}_i'\lambda_{32} + P\lambda_{22} + \hat{p}_0^2\left(R\epsilon_{22} + \frac{C_{Le}}{2}\delta_{12} + \hat{W}_0\hat{p}_0^2\chi + \hat{r}_i\gamma_{12} - \delta_y\hat{p}_0 Q + \hat{p}_i'\lambda_{12}\right)$$

$$H_{52} = \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{42} + U\delta_{32} + S\psi_{32} + T\psi_{22} + Q\gamma_{32} + P\lambda_{32} + \hat{W}_0\hat{p}_0^3\psi_{12} + \\ + \hat{p}_0^2\left(R\epsilon_{32} - \frac{C_{Le}}{2}\delta_{22} + \hat{r}_i\gamma_{22} + Q\gamma_{12} + \hat{p}_i'\lambda_{22} + P\lambda_{12}\right)$$

$$H_{62} = U\delta_{42} + T\psi_{32} + \hat{W}_0\hat{p}_0^3\psi_{22} + \hat{p}_0^2\left(R\epsilon_{42} - \frac{C_{Le}}{2}\delta_{32} + \hat{r}_i\gamma_{32} + Q\gamma_{22} + \hat{p}_i'\lambda_{32} + P\lambda_{22}\right)$$

$$H_{72} = \hat{p}_0^2\left(\hat{W}_0\hat{p}_0\psi_{32} - \frac{C_{Le}}{2}\delta_{42} + Q\gamma_{32} + P\lambda_{32}\right)$$

where

$$\psi_{12} = \hat{p}_0\{-\omega_n\delta_y + \chi(\nu_l + \nu_n) + \omega\}$$

$$\psi_{22} = \hat{p}_0\{-\delta_y(\omega_l\nu_{np} + \omega_n\nu_l - \omega_n z_w) + \omega(\nu_l + \nu_n) + \chi(\nu_l\nu_n + \nu_{lr}\nu_{np})\}$$

$$\psi_{32} = \hat{p}_0\{\omega(\nu_l\nu_n + \nu_{lr}\nu_{np}) + z_w(\omega_l\nu_{np}\delta_y + \omega_n\nu_l\delta_y)\}$$

$$\delta_{12} = -(\chi k_{10} + \omega)$$

$$\delta_{22} = -(\chi k_{20} + \omega k_{10}) - \hat{W}_0\omega_l\chi$$

$$\delta_{32} = -(\chi k_{30} + \omega k_{20} + \hat{p}_0^2\omega_n\delta_y) + \hat{W}_0(\omega_n\nu_{lr}\chi - \omega_l\nu_n\chi - \omega_l\omega)$$

$$\delta_{42} = -\{\omega k_{30} + \hat{p}_0^2\delta_y(\omega_l\nu_{np} + \omega_n\nu_l)\} - \hat{W}_0\omega(\omega_l\nu_n - \omega_n\nu_{lr})$$

$$\lambda_{12} = \hat{p}_0(\hat{W}_0\chi + \delta_y\nu_{np})$$

$$\lambda_{22} = \hat{p}_0\{\nu_{np}(-\delta_y z_w + \chi + \delta_y\bar{y}_v) + \hat{W}_0(\omega + \nu_n\chi - \omega_n\delta_y)\}$$

$$\lambda_{32} = \hat{p}_0\{\nu_{np}(-\delta_y\bar{y}_v z_w + \omega + \delta_y\hat{p}_0^2) + \hat{W}_0(\omega\nu_n + \omega_n\delta_y z_w)\}$$

$$\gamma_{12} = \hat{p}_0\{-\delta_y(\bar{y}_v + \nu_l - z_w) - \chi\}$$

$$\gamma_{22} = \hat{p}_0\{-\delta_y(\bar{y}_v\nu_l - \bar{y}_v z_w - \nu_l z_w + \hat{p}_0^2) - \nu_l\chi - \omega + \hat{W}_0(-\omega_l\delta_y + \nu_{lr}\chi)\}$$

$$\gamma_{32} = \hat{p}_0\{-\delta_y\nu_l(\hat{p}_0^2 - \bar{y}_v z_w) - \nu_l\omega + \hat{W}_0(\nu_{lr}\omega + \delta_y\omega_l z_w)\}$$

$$\epsilon_{12} = k_{10} - z_w$$

$$\epsilon_{22} = k_{20} - z_w k_{10} + \hat{p}_0^2 + \hat{W}_0\omega_l$$

$$\epsilon_{32} = k_{30} - z_w k_{20} + \hat{p}_0^2(\nu_l + \nu_n) + \hat{W}_0\{\omega_l(\nu_n - z_w) - \omega_n\nu_{lr}\}$$

$$\epsilon_{42} = \hat{p}_0^2(\nu_l\nu_n + \nu_{lr}\nu_{np}) - z_w\{k_{30} - \hat{W}_0(\omega_n\nu_{lr} - \omega_n\nu_n)\}.$$

Numerator for $\dot{r}$.

$$H_{03} = \hat{r}_i$$
$$H_{13} = \gamma_{13}\hat{r}_i + Q + \omega_n\hat{v}_i - \nu_{np}\hat{p}_i' + \delta_z\hat{p}_0(\chi\hat{w}_i - R)$$
$$H_{23} = \gamma_{23}\hat{r}_i + Q\gamma_{13} + \hat{v}_i\psi_{13} + S\omega_n + \hat{p}_i'\lambda_{13} - \nu_{np}P + R\epsilon_{13} + \hat{w}_i\delta_{13} + \\ + \hat{p}_0\delta_z\chi\frac{C_{Le}}{2}(\cos\phi_i - 1) + \hat{p}_0^2\hat{r}_i$$
$$H_{33} = \gamma_{33}\hat{r}_i + Q\gamma_{23} + \hat{v}_i\psi_{23} + S\psi_{13} + T\omega_n + \hat{p}_i'\lambda_{23} + P\lambda_{13} + R\epsilon_{23} + \hat{w}_i\delta_{23} + \\ + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{13} + U\delta_z\hat{p}_0\chi + \hat{p}_0(\hat{r}_i\gamma_{13} + Q - \hat{p}_i'\nu_{np} - \delta_z\hat{p}_0R)$$
$$H_{43} = \gamma_{43}\hat{r}_i + Q\gamma_{33} + \hat{v}_i\psi_{33} + S\psi_{23} + T\psi_{13} + \hat{p}_i'\lambda_{33} + P\lambda_{23} + R\epsilon_{33} + \hat{w}_i\delta_{33} + \\ + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{23} + U\delta_{13} + \hat{p}_0^2(\hat{r}_i\gamma_{23} + Q\gamma_{13} + \hat{p}_i'\lambda_{13} - \nu_{np}P + R\epsilon_{13}) + \\ + \hat{p}_0^3\left(\hat{W}_0\omega_n - \delta_z\chi\frac{C_{Le}}{2}\right)$$
$$H_{53} = Q\gamma_{43} + S\psi_{33} + T\psi_{23} + P\lambda_{33} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{33} + U\delta_{23} + \\ + \hat{p}_0^2\left(\hat{r}_i\gamma_{33} + Q\gamma_{23} + \hat{p}_i'\lambda_{23} + P\lambda_{13} + R\epsilon_{23} - \frac{C_{Le}}{2}\delta_{13}\right) + \hat{W}_0\hat{p}_0^3\psi_{13}$$
$$H_{63} = T\psi_{33} + U\delta_{33} + \hat{p}_0^2\left(\hat{r}_i\gamma_{43} + Q\gamma_{33} + \hat{p}_i'\lambda_{33} + P\lambda_{23} + R\epsilon_{33} - \frac{C_{Le}}{2}\delta_{23}\right) + \hat{W}_0\hat{p}_0^3\psi_{23}$$
$$H_{73} = \hat{p}_0^2\left(Q\gamma_{43} + P\lambda_{33} - \frac{C_{Le}}{2}\delta_{33}\right) + \hat{W}_0\hat{p}_0^3\psi_{33}$$

where

$$\psi_{13} = \omega_n(k_{11} + \nu_l) + \omega_l\nu_{np} - \delta_z\hat{p}_0^2\chi$$
$$\psi_{23} = \omega_nk_{21} + (\omega_l\nu_{np} + \omega_n\nu_l)k_{11} - \delta_z\hat{p}_0^2(\nu_l\chi + \omega)$$
$$\psi_{33} = (\omega_l\nu_{np} + \omega_n\nu_l)k_{21} - \delta_z\hat{p}_0^2\nu_l\omega$$
$$\lambda_{13} = -\nu_{np}(\bar{y}_v + k_{11}) + \hat{W}_0\omega_n$$
$$\lambda_{23} = -\nu_{np}(k_{21} + \bar{y}_vk_{11} + \hat{p}_0^2) + \hat{W}_0(\omega_nk_{11} - \delta_z\hat{p}_0^2\chi)$$
$$\lambda_{33} = -\nu_{np}(\bar{y}_vk_{21} + \nu\hat{p}_0^2) + \hat{W}_0(\omega_nk_{21} - \delta_z\hat{p}_0^2\omega)$$
$$\gamma_{13} = \bar{y}_v + \nu_l + k_{11}$$
$$\gamma_{23} = (\bar{y}_v + \nu_l)k_{11} + k_{21} + \bar{y}_v\nu_l + \hat{p}_0^2 + \hat{W}_0\omega_l$$
$$\gamma_{33} = (\bar{y}_v + \nu_l)k_{21} + k_{11}\bar{y}_v\nu_l + \hat{p}_0^2(\nu + \nu_l) + \hat{W}_0\omega_lk_{11}$$
$$\gamma_{43} = k_{21}\bar{y}_v\nu_l + \hat{p}_0^2\nu\nu_l + \hat{W}_0\omega_lk_{21}$$
$$\epsilon_{13} = \delta_z\hat{p}_0(z_w - \bar{y}_v - \nu_l)$$
$$\epsilon_{23} = \hat{p}_0\{\delta_z(\bar{y}_vz_w - \bar{y}_v\nu_l + \nu_lz_w - \hat{p}_0^2) + \omega_n - \hat{W}_0\omega_l\delta_z\}$$
$$\epsilon_{33} = \hat{p}_0\{\delta_z(\bar{y}_v\nu_lz_w - \hat{p}_0^2\nu_l + \hat{W}_0\omega_lz_w) + \omega_l\nu_{np} + \omega_n\nu_l\}$$
$$\delta_{13} = \hat{p}_0\{\delta_z(\omega + \chi\bar{y}_v + \chi\nu_l) + \omega_n\}$$
$$\delta_{23} = \hat{p}_0\{\delta_z(\omega\bar{y}_v + \omega\nu_l + \chi\bar{y}_v\nu_l + \hat{W}_0\omega_l\chi) + \omega_l\nu_{np} + \omega_n\nu_l + \omega_n\nu\}$$
$$\delta_{33} = \hat{p}_0\{\delta_z(\omega\bar{y}_v\nu_l + \hat{W}_0\omega_l\omega) + \nu(\omega_l\nu_{np} + \omega_n\nu_l)\}.$$

Numerator for $\hat{w}$.

$$H_{04} = \hat{w}_i$$

$$H_{14} = \hat{w}_i\delta_{14} + \frac{C_{Le}}{2}(\cos\phi_i - 1) + R - \hat{p}_0\hat{v}_i$$

$$H_{24} = \hat{w}_i\delta_{24} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{14} + U + Rk_{20} + \hat{r}_i\gamma_{14} + \hat{v}_i\psi_{14} - S\hat{p}_0 - \hat{W}_0\hat{p}_0\hat{p}_i'$$

$$H_{34} = \hat{w}_i\delta_{34} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{24} + U\delta_{14} + R\epsilon_{14} + \hat{r}_i\gamma_{24} + Q\gamma_{14} + \hat{p}_i'\lambda_{14} + \hat{v}_i\psi_{24} + \\ + S\psi_{14} + \hat{p}_0^2\left(R - \frac{C_{Le}}{2}\right) - T\hat{p}_0 - P\hat{W}_0\hat{p}_0$$

$$H_{44} = \hat{w}_i\delta_{44} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{34} + U\delta_{24} + R\epsilon_{24} + \gamma_{34}\hat{r}_i + Q\gamma_{24} + \hat{p}_i'\lambda_{24} + P\lambda_{14} + \\ + \hat{v}_i\psi_{34} + S\psi_{24} + T\psi_{14} + \hat{p}_0^2\left(Rk_{20} - \frac{C_{Le}}{2}\delta_{14} + \hat{r}_i\gamma_{14} - \hat{W}_0\hat{p}_0^2 - \hat{W}_0\hat{p}_0\hat{p}_i'\right)$$

$$H_{54} = \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{44} + U\delta_{34} + Q\gamma_{34} + P\lambda_{24} + S\psi_{34} + T\psi_{24} + \\ + \hat{p}_0^2\left(\hat{r}_i\gamma_{24} - \frac{C_{Le}}{2}\delta_{24} + R\epsilon_{14} + Q\gamma_{14} + \hat{p}_i'\lambda_{14} + \hat{W}_0\hat{p}_0\psi_{14}\right) - \hat{W}_0\hat{p}_0^3P$$

$$H_{64} = U\delta_{44} + T\psi_{34} + \hat{p}_0^2\left(R\epsilon_{24} - \frac{C_{Le}}{2}\delta_{34} + \hat{r}_i\gamma_{34} + Q\gamma_{24} + P\lambda_{14} + \hat{p}_i'\lambda_{24} + \hat{W}_0\hat{p}_0\psi_{24}\right)$$

$$H_{74} = \hat{p}_0^2\left(Q\gamma_{34} - \frac{C_{Le}}{2}\delta_{44} + P\lambda_{24} + \hat{W}_0\hat{p}_0\psi_{34}\right)$$

where

$$\psi_{14} = -\hat{p}_0(\nu + \nu_n + \nu_l)$$

$$\psi_{24} = -\hat{p}_0(\nu\nu_n + \nu_n\nu_l + \nu\nu_l + \nu_{lr}\nu_{np} + \delta_y\omega_n - \delta_y\delta_z\hat{p}_0^2)$$

$$\psi_{34} = -\hat{p}_0(\nu\nu_n\nu_l + \omega_l\delta_y\nu_{np} + \nu\nu_{lr}\nu_{np} + \delta_y\omega_n\nu_l - \delta_y\delta_z\hat{p}_0^2\nu_l)$$

$$\lambda_{14} = -\hat{p}_0\{\nu_{np}(1-\delta_y) + \hat{W}_0(\nu + \nu_n)\}$$

$$\lambda_{24} = -\hat{p}_0\{\nu_{np}(\delta_y\bar{y}_v + \nu) + \hat{W}_0(-\delta_y\delta_z\hat{p}_0^2 + \nu\nu_n + \delta_y\omega_n)\}$$

$$\gamma_{14} = \hat{p}_0(1-\delta_y)$$

$$\gamma_{24} = \hat{p}_0\{-\delta_y(\nu_l + \bar{y}_v) + \nu_l + \nu - \hat{W}_0\nu_{lr}\}$$

$$\gamma_{34} = \hat{p}_0\{\nu_l(\nu - \delta_y\bar{y}_v) - \hat{W}_0(\nu\nu_{lr} + \delta_y\omega_l)\}$$

$$\epsilon_{14} = k_{20} - \delta_z\hat{p}_0^2 + \hat{W}_0\omega_l$$

$$\epsilon_{24} = k_{30} - \delta_z\hat{p}_0^2\nu_l - \hat{W}_0(\omega_n\nu_{lr} - \omega_l\nu_n - \delta_z\nu_{lr}\hat{p}_0^2)$$

$$\delta_{14} = k_{10} + \nu$$

$$\delta_{24} = k_{20} + \nu k_{10} - \delta_y\delta_z\hat{p}_0^2 + \hat{W}_0\omega_l$$

$$\delta_{34} = k_{30} + \nu k_{20} - \delta_y\delta_z\hat{p}_0^2(\nu_l + \bar{y}_v) + \hat{W}_0(\omega_l\nu + \omega_l\nu_n - \nu_{lr}\omega_n)$$

$$\delta_{44} = \nu k_{30} - \delta_y\delta_z\hat{p}_0^2\bar{y}_v\nu_l + \hat{W}_0(\omega_l\nu\nu_n - \omega_n\nu\nu_{lr} - \delta_y\delta_z\hat{p}_0^2\omega_l).$$

Numerator for $\hat{\vartheta}$ or β.

$$H_{05} = \hat{\vartheta}_i$$

$$H_{15} = \hat{\vartheta}_i\psi_{15} + S - \hat{r}_i + \hat{\omega}_i\hat{p}_0 + \hat{W}_0\hat{p}_i'$$

$$H_{25} = \hat{\vartheta}_i\psi_{25} + S\psi_{15} + T + \hat{p}_i'\nu_{np} - \hat{r}_i\gamma_{15} - Q + \hat{p}_0(1+\delta_z)R + \hat{\omega}_i\delta_{15} + \\ + \frac{C_{Le}}{2}\hat{p}_0(\cos\phi_i - 1) + \hat{W}_0\{P + \hat{p}_i'(\nu_n + k_{21})\}$$

$$H_{35} = \hat{\vartheta}_i\psi_{35} + S\psi_{25} + T\psi_{15} + \hat{p}_i'\lambda_{15} + \nu_{np}P - \hat{r}_i\gamma_{25} - Q\gamma_{15} + R\epsilon_{15} + \hat{\omega}_i\delta_{25} + \\ + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{15} + U\hat{p}_0 + P\hat{W}_0(\nu_n + k_{21}) + \hat{W}_0\hat{p}_0^3 + \hat{p}_0^2(\hat{p}_i'\hat{W}_0 - \hat{r}_i)$$

$$H_{45} = \hat{\vartheta}_i\psi_{45} + S\psi_{35} + T\psi_{25} + \hat{W}_0\hat{p}_0^3\psi_{15} + \hat{p}_i'\lambda_{25} + P\lambda_{15} - \hat{r}_i\gamma_{35} - Q\gamma_{25} + R\epsilon_{25} + \\ + \hat{\omega}_i\delta_{35} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{25} + U\delta_{15} - \frac{C_{Le}}{2}\hat{p}_0^3 + R\hat{p}_0^3(1+\delta_z) + \\ + \hat{p}_0^2\{\nu_{np}\hat{p}_i' - \hat{r}_i\gamma_{15} - Q + \hat{W}_0P + \hat{W}_0\hat{p}_i'(\nu_n + k_{21})\}$$

$$H_{55} = S\psi_{45} + T\psi_{35} + P\lambda_{25} - Q\gamma_{35} + \frac{C_{Le}}{2}(\cos\phi_i - 1)\delta_{35} + U\delta_{25} + \\ + \hat{p}_0^2\left\{\hat{W}_0\hat{p}_0\psi_{25} + \hat{p}_i'\lambda_{15} + P\nu_{np} - \hat{r}_i\gamma_{25} - Q\gamma_{15} + R\epsilon_{15} - \frac{C_{Le}}{2}\delta_{15} + \hat{W}_0(\nu_n + k_{21})P\right\}$$

$$H_{65} = T\psi_{45} + U\delta_{35} + \hat{p}_0^2\left\{\hat{W}_0\hat{p}_0\psi_{35} + \hat{p}_i'\lambda_{25} + P\lambda_{15} - \hat{r}_i\gamma_{35} - Q\gamma_{25} + R\epsilon_{25} - \frac{C_{Le}}{2}\delta_{25}\right\}$$

$$H_{75} = \hat{p}_0^2\left\{\hat{W}_0\hat{p}_0\psi_{45} + P\lambda_{25} - Q\gamma_{35} - \frac{C_{Le}}{2}\delta_{35}\right\}$$

where

$$\psi_{15} = \nu_l + \nu_n + k_{11}$$

$$\psi_{25} = k_{11}(\nu_l + \nu_n) + k_{21} + \nu_{lr}\nu_{np} + \nu_l\nu_n - \delta_y\delta_z\hat{p}_0^2$$

$$\psi_{35} = k_{21}(\nu_l + \nu_n) + k_{11}(\nu_{lr}\nu_{np} + \nu_l\nu_n) + \delta_y\delta_z\hat{p}_0^2(z_w - \nu_l)$$

$$\psi_{45} = k_{21}(\nu_{lr}\nu_{np} + \nu_l\nu_n) + \delta_y\delta_z\hat{p}_0^2\nu_l z_w$$

$$\lambda_{15} = \nu_{np}k_{11} + \hat{W}_0(\nu_n k_{11} + k_{21} - \delta_y\delta_z\hat{p}_0^2)$$

$$\lambda_{25} = \nu_{np}(k_{21} + \delta_y\hat{p}_0^2) + \hat{W}_0(\nu_n k_{21} + z_w\delta_y\delta_z\hat{p}_0^2)$$

$$\gamma_{15} = k_{11} + \nu_l - \hat{W}_0\nu_{lr}$$

$$\gamma_{25} = k_{21} + \nu_l k_{11} + \delta_y\hat{p}_0^2 - \hat{W}_0\nu_{lr}(\nu + \chi - z_w)$$

$$\gamma_{35} = \nu_l(k_{21} + \delta_y\hat{p}_0^2) - \hat{W}_0\nu_{lr}k_{21}$$

$$\epsilon_{15} = \hat{p}_0\{\delta_z(\nu_l - z_w - \hat{W}_0\nu_{lr}) + \nu_l + \nu_n\}$$

$$\epsilon_{25} = \hat{p}_0\{\nu_l(\nu_n - \delta_z z_w) + \nu_{lr}(\nu_{np} + \hat{W}_0\delta_z z_w)\}$$

$$\delta_{15} = \hat{p}_0(\nu_l + \nu + \nu_n - \delta_z\chi)$$

$$\delta_{25} = \hat{p}_0\{\delta_z(-\delta_y\hat{p}_0^2 - \omega - \chi\nu_l + \hat{W}_0\nu_{lr}\chi) + \nu_l(\nu_n + \nu) + \nu\nu_n + \nu_{lr}\nu_{np}\}$$

$$\delta_{35} = \hat{p}_0\{\delta_z(-\delta_y\hat{p}_0^2\nu_l - \omega\nu_l + \hat{W}_0\nu_{lr}\omega) + \nu(\nu_l\nu_n + \nu_{lr}\nu_{np})\}.$$

