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Summary.—Calculations of the pressure on a flat elliptic cone and on a flat elliptic hyper-cone at supersonic speeds and zero incidence are made for the case when the cones lie inside the Mach cone of the apex. The results are combined to give the pressure distribution and drag of a wing-like surface at zero incidence in a supersonic stream (*see* Fig. 6). It is found that the pressure is constant along straight lines on this surface which are normal to the wind direction (*see* Fig. 7). The drag results (Fig. 8) show the effect of sweepback on drag at supersonic speeds.

1. *Introductory Account.*—The lift and drag of a pointed triangular plate or 'delta wing' at supersonic speeds is calculated in Ref. 1. For the derivation a special system of curvilinear co-ordinates is introduced, which are closely linked with the plate and with the Mach cone from its apex. The differential equation of linearised supersonic flow is then solved in this special set of co-ordinates by standard methods, and it is found that one of the simplest colutions corresponds to the flat delta wing at incidence. No discussion was given in Ref. 1 of the significance of the other simple solutions of the equation in the special hyperboloido-conal co-ordinates. However, it became apparent that other interesting cases could be solved in this way, and the present report gives an example of the procedure.

We first determine the pressure distribution for the thin elliptic cone with the equation

$$\frac{z}{2t_0} = \left(\frac{x^2 - y^2 \cot^2 \gamma}{c^2}\right)^{1/2},$$

where x is measured down stream along the wind direction from the apex,
y is measured to starboard,
z is measured upward,
c is the chord in the vertical plane of symmetry,
γ is the apex semi-angle in the horizontal plane of symmetry,
and t_0 is a constant determining the thickness.

The notation and shape are shown in Figs. 1 and 2.

The cone is set symmetrically to the wind direction so that the pressure on it is symmetrical with respect to y and z.

The solution is only valid if the cone lies wholly within the Mach cone of the apex, so that the Mach angle $\mu = \sin^{-1}(1/M)$ is greater than the apex semi-angle γ.

It is shown that the pressure is constant over the cone, and that the pressure coefficient C_p is given by

$$C_p \sqrt{(M^2 - 1)} = \left(\frac{4t_0}{c}\right) f_1 \left(\frac{\tan \gamma}{\tan \mu}\right),$$

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where f_1 is a function which is given in Table 1 and Fig. 4. Values of C_p are shown plotted against M in Fig. 5 for $\gamma = 15$ deg, 30 deg and 45 deg.

We next determine the pressure distribution for the surface with the equation

$$\frac{z}{2t_0} = \frac{x}{c}\left(\frac{x^2 - y^2 \cot^2 \gamma}{c^2}\right)^{1/2}.$$

The shape of this surface is shown in Fig. 3. It will be referred to as an 'elliptic hyper-cone' as it has some resemblance to an elliptic cone. This hyper-cone is also set along the wind and again the solution is only valid if the surface lies wholly within the Mach cone of the apex. It is found that the pressure coefficient on this surface is given by

$$C_p \sqrt{(M^2 - 1)} = \left(\frac{4t_0\, x}{c^2}\right) f_2 \left(\frac{\tan \gamma}{\tan \mu}\right),$$

where f_2 is given in Table 1 and Fig. 4.

Finally the solutions are combined to give the surface

$$\frac{z}{2t_0} = \left(1 - \frac{x}{c}\right)\left(\frac{x^2 - y^2 \cot^2 \gamma}{c^2}\right)^{1/2},$$

with the pressure distribution

$$C_p \sqrt{(M^2 - 1)} = \left(\frac{4t_0}{c}\right)\left(f_1 - \frac{x}{c} f_2\right),$$

where f_1 and f_2 are the functions defined above. The surface is shown in Fig. 6 and is wing-like in shape. It has a straight sharp trailing edge on the line $x = c$, $z = 0$, and leading edges $x = \pm y \cot \gamma$, $z = 0$ which are swept-back and are rounded except at the apex. The wing section in the plane of symmetry $y = 0$ is biconvex, with the equation

$$\frac{z}{2t_0} = \pm \frac{x}{c}\left(1 - \frac{x}{c}\right).$$

The maximum value of z is $t_0/2$ at $x = c/2$, and the thickness/chord ratio of the centre section is therefore t_0/c.

Fig. 7 gives the example of the pressure distribution for a wing of this form, of apex semi-angle $\gamma = 30$ deg and of centre section thickness/chord ratio $t_0/c = 0{\cdot}10$ at a Mach number $M = \sqrt{2}$ ($\mu = 45$ deg).

In calculating the wing drag it is necessary to allow for the leading-edge force (Appendix II) as well as for the pressure distribution given by the theory. It is found that the drag coefficient C_D, based on wing area, is given by

$$C_D \sqrt{(M^2 - 1)} = \frac{2D\sqrt{(M^2 - 1)}}{\rho V^2 c^2 \tan \gamma} = \frac{2\pi}{3}\left(\frac{t_0}{c}\right)^2 \left[f_2\left(\frac{\tan \gamma}{\tan \mu}\right) + \frac{\tan \gamma}{\tan \mu}\left(1 - \frac{\tan^2 \gamma}{\tan^2 \mu}\right)^{1/2}\right],$$

where f_2 is the function given in Table 1 and Fig. 4. The values of C_D, for a centre-section thickness/chord ratio t_0/c of 10 per cent are shown plotted against M for $\gamma = 15$ deg, 30 deg, and 45 deg in Fig. 8(a). This gives an indication of the effect of sweepback on drag. The strip theory values for the centre-section are also shown in Fig. 8(a), but a direct strip theory comparison is not possible on account of the rounded leading edge of the wing.