APPENDIX II

A Study of the Contribution of the Gravity Terms to the Solution in Steady Rolling Flight

As mentioned in the text, the question of the importance of the gravity terms has to be resolved before we can proceed with the approximate solutions proposed herein. To examine this more fully, we turn to the steady rolling flight in which we assume that the aileron is operated so as to maintain a constant rate of roll ($\hat{p}_0$) throughout. In this special case the inertia cross-coupling terms are accurately represented and the gravity terms with improved accuracy, but not exactly. The equations of motion in operational form are (cf. Section 3 and Appendix VI):

$$\left.\begin{aligned}
&(D+\nu)\hat{q} + \delta_y\hat{p}_0\hat{r} + (\omega+\chi D)\hat{w} = (\hat{q}_i+\chi\hat{w}_i)D \\
&\delta_z\hat{p}_0\hat{q} + (D+\nu_n{}^*)\hat{r} - \omega_n{}^*\hat{v} = \hat{r}_iD - \nu_{np}{}^*\hat{p}_0 \\
&\hat{r} + (D+\bar{y}_v)\hat{v} - \hat{p}_0\hat{w} = \hat{v}_iD + \hat{W}_0\hat{p}_0 + \\
&\qquad + \frac{C_{Le}}{2}\left\{\sin\phi_i\frac{D^2}{D^2+\hat{p}_0{}^2} + \cos\phi_i\frac{\hat{p}_0D}{D^2+\hat{p}_0{}^2}\right\} \\
&-\hat{q} + \hat{p}_0\hat{v} + (D-z_w)\hat{w} = \hat{w}_iD + \frac{C_{Le}}{2}\left\{\cos\phi_i\frac{D^2}{D^2+\hat{p}_0{}^2} - \sin\phi_i\frac{\hat{p}_0D}{D^2+\hat{p}_0{}^2}\right\} - \\
&\qquad - \frac{C_{Le}}{2}.
\end{aligned}\right\} \quad \ldots.(18)$$

where

$$\omega_n{}^* = \omega_n + \omega_l\frac{i_A}{i_C}\frac{n_\xi}{l_\xi},$$

$$\nu_n{}^* = \nu_n + \nu_{lr}\frac{i_A}{i_C}\frac{n_\xi}{l_\xi},$$

$$\nu_{np}{}^* = \nu_{np} - \nu_l\frac{i_A}{i_C}\frac{n_\xi}{l_\xi},$$

with the aileron angle being defined by:

$$-\delta_{l\xi}\xi = (D+\nu_l)\hat{p} - \nu_{lr}\hat{r} + \omega_l\hat{v} = \nu_l\hat{p}_0 - \nu_{lr}\hat{r} + \omega_l\hat{v}.$$

The operational determinant of the above system of equations is:

$$\begin{vmatrix} D+\nu & \delta_y\hat{p}_0 & 0 & \chi D+\omega \\ \delta_z\hat{p}_0 & D+\nu_n{}^* & -\omega_n{}^* & 0 \\ 0 & 1 & D+\bar{y}_v & -\hat{p}_0 \\ -1 & 0 & \hat{p}_0 & D-z_w \end{vmatrix}$$
$$= G_0D^4 + G_1D^3 + G_2D^2 + G_3D + G_4 \quad (19)$$

if we write:

$$k_{10}{}^* = \bar{y}_v + \nu_n{}^*, \qquad k_{20}{}^* = \omega_n{}^* + \bar{y}_v\nu_n{}^*,$$

and

$$k_{11} = \nu + \chi - z_w, \qquad k_{21} = \omega - \nu z_w,$$

$$G_0 = 1$$

$$G_1 = k_{10}{}^* + k_{11}$$

$$G_2 = k_{20}{}^* + k_{21} + k_{10}{}^*k_{11} + \hat{p}_0{}^2(1-\delta_y\delta_z)$$

$$G_3 = k_{20}{}^*k_{11} + k_{10}{}^*k_{21} + \hat{p}_0{}^2\{\nu + \nu_n{}^* - \delta_z\chi - \delta_y\delta_z(\bar{y}_v - z_w)\}$$

$$G_4 = k_{20}{}^*k_{21} + \hat{p}_0{}^2\{\nu\nu_n{}^* - (\delta_z\omega - \delta_y\omega_n{}^*) - \delta_y\delta_z(\hat{p}_0{}^2 - \bar{y}_vz_w)\}.$$

In general, the numerator for $\hat{w}$ can be written:

$$\frac{1}{D^2 + \hat{p}_0^{\,2}} \{H_{04}D^6 + H_{14}D^5 + H_{24}D^4 + H_{34}D^3 + H_{44}D^2 + H_{54}D + H_{64}\}.$$

However, when $\hat{q}_i = \hat{r}_i = \hat{w}_i = \hat{v}_i = 0$ and $\phi_i = 0$ the coefficients become:

$$H_{04} = H_{14} = 0$$

$$H_{24} = -\hat{p}_0^{\,2}\hat{W}_0$$

$$H_{34} = -C_{Le}\hat{p}_0^{\,2} - \hat{p}_0^{\,2}\{\nu_{np}^*(1-\delta_y) + \hat{W}_0(\nu+\nu_n^*)\}$$

$$H_{44} = -\frac{C_{Le}}{2}\hat{p}_0^{\,2}(2\nu+\nu_n^*+k_{10}^*) - \hat{p}_0^{\,2}\{\nu_{np}^*(\nu-\delta_y\bar{y}_v) + \hat{W}_0(\hat{p}_0^{\,2}[1-\delta_y\delta_z] + \nu\nu_n^* + \delta_y\omega_n^*)\}$$

$$H_{54} = -\frac{C_{Le}}{2}\hat{p}_0^{\,2}(k_{20}^* + \nu k_{10}^* - 2\delta_y\delta_z\hat{p}_0^{\,2} + \delta_y\omega_n^* + \nu\nu_n^*) - \hat{p}_0^{\,4}\{\nu_{np}^*(1-\delta_y) + \hat{W}_0(\nu+\nu_n^*)\}$$

$$H_{64} = -\frac{C_{Le}}{2}\hat{p}_0^{\,2}(\nu k_{20}^* - \delta_y\delta_z\hat{p}_0^{\,2}\bar{y}_v) - \hat{p}_0^{\,4}\{\nu_{np}^*(\nu-\delta_y\bar{y}_v) + \hat{W}_0(\nu\nu_n^* + \delta_y\omega_n^* - \delta_y\delta_z\hat{p}_0^{\,2})\}.$$

Similarly, we have for $\hat{v}$:

$$\frac{1}{D^2 + \hat{p}_0^{\,2}} \{H_{05}D^6 + H_{15}D^5 + H_{25}D^4 + H_{35}D^3 + H_{45}D^2 + H_{55}D + H_{65}\}$$

where again for the above initial conditions:

$$H_{05} = 0$$

$$H_{15} = \hat{W}_0\hat{p}_0$$

$$H_{25} = \frac{C_{Le}}{2}\hat{p}_0 + \hat{p}_0\{\hat{W}_0(\nu_n^* + k_{11}) + \nu_{np}^*\}$$

$$H_{35} = \frac{C_{Le}}{2}\hat{p}_0(k_{11}+\nu_n^*) + \hat{p}_0[\hat{W}_0\{k_{21} + \nu_n^*k_{11} + (1-\delta_y\delta_z)\hat{p}_0^{\,2}\} + \nu_{np}^*k_{11}]$$

$$H_{45} = \frac{C_{Le}}{2}\hat{p}_0\{k_{21} + k_{11}\nu_n^* - \hat{p}_0^{\,2}(1+\delta_y\delta_z)\} + \\ + \hat{p}_0\{\hat{W}_0[\nu_n^*k_{21} + \hat{p}_0^{\,2}(\nu_n^* + k_{11} + \delta_y\delta_z z_w)] + \nu_{np}^*[k_{21} + (1+\delta_y)\hat{p}_0^{\,2}]\}$$

$$H_{55} = \frac{C_{Le}}{2}\hat{p}_0\{\nu_n^*k_{21} + \delta_y\delta_z\hat{p}_0^{\,2}z_w - \hat{p}_0^{\,2}(\nu+\nu_n^* - \delta_z\chi)\} + \hat{p}_0^{\,3}\{\hat{W}_0(k_{21} + \nu_n^*k_{11} - \delta_y\delta_z\hat{p}_0^{\,2}) + \\ + \nu_{np}^*k_{11}\}$$

$$H_{65} = \frac{C_{Le}}{2}\hat{p}_0^{\,3}(\delta_z\omega + \delta_y\delta_z\hat{p}_0^{\,2} - \nu\nu_n^*) + \hat{p}_0^{\,3}\{\hat{W}_0(k_{21}\nu_n^* + \delta_y\delta_z\hat{p}_0^{\,2}z_w) + \nu_{np}^*(k_{21} + \delta_y\hat{p}_0^{\,2})\}.$$

These solutions are readily transformed by means of the tables of Ref. 8. The response in $\hat{w}$ and $\hat{v}$ for our example aircraft is calculated including gravity terms ($C_{Le} = 0{\cdot}358$) and neglecting gravity terms (equivalent to setting $C_{Le} = 0$).

Two values of $\hat{p}_0$ are chosen for these calculations; $\hat{p}_0 = 4$ which gives a stable motion, and $\hat{p}_0 = 6$ which yields a divergent motion (*see* Figs. 3 to 10). In both cases the effect of the gravity terms is small. Furthermore, the effect for a given ϵ_0 (or $\hat{W}_0$) is proportional to C_{Le}, and so we may conclude that as a reasonable approximation these terms may be ignored for modest values of C_{Le}. The approximation used in the main text, which includes the main part of these terms, can also be accepted.

APPENDIX III

Simplified Response Calculation

The analysis of Appendix II and the results of other investigations such as References 3 to 6 indicate that for modest C_L values the contribution of the gravity terms to the response is small and can be neglected. The linearised equations of motion in operational form are then:

$$\left.\begin{array}{rrrrrl}
(D+\bar{y}_v)\hat{v} & -\hat{W}_0\hat{p} & +\hat{r} & & -\hat{p}_0\hat{w} & = \hat{v}_i D \\
\omega_l\hat{v} & +(D+\nu_l)\hat{p} & -\nu_{lr}\hat{r} & & & = -\delta_{l\xi}\bar{\xi}+\hat{p}_i D \\
-\omega_n\hat{v} & +\nu_{np}\hat{p} & +(D+\nu_n)\hat{r} & +\delta_z\hat{p}_0\hat{q} & & = -\delta_{n\xi}\bar{\xi}+\hat{r}_i D \\
 & & \delta_y\hat{p}_0\hat{r} & +(D+\nu)\hat{q} & +(\omega+\chi D)\hat{w} & = (\hat{q}_i+\chi\hat{w}_i)D \\
\hat{p}_0\hat{v} & & & -\hat{q} & +(D-z_w)\hat{w} & = \hat{w}_i D.
\end{array}\right\} \tag{20}$$

The characteristic equation can be written:

$$G_0\lambda^5 + G_1\lambda^4 + G_2\lambda^3 + G_3\lambda^2 + G_4\lambda + G_5 = 0 \tag{21}$$

where the expressions for the coefficients G_n are those of Appendix I. For high roll rates the factors of the characteristic equation are of two types:

(a)
$$(\lambda^2+a_1\lambda+b_1)(\lambda^2+a_2\lambda+b_2)(\lambda+b_3) = 0 \tag{22}$$

and

(b)
$$(\lambda^2+a_1\lambda+b_1)(\lambda+b_3)(\lambda+b_4)(\lambda+b_5) = 0. \tag{23}$$

The factor $D^2 + a_1D + b_1$ in both cases is associated with a high frequency mode of small amplitude ($b_1 \gg a_1$, a_2, b_2, b_3, b_4, and b_5).

Approximate Roots.

Equating the coefficients of equations (22) and (23) to those of equation (21) the following relationships are obtained:

$$\left.\begin{aligned}
a_1 + P &= G_1 \\
b_1 + a_1P + Q &= G_2 \\
Pb_1 + Qa_1 + R &= G_3 \\
Qb_1 + Ra_1 &= G_4 \\
Rb_1 &= G_5
\end{aligned}\right\} \tag{24}$$

where

$$\begin{aligned}
P &= a_2 + b_3 \quad \text{or } b_3 + b_4 + b_5 \\
Q &= b_3a_2 + b_2 \text{ or } b_3b_4 + b_4b_5 + b_5b_3 \\
R &= b_2b_3 \qquad \text{or } b_3b_4b_5.
\end{aligned}$$

From these equations:

$$a_1 = G_1 - P \tag{25}$$

$$b_1 = G_2 - Pa_1 - Q \tag{26}$$

$$R = \frac{G_5}{b_1} \tag{27}$$

$$Q = \frac{G_4 - Ra_1}{b_1} \tag{28}$$

$$P = \frac{G_3 - Qa_1 - R}{b_1} \tag{29}$$

and if the approximation $P \approx G_3/b_1$ is made, and if $b_1^2 \gg G_4, G_5$

$$a_1 \approx G_1 - \frac{G_3}{b_1} \tag{30}$$

$$b_1 \approx G_2 - \left(G_1 - \frac{G_3}{b_1}\right) \frac{G_3}{b_1} - \frac{G_4}{b_1}$$

$$\approx G_2 - \left(G_1 - \frac{G_3}{G_2}\right) \left(\frac{G_3}{G_2}\right) - \frac{G_4}{G_2} \tag{31}$$

$$R = \frac{G_5}{b_1} \tag{32}$$

$$Q \approx \frac{G_4}{b_1} - \frac{R}{b_1} \left(G_1 - \frac{G_3}{b_1}\right) \tag{33}$$

$$P = \frac{G_3}{b_1} - \frac{R}{b_1} - \frac{Q}{b_1} \left(G_1 - \frac{G_3}{b_1}\right) \tag{34}$$

and from equation (25) we have:

$$a_1 = G_1 - P.$$

Equations (30) to (34) give approximate values of a_1, b_1, P, Q, and R of sufficient accuracy for moderate to large values of $\hat{p}_0$. For small $\hat{p}_0$ values, which are unlikely to enter into practical calculations, use must be made of an iterative solution of equations (25) to (29) if reasonable accuracy is to be achieved. The values of a_2, b_2, b_3, b_4 and b_5 can be calculated from the following expressions, derived from equation (24), when a_1, b_1, P, Q and R are known:

$$b_3^3 - Pb_3^2 + Qb_3 - R = 0 \tag{35}$$

$$b_4^2 + \left(\frac{R}{b_3^2} - \frac{Q}{b_3}\right) b_4 + \frac{R}{b_3} = 0 \tag{36}$$

$$b_5 = \frac{R}{b_3 b_4} \tag{37}$$

$$a_2 = P - b_3 \tag{38}$$

$$b_2 = Q - a_2 b_3. \tag{39}$$

A comparison of exact and approximate roots and root coefficients is given in Table 2, for various $\hat{W}_0$ and $\hat{p}_0$ values. The agreement is excellent for moderate and high rates of roll.

Charts (Figs. 22 to 25) are provided to expedite the solution of the b_3 cubic {equation (35)}. They are derived using the following method which is that of Reference 10. If b_3 is replaced by Lm where:

$$L = k\,{}^3\!\sqrt{-R}, \tag{40}$$

the cubic can be written in the form:

$$m^3 + Am^2 + Bm + \frac{1}{k^3} = 0 \tag{41}$$

where

$$A = \frac{-P}{k(-R)^{1/3}} \tag{42}$$

$$B = \frac{Q}{k^2(-R)^{2/3}} \tag{43}$$

and where

$$k = +1 \quad \text{for} \quad R > 0$$

$$k = -1 \quad \text{for} \quad R < 0.$$

Equation (41) will factor into either one real root and one complex pair of roots:

$$\left[m \pm \frac{1}{(\omega')^2}\right][m^2 + 2\zeta\omega' m + (\omega')^2] = 0 \tag{44}$$

or three real roots:

$$\left[m \pm \frac{1}{b_6 b_7}\right][m + b_6][m + b_7] = 0 \tag{45}$$

where the plus sign in the first factor of equations (44) and (45) is associated with $k = +1$ and the minus sign with $k = -1$.

If equations (44) and (45) are equated to equation (41) the following relationships between coefficients are obtained:

$$\left.\begin{aligned} A &= 2\zeta\omega' \pm \frac{1}{(\omega')^2} \\ B &= (\omega')^2 \pm \frac{2\zeta}{\omega'} \end{aligned}\right\} \tag{46}$$

or

$$\left.\begin{aligned} A &= b_6 + b_7 \pm \frac{1}{b_6 b_7} \\ B &= b_6 b_7 \pm \frac{b_6 + b_7}{b_6 b_7}. \end{aligned}\right\} \tag{47}$$

Figs. 22, 23, 24 and 25 are graphical representations of expressions (46) and (47)

Thus, to determine the real roots of equation (35), A and B are computed using expressions (42) and (43), choosing the value of k appropriate to the sign of R. The values of ω' and ζ or b_6 and $b_6 b_7$ are then obtained from the appropriate chart and the roots computed using the relationships:

$$b_3 = \frac{k|R^{1/3}|}{(\omega')^2}$$

or

$$b_3 = \frac{k|R^{1/3}|}{b_6 b_7}, \quad |R^{1/3}|b_6, \quad |R^{1/3}|b_7.$$

Response.

The response in $\hat{p}$, $\hat{q}$, $\hat{r}$, $\hat{w}$ and $\hat{v}$ can be represented by the operational fraction:

$$F(D) = \frac{H_{0n}'D^5 + H_{1n}'D^4 + H_{2n}'D^3 + H_{3n}'D^2 + H_{4n}'D + H_{5n}'}{G_0D^5 + G_1D^4 + G_2D^3 + G_3D^2 + G_4D + G_5} \tag{48}$$

if the effect of gravity is neglected. If the factors of the denominator are of the type of equation (22) the response can be expressed:

$$\begin{aligned} F_A(\tau) &= A_0 + A_1e^{-b_3\tau} + e^{-r_2\tau}(A_2\cos s_2\tau + A_3\sin s_2\tau) \\ &\quad + e^{-r_1\tau}(A_4\cos s_1\tau + A_5\sin s_1\tau) \end{aligned} \tag{49}$$

or if the factors are of the type of equation (23) the response expression will be:

$$F_B(\tau) = B_0 + B_1e^{-b_3\tau} + B_2e^{-b_4\tau} + B_3e^{-b_5\tau} + e^{-r_1\tau}(B_4\cos s_1{}^{\tau} + B_5\sin s_1\tau) \tag{50}$$

The coefficients A_n and B_n in the above expressions can be calculated directly using the method given in Section 4 of Reference 9, or by use of the tables of Reference 8.

The A_n and B_n expressions are as follows if $b_1 \gg a_1$. (It should be noted that the formulae given in Reference 9 are in the notation of the Laplace transform and must be re-expressed if the Heaviside notation is used.)

$$\left.\begin{aligned}
A_0 &= \frac{H_{5n}'}{b_1b_2b_3} \\
A_1 &= \frac{-H_{0n}'b_3^5 + H_{1n}'b_3^4 - H_{2n}'b_3^3 + H_{3n}'b_3^2 - H_{4n}'b_3 + H_{5n}'}{(-b_3)(b_3^2 - a_1b_3 + b_1)(b_3^2 - a_2b_3 + b_2)} \\
A_2 &= F_A(0) - (A_0 + A_1 + A_4) \\
A_3 &= \frac{1}{s_2}[F_A'(0) + b_3A_1 + r_1A_4 + r_2A_2 - s_1A_5] \\
A_4 &= \frac{\dfrac{G_3}{b_1}\left[b_1H_{1n}' - r_1H_{2n}' + \dfrac{r_1H_{4n}' + (r_1^2 - s_1^2)H_{3n}'}{b_1}\right] - s_1\sqrt{G_2}\left[H_{2n}' - \dfrac{2r_1H_{3n}' + H_{4n}'}{b_1}\right]}{b_1\left(\dfrac{G_3^2}{b_1^2} + G_2\right)} \\
A_5 &= \frac{\sqrt{G_2}\left[b_1H_{1n}' - rH_{2n}' + \dfrac{r_1H_{4n}' + (r_1^2 - s_1^2)H_{3n}'}{b_1}\right] + \dfrac{G_3}{b_1}s_1\left[H_{2n}' - \dfrac{2r_1H_{3n}' + H_{4n}'}{b_1}\right]}{b_1\left(\dfrac{G_3^2}{b_1^2} + G_2\right)} \\
B_0 &= \frac{H_{5n}'}{b_1b_3b_4b_5} \\
B_1 &= \frac{-H_{0n}'b_3^5 + H_{1n}'b_3^4 - H_{2n}'b_3^3 + H_{3n}'b_3^2 - H_{4n}'b_3 + H_{5n}'}{(-b_3)(b_3^2 - a_1b_3 + b_1)(b_4 - b_3)(b_5 - b_3)} \\
B_2 &= \frac{-H_{0n}'b_4^5 + H_{1n}'b_4^4 - H_{2n}'b_4^3 + H_{3n}'b_4^2 - H_{4n}'b_4 + H_{5n}'}{(-b_4)(b_4^2 - a_1b_4 + b_1)(b_3 - b_4)(b_5 - b_4)} \\
B_3 &= \frac{-H_{0n}'b_5^5 + H_{1n}'b_5^4 - H_{2n}'b_5^3 + H_{3n}'b_5^2 - H_{4n}'b_5 + H_{5n}'}{(-b_5)(b_5^2 - a_1b_5 + b_1)(b_3 - b_5)(b_4 - b_5)} \\
B_4 &= F_B(0) - (B_0 + B_1 + B_2 + B_3) \\
B_5 &= \frac{1}{s_1}[F_B'(0) + b_3B_1 + b_4B_2 + b_5B_3 + r_1B_4].
\end{aligned}\right\} \tag{51}$$

The numerators of the expressions for A_1, B_1, B_2 and B_3 can be computed rapidly by a desk calculating machine using the routine of Section 2.1 of Reference 9.

The numerator coefficients H_{jn}' of the operational fraction can be derived from the equations of motion and are as follows (cf. Appendix I):

$\underline{\hat{p}}$:

$$\left.\begin{aligned}
H_{01}' &= \hat{p}_i \\
H_{11}' &= \lambda_{11}\hat{p}_i - \omega_l\hat{v}_i + \nu_{lr}\hat{r}_i + \mu\bar{\xi}\,\frac{l_\xi}{i_A} \\
H_{21}' &= \lambda_{21}\hat{p}_i - \psi_{11}\hat{v}_i + \gamma_{11}\hat{r}_i - \delta_z\hat{p}_0\nu_{lr}\hat{q}_i + (\delta_{11} - \delta_z\hat{p}_0\nu_{lr}\chi)\hat{w}_i + \\
&\quad + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{11} + \frac{n_\xi}{i_c}\,\nu_{lr}\right) \\
H_{31}' &= \lambda_{31}\hat{p}_i - \psi_{21}\hat{v}_i + \gamma_{21}\hat{r}_i + \epsilon_{11}\hat{q}_i + (\epsilon_{11}\chi + \delta_{21})\hat{w}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{21} + \frac{n_\xi}{i_C}\,\gamma_{11}\right) \\
H_{41}' &= \lambda_{41}\hat{p}_i - \psi_{31}\hat{v}_i + \gamma_{31}\hat{r}_i + \epsilon_{21}\hat{q}_i + (\epsilon_{21}\chi + \delta_{31})\hat{w}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{31} + \frac{n_\xi}{i_C}\,\gamma_{21}\right) \\
H_{51}' &= \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{41} + \frac{n_\xi}{i_C}\,\gamma_{31}\right).
\end{aligned}\right\} \qquad (52)$$

$\underline{\hat{q}}$:

$$\left.\begin{aligned}
H_{02}' &= \hat{q}_i \\
H_{12}' &= \epsilon_{12}\hat{q}_i + (\epsilon_{12}\chi + \delta_{12})\hat{w}_i - \delta_y\hat{p}_0\hat{r}_i + \hat{p}_0\chi\hat{v}_i \\
H_{22}' &= \epsilon_{22}\hat{q}_i + (\epsilon_{22}\chi + \delta_{22})\hat{w}_i + \gamma_{12}\hat{r}_i + \psi_{12}\hat{v}_i + \lambda_{12}\hat{p}_i - \mu\bar{\xi}\,\frac{n_\xi}{i_C}\,\delta_y\hat{p}_0 \\
H_{32}' &= \epsilon_{32}\hat{q}_i + (\epsilon_{32}\chi + \delta_{32})\hat{w}_i + \gamma_{22}\hat{r}_i + \psi_{22}\hat{v}_i + \lambda_{22}\hat{p}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{12} + \frac{n_\xi}{i_C}\,\gamma_{12}\right) \\
H_{42}' &= \epsilon_{42}\hat{q}_i + (\epsilon_{42}\chi + \delta_{42})\hat{w}_i + \gamma_{32}\hat{r}_i + \psi_{32}\hat{v}_i + \lambda_{32}\hat{p}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{22} + \frac{n_\xi}{i_C}\,\gamma_{22}\right) \\
H_{52}' &= \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{32} + \frac{n_\xi}{i_C}\,\gamma_{32}\right).
\end{aligned}\right\} \qquad (53)$$

$\underline{\hat{r}}$:

$$\left.\begin{aligned}
H_{03}' &= \hat{r}_i \\
H_{13}' &= \gamma_{13}\hat{r}_i + \omega_n\hat{v}_i - \nu_{np}\hat{p}_i - \delta_z\hat{p}_0\hat{q}_i + \mu\bar{\xi}\,\frac{n_\xi}{i_C} \\
H_{23}' &= \gamma_{23}\hat{r}_i + \psi_{13}\hat{v}_i + \lambda_{13}\hat{p}_i + \epsilon_{13}\hat{q}_i + (\epsilon_{13}\chi + \delta_{13})\hat{w}_i + \mu\bar{\xi}\left(-\frac{l_\xi}{i_A}\,\nu_{np} + \frac{n_\xi}{i_C}\,\gamma_{13}\right) \\
H_{33}' &= \gamma_{33}\hat{r}_i + \psi_{23}\hat{v}_i + \lambda_{23}\hat{p}_i + \epsilon_{23}\hat{q}_i + (\epsilon_{23}\chi + \delta_{23})\hat{w}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{13} + \frac{n_\xi}{i_C}\,\gamma_{23}\right) \\
H_{43}' &= \gamma_{43}\hat{r}_i + \psi_{33}\hat{v}_i + \lambda_{33}\hat{p}_i + \epsilon_{33}\hat{q}_i + (\epsilon_{33}\chi + \delta_{33})\hat{w}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{23} + \frac{n_\xi}{i_C}\,\gamma_{33}\right) \\
H_{53}' &= \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\,\lambda_{33} + \frac{n_\xi}{i_C}\,\gamma_{43}\right).
\end{aligned}\right\} \qquad (54)$$

$\underline{\hat{w}}$:

$$\left.\begin{aligned}
H_{04}' &= \hat{w}_i \\
H_{14}' &= (\delta_{14}+\chi)\hat{w}_i + \hat{q}_i - \hat{p}_0\hat{v}_i \\
H_{24}' &= (\delta_{24}+\chi k_{20})\hat{w}_i + k_{20}\hat{q}_i + \psi_{14}\hat{v}_i - \hat{W}_0\hat{p}_0\hat{p}_i + \gamma_{14}\hat{r}_i \\
H_{34}' &= (\delta_{34}+\chi\epsilon_{14})\hat{w}_i + \epsilon_{14}\hat{q}_i + \psi_{24}\hat{v}_i + \lambda_{14}\hat{p}_i + \gamma_{24}\hat{r}_i + \mu\bar{\xi}\left(-\frac{l_\xi}{i_A}\hat{W}_0\hat{p}_0 + \frac{n_\xi}{i_C}\gamma_{14}\right) \\
H_{44}' &= (\delta_{44}+\chi\epsilon_{24})\hat{w}_i + \epsilon_{24}\hat{q}_i + \psi_{34}\hat{v}_i + \lambda_{24}\hat{p}_i + \gamma_{34}\hat{r}_i + \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\lambda_{14} + \frac{n_\xi}{i_C}\gamma_{24}\right) \\
H_{54}' &= \mu\bar{\xi}\left(\frac{l_\xi}{i_A}\lambda_{24} + \frac{n_\xi}{i_C}\gamma_{34}\right).
\end{aligned}\right\} \qquad (55)$$

$\underline{\hat{v}}$:

$$\left.\begin{aligned}
H_{05}' &= \hat{v}_i \\
H_{15}' &= \psi_{15}\hat{v}_i - \hat{r}_i + \hat{W}_0\hat{p}_i + \hat{p}_0\hat{w}_i \\
H_{25}' &= \psi_{25}\hat{v}_i - \gamma_{15}\hat{r}_i + [\hat{W}_0(\nu_n+k_{21}) + \nu_{np}]\hat{p}_i + [\hat{p}_0(1+\delta_z)\chi + \delta_{15}]\hat{w}_i + \\
&\qquad + \hat{p}_0(1+\delta_z)\hat{q}_i + \mu\bar{\xi}\left[\frac{l_\xi}{i_A}\hat{W}_0 - \frac{n_\xi}{i_C}\right] \\
H_{35}' &= \psi_{35}\hat{v}_i - \gamma_{25}\hat{r}_i + \lambda_{15}\hat{p}_i + (\epsilon_{15}\chi+\delta_{25})\hat{w}_i + \epsilon_{15}\hat{q}_i + \\
&\qquad + \mu\bar{\xi}\left[\frac{l_\xi}{i_A}\{\hat{W}_0(\nu_n+k_{21}) + \nu_{np}\} - \frac{n_\xi}{i_C}\delta_{15}\right] \\
H_{45}' &= \psi_{45}\hat{v}_i - \gamma_{35}\hat{r}_i + \lambda_{25}\hat{p}_i + (\epsilon_{25}\chi+\delta_{35})\hat{w}_i + \epsilon_{25}\hat{q}_i + \mu\bar{\xi}\left[\frac{l_\xi}{i_A}\lambda_{15} - \frac{n_\xi}{i_C}\gamma_{25}\right] \\
H_{55}' &= \mu\bar{\xi}\left[\frac{l_\xi}{i_A}\lambda_{25} - \frac{n_\xi}{i_C}\gamma_{35}\right].
\end{aligned}\right\} \qquad (56)$$

The calculated response, using this simplified method and neglecting the fast oscillatory mode, is compared with the solution of the equations linearised as in main text and a digital-computer solution of the original non-linear equations in Figs. 13, 14 and 15. It can be seen that the agreement is quite good.

In the simplified-method solutions shown in these figures the calculation has been broken down into three sections. In the first $\hat{p}_0$ has a constant non-zero value and ξ the three values $\xi_1(0 < t < t_1)$, $\xi_2(t_1 < t < t_2)$ and $\xi_3(t_2 < t < t_2')$.

As the initial conditions $\hat{p}_i$, $\hat{q}_i$, $\hat{r}_i$, $\hat{w}_i$ and $\hat{v}_i$ for this first interval are all zero, the numerator-coefficient expressions {equations (52) to (56)} can be greatly simplified and become proportional to $\bar{\xi}$. It is therefore possible to superimpose solutions to obtain the response to the varying ξ in this interval, thereby considerably shortening the computations.

In the second interval ($t_2' < t < t_3$), $\hat{p}_0 = 0$, but $\xi = \xi_3$. The equations of motion in operational form are thus reduced to the familiar uncoupled form (without gravity terms):

$$\left.\begin{aligned}
(D+\bar{y}_v)\hat{v} - \hat{W}_0\hat{p} + \hat{r} &= \hat{v}_i D \\
\omega_l\hat{v} + (D+\nu_l)\hat{p} - \nu_{lr}\hat{r} &= \frac{\mu l_\xi}{i_A}\bar{\xi} + \hat{p}_i D \\
-\omega_n\hat{v} + \nu_{np}\hat{p} + (D+\nu_n)\hat{r} &= \frac{\mu n_\xi}{i_C}\bar{\xi} + \hat{r}_i D \\
(D+\nu)\hat{q} + (\omega+\chi D)\hat{w} &= (\hat{q}_i+\chi\hat{w}_i)D \\
-\hat{q} + (D-z_w)\hat{w} &= \hat{w}_i D.
\end{aligned}\right\} \tag{57}$$

The operational fraction representing the lateral response is therefore:

$$\varphi(D) = \frac{H_{0n}''D^3 + H_{1n}''D^2 + H_{2n}''D + H_{3n}''}{D^3 + k_{10}D^2 + (k_{20}+\hat{W}_0\omega_l)D + \{k_{30} + \hat{W}_0(\omega_l\nu_n - \omega_n\nu_{lr})\}}, \tag{58}$$

and for the longitudinal response:

$$\varphi(D) = \frac{H_{0n}''D^2 + H_{1n}''D + H_{2n}''}{D^2 + k_{11}D + k_{21}} \tag{59}$$

where the fraction numerator coefficients H_{jn}'' are:

$\underline{\hat{p}}$:

$$\left.\begin{aligned}
H_{01}'' &= \hat{p}_i \\
H_{11}'' &= \hat{p}_i(\bar{y}_v+\nu_n) + \nu_{lr}\hat{r}_i - \hat{v}_i\omega_l + \frac{\mu l_\xi}{i_A}\bar{\xi} \\
H_{21}'' &= \hat{p}_i(\bar{y}_v\nu_n+\omega_n) + (\omega_l+\nu_{lr}\bar{y}_v)\hat{r}_i + \hat{v}_i(\omega_n\nu_{lr}-\omega_l\nu_n) + \mu\bar{\xi}\left[\frac{l_\xi}{i_A}(\bar{y}_v+\nu_n) + \frac{n_\xi}{i_C}\nu_{lr}\right] \\
H_{31}'' &= \mu\bar{\xi}\left[\frac{l_\xi}{i_A}(\bar{y}_v\nu_n+\omega_n) + \frac{n_\xi}{i_C}(\omega_l+\nu_{lr}\bar{y}_v)\right]
\end{aligned}\right\} \tag{60}$$

$\underline{\hat{q}}$:

$$\left.\begin{aligned}
H_{02}'' &= \hat{q}_i \\
H_{12}'' &= -z_w\hat{q}_i - \hat{w}_i(\omega+\chi z_w) \\
H_{22}'' &= 0
\end{aligned}\right\} \tag{61}$$

$\underline{\hat{r}}$:

$$\left.\begin{aligned}
H_{03}'' &= \hat{r}_i \\
H_{13}'' &= \hat{r}_i(\bar{y}_v+\nu_l) + \omega_n\hat{v}_i - \nu_{np}\hat{p}_i + \frac{\mu n_\xi}{i_C}\bar{\xi} \\
H_{23}'' &= \hat{r}_i(\bar{y}_v\nu_l+\hat{W}_0\omega_l) + \hat{v}_i(\omega_l\nu_{np}+\omega_n\nu_l) + \hat{p}_i(\hat{W}_0\omega_n-\bar{y}_v\nu_{np}) + \\
&\quad + \mu\bar{\xi}\left[\frac{n_\xi}{i_C}(\bar{y}_v+\nu_l) - \frac{l_\xi}{i_A}\nu_{np}\right] \\
H_{33}'' &= \mu\bar{\xi}\left[\frac{l_\xi}{i_A}(\hat{W}_0\omega_n-\bar{y}_v\nu_{np}) + \frac{n_\xi}{i_C}(\bar{y}_v\nu_l+\hat{W}_0\omega_l)\right]
\end{aligned}\right\} \tag{62}$$

$\underline{\hat{w}}$:

$$\left.\begin{aligned} H_{04}'' &= \hat{w}_i \\ H_{14}'' &= \hat{w}_i(\chi+\nu) + \hat{q}_i \\ H_{24}'' &= 0 \end{aligned}\right\} \quad (63)$$

$\underline{\hat{v}}$:

$$\left.\begin{aligned} H_{05}'' &= \hat{v}_i \\ H_{15}'' &= \hat{v}_i(\nu_l+\nu_n) - \hat{r}_i + \hat{W}_0\hat{p}_i \\ H_{25}'' &= \hat{v}_i(\nu_l\nu_n+\nu_{lr}\nu_{np}) + \hat{r}_i(\hat{W}_0\nu_{lr}-\nu_l) + \hat{p}_i(\hat{W}_0\nu_n+\nu_{np}) + \mu\bar{\xi}\left[\frac{l_\xi}{i_A}\hat{W}_0 - \frac{n_\xi}{i_C}\right] \\ H_{35}'' &= \mu\bar{\xi}\left[\frac{l_\xi}{i_A}(\nu_{np}+\hat{W}_0\nu_n) + \frac{n_\xi}{i_C}(\hat{W}_0\nu_{lr}-\nu_l)\right]. \end{aligned}\right\} \quad (64)$$

In these expressions $\hat{p}_i$, $\hat{q}_i$, $\hat{r}_i$, $\hat{w}_i$, and $\hat{v}_i$ are the final values in the preceding interval.

During the third interval $\hat{p}_0$ and ξ are zero. The calculations for the lateral response will therefore be similar to those in the second interval, the ξ terms in the H_{jn}'' expressions being dropped. The $\hat{p}_i$, $\hat{q}_i$, $\hat{r}_i$, $\hat{w}_i$ and $\hat{v}_i$ values are again the final values of the last interval. It can be seen that new coefficients will not have to be computed for the longitudinal response which is, in the uncoupled regime, independent of ξ. The calculations of the second interval should therefore be continued into the third interval.

APPENDIX IV

Alternative Steady States, and their Stability

When gravity can be considered as making a negligible contribution to the motion we have a set of equations yielding steady states other than that corresponding to $p = q = r = w = v = 0$. Rough approximations to the values defining these alternative steady states have been given by Pinsker[3, 6]. We shall begin by calculating the steady-state values $\hat{p}_s$, $\hat{q}_s$, $\hat{r}_s$, $\hat{w}_s$ and $\hat{v}_s$, with no additional approximations.