There is no basic difficulty in making similar calculations for more complicated wing shapes, though the process would becomes increasingly complex.

The effect of incidence is given by the theory of Ref. 1. This will not be a good approximation very close to the leading edges but should be satisfactory elsewhere.

2. *Notation.*

ρ	air density
V	free-stream velocity
M	Mach number
μ	Mach angle
$\triangle p$	excess pressure on the surface
C_p	pressure coefficient $2\triangle p/\rho V^2$
C_D	drag coefficient based on wing area
C_{Df}	,, ,, ,, ,, frontal area
c	centre section chord
t_0	maximum thickness of wing
γ	apex semi-angle of wing
x	distance measured from the apex in the direction of the stream
y	distance measured to starboard from the centre section
z	distance measured upwards from the horizontal plane of symmetry
n	$(M^2 - 1)^{1/2}$
$x'\ y'\ z'$	defined by equation (1)
h, k	,, ,, ,, (2)
r, μ, v	,, ,, ,, (3)
Φ	induced velocity potential

3. *Method of Solution.*—From this point it will be assumed that the theory developed in Ref. 1 is familiar to the reader, as the present investigation is a continuation of the earlier one. The notation of Ref. 1 will be followed and the complete set of definitions will not be repeated here.

We work mainly with the r, μ, v co-ordinate system where

$$x = nx', \quad y = y', \quad z = z', \qquad \text{(1)}$$

$$\left.\begin{aligned} n^2 &= M^2 - 1 = \cot^2 \mu = k^2 - h^2, \\ k^2 &= \cot^2 \gamma,\ h^2 = \cot^2 \gamma - \cot^2 \mu, \end{aligned}\right\} \qquad \text{(2)}$$

$$x' = r\frac{\mu v}{hk},\ y' = r\left[\frac{(\mu^2 - h^2)(v^2 - h^2)}{h^2(k^2 - h^2)}\right]^{1/2},\ z' = r\left[\frac{(\mu^2 - k^2)(k^2 - v^2)}{k^2(k^2 - h^2)}\right]^{1/2} \qquad \text{(3)}$$

We assume that the surfaces under investigation all lie close to the basic plate whose equation is $\mu = k$, and that the induced velocities on these surfaces are small and equal to the induced velocities on the plate. This leads to a relation between the shape of the body $z = z(x, y)$ and its induced velocity potential Φ of the form

$$\frac{\partial_z}{\partial x} = + \frac{1}{V}\left(\frac{\partial \Phi}{\partial z}\right)_{\mu = k}, \qquad \text{(4)}$$

where V is the stream velocity. In addition, for the linearised theory, the excess pressure $\triangle p$ and the pressure coefficient C_p are given by

$$\triangle p = -\rho V\left(\frac{\partial \Phi}{\partial x}\right)_{\mu = k},\ C_p = \frac{2\triangle p}{\rho V^2} = -\frac{2}{V}\left(\frac{\partial \Phi}{\partial x}\right)_{\mu = k}. \qquad \text{(5)}$$

The solutions of problems of the type under consideration are given by selection or combination of the solutions for the potential of the form

$$\Phi_n^m = Cr^n F_n^m(\mu)\, E_n^m(\nu), \qquad (6)$$

where $E_n^m(\mu)$ is the standard Lamé function, and $F_n^m(\mu)$ is the second Lamé function, which is related to $E_n^m(\mu)$ by the equation

$$F_n^m(\mu) = E_n^m(\mu)\int_\mu^\infty \frac{dt}{[E_n^m(t)]^2\,[(t^2 - h^2)\,(t^2 - k^2)]^{1/2}}. \qquad (7)$$

The reader is referred to Appendix V of Ref. 1, and to Hobson's book[2] for further information.

4. *The Elliptic Cone at Zero Incidence.*—We first solve the problem of the flow past a thin elliptic cone, set at zero incidence, whose equation is

$$\frac{z}{2t_0} = \left(\frac{x^2 - y^2\cot^2\gamma}{c^2}\right)^{1/2}. \qquad (8)$$

This is obtained by taking the expression (6) for the induced potential, with $m=n=1$, so that[1,2]

$$\left.\begin{aligned} E_1^1(\mu) &= \mu,\ E_1^1(\nu) = \nu, \\ F_1^1(\mu) &= \mu\int_\mu^\infty \frac{dt}{t^2\,[(t^2 - h^2)\,(t^2 - k^2)]^{1/2}} = \mu\, I(\mu), \end{aligned}\right\} \qquad (9)$$

and

$$\Phi_1 = C_1 r\mu\nu\, I(\mu) = C_1\, h\, k\, x'\, I(\mu). \qquad (10)$$

At the plate $\mu = k$ we have

$$\frac{\partial\Phi_1}{\partial z} = \frac{\partial\Phi_1}{\partial\mu}\cdot\frac{\partial\mu}{\partial z'}, \qquad (11)$$

since $z = z'$ and r and ν do not vary along the normal at the plate. From (10) and (11) and from (22) of Ref. 1 we get

$$\frac{\partial\Phi_1}{\partial z} = \left[C_1 r\nu I(\mu) - C_1\frac{r\,\mu\,\nu}{\mu^2(\mu^2 - h^2)^{1/2}\,(\mu^2 - k^2)^{1/2}}\right]\left[\frac{\mu(\mu^2 - h^2)\,(\mu^2 - k^2)^{1/2}(k^2 - \nu^2)^{1/2}}{k(k^2 - h^2)^{1/2} r(\mu^2 - \nu^2)}\right].$$

As $\mu \rightarrow k$ this becomes

$$\frac{\partial\Phi_1}{\partial z} = -\frac{C_1\nu}{k(k^2 - \nu^2)^{1/2}}$$

Also on the plate, from (3),

$$\left.\begin{aligned} r^2 &= x'^2 - y'^2 = (x^2 - n^2y^2)/n^2, \\ r^2\nu^2 &= h^2x'^2 = h^2x^2/n^2, \end{aligned}\right\}, \qquad (12)$$

and hence

$$\begin{aligned}\frac{\partial\Phi_1}{\partial z} &= -C_1\frac{hx'}{k[x'^2(k^2 - h^2) - k^2y'^2]^{1/2}} \\ &= -C_1\frac{hx}{k(k^2 - h^2)^{1/2}\,(x^2 - k^2y^2)^{1/2}}, \end{aligned} \qquad (13)$$

From (4) and (12) the equation of the body inducing the potential Φ_1 is

$$\frac{\partial z}{\partial x} = -\frac{C_1}{V}\cdot\frac{hx}{k(k^2 - h^2)^{1/2}\,(x^2 - k^2y^2)^{1/2}},$$

which integrates to

$$z = -\frac{C_1 h}{Vk}\left(\frac{x^2 - k^2y^2}{k^2 - h^2}\right)^{1/2},$$

if the constant of integration vanishes. Since $k = \cot\gamma$ and $k^2 - h^2 = n^2$ this is identical with

(8), provided that we put

$$C_1 = - V\left(\frac{2t_0}{c}\right)\frac{k(k^2 - h^2)^{1/2}}{h} = - V\left(\frac{2t_0}{c}\right)\frac{kn}{h}. \quad \cdots \quad (14)$$

The longitudinal induced velocity on the plate is

$$\frac{\partial \Phi_1}{\partial x} = \frac{1}{n}\frac{\partial \Phi_1}{\partial x'} = C_1 \frac{hk}{n} I(k) , \quad \cdots \quad (15)$$

since it may be verified[1] that $\frac{\partial \mu}{\partial x'}$ vanishes on the plate. Hence, from (14) and (15),

$$\frac{\partial \Phi_1}{\partial x} = - V\left(\frac{2t_0}{c}\right)k^2 I(k) .$$

Now, from (9),

$$k^3 I(k) = k^3 \int_h^\infty \frac{dt}{t^2(t^2 - h^2)^{1/2}(t^2 - k^2)^{1/2}}$$

$$= \int_0^{\pi/2} \frac{\sin^2\theta \, d\theta}{(1 - h^2 \sin^2\theta/k^2)^{1/2}} ,$$

on putting $t = k$ cosec θ, and hence

$$k^3 I(k) = D\left(\frac{h}{k}\right) = \frac{K(h/k) - E(h/k)}{(h/k)^2} ,$$

where K and E are the standard complete elliptic integrals, so that

$$\frac{\partial \Phi_1}{\partial x} = - V\left(\frac{2t_0}{c}\right)\frac{D(h/k)}{k}.$$

Values of the function D are given in Ref. 3.

The excess pressure on the cone is

$$\triangle p = - \rho V \frac{\partial \Phi_1}{\partial x} = \rho V^2 \left(\frac{2t_0}{c}\right)\frac{D(h/k)}{k} ,$$

and hence

$$C_p \sqrt{(M^2 - 1)} = \left(\frac{4t_0}{c}\right)\left(1 - \frac{h^2}{k^2}\right)^{1/2} D\left(\frac{h}{k}\right),$$

since $(M^2 - 1) = n^2 = (k^2 - h^2)$. We write this as

$$C_p \sqrt{(M^2 - 1)} = \left(\frac{4t_0}{c}\right) f_1\left(\frac{\tan\gamma}{\tan\mu}\right), \quad \cdots \quad (16)$$

where

$$\left.\begin{aligned} f_1\left(\frac{\tan\gamma}{\tan\mu}\right) &= \left(1 - \frac{h^2}{k^2}\right)^{1/2} D\left(\frac{h}{k}\right), \\ \text{and} \quad \frac{h^2}{k^2} &= 1 - \frac{\tan^2\gamma}{\tan^2\mu}. \end{aligned}\right\} \quad \cdots \quad (17)$$

The function f_1 is given in Table 1 and Fig. 4. The values of $C_p/(4t_0/c)$ are plotted in Fig. 5 against M for $\gamma = 15$ deg, 30 deg, and 45 deg. It should be noted that $(4t_0/c)$ is equal to the total angle between the upper and lower generators of the cone in the vertical plane of symmetry $y = 0$. For the infinite wedge of total angle $4t_0/c$ set symmetrically to the wind the value of $C_p/(4t_0/c)$ is equal to $(M^2 - 1)^{1/2}$ and this quantity is also plotted in Fig. 5 to show the reduction in pressure on the cones in comparison with this wedge.