The equations defining the steady states, with aileron centralised, are obtained from the equations of motion by dropping all terms in $d/d\tau$. They are:

$$\left.\begin{aligned} \nu_l\hat{p}_s - \nu_{lr}\hat{r}_s + \omega_l\hat{v}_s + \delta_x\hat{q}_s\hat{r}_s &= 0 \\ \nu\hat{q}_s + \omega\hat{w}_s + \delta_y\hat{p}_s\hat{r}_s &= 0 \\ \nu_{np}\hat{p}_s + \nu_n\hat{r}_s + \delta_z\hat{p}_s\hat{q}_s &= 0 \\ -\hat{W}_0\hat{p}_s + \hat{r}_s - \hat{p}_s\hat{w}_s + \bar{y}_v\hat{v}_s &= 0 \\ \hat{q}_s - z_w\hat{w}_s + \hat{p}_s\hat{v}_s &= 0\,. \end{aligned}\right\} \tag{65}$$

To solve these equations we treat the last four as equations for $\hat{q}_s$, $\hat{r}_s$, $\hat{w}_s$ and $\hat{v}_s$ in terms of $\hat{p}_s$. Their solution can be written:

$$\left.\begin{aligned} (\mathscr{C}_0+\mathscr{C}_1\hat{p}_s^{\,2}+\mathscr{C}_2\hat{p}_s^{\,4})\hat{q}_s &= \mathscr{Q}_1\hat{p}_s^{\,2} + \mathscr{Q}_2\hat{p}_s^{\,4} \\ (\mathscr{C}_0+\mathscr{C}_1\hat{p}_s^{\,2}+\mathscr{C}_2\hat{p}_s^{\,4})\hat{r}_s &= \mathscr{R}_1\hat{p}_s + \mathscr{R}_2\hat{p}_s^{\,3} \\ (\mathscr{C}_0+\mathscr{C}_1\hat{p}_s^{\,2}+\mathscr{C}_2\hat{p}_s^{\,4})\hat{v}_s &= \beta_1\hat{p}_s + \beta_2\hat{p}_s^{\,3} \\ (\mathscr{C}_0+\mathscr{C}_1\hat{p}_s^{\,2}+\mathscr{C}_2\hat{p}_s^{\,4})\hat{w}_s &= \alpha_1\hat{p}_s^{\,2} + \alpha_2\hat{p}_s^{\,4} \end{aligned}\right\} \tag{66}$$

where

$$\begin{aligned} \mathscr{C}_0 &= (\omega - \nu z_w)(\omega_n + \bar{y}_v\nu_n) \\ \mathscr{C}_1 &= \omega_n\delta_y + \delta_y\delta_z z_w\bar{y}_v + \nu\nu_n - \omega\delta_z \\ \mathscr{C}_2 &= -\delta_y\delta_z \\ \mathscr{Q}_1 &= \omega(\nu_n\hat{W}_0 + \nu_{np}) + \delta_y\omega_n z_w\hat{W}_0 - \delta_y z_w\nu_{np}\bar{y}_v \\ \mathscr{Q}_2 &= \delta_y\nu_{np} \\ \mathscr{R}_1 &= \omega_n\hat{W}_0(\omega - z_w\nu) - \omega\nu_{np}\bar{y}_v + z_w\nu\nu_{np}\bar{y}_v \\ \mathscr{R}_2 &= -\delta_z\omega\hat{W}_0 - \nu\nu_{np} \\ \beta_1 &= (\omega - \nu z_w)\nu_n\hat{W}_0 - \omega\nu_{np} - z_w\nu\nu_{np} \\ \beta_2 &= \delta_y(\delta_z z_w\hat{W}_0 + \nu_{np}) \\ \alpha_1 &= \delta_y(\nu_{np}\bar{y}_v - \omega_n\hat{W}_0) - \nu\nu_{np} - \nu\nu_n\hat{W}_0 \\ \alpha_2 &= \delta_y\delta_z\hat{W}_0\,. \end{aligned}$$

Inserting these in the first of the equations we have a quartic in $\hat{p}_s^2$:

$$\nu_l \mathscr{C}_2^2 \hat{p}_s^8 + (2\nu_l \mathscr{C}_1 \mathscr{C}_2 + \delta_x \mathscr{Q}_2 \mathscr{R}_2 + \mathscr{C}_2 \beta_2 \omega_l + \mathscr{C}_2 \mathscr{R}_2 \nu_{lr}) \hat{p}_s^6 +$$

$$+ (2\nu_l \mathscr{C}_0 \mathscr{C}_2 + \nu_l \mathscr{C}_1^2 + \mathscr{C}_2 \beta_1 \omega_l + \mathscr{C}_1 \beta_2 \omega_l + \delta_x \mathscr{Q}_1 \mathscr{R}_2 + \delta_x \mathscr{Q}_2 \mathscr{R}_1 + \nu_{lr} \mathscr{R}_1 \mathscr{C}_2 + \nu_{lr} \mathscr{R}_2 \mathscr{C}_1) \hat{p}_s^4 +$$

$$+ (2\nu_l \mathscr{C}_0 \mathscr{C}_1 + \mathscr{C}_1 \beta_1 \omega_l + \mathscr{C}_0 \beta_2 \omega_l + \delta_x \mathscr{Q}_1 \mathscr{R}_1 + \nu_{lr} \mathscr{C}_1 \mathscr{R}_1 + \nu_{lr} \mathscr{C}_0 \mathscr{R}_2) \hat{p}_s^2 +$$

$$+ \mathscr{C}_0 (\nu_l \mathscr{C}_0 + \omega_l \beta_1 + \mathscr{R}_1 \nu_{lr}) = 0. \tag{67}$$

This yields four values of $\hat{p}_s^2$, with corresponding values of $\hat{q}_s$, $\hat{r}_s$, $\hat{v}_s$, $\hat{w}_s$ and $\hat{\alpha}_s$.

In as much as we have in the main text regarded q and r as small of first order, and so neglected the $\hat{q}\hat{r}$ term we shall examine the effect of this approximation on the steady-state solutions. The above quartic in $\hat{p}_s^2$ simplifies to a quadratic:

$$\nu_l \mathscr{C}_2 \hat{p}_s^4 + (\nu_l \mathscr{C}_1 + \omega_l \beta_2 + \nu_{lr} \mathscr{R}_2) \hat{p}_s^2 + (\mathscr{C}_0 \nu_l + \omega_l \beta_1 + \nu_{lr} \mathscr{R}_1) = 0. \tag{68}$$

It may be noted that this is equivalent to the equation:

$$(G_5)_{\hat{p}_0 = \hat{p}_s} = 0. \tag{69}$$

The other relationships remain unaltered. Numerical solutions have been obtained for the aeroplane used in our response calculations, and are given in the table below.

Steady-State Conditions

	$\hat{p}_s$	$\hat{q}_s$	$\hat{r}_s$	$\hat{w}_s$	$\hat{v}_s$
Including qr term	− 10·1965	− 1·1533	− 0·2126	0·1038	0·1353
	− 4·8788	− 0·9585	1·1738	− 0·2690	0·0765
	− 9·215	−120·28	15·34	− 2·02	12·56
	− 5·5146	− 7·124	10·382	− 1·8285	0·5706
Neglecting qr term	− 10·1864	− 1·1676	− 0·2115	0·1037	0·1368
	− 4·7705	− 0·8010	1·4825	− 0·2278	0·0640

Having determined these other steady states we consider the linearised motion around these.

We consider a small perturbation represented by $\delta\hat{p}$, $\delta\hat{q}$, $\delta\hat{r}$, $\delta\hat{v}$ and $\delta\hat{w}$. The equations of motion are, again retaining in the first instance the term in qr:

$$\left.\begin{aligned}
(D+\nu_l)(\hat{p}_s+\delta\hat{p}) + \omega_l(\hat{v}_s+\delta\hat{v}) - \nu_{lr}(\hat{r}_s+\delta\hat{r}) + \delta_x(\hat{q}_s+\delta\hat{q})(\hat{r}_s+\delta\hat{r}) &= 0 \\
\nu_{np}(\hat{p}_s+\delta\hat{p}) - \omega_n(\hat{v}_s+\delta\hat{v}) + (D+\nu_n)(\hat{r}_s+\delta\hat{r}) + \delta_z(\hat{p}_s+\delta\hat{p})(\hat{q}_s+\delta\hat{q}) &= 0 \\
(D+\nu)(\hat{q}_s+\delta\hat{q}) + (\omega+\chi D)(\hat{w}_s+\delta\hat{w}) + \delta_y(\hat{p}_s+\delta\hat{p})(\hat{r}_s+\delta\hat{r}) &= 0 \\
-(\hat{p}_s+\delta\hat{p})(\hat{W}_0+\hat{w}_s+\delta\hat{w}) + (D+\bar{y}_v)(\hat{v}_s+\delta\hat{v}) + (\hat{r}_s+\delta\hat{r}) &= 0 \\
-(\hat{q}_s+\delta\hat{q}) + (D-z_w)(\hat{w}_s+\delta\hat{w}) + (\hat{p}_s+\delta\hat{p})(\hat{v}_s+\delta\hat{v}) &= 0.
\end{aligned}\right\} \tag{70}$$

By virtue of the fact that $\hat{p}_s$, $\hat{q}_s$, $\hat{r}_s$, $\hat{v}_s$ and $\hat{w}_s$ satisfy the steady-state equations we have on neglecting products of the perturbations:

$$\left.\begin{array}{llllll}
(D+\nu_l)\delta\hat{p} & +\,\omega_l\delta\hat{v} & +\,(\delta_x\hat{q}_s-\nu_{lr})\delta\hat{r} & +\,\delta_x\hat{r}_s\delta\hat{q} & & = 0 \\
(\nu_{np}+\delta_z\hat{q}_s)\delta\hat{p} & -\,\omega_n\delta\hat{v} & +\,(D+\nu_n)\delta\hat{r} & +\,\delta_z\hat{p}_s\delta\hat{q} & & = 0 \\
\delta_y\hat{r}_s\delta\hat{p} & & +\,\delta_y\hat{p}_s\delta\hat{r} & +\,(D+\nu)\delta\hat{q} & +\,(\omega+\chi D)\delta\hat{w} & = 0 \\
-\,(\hat{W}_0+\hat{w}_s)\delta\hat{p} & +\,(D+\bar{y}_v)\delta\hat{v} & +\,\delta\hat{r} & & -\,\hat{p}_s\delta\hat{w} & = 0 \\
\beta_s\delta\hat{p} & +\,\hat{p}_s\delta\hat{v} & & -\,\delta\hat{q} & +\,(D-z_w)\delta\hat{w} & = 0.
\end{array}\right\} \tag{71}$$

The stability equation can be written, $(\alpha_s = \hat{W}_0+\hat{w}_s)$,

$$\Delta = \begin{vmatrix}
\lambda+\bar{y}_v & -\,\alpha_s & 1 & 0 & -\,\hat{p}_s \\
\omega_l & \lambda+\nu_l & (-\,\nu_{lr}+\delta_x\hat{q}_s) & \delta_x\hat{r}_s & 0 \\
-\,\omega_n & (\nu_{np}+\delta_z\hat{q}_s) & \lambda+\nu_n & \delta_z\hat{p}_s & 0 \\
0 & +\,\delta_y\hat{r}_s & +\,\delta_y\hat{p}_s & \lambda+\nu & \omega+\chi\lambda \\
\hat{p}_s & \hat{v}_s & 0 & -1 & \lambda-z_w
\end{vmatrix} = 0. \tag{72}$$

In this form we can express it as the sum of three determinants, the first of which is identical in form with the denominator of the main text. We thus have:

$$\begin{vmatrix}
\lambda+\bar{y}_v & -\,\alpha_s & 1 & 0 & -\,\hat{p}_s \\
\omega_l & \lambda+\nu_l & -\,\nu_{lr}+\delta_x\hat{q}_s & 0 & 0 \\
-\,\omega_n & \nu_{np}+\delta_z\hat{q}_s & \lambda+\nu_n & \delta_z\hat{p}_s & 0 \\
0 & 0 & \delta_y\hat{p}_s & \lambda+\nu & \omega+\chi\lambda \\
\hat{p}_s & 0 & 0 & -1 & \lambda-z_w
\end{vmatrix} + \begin{vmatrix}
\lambda+\bar{y}_v & -\,\alpha_s & 1 & 0 & -\,\hat{p}_s \\
\omega_l & \lambda+\nu_l & -\,\nu_{lr}+\delta_x\hat{q}_s & \delta_x\hat{r}_s & 0 \\
-\,\omega_n & \nu_{np}+\delta_z\hat{q}_s & \lambda+\nu_n & 0 & 0 \\
0 & 0 & \delta_y\hat{p}_s & 0 & \omega+\chi\lambda \\
\hat{p}_s & 0 & 0 & 0 & \lambda-z_w
\end{vmatrix} +$$

$$+ \begin{vmatrix}
\lambda+\bar{y}_v & 0 & 1 & 0 & -\,\hat{p}_s \\
\omega_l & 0 & -\,\nu_{lr}+\delta_x\hat{q}_s & \delta_x\hat{r}_s & 0 \\
-\,\omega_n & 0 & \lambda+\nu_n & \delta_z\hat{p}_s & 0 \\
0 & \delta_y\hat{r}_s & \delta_y\hat{p}_s & \lambda+\nu & \omega+\chi\lambda \\
\hat{p}_s & \hat{v}_s & 0 & -1 & \lambda-z_w
\end{vmatrix} = 0. \tag{73}$$

Expanding these, we have a quintic with the following coefficients:

λ^5:

$$1 = G_0 \tag{74}$$

λ^4:

$$\nu+\chi-z_w+\nu_l+\nu_n+\bar{y}_v = G_1 \tag{75}$$

λ^3:

$$(G_2)+(\delta_2\nu_{lr}-\delta_x\nu_{np})\hat{q}_s-\delta_x\delta_z\hat{q}_s^{\,2}-\delta_x\delta_y\hat{r}_s^{\,2}+\omega_l\hat{w}_s+\delta_n\chi\hat{v}_s\hat{r}_s$$

λ^2:

$$(G_3) + \delta_2\omega_l\hat{q}_s + (\nu+\chi-z_w+\bar{y}_v)(\delta_z\nu_{lr}-\delta_x\nu_{np})\hat{q}_s - \delta_x\delta_z(\nu+\chi-z_w+\bar{y}_v)\hat{q}_s^2 -$$
$$- \delta_x\delta_y(\nu_n+\bar{y}_v-z_w)\hat{r}_s^2 + \omega_l(\nu_n+\nu+\chi-z_w)\hat{w}_s - \nu_{lr}\omega_n\hat{w}_s +$$
$$+ \delta_y(\delta_x\nu_{np}-\delta_z\nu_{lr})\hat{p}_s\hat{r}_s + (\delta_z\chi\nu_{lr}-\omega_l)\hat{p}_s\hat{v}_s + 2\delta_x\delta_y\delta_z\hat{p}_s\hat{q}_s\hat{r}_s -$$
$$- \delta_x\delta_z\chi\hat{p}_s\hat{q}_s\hat{v}_s + \delta_x(\omega+\chi\bar{y}_v+\chi\nu_n)\hat{v}_s\hat{r}_s + \delta_x\omega_n\alpha_s\hat{q}_s + \delta_x\chi\alpha_s\hat{p}_s\hat{r}_s$$

λ:

$$(G_4) + (\delta_z\nu_{lr}-\delta_x\nu_{np})\{\omega - \nu z_w + \bar{y}_v(\nu+\chi-z_w)\}\hat{q}_s + \delta_z\omega_l(\nu+\chi-z_w)\hat{q}_s -$$
$$- \delta_x\delta_z\{\omega - \nu z_w + \bar{y}_v(\nu+\chi-z_w)\}\hat{q}_s^2 - \delta_x\delta_y\{\omega_n + \bar{y}_v - z_w(\nu_n+\bar{y}_v)\}\hat{r}_s^2 +$$
$$+ \omega_l(\omega-\nu z_w)\hat{w}_s + (\omega_l\nu_n-\omega_n\nu_{lr})(\nu+\chi-z_w)\hat{w}_s + (\delta_z\nu_{lr}-\delta_x\nu_{np})\hat{p}_s^2\hat{q}_s +$$
$$+ \delta_z(\chi\nu_{lr}-\delta_y\omega_l)\hat{p}_s^2\hat{w}_s + \{\delta_y(\bar{y}_v-z_w)(\delta_x\nu_{np}-\delta_z\nu_{lr}) - \delta_y(1+\delta_z)\omega_l + \delta_x\nu_{np}\chi\}\hat{p}_s\hat{r}_s -$$
$$- \delta_x\delta_y\hat{p}_s^2\hat{r}_s^2 + \{\omega_l(\delta_z\chi-\nu-\nu_n) + \nu_{lr}(\omega_n+\delta_z\omega+\delta_z\chi\bar{y}_v)\}\hat{p}_s\hat{v}_s +$$
$$+ \delta_x\delta_z\{2\delta_y(\bar{y}_v-z_w) + \chi\}\hat{p}_s\hat{q}_s\hat{r}_s - \delta_x\{\delta_z(\omega+\chi\bar{y}_v) + \omega_n\}\hat{p}_s\hat{q}_s\hat{v}_s -$$
$$- \delta_x\delta_z\hat{p}_s\hat{q}_s^2 + \delta_x\{\omega(\bar{y}_v+\nu_n) + \chi(\omega_n+\bar{y}_v\nu_n)\}\hat{v}_s\hat{r}_s + \delta_x\omega_n(\nu+\chi-z_w)\alpha_s\hat{q}_s +$$
$$+ \delta_x(\omega+\nu_n\chi-\delta_y\omega_n)\alpha_s\hat{p}_s\hat{r}_s - \delta_x\delta_z\alpha_s\hat{p}_s^2\hat{q}_s.$$

Const.:

$$(G_5) + (\omega-\nu z_w)\{\delta_z(\omega_l+\bar{y}_v\nu_{lr}) - \delta_x\bar{y}_v\nu_{np}\}\hat{q}_s - \delta_x\delta_z\bar{y}_v(\omega-\nu z_w)\hat{q}_s^2 +$$
$$+ \delta_x\delta_y z_w(\omega_n+\bar{y}_v\nu_n)\hat{r}_s^2 + (\omega-\nu z_w)(\omega_l\nu_n-\omega_n\nu_{lr})\hat{w}_s + \{\delta_y\delta_z\omega_l + \nu(\delta_z\nu_{lr}-\delta_x\nu_{np})\}\hat{p}_s\hat{q}_s^2 +$$
$$+ \delta_z(\omega\nu_{lr}+\delta_y\omega_l z_w)\hat{p}_s^2\hat{w}_s + \{\delta_x\omega\nu_{np} + \delta_y\delta_z z_w\omega_l - \delta_y(\omega_l\nu_n-\omega_n\nu_{lr}) + \delta_y\bar{y}_v z_w(\delta_z\nu_{lr}-\delta_x\nu_{np})\}\hat{p}_s\hat{r}_s -$$
$$- \delta_x\delta_y\nu_n\hat{p}_s^2\hat{r}_s^2 + \delta_y(\delta_x\nu_{np}-\delta_z\nu_{lr})\hat{p}_s^3\hat{r}_s + \{\delta_z\omega(\omega_l+\bar{y}_v\nu_{lr}) - \nu(\omega_l\nu_n-\omega_n\nu_{lr})\}\hat{p}_s\hat{v}_s +$$
$$+ \delta_y\delta_z\omega_l\hat{p}_s^3\hat{v}_s + \delta_x\delta_y\omega_n\hat{p}_s^2\hat{v}_s\hat{r}_s + \delta_x\{\delta_z\omega - \delta_y\omega_n - 2\delta_y\delta_z z_w\bar{y}_v\}\hat{p}_s\hat{q}_s\hat{r}_s -$$
$$- \delta_x(\omega_n\nu+\delta_z\omega\bar{y}_v)\hat{p}_s\hat{q}_s\hat{v}_s + 2\delta_x\delta_y\delta_z\hat{p}_s^3\hat{q}_s\hat{r}_s - \delta_x\delta_z\nu\hat{p}_s^2\hat{q}_s^2 + \delta_x\omega(\omega_n+\bar{y}_n\nu_n)\hat{v}_s\hat{r}_s +$$
$$+ \delta_x\omega_n(\omega-\nu z_w)\alpha_s\hat{q}_s + \delta_x(\omega\nu_n+\delta_y z_w\omega_n)\alpha_s\hat{p}_s\hat{r}_s - \delta_x\delta_z\omega\alpha_s\hat{p}_s^2\hat{q}_s^2,$$

where (G_2), (G_3) etc. are the G_2, G_3 etc. of Appendix I with p_s in place of p_0.

A complete study of the stability of the linearised perturbation motion around the steady states would require solution of this complicated quintic for the various steady-state conditions given earlier. It was decided that rather than pursue this matter further it would be more interesting to see how nearly the linearised solution, neglecting the term qr in the equations of motion, followed the solution of the full equations.

An instant of time in the neighbourhood of steady state $p_s = -10{\cdot}186$ was chosen, for which there would be practically no disturbance other than in p, w and r.

The linearised response is compared with motion given by the digital-computer solution in Fig. 19, which shows that the two solutions are in poor agreement in the later stages. To rule out the possibility of this being the result of neglecting gravity, the digital-computer calculation has been repeated neglecting gravity (*see* Figs. 19 and 20). We may thus conclude that the motion around these steady states is also essentially non-linear, and its calculation would require a procedure similar to that used in the main text or Appendix III.

APPENDIX V

Derivation of Aileron Input to Give Approximately Trapezoidal Rate-of-Roll Response

The single-square wave aileron input used in the earlier stages of the calculations yields a response in rate of roll which was not strictly typical of the flight records available. Furthermore, it is unrepresentative in that the pilot makes no effort to terminate the roll manoeuvre. Accordingly a more realistic manoeuvre may be achieved if the rate-of-roll time history is specified and the aileron required to produce this calculated on the basis of the simple roll equation:

$$(D+\nu_l)\hat{p} = -\delta_{l\xi}\xi . \tag{76}$$

This procedure will yield a variation of ξ. This will be realistic to a degree which depends on how soon and how rapidly the pilot has to check the stick deflection and this in turn depends on the inertia and the damping-in-roll characteristics of the aeroplane (Reference 12). It can never be truly realistic in that instantaneous application of aileron is implied initially by the finite slope of the rate-of-roll curve.

The required aileron movement is readily obtained from the above equation (76). Suppose the rate of roll is such that it has a value $\hat{p}_1$ at time τ_1, $\hat{p}_2 = k\hat{p}_1$ at time τ_2, zero at time τ_3 and varies linearly between these points. (The trapezoidal variation $k = 1$ is illustrated in Fig. 21.) We then have for the aileron angle:

$$\left.\begin{aligned}
\xi &= \frac{\hat{p}_1}{\tau_1\delta_{l\xi}}(1+\nu_l\tau), \quad 0 < \tau \leqslant \tau_1 \\
\xi &= \frac{\hat{p}_1(k-1)}{(\tau_2-\tau_1)\delta_{l\xi}}\{1+\nu_l(\tau-\tau_1)\} + \frac{\nu_l\hat{p}_1}{\delta_{l\xi}}, \quad \tau_1 \leqslant \tau \leqslant \tau_2 \\
\xi &= \frac{-k\hat{p}_1}{\delta_{l\xi}}\left\{\frac{1}{\tau_3-\tau_2}[1+\nu_l(\tau-\tau_2)-\nu_l]\right\}, \quad \tau_2 \leqslant \tau \leqslant \tau_3
\end{aligned}\right\} . \tag{77}$$

The aileron angles required to produce a trapezoidal variation of rate of roll are obtained from these expressions by setting $k = 1$, *see* Fig. 21.

It is seen that within each interval ξ does not vary much and for the inverse problem of calculating the response to a given aileron input it would be an advantage to assume ξ constant within each interval.

The solutions to the equation of motion (76) for each interval give:

$$\left.\begin{aligned}
\frac{-\delta_{l\xi}}{\nu_l}\xi_1(1-e^{-\nu_l\tau_1}) &= \hat{p}_1 \\
\frac{-\delta_{l\xi}\xi_2}{\nu_l}\{1-e^{-\nu_l(\tau_2-\tau_1)}\} &= \hat{p}_1\{k-e^{-\nu_l(\tau_2-\tau_1)}\} \\
\frac{+\delta_{l\xi}\xi_3}{\nu_l}\{1-e^{-\nu_l(\tau_3-\tau_2)}\} &= \hat{p}_2 e^{-\nu_l(\tau_3-\tau_2)}
\end{aligned}\right\} . \tag{78}$$

Again integrating the equation of motion with respect to τ we have:

$$-\delta_{l\xi}\int_0^\tau \xi\, d\tau = |\hat{p}|_0^\tau + \nu_l\int_0^\tau \hat{p}\, d\tau \tag{79}$$

or

$$\frac{-\delta_{l\xi}}{\nu_l}[\tau_1\xi_1 + (\tau_2-\tau_1)\xi_2 + (\tau_3-\tau_2)\xi_3] = \phi . \tag{80}$$

This yields four relationships {(78) and (80)} between the quantities ξ_1, ξ_2, ξ_3, $\hat{p}_1$, $\hat{p}_2$, τ_1, τ_2, τ and ϕ, and so enable any four of them to be determined if the other five are specified.

APPENDIX VI

Calculation of Aircraft Response when Rate-of-Roll Time History is Specified

In the main text and elsewhere, we have touched on the two alternative approaches to the problem, namely the response in rate of roll, incidence and sideslip etc. may be calculated for a given aileron input, or the rate-of-roll time history may be specified and the appropriate aileron input has to be determined.

As both approaches are likely to find application in design, it is worthwhile to outline the analysis in terms of the second approach. No numerical examples are given.

We return to equations (13), and now write $\hat{p} = f(\tau)$, where $f(\tau)$ is a prescribed function. In particular, we may take a trapezoidal variation of $\hat{p}$ specifying, for example, the initial rate of growth of $\hat{p}$, its maximum value, and final rate at which $\hat{p}$ is reduced to zero, together with the value of τ at which it becomes zero.

We now write the equations of motion in the form:

$$\left.\begin{aligned}
-\delta_{l\xi}\xi &= (D+\nu_l)\hat{p} - \nu_{lr}\hat{r} + \omega_l\hat{v} \\
(D+\nu)\hat{q} + \delta_y\hat{p}_0\hat{r} + (\omega+\chi D)\hat{w} &= 0 \\
\nu_{np}\hat{p} + \delta_z\hat{p}_0\hat{q} + (D+\nu_n)\hat{r} - \omega_n\hat{v} &= -\delta_{n\xi}\xi \\
-\hat{W}_0\hat{p} + \hat{r} + (D+\bar{y}_v)\hat{v} - \hat{p}_0\hat{w} &= \frac{C_{Le}}{2}\sin\phi \\
-\hat{q} + \hat{p}_0\hat{v} + (D-z_w)\hat{w} &= \frac{C_{Le}}{2}(\cos\phi-1)
\end{aligned}\right\} \qquad (81)$$

in which $\hat{p}_0$ is some constant specified value of $\hat{p}$ as in the analysis of Section 3 of the main text. These equations may be re-written as follows:

$$\left.\begin{aligned}
(D+\nu)\hat{q} + \delta_y\hat{p}_0\hat{r} + (\omega+\chi D)\hat{w} &= 0 \\
\delta_z\hat{p}_0\hat{q} + (D+\nu_n{}^*)\hat{r} - \omega_n{}^*\hat{v} &= \left(\frac{\delta_{n\xi}}{\delta_{l\xi}}D - \nu_{np}{}^*\right)f(\tau) \\
\hat{r} + (D+\bar{y}_v)\hat{v} - \hat{p}_0\hat{w} &= \frac{C_{Le}}{2}\sin\phi + \hat{W}_0 f(\tau) \\
-\hat{q} + \hat{p}_0\hat{v} - (D-z_w)\hat{w} &= \frac{C_{Le}}{2}(\cos\phi-1)
\end{aligned}\right\} \qquad (82)$$

with the equation for the rolling moment giving the aileron angle ξ as function of τ:

$$-\delta_{lr}\xi = (D+\nu_l)f(\tau) - \nu_{lr}\hat{r} + \omega_l\hat{v}, \qquad (83)$$

where

$$\nu_n{}^* = \nu_n + \frac{\delta_{n\xi}}{\delta_{l\xi}}\nu_{lr} = \nu_n + \nu_{lr}\frac{i_A n_\xi}{i_C l_\xi}$$

$$\nu_{np}{}^* = \nu_{np} - \frac{\delta_{n\xi}}{\delta_{l\xi}}\nu_l = \nu_{np} - \nu_l\frac{i_A n_\xi}{i_C l_\xi}$$

and

$$\omega_n{}^* = \omega_n + \frac{\delta_{n\xi}}{\delta_{l\xi}}\omega_l = \omega_n + \omega_l\frac{i_A n_\xi}{i_C l_\xi}.$$

The above equations (82) are readily solved for $\hat{q}$, $\hat{r}$, $\hat{v}$ or β and $\hat{w}$, either with an approximation of the gravity terms as in the main text, or neglecting these terms (cf. Appendix III).

Substitution of the solutions in the last equation (83) gives the aileron angle, ξ, as a function of τ or time.

The characteristic equation is a quartic and can be written:

$$G_0\lambda^4 + G_1\lambda^3 + G_2\lambda^2 + G_3\lambda + G_4 = 0 \tag{84}$$

where

$$\begin{aligned}
G_0 &= 1 \\
G_1 &= k_{10}^* + k_{11} \\
G_2 &= k_{20}^* + k_{21} + k_{10}^*k_{11} + \hat{p}_0^2(1-\delta_y\delta_z) \\
G_3 &= k_{21}k_{10}^* + k_{20}^*k_{11} - \delta_z\hat{p}_0^2\chi + \hat{p}_0^2(\nu+\nu_n^*) - \delta_y\delta_z\hat{p}_0^2(\bar{y}_v - z_w) \\
G_4 &= k_{20}^*k_{21} + \hat{p}_0^2(\delta_y\omega_n^* - \delta_z\omega) - \delta_y\delta_z\hat{p}_0^2(\hat{p}_0^2 - \bar{y}_v z_w)
\end{aligned}$$

where

$$\begin{aligned}
k_{10}^* &= \bar{y}_v + \nu_n^* \\
k_{20}^* &= \omega_n^* + \bar{y}_v\nu_n^* \\
k_{11} &= \nu + \chi - z_w \\
k_{21} &= \omega - \nu z_w \quad \text{(cf. Appendices I and II).}
\end{aligned}$$

Let us consider the solution of the equations neglecting gravity terms. In operational form, the equations of motion can now be written:

$$\left.\begin{aligned}
(D+\nu)\hat{q} + \delta_y\hat{p}_0\hat{r} + (\omega+\chi D)\hat{w} &= (\hat{q}_i + \chi\hat{w}_i)D \\
\delta_z\hat{p}_0\hat{q} + (D+\nu_n)\hat{r} - \omega_n^*\hat{v} &= \hat{r}_iD + \left(\frac{\delta_{n\xi}}{\delta_{l\xi}}D - \nu_{np}^*\right)\bar{f}(D) \\
\hat{r} + (D+\bar{y}_v)\hat{v} - \hat{p}_0\hat{w} &= \hat{v}_iD + \hat{W}_0\bar{f}(D) \\
-\hat{q} + \hat{p}_0\hat{v} - (D-z_w)\hat{w} &= \hat{w}_iD
\end{aligned}\right\} \tag{85}$$

where $\bar{f}(D)$ is operational equivalent of $f(\tau)$ and thus $D\bar{f}(D)$ is operational equivalent of $df(\tau)/d\tau$ since $f(0) = 0$.

To proceed further, it is necessary to specify $f(\tau)$. A relatively simple and reasonable choice is a trapezoidal variation as mentioned earlier. The calculation in this particular case would proceed along lines very similar to those indicated in the text immediately following equation (56). As no numerical examples are available for illustration, complete details of the calculation are omitted.

TABLE 1

Geometric, Inertia and Aerodynamic Derivatives (with Respect to Principal Inertia Axes) Assumed for the Aircraft Used as Example

$S = 400$ sq ft $\qquad b = 35$ ft $\qquad W = 25{,}000$ lb

$I_x = 900{,}000$ lb ft^2 $\qquad i_A = 0\cdot12$

$I_y = 4{,}100{,}000$ lb ft^2 $\qquad i_B = 0\cdot54$

$I_z = 5{,}000{,}000$ lb ft^2 $\qquad i_C = 0\cdot65$

Mach No. $0\cdot8$ $\qquad$ Height 40,000 ft

$\mu = 186\cdot2$ $\qquad \hat{t} = 4\cdot2318$ sec

$y_v = -0\cdot32$

$l_\xi = -0\cdot25$

$l_p = -0\cdot25$

$l_v = -0\cdot10$

$n_\xi = -0\cdot07\alpha$

$n_p = 0\cdot05 - 0\cdot3\alpha$

$n_v = 0\cdot20$

$n_r = -0\cdot46$

$\frac{\bar{c}}{s} m_w = -0\cdot083$

$\left(\frac{\bar{c}}{s}\right)^2 m_{\dot{w}} = -0\cdot218$

$\left(\frac{\bar{c}}{s}\right)^2 m_q = -0\cdot376$

$z_w = -2\cdot175$

$l_r = 0$

$y_p = 0$

TABLE 2

Exact and Approximate Roots of Stability Quintics for varying $\hat{p}_0$ and $\hat{W}_0$

$\pm\hat{p}_0$	$\hat{W}_0$	G_0	G_1	G_2	G_3	G_4	G_5		b_3	a_1	b_1	a_2	b_2	b_4	b_5
2·96	0·0873	1·0	6·3024	126·9737	500·3804	2090·4965	2573·2332	Exact Approx.	1·6144 1·625	1·85319 1·88	97·8655 98·3	2·8348 2·79	16·2865 15·8		
6·76	0·0873	1·0	6·3024	187·5488	722·8127	1130·3395	502·9505	Exact Approx.	0·7059 0·73	2·2003 2·20	171·9881 172·0	3·3962 3·37	4·1373 4·07		
10·0	0·0873	1·0	6·3024	276·6047	1049·8272	2889·5993	3800·9897	Exact Approx.	2·0405 2·03	2·3647 2·37	256·1492 256·2	1·8972 1·91	7·2727 7·26		
2·96	−0·0873	1·0	6·3024	100·9653	396·8045	1101·1672	1701·7464	Exact Approx.	2·4998 2·46	1·8849 1·88	79·2555 79·37	1·9176 1·97	8·5895 8·43		
6·76	−0·0873	1·0	6·3024	161·5404	619·2368	−473·8013	−1705·7514	Exact Approx.	−1·6555 −1·65	2·2058 2·19	155·3971 155·4			4·1575 4·18	1·5946 1·60
10·0	−0·0873	1·0	6·3024	250·5963	946·2512	381·5739	− 373·6607	Exact Approx.	−0·4405 −0·44	2·3637 2·36	239·6788 239·7			3·3111 3·31	1·0681 1·07

$$G_0\lambda_5 + G_1\lambda_4 + G_2\lambda_3 + G_3\lambda^2 + G_4\lambda + G_5 = (\lambda^2+a_1\lambda+b_1)\,(\lambda^2+a_2\lambda+b_2)\,(\lambda+b_3)$$

$$\text{or } (\lambda^2+a_1\lambda+b_1)\,(\lambda+b_3)\,(\lambda+b_4)\,(\lambda+b_5)$$

(88722)

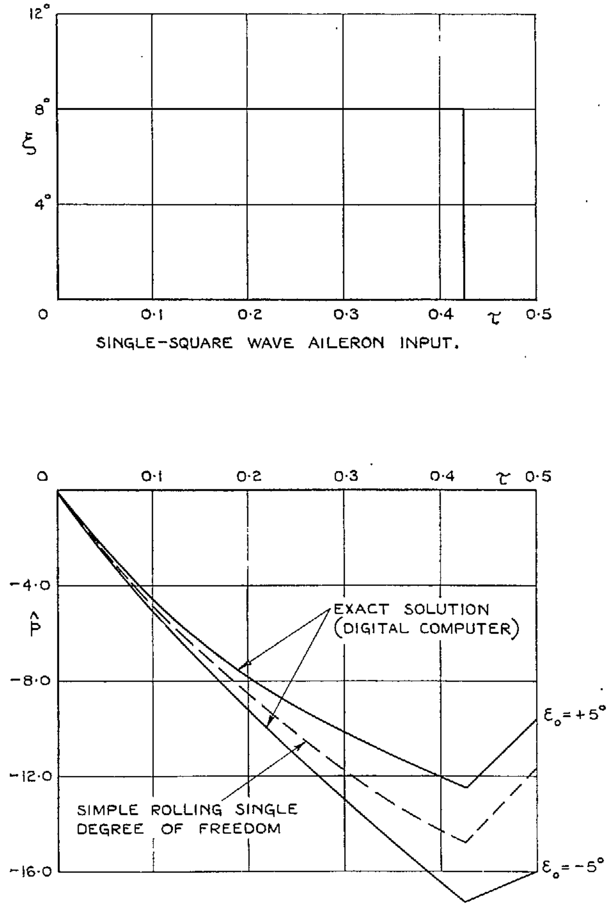


FIG. 1. Comparison of rolling motion single degree of freedom with that calculated on digital computer using full equations.

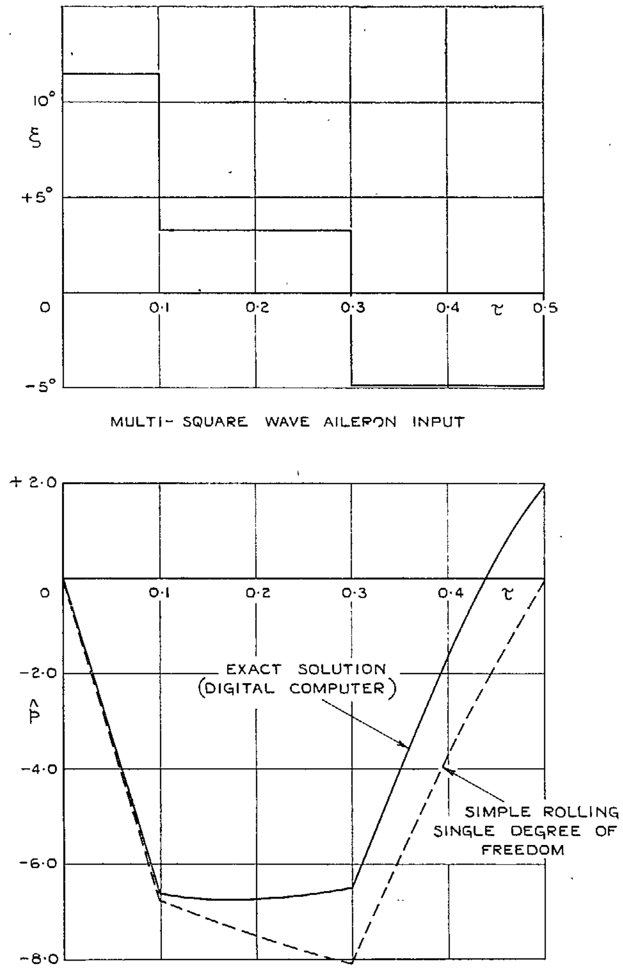


FIG. 2. Comparison of rolling motion single degree of freedom with that calculated on digital computer using full equations.