5. *The Elliptic Hyper-cone at Zero Incidence.*—We next solve the problem of the flow past a thin elliptic hyper-cone, at zero incidence, whose equation is

$$\frac{z}{2t_0} = \frac{x}{c}\left(\frac{x^2 - y^2 \cot^2 \gamma}{c^2}\right)^{1/2}. \qquad (x > 0) \quad \ldots \quad (18)$$

This is obtained[2] by combining the two solutions for $n = 2$ of type K, which are given by

$$E_2^m(\mu) = \mu^2 - a_m, \qquad (m = 1, 2)$$

where a_1 and a_2 are constants. Substituting this in Lamé's equation[1,2] with $n = 2$ we get

$$(\mu^2 - a_m)[6\mu^2 - p(h^2 + k^2)] - 2\mu^2(2\mu^2 - h^2 - k^2) - 2(\mu^2 - h^2)(\mu^2 - k^2) = 0.$$

Equating the coefficients of the powers of μ to zero and eliminating the constant p we obtain

$$3a_m^2 - 2a_m(h^2 + k^2) + h^2k^2 = 0. \quad \ldots \quad (19)$$

The roots of this equation are a_1 and a_2 and hence

$$3(a_1 + a_2) = 2(h^2 + k^2),\ 3a_1 a_2 = h^2k^2. \quad \ldots \quad (20)$$

The second solution of Lamé's equation, corresponding to $E_2^m(\mu)$, is

$$F_2^m(\mu) = (\mu^2 - a_m)\int_\mu^\infty \frac{dt}{(t^2 - a_m)^2 (t^2 - h^2)^{1/2} (t^2 - k^2)^{1/2}}.$$

We write this in the form

$$F_2^m(\mu) = (\mu^2 - a_m)\ J_m(\mu),$$

which defines the quantity $J_m(\mu)$.

Consider the potential

$$\begin{aligned}\phi_m &= C_2 r^2 F_2^m(\mu) E_2^m(v), \\ &= C_2 r^2(\mu^2 - a_m)(v^2 - a_m) J_m(\mu), \quad \ldots \quad (21)\end{aligned}$$

where C_2 is a constant. The shape of the body producing this induced velocity potential is given by (4). As before, at the plate $\mu = k$, we have

$$\frac{\partial\phi_m}{\partial z} = \frac{\partial\phi_m}{\partial\mu}\frac{\partial\mu}{\partial z'}.$$

Differentiating (21) with respect to μ and substituting from (22) of Ref. 1 for $\frac{\partial\mu}{\partial z'}$ we get

$$\frac{\partial\phi_m}{\partial z} = C_2 r^2(v^2 - a_m)\left[2\mu J_m(\mu) - \frac{1}{(\mu^2 - a_m)(\mu^2 - h^2)^{1/2}(\mu^2 - k^2)^{1/2}}\right] \times \left[\frac{\mu(\mu^2 - h^2)(\mu^2 - k^2)^{1/2}(k^2 - v^2)^{1/2}}{k(k^2 - h^2)^{1/2} r(\mu^2 - v^2)}\right].$$

As μ tends to k this becomes

$$\frac{\partial\phi_m}{\partial z} = -\frac{C_2 r(v^2 - a_m)}{(k^2 - a_m)(k^2 - v^2)^{1/2}}. \quad \ldots \quad (22)$$

We now construct the potential Φ_2 by combining ϕ_1 and ϕ_2 such that

$$\Phi_2 = \frac{\phi_1}{a_1} - \frac{\phi_2}{a_2}. \qquad (23)$$

Then from (22) and (23)

$$\frac{\partial \Phi_2}{\partial z} = - \frac{C_2 r}{(k^2 - v^2)^{1/2}} \left[\frac{(v^2 - a_1)}{a_1(k^2 - a_1)} - \frac{(v^2 - a_2)}{a_2(k^2 - a_2)}\right],$$

$$= - \frac{C_2 r (a_2 - a_1) [v^2(k^2 - a_1 - a_2) + a_1 a_2]}{a_1 a_2 (k^2 - a_1)(k^2 - a_2)(k^2 - v^2)^{1/2}}.$$

From (20) we get

$$v^2(k^2 - a_1 - a_2) + a_1 a_2 = \tfrac{1}{3} [v^2(k^2 - 2h^2) + h^2 k^2],$$

$$(k^2 - a_1)(k^2 - a_2) = \tfrac{1}{3} k^2 (k^2 - h^2),$$

so that

$$\frac{\partial \Phi_2}{\partial z} = C_2' \frac{r[v^2(k^2 - 2h^2) + h^2 k^2]}{h^2 (k^2 - v^2)^{1/2}}, \qquad (24)$$

where $$C_2' = - \frac{3C_2 (a_2 - a_1)}{k^4(k^2 - h^2)}.$$

From (12) and (24) it follows that

$$\frac{\partial \Phi_2}{\partial z} = C_2' \frac{x^2(k^2 - 2h^2) + k^2(x^2 - n^2 y^2)}{n[k^2(x^2 - n^2y^2) - h^2x^2]^{1/2}}$$

$$= C_2' \frac{(2x^2 - k^2y^2)}{(x^2 - k^2y^2)^{1/2}}, \text{ since } n^2 = k^2 - h^2.$$