E

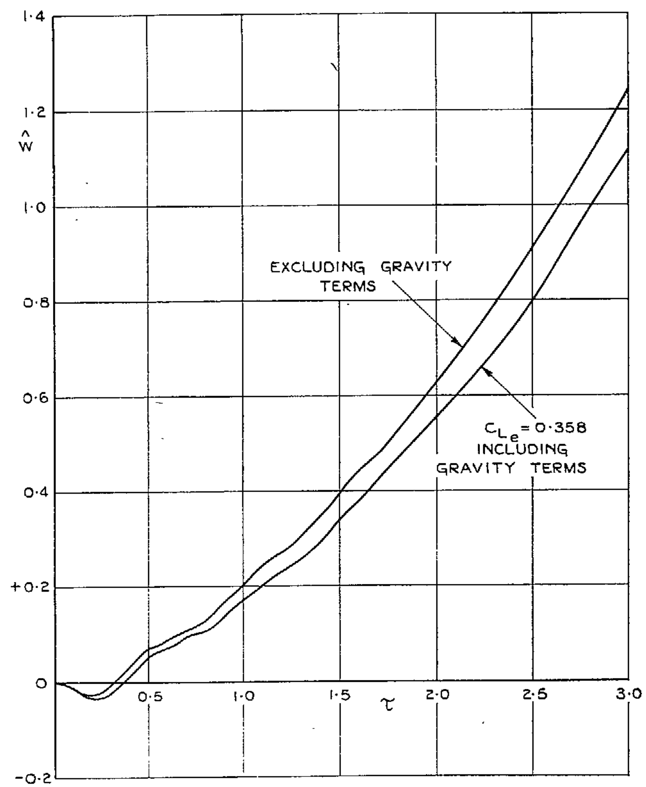


FIG. 3. Effect of gravity terms on response in a steady rolling motion (incidence response). $\hat{p}_0 = 6$, $\epsilon_0 = 5°$.

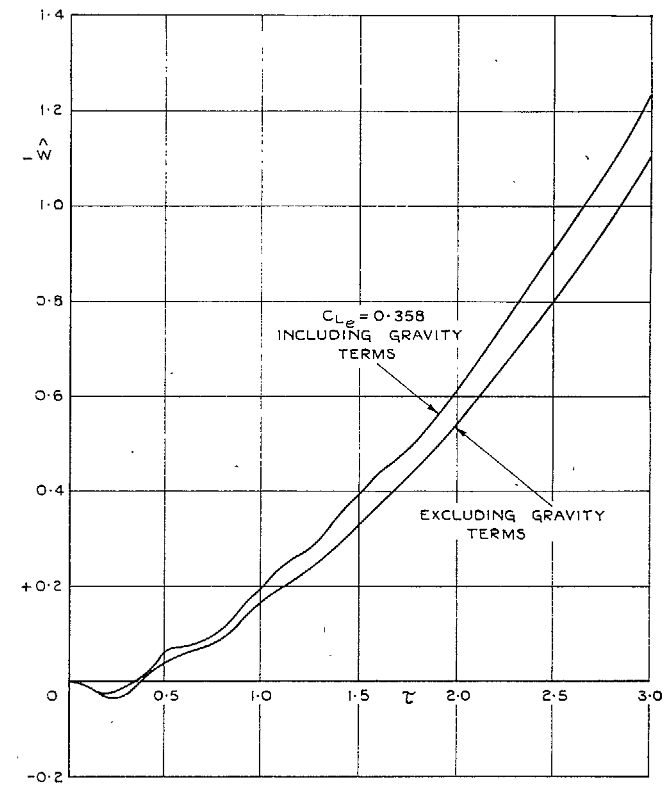


FIG. 4. Effect of gravity terms on response in a steady rolling motion (incidence response). $\hat{p}_0 = 6$, $\epsilon_0 = -5°$.

(88722)

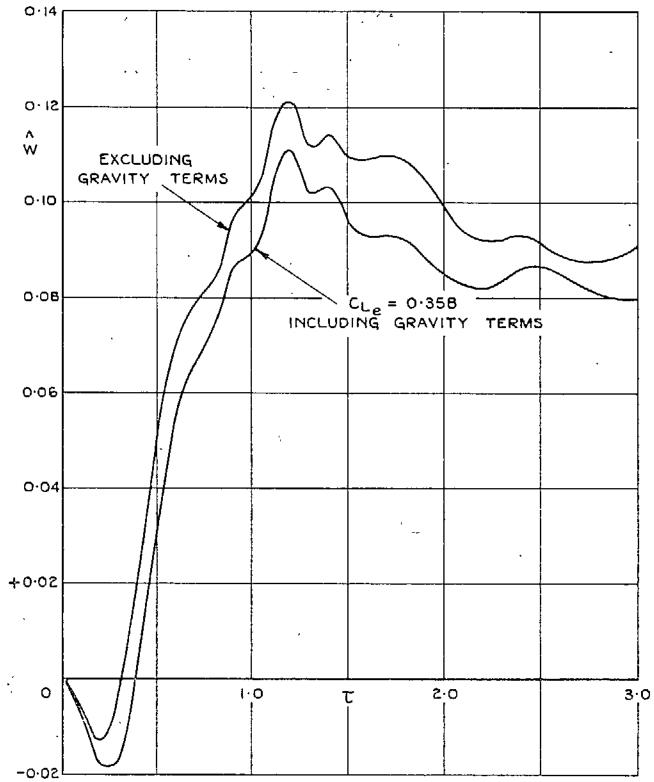


FIG. 5. Effect of gravity terms on response in a steady rolling motion (incidence response). $\hat{p}_0 = 4$, $\epsilon_0 = 5°$.

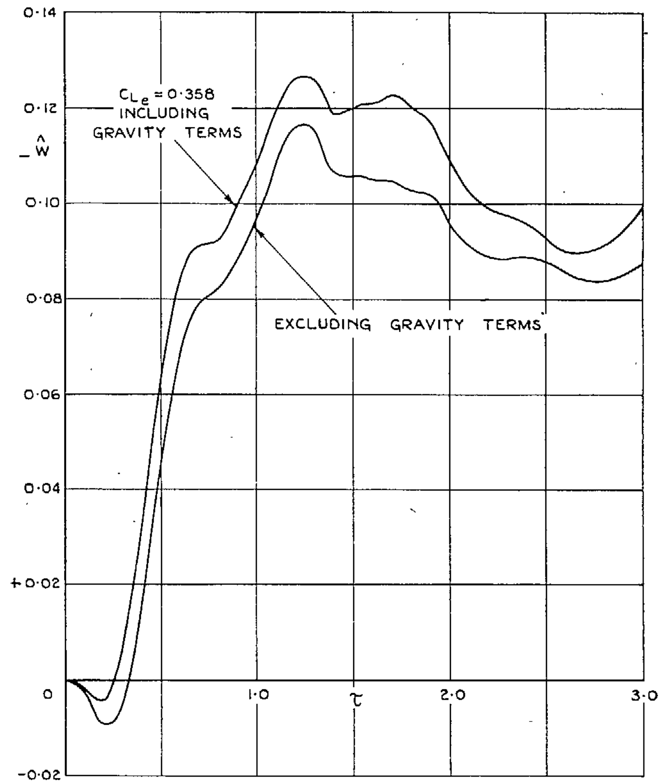


FIG. 6. Effect of gravity terms on response in a steady rolling motion (incidence response). $\hat{p}_0 = 4$, $\epsilon_0 = -5°$.

F

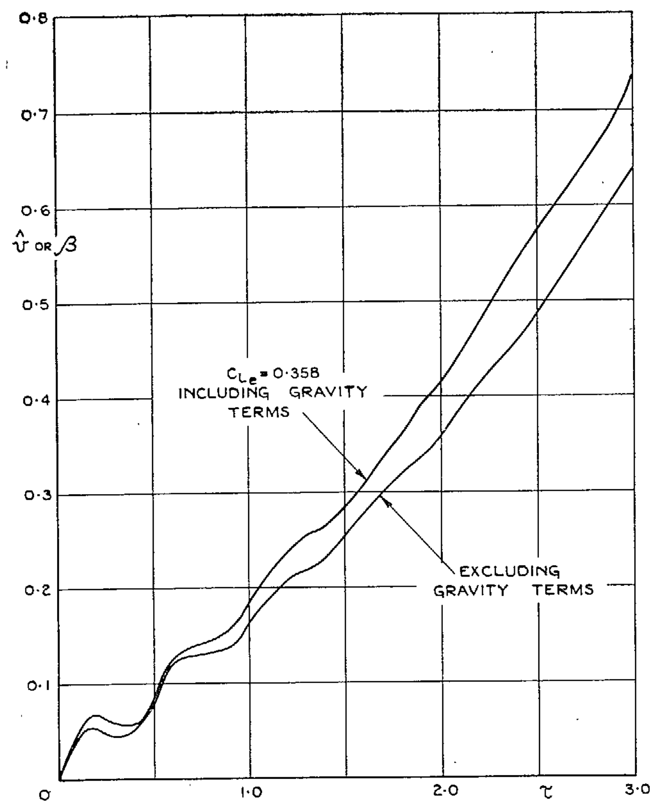


FIG. 7. Effect of gravity terms on response in a steady rolling motion (sideslip response). $\hat{p}_0 = 6$, $\epsilon_0 = 5$.

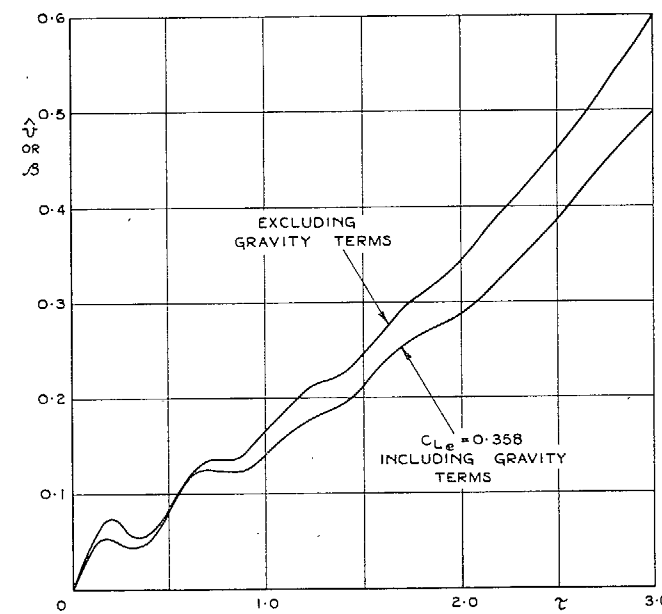


FIG. 8. Effect of gravity terms on response in a steady rolling motion (sideslip response). $\hat{p}_0 = 6$, $\epsilon_0 = -5°$.

(88277)

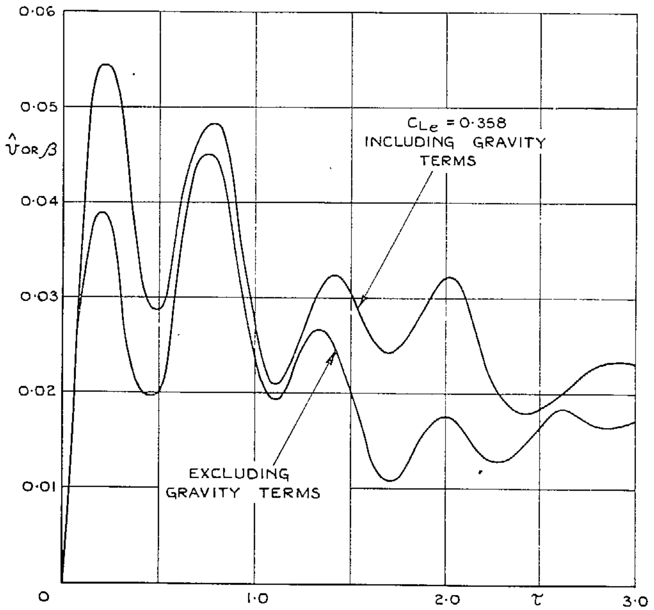


FIG. 9. Effect of gravity terms on response in a steady rolling motion (sideslip response). $\hat{p}_0 = 4$, $\epsilon_0 = 5$.

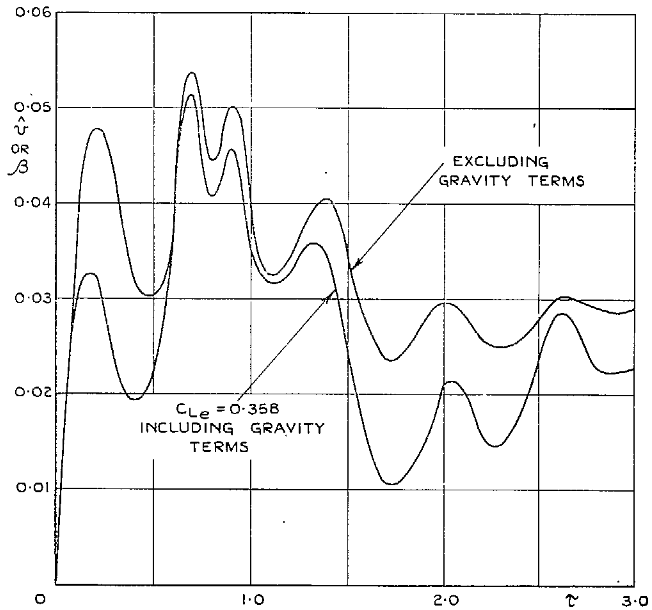


FIG. 10. Effect of gravity terms on response in a steady rolling motion (sideslip response). $\hat{p}_0 = 4$, $\epsilon_0 = -5°$.

F 2

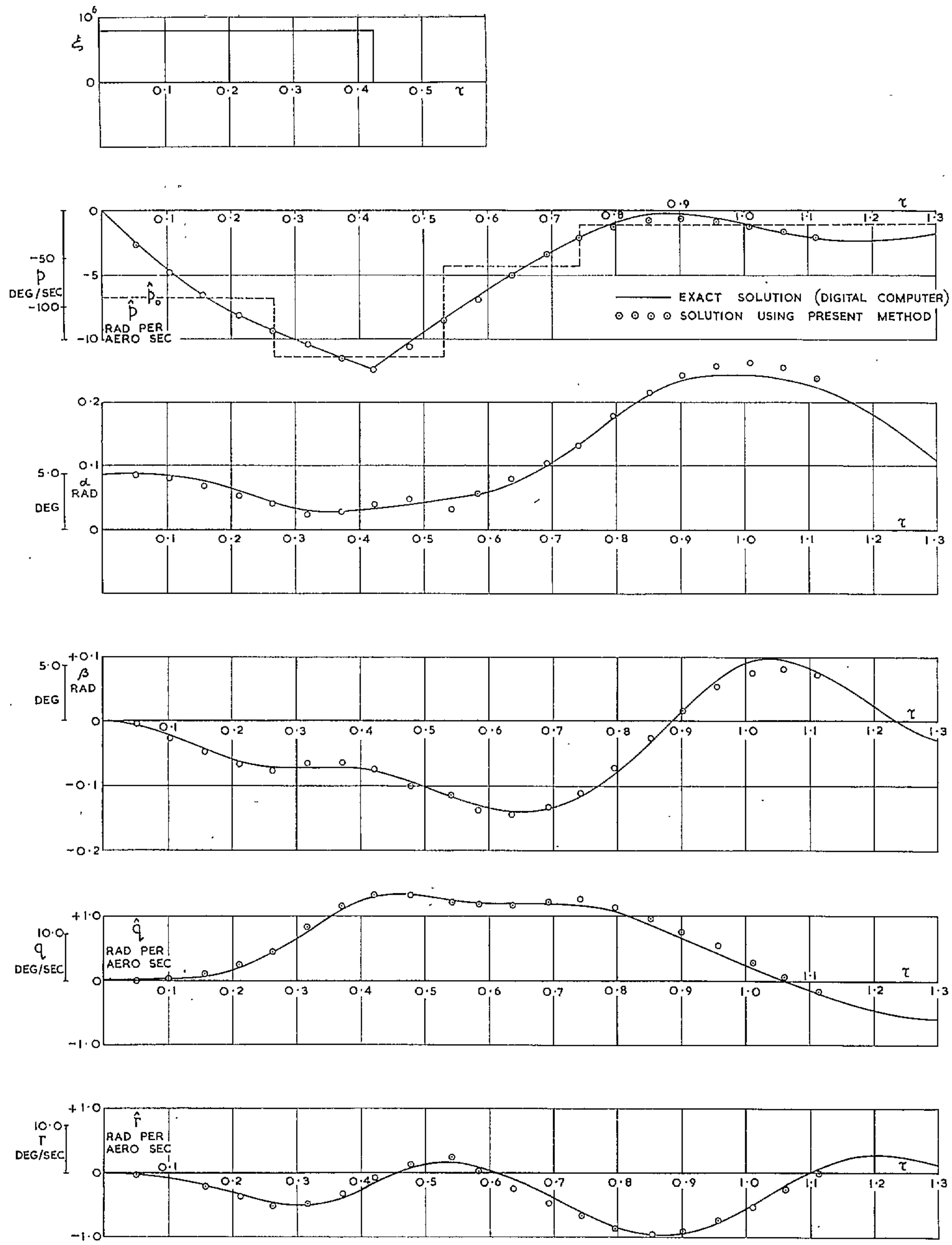


FIG. 11. Time histories of p, q, r, α and β (single-square wave ξ input). $p_{max} = 168°/sec$, $\epsilon_0 = 5°$.

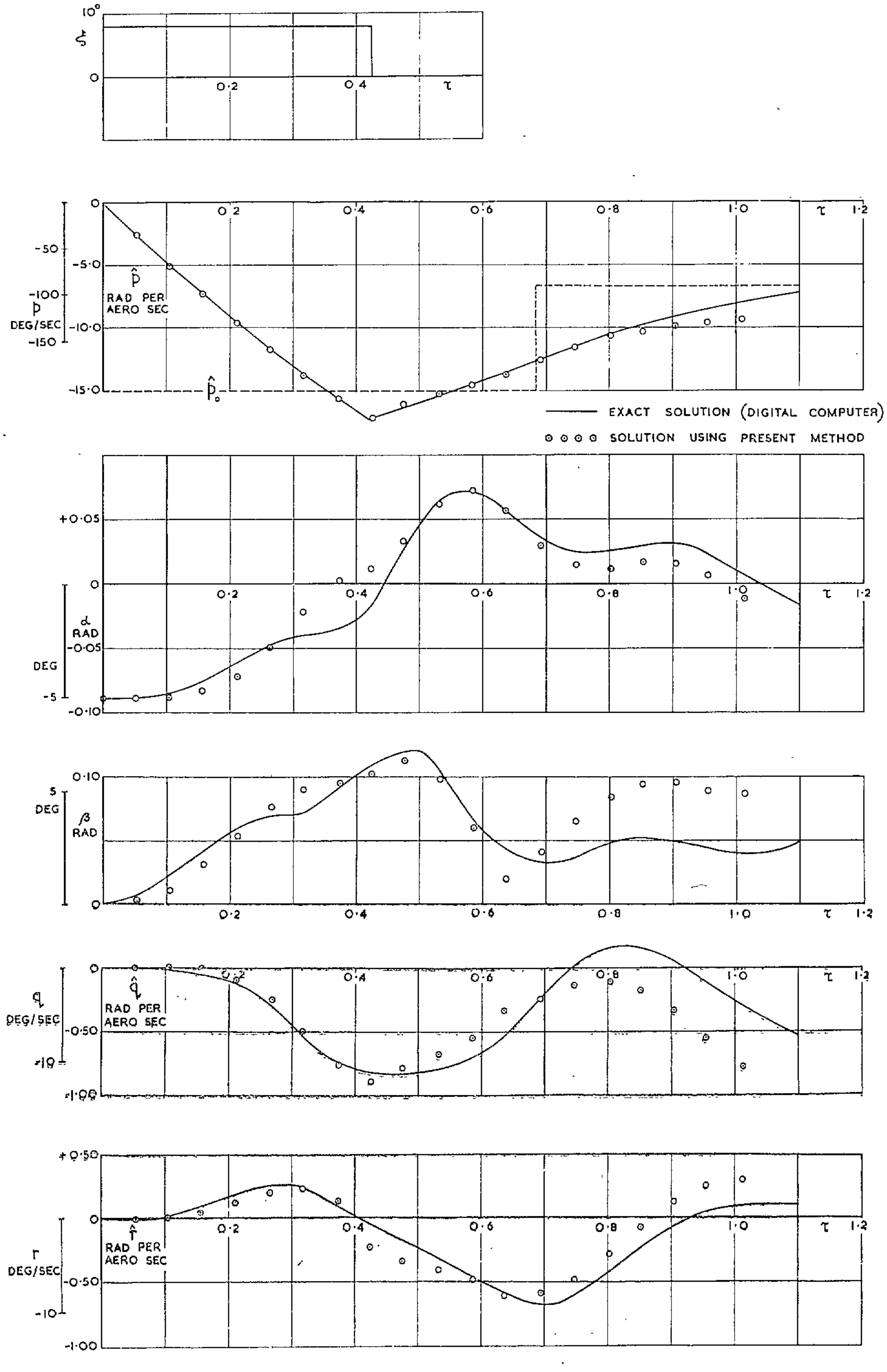


FIG. 12. Time histories of p, q, r, α and β (single-square wave ξ input). $p_{max} = 233°/sec$, $\epsilon_0 = -5°$.