Therefore, from (4), the shape of body giving rise to Φ_2 is given by

$$\frac{\partial z}{\partial x} = \frac{C_2'}{V} \cdot \frac{(2x^2 - k^2y^2)}{(x^2 - k^2y^2)^{1/2}}.$$

Integrating we obtain

$$z = \frac{C_2'}{V} x (x^2 - k^2y^2)^{1/2},$$

if the constant of integration is zero. This is equivalent to (18), since $k = \cot \gamma$, if

$$C_2' = V \frac{2t_0}{c^2},$$

so that $$C_2 = - V \cdot \frac{2t_0}{3c^2} \cdot \frac{k^2 (k^2 - h^2)}{(a_2 - a_1)}. \qquad (25)$$

At the plate $\mu = k$ we have from (21)

$$\phi_m = C_2 r^2 (k^2 - a_m)(v^2 - a_m) J_m(k)$$

$$= \frac{C_2 (k^2 - a_m)}{n^2} \left[x^2 (h^2 - a_m) + n^2 a_m y^2\right] J_m(k).$$

Hence the longitudinal induced velocity is

$$\frac{\partial \phi_m}{\partial x} = 2C_2 x \cdot \frac{(h^2 - a_m)(k^2 - a_m)}{n^2} \cdot J_m(k).$$

For the potential Φ_2 we get, from (23),

$$\frac{\partial \Phi_2}{\partial x} = 2C_2 \frac{x}{n^2} \left[\frac{(h^2 - a_1)(k^2 - a_1)}{a_1} J_1(k) - \frac{(h^2 - a_2)(k^2 - a_2)}{a_2} J_2(k)\right] . \quad \ldots \quad (26)$$

The integrals J_1 and J_2 are evaluated by W. Mangler in the Appendix and substituting from the last equation of the Appendix in (26) we obtain

$$\frac{\partial \Phi_2}{\partial x} = \frac{C_2 x}{k n^2} \left[\frac{(k^2 - a_1) K - k^2 E}{a_1^2} - \frac{(k^2 - a_2) K - k^2 E}{a_2^2}\right] ,$$

$$= \frac{C_2 x}{k n^2} \left(\frac{1}{a_1} - \frac{1}{a_2}\right) \left[k^2 (K - E) \left(\frac{1}{a_1} + \frac{1}{a_2}\right) - K\right] .$$

Substituting for C_2 from (25) and for a_1, and a_2 from (20) we get

$$\frac{\partial \Phi_2}{\partial x} = - V \cdot \frac{2t_0 x}{n^2 c^2} \frac{(k^2 - h^2)k}{h^2} \left[\frac{2(h^2 + k^2)}{h^2} (K - E) - K\right] . \quad \ldots \quad (27)$$

Now $n^2 = k^2 - h^2$, and also by the standard definitions of the complete elliptic integrals[3] we have

$$\frac{K(h/k) - E(h/k)}{(h/k)^2} = D\left(\frac{h}{k}\right) ,$$

and $$K(h/k) = 2D(h/k) - (h/k)^2 \, C(h/k) .$$

Hence (27) may be written

$$\frac{\partial \Phi_2}{\partial x} = - V \frac{2t_0 x}{k c^2} \left[2D(h/k) + C(h/k)\right] .$$

The excess pressure on the surface is therefore

$$\Delta p = - \rho V \frac{\partial \Phi_2}{\partial x} = \rho V^2 \frac{2t_0 x}{k c^2} \left[2D + C\right] ,$$

and hence

$$C_p \sqrt{(M^2 - 1)} = \frac{4t_0 x}{c^2} f_2 \left(\frac{\tan \gamma}{\tan \mu}\right) , \quad \ldots \quad (28)$$

where

$$\left.\begin{aligned} f_2\left(\frac{\tan \gamma}{\tan \mu}\right) &= \left(1 - \frac{h^2}{k^2}\right)^{1/2} \left[2D(h/k) + C(h/k)\right] , \\ \frac{h^2}{k^2} &= 1 - \frac{\tan^2 \gamma}{\tan^2 \mu} . \end{aligned}\right\} \quad \ldots \quad (29)$$

The function f_2 is given in Table 1 and Fig. 4.

6. *Flow Over a Wing-like Surface.*—We can combine the results for the cone and the hyper-cone to give the flow over the surface whose equation is obtained as the difference between (8) and (18) and is therefore

$$\frac{z}{2t_0} = \left(1 - \frac{x}{c}\right) \left(\frac{x^2 - y^2 \cot^2 \gamma}{c^2}\right)^{1/2} . \quad \ldots \quad (30)$$

The pressure distribution for this surface is obtained from (16) and (28) to be

$$C_p \sqrt{(M^2 - 1)} = \left(\frac{4t_0}{c}\right) \left[f_1 \left(\frac{\tan \gamma}{\tan \mu}\right) - \frac{x}{c} f_2 \left(\frac{\tan \gamma}{\tan \mu}\right)\right] . \quad \ldots \quad (31)$$

The surface is wing-like in shape and its geometrical characteristics are given in Fig. 6. As an example the pressure distribution for such a wing of apex semi-angle $\gamma = 30$ deg at a Mach number of $\sqrt{2}$ ($\mu = 45$ deg) and of centre section thickness-chord ratio 10 per cent ($t_0/c = 0{\cdot}10$) is shown in Fig. 7.

To calculate the drag of the wing it is necessary to allow both for the effect of the pressure distribution given by (31) and for the leading-edge force, as pointed out by R. T. Jones[4] (*see* Appendix II). After some calculation we obtain the formula :—

$$C_D\sqrt{(M^2 - 1)} = \frac{2\pi}{3}\left(\frac{t_0}{c}\right)^2 \left[f_2\left(\frac{\tan\gamma}{\tan\mu}\right) + \frac{\tan\gamma}{\tan\mu}\left(1 - \frac{\tan^2\gamma}{\tan^2\mu}\right)^{-1/2}\right].$$

for the drag coefficient C_D, based on wing area, where f_2 is the function given in Table 1 and Fig. 4. In this formula the first term in the square brackets represents the contribution from (31) and the second term the contribution from the leading edge force. The drag coefficient based on frontal area, C_{Df}, is given by

$$C_{Df}\sqrt{(M^2 - 1)} = \frac{4\sqrt{2}}{3}\left(\frac{t_0}{c}\right) \left[f_2\left(\frac{\tan\gamma}{\tan\mu}\right) + \frac{\tan\gamma}{\tan\mu}\left(1 - \frac{\tan^2\gamma}{\tan^2\mu}\right)^{-1/2}\right].$$

As an example the drag coefficients C_D and C_{Df} for a wing of this shape, with centre section thickness-chord ratio $t_0/c = 0{\cdot}10$ is plotted against M for $\gamma = 15$ deg, 30 deg and 45 deg, in Fig. 8. The strip theory values for the biconvex centre section are also shown in Fig. 8.

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APPENDIX I

Evaluation of a Definite Integral

It is required to calculate the value of the integral

$$J_m(k) = \int_k^\infty \frac{dt}{(t^2 - a_m)^2\,(t^2 - h^2)^{1/2}\,(t^2 - k^2)^{1/2}},$$

where $\quad 3a_m^2 - 2a_m(h^2 + k^2) + h^2k^2 = 0.$

Putting $t = k \operatorname{cosec}\theta$, $\alpha^2 = a_m/k^2$, and $\varkappa^2 = h^2/k^2$ we get

$$L = k^5 J_m(k) = \int_0^{\pi/2} \frac{\sin^4\theta\, d\theta}{(1 - \alpha^2\sin^2\theta)^2\,(1 - \varkappa^2\sin^2\theta)^{1/2}},$$

where $\quad 3\alpha^4 - 2\alpha^2(1 + \varkappa^2) + \varkappa^2 = 0.$ .. .. (i)

With $$u = \int_0^{\theta} \frac{d\theta}{(1 - k^2 \operatorname{sn}^2 \theta)^2}, \quad \operatorname{sn} u = \sin\theta,$$

this becomes $$L = \int \frac{\operatorname{sn}^4 u \, du}{(1 - \alpha^2 \operatorname{sn}^2 u)^2},$$

where the lower limit of integration is zero and the upper limit is $K(\varkappa)$, the complete elliptic integral of the first kind. It is convenient to consider the indefinite integral first.

On account of the relations

$$\frac{d}{du}\left(\frac{1}{1 - \alpha^2 \operatorname{sn}^2 u}\right) = \frac{2\alpha^2 \operatorname{sn} u.\ \operatorname{cn} u.\ \operatorname{dn} u}{(1 - \alpha^2 \operatorname{sn}^2 u)^2},$$

$$\frac{d}{du}\left(\frac{\operatorname{sn}^3 u}{\operatorname{cn} u.\ \operatorname{dn} u}\right) = \operatorname{sn}^2 u \left[3 + \varkappa^2 \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u} + \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u}\right],$$

It follows that

$$2\alpha^2 L = \int \frac{\operatorname{sn}^3 u}{\operatorname{cn} u.\ \operatorname{dn} u} \cdot \frac{d}{du}\left(\frac{1}{1 - \alpha^2 \operatorname{sn}^2 u}\right) du,$$

$$= \frac{\operatorname{sn}^3 u}{\operatorname{cn} u \operatorname{dn} u (1 - \alpha^2 \operatorname{sn}^2 u)} - \int \left[3 + \frac{\varkappa^2 \operatorname{sn}^2 u}{\operatorname{dn}^2 u} + \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u}\right] \frac{\operatorname{sn}^2 u \, du}{(1 - \alpha^2 \operatorname{sn}^2 u)}.$$

Also, since

$$\frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u (1 - \alpha^2 \operatorname{sn}^2 u)} = \frac{1}{(\alpha^2 - \varkappa^2)}\left[\frac{1}{1 - \alpha^2 \operatorname{sn}^2 u} - \frac{1}{\operatorname{dn}^2 u}\right],$$

and

$$\frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u (1 - \alpha^2 \operatorname{sn}^2 u)} = \frac{1}{(\alpha^2 - 1)}\left[\frac{1}{1 - \alpha^2 \operatorname{sn}^2 u} - \frac{1}{\operatorname{cn}^2 u}\right],$$

we get

$$2\alpha^2 L = \frac{\operatorname{sn}^3 u}{\operatorname{cn} u \operatorname{dn} u (1 - \alpha^2 \operatorname{sn}^2 u)} + \frac{\varkappa^2}{(\alpha^2 - \varkappa^2)} \int \frac{\operatorname{sn}^2 u \, du}{\operatorname{dn}^2 u}$$

$$+ \frac{1}{(\alpha^2 - 1)} \int \frac{\operatorname{sn}^2 u \, du}{\operatorname{cn}^2 u} - \left(3 + \frac{\varkappa^2}{\alpha^2 - \varkappa^2} + \frac{1}{\alpha^2 - 1}\right) \int \frac{\operatorname{sn}^2 u \, du}{(1 - \alpha^2 \operatorname{sn}^2 u)}.$$

The last term vanishes because of (i).