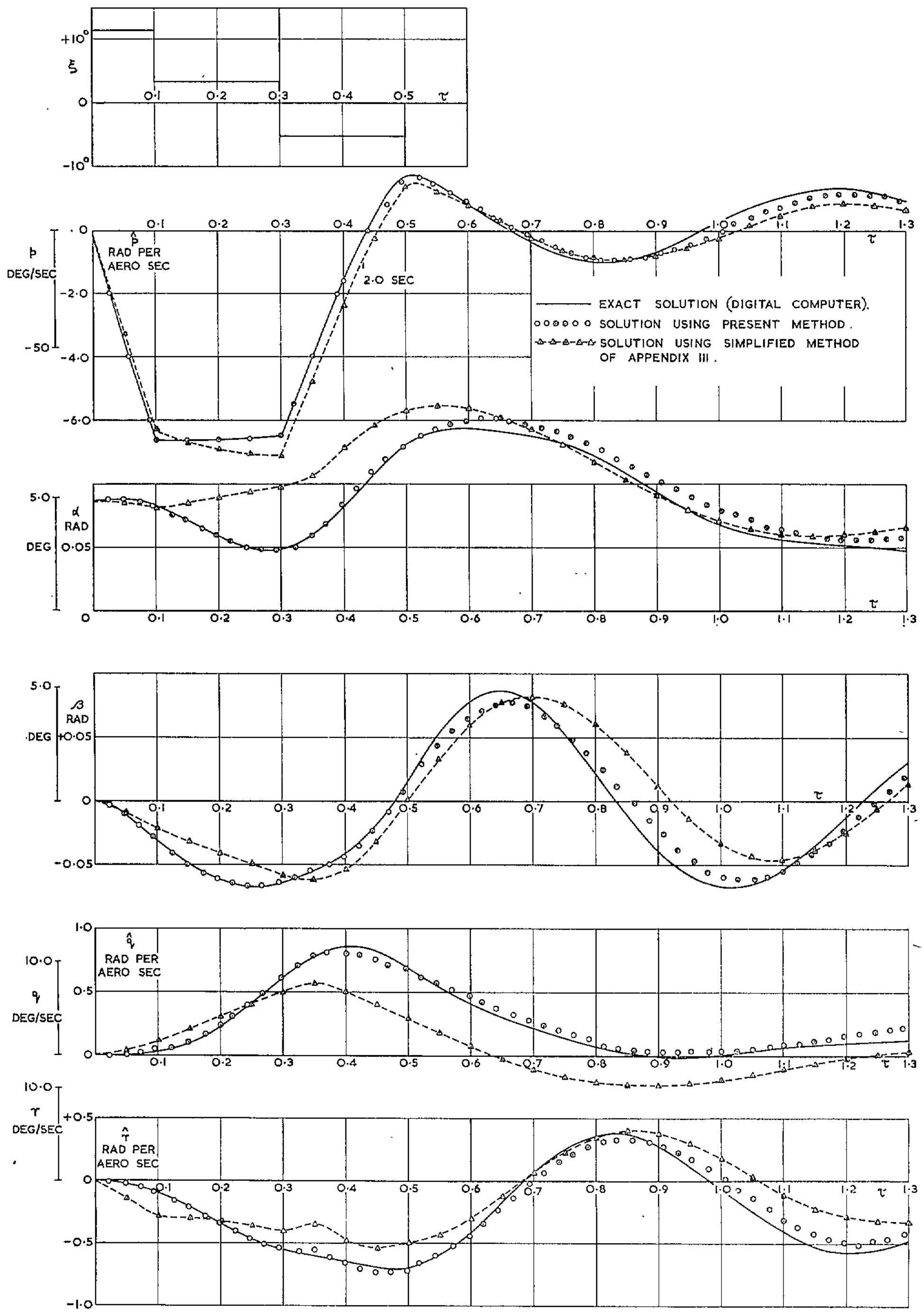


FIG. 13. Time histories of p, q, r, α and β (multi-square wave ξ input). $p_{\max} = 90°/\text{sec}$, $\epsilon_0 = 5°$.

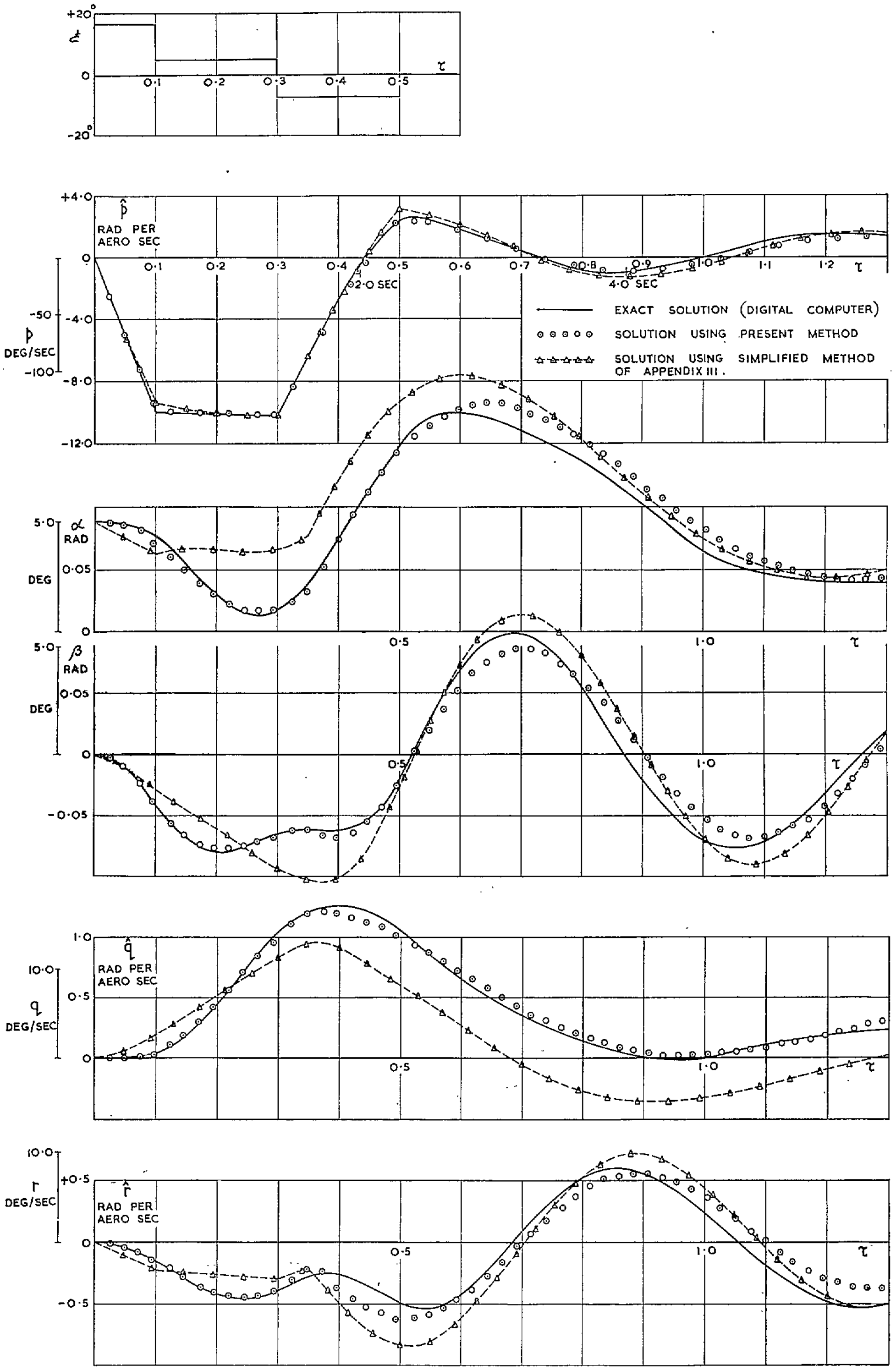


FIG. 14. Time histories of p, q, r, α and β (multi-square wave ξ input). $p_{\max} = 135°/\text{sec}$, $\epsilon_0 = 5°$.

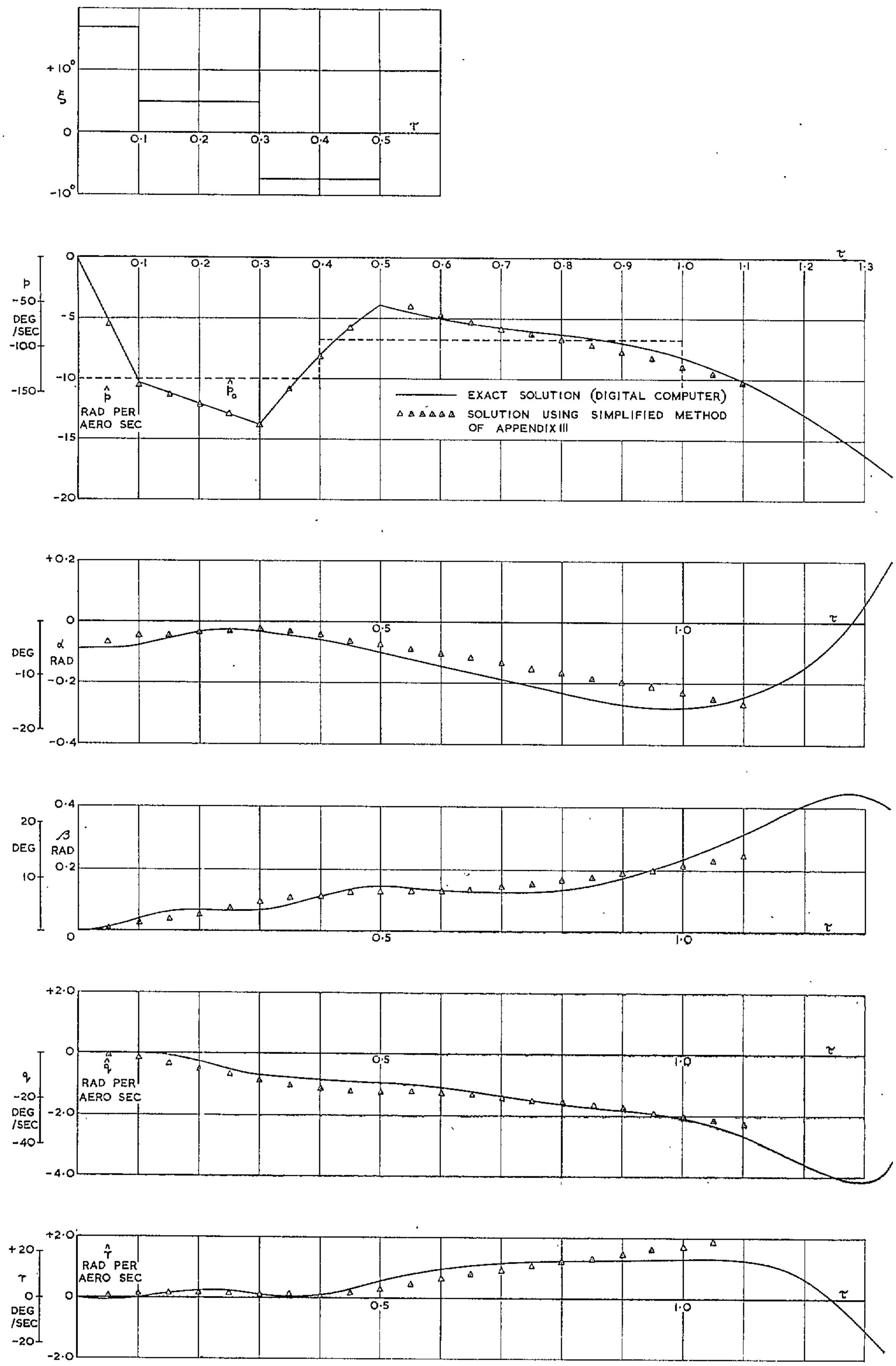


FIG. 15. Time histories of p, q, r, α and β (multi-square wave ξ input). $p_{max} = 185°/sec$, $\epsilon_0 = -5°$.

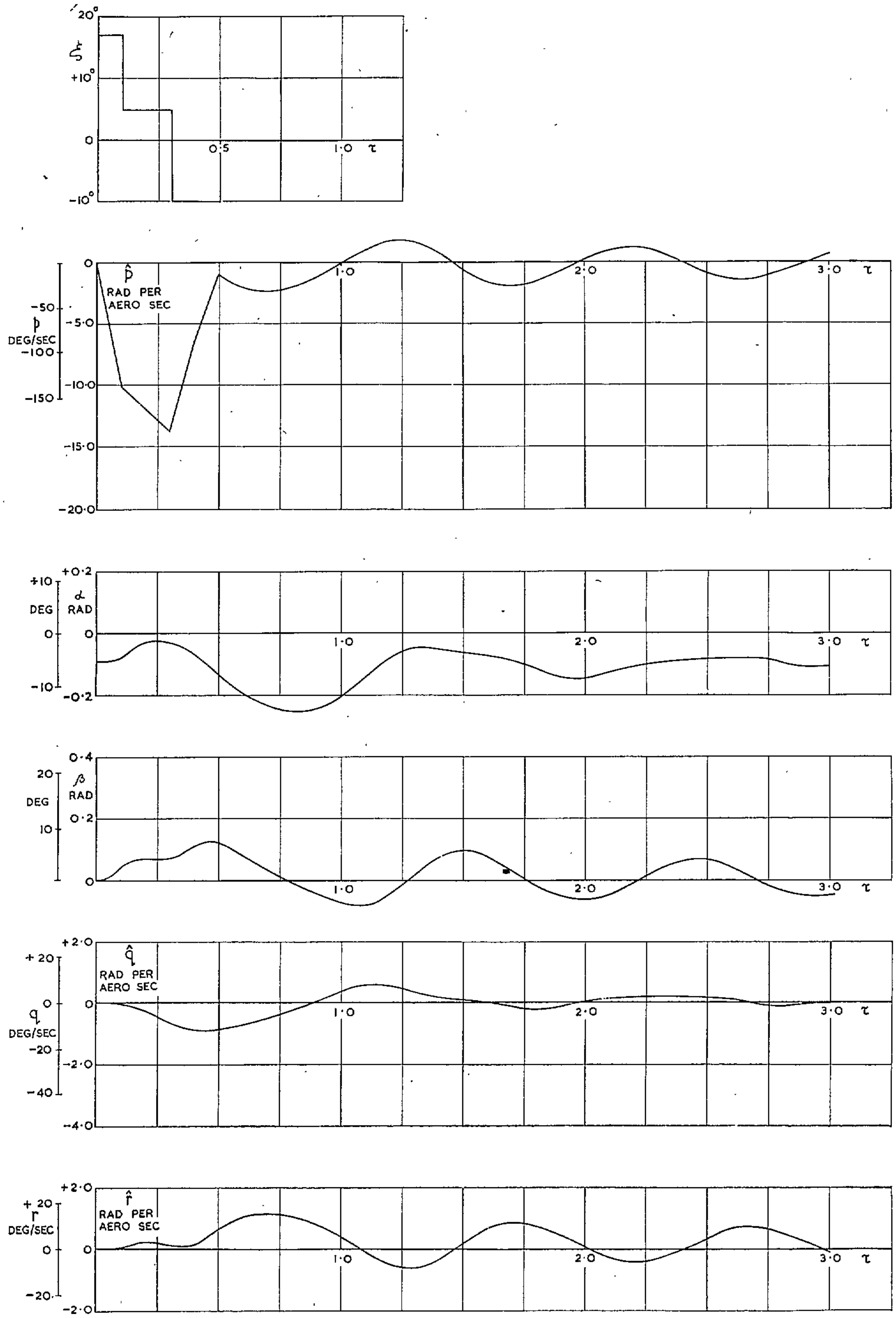


FIG. 16. Time histories of p, q, r, α and β (multi-square wave ξ input with increased amount of reverse ξ). $p_{\max} = 185°/\mathrm{sec}$, $\epsilon_0 = -5°$, as given by digital-computer calculations.

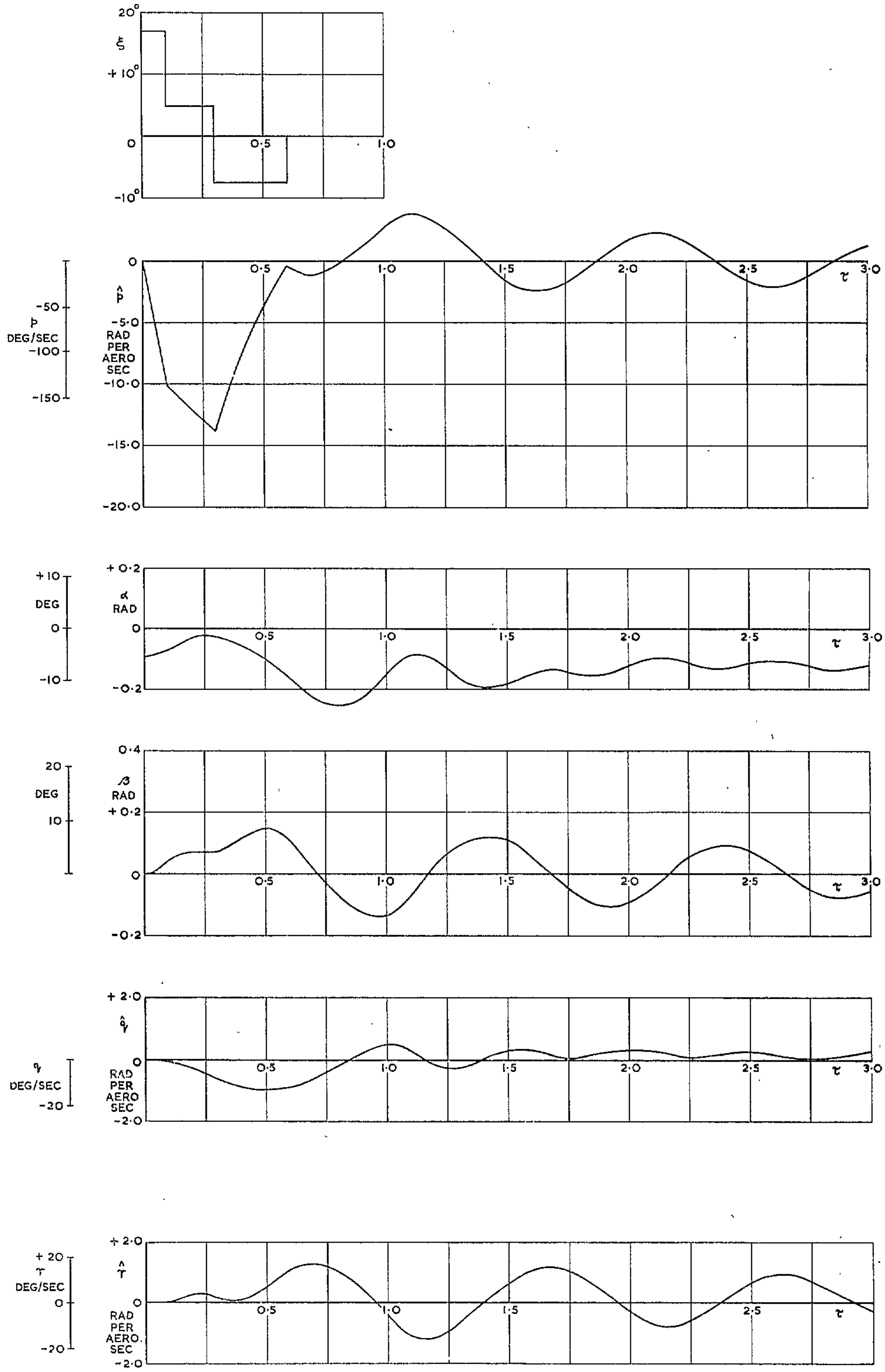


FIG. 17. Time histories of p, q, r, α and β (multi-square wave ξ input with increased duration of original reverse ξ). $p_{\max} = 185°/\text{sec}$, $\epsilon_0 = -5°$, as given by digital-computer calculations.