By the properties of the Jacobian elliptic functions we can verify that

$$(1 - \varkappa^2) \int \frac{\operatorname{sn}^2 u}{\operatorname{cn}^2 u} du = \int \frac{d}{du}\left(\frac{\operatorname{dn} u}{\operatorname{cn} u}\right) \operatorname{sn} u \, du = \frac{\operatorname{sn} u \operatorname{dn} u}{\operatorname{cn} u} - \int \operatorname{dn}^2 u \, du,$$

$$(1 - \varkappa^2) \int \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u} du = -\int \frac{d}{du}\left(\frac{\operatorname{cn} u}{\operatorname{dn} u}\right) \operatorname{sn} u \, du = -\frac{\operatorname{sn} u.\ \operatorname{cn} u}{\operatorname{dn} u} + \int \operatorname{cn}^2 u \, du,$$

$$= -\frac{\operatorname{sn} u.\ \operatorname{cn} u}{\operatorname{dn} u} - \left(\frac{1 - \varkappa^2}{\varkappa^2}\right) u + \frac{1}{\varkappa^2} \int \operatorname{dn}^2 u \, du.$$

Substituting these relations in the above expression for L we obtain, after some reduction,

$$2\alpha^2 L = \frac{1}{(\varkappa^2 - \alpha^2)}\left[u - \frac{1}{(1 - \alpha^2)} \int \operatorname{dn}^2 u \, du\right]$$

$$+ \frac{\alpha^2 \operatorname{sn} u.\ \operatorname{cn} u.\ \operatorname{dn} u}{(\varkappa^2 - \alpha^2)(1 - \alpha^2)(1 - \alpha^2 \operatorname{sn}^2 u)}.$$

This is the value of the indefinite integral and, inserting the limits of integration, zero and $K(\varkappa)$, we obtain

$$L = \int_0^{K(\varkappa)} \frac{\mathrm{sn}^4 u\, du}{(1 - \alpha^2\, \mathrm{sn}^2 u)^2} = \frac{(1 - \alpha^2)\, K(\varkappa) - E(\varkappa)}{2\alpha^2(\varkappa^2 - \alpha^2)\,(1 - \alpha^2)} .$$

Substituting the values of α and $\varkappa$ this becomes

$$L = \frac{k^4\, [(k^2 - a_m)\, K(h/k) - k^2\, E(h/k)]}{2a_m(h^2 - a_m)\,(k^2 - a_m)},$$

so that finally

$$J_m(k) = \frac{(k^2 - a_m)\, K(h/k) - k^2 E(h/k)}{2a_m\, k(h^2 - a_m)\,(k^2 - a_m)} .$$

APPENDIX II

The Leading-edge Force

R. T. Jones[4] has shown that linear theory may lead to incorrect results if applied to calculate the drag of bodies with rounded leading edges, unless allowance is made for the local high pressure there. Jones' formula for the leading-edge drag will be derived here by a method which is different from his method.

Consider the flow past a parabolic cylinder with its axis parallel to a stream of velocity V (Fig. 9). The equation of the cylinder is

$$r\,(1 + \cos\theta) = r_0$$

or

$$r^{1/2} \cos(\theta/2) = (r_0/2)^{1/2},$$

where r_0 is the nose radius. The complex potential of the motion is

$$w = \phi + i\psi = -\,Vz + 2\,V\,(r_0/2)^{1/2}\, z^{1/2},$$

so that

$$\psi = -\,2\,Vr^{1/2} \sin(\theta/2)\,[r^{1/2} \cos(\theta/2) - (r_0/2)^{1/2}],$$

which verifies that the cylinder is a streamline. We calculate the force on the cylinder from Blasius's theorem :—

$$X - iY = \tfrac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 d_z,$$

where X and Y are the component forces per unit length along the axes and the integration is along the contour C (Fig. 9). This formula gives immediately

$$X = -\,\pi\, r_0\, (\tfrac{1}{2}\rho V^2), \qquad \text{(i)}$$

$$Y = 0,$$

so that the resultant force on the cylinder is a drag force given by the above expression.