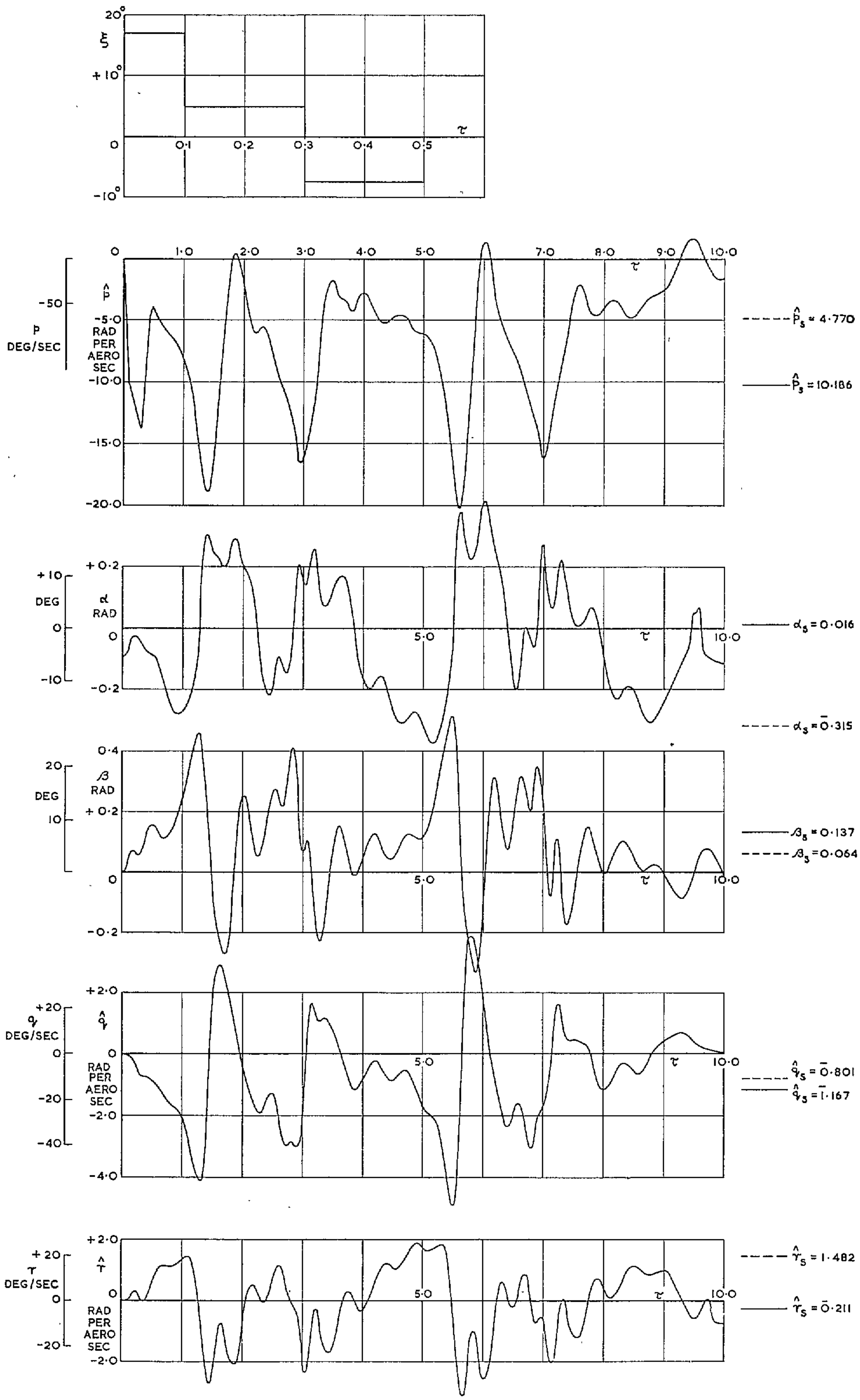


FIG. 18. Time histories of p, q, r, α and β for same conditions as Fig. 15, for long time interval (digital-computer calculations).

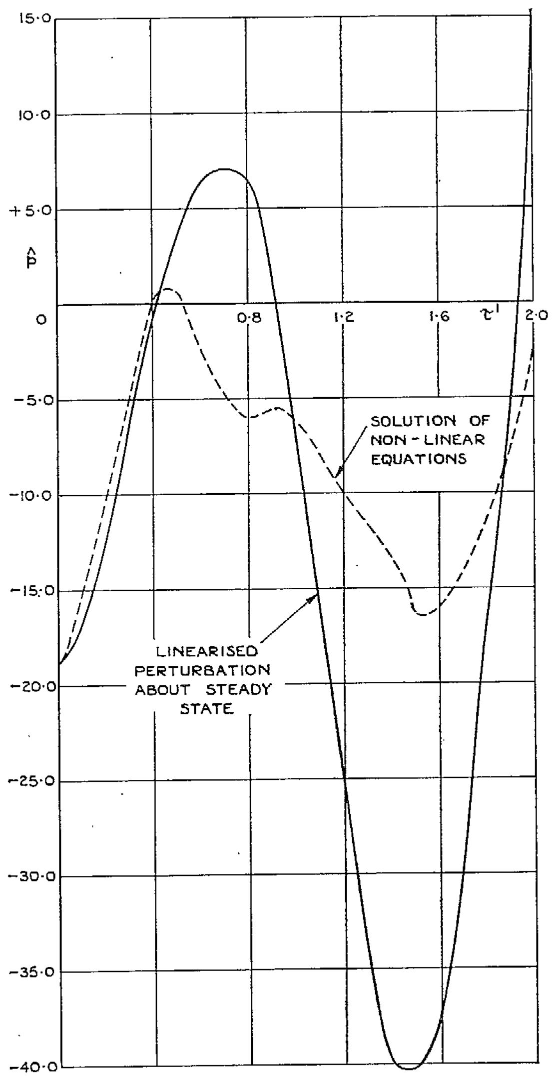


FIG. 19. Perturbation around steady state. ($\hat{p} = 10 \cdot 186$), in example of Fig. 18.

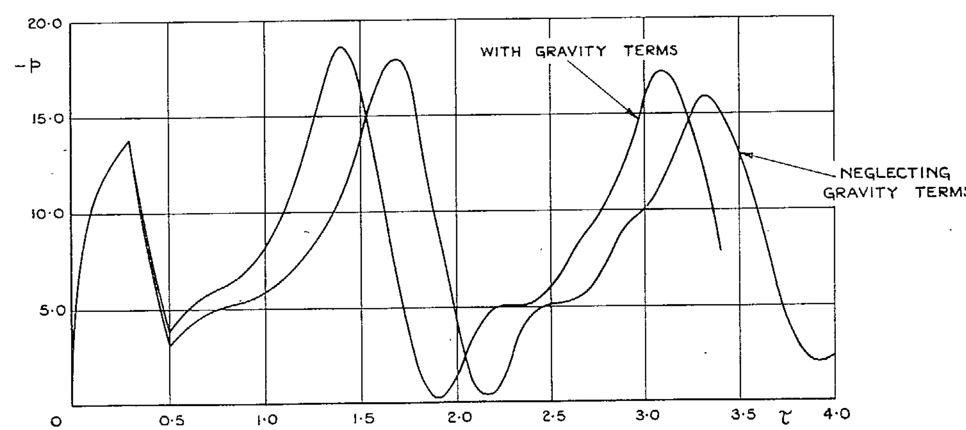
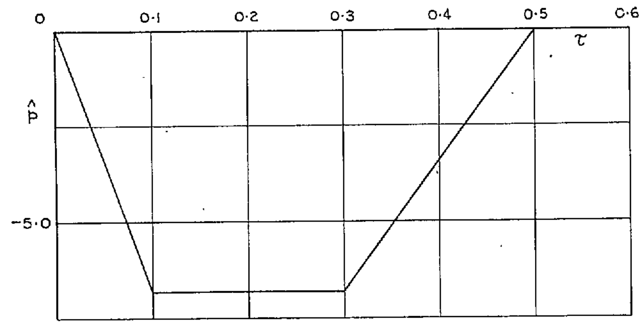
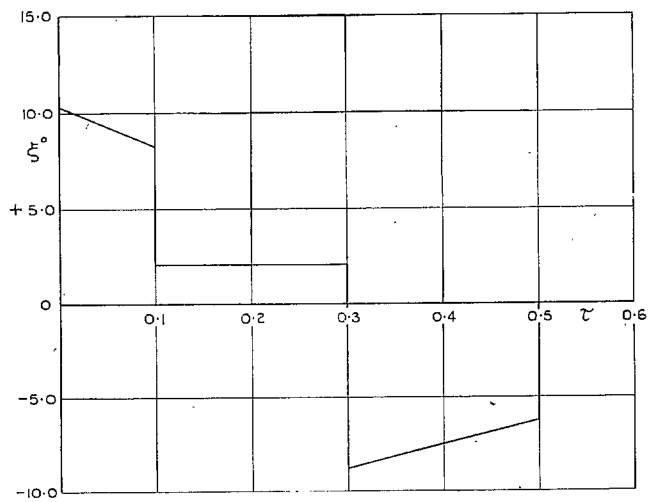


FIG. 20. Effect of the gravity terms in the response in rate of roll after long interval of time (for conditions of Fig. 15).



(a) SPECIFIED RATE OF ROLL.



(b) THE REQUIRED AILERON INPUT AS DETERMINED BY SINGLE DEGREE OF FREEDOM.

FIG. 21. Trapezoidal variation of rate of roll and the related aileron angle as given by single degree of freedom calculation.

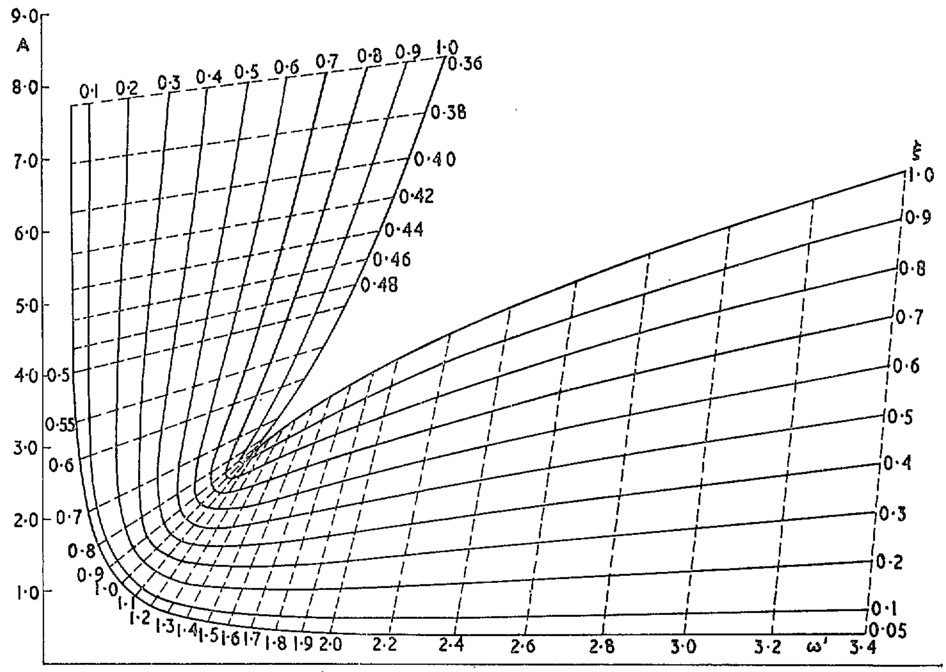


FIG. 22. Chart for determining roots of the cubic equation ($k = +1\cdot0$, one real root, one pair of complex roots).

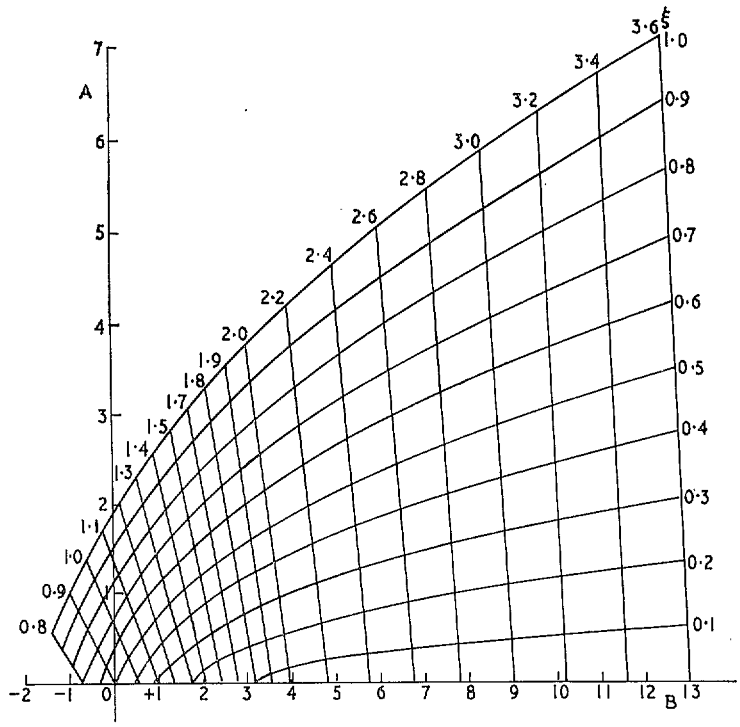


FIG. 23. Chart for determining roots of the cubic equation ($k = -1\cdot0$, one real root, one pair of complex roots).

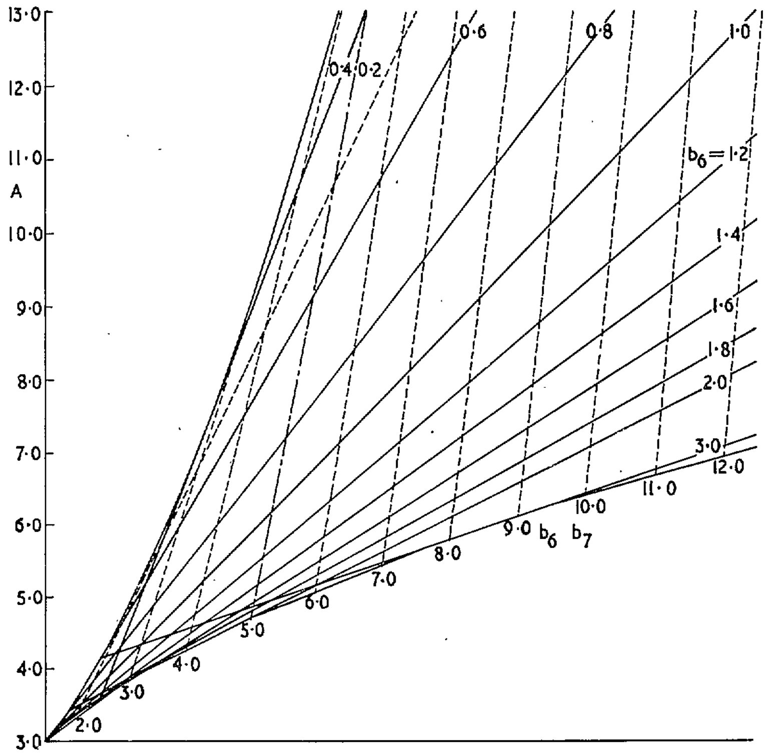


FIG. 24. Chart for determining roots of the cubic equation ($k = 1 \cdot 0$, three real roots).

(88722) Wt. 65/1418 K.5 6/64 Hw.

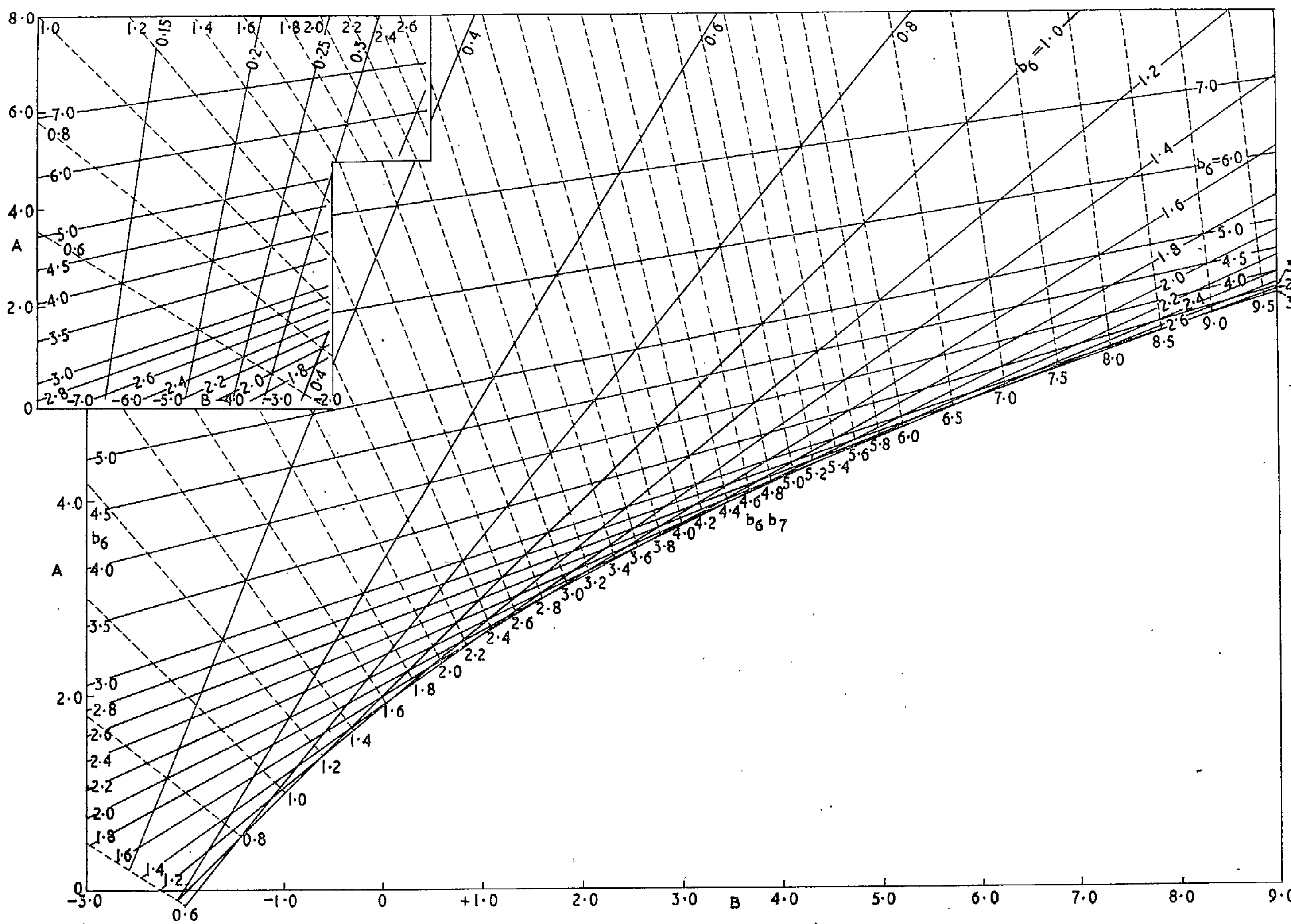


FIG. 25. Chart for determining roots of the cubic equation ($k = -1 \cdot 0$, three real roots).

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