It is necessary to extend this formula to allow for the effect of sweepback and compressibility. This is done by assuming that only the component velocity normal to the leading edge, $V \sin\gamma$ (Fig. 2) is effective for sweptback wings and that compressibility can be allowed for by the Prandtl-Glauert formula applied to this normal component Mach number $M \sin\gamma$. When extended in this way (i) becomes

$$X = -\,\pi\, r_0\, (\tfrac{1}{2}\rho V^2)\, \frac{\sin^2\gamma}{(1 - M^2 \sin^2\gamma)^{1/2}} .$$

TABLE 1

Values of The Functions f_1 and f_2

$\dfrac{\tan\gamma}{\tan\mu}$	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
f_1	0	0·2707	0·4095	0·5048	0·5755	0·6303	0·6740	0·7097	0·7393	0·7642	0·7854
f_2	0	0·7148	1·0438	1·2528	1·3979	1·5038	1·5838	1·6458	1·6927	1·7347	1·7672

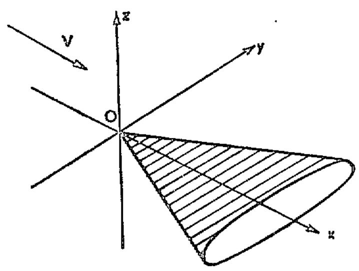

FIG. 1. Diagram showing notation.

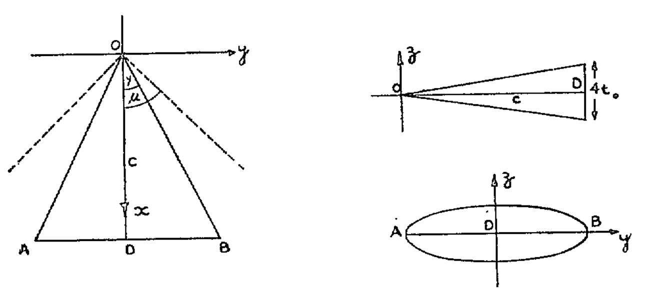

FIG. 2. Elliptic cone.

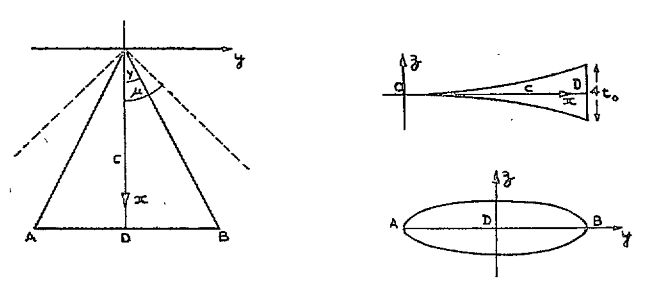

FIG. 3. Elliptic hyper-cone.

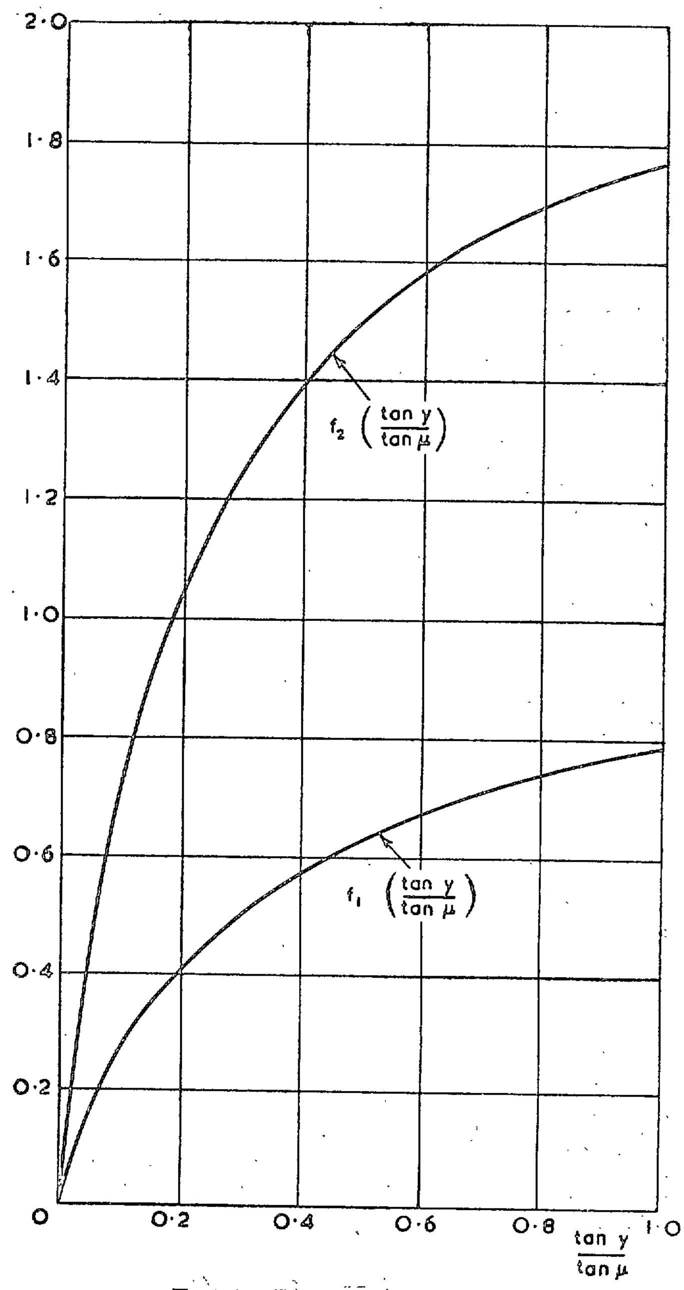

FIG. 4. Functions f_1 and f_2.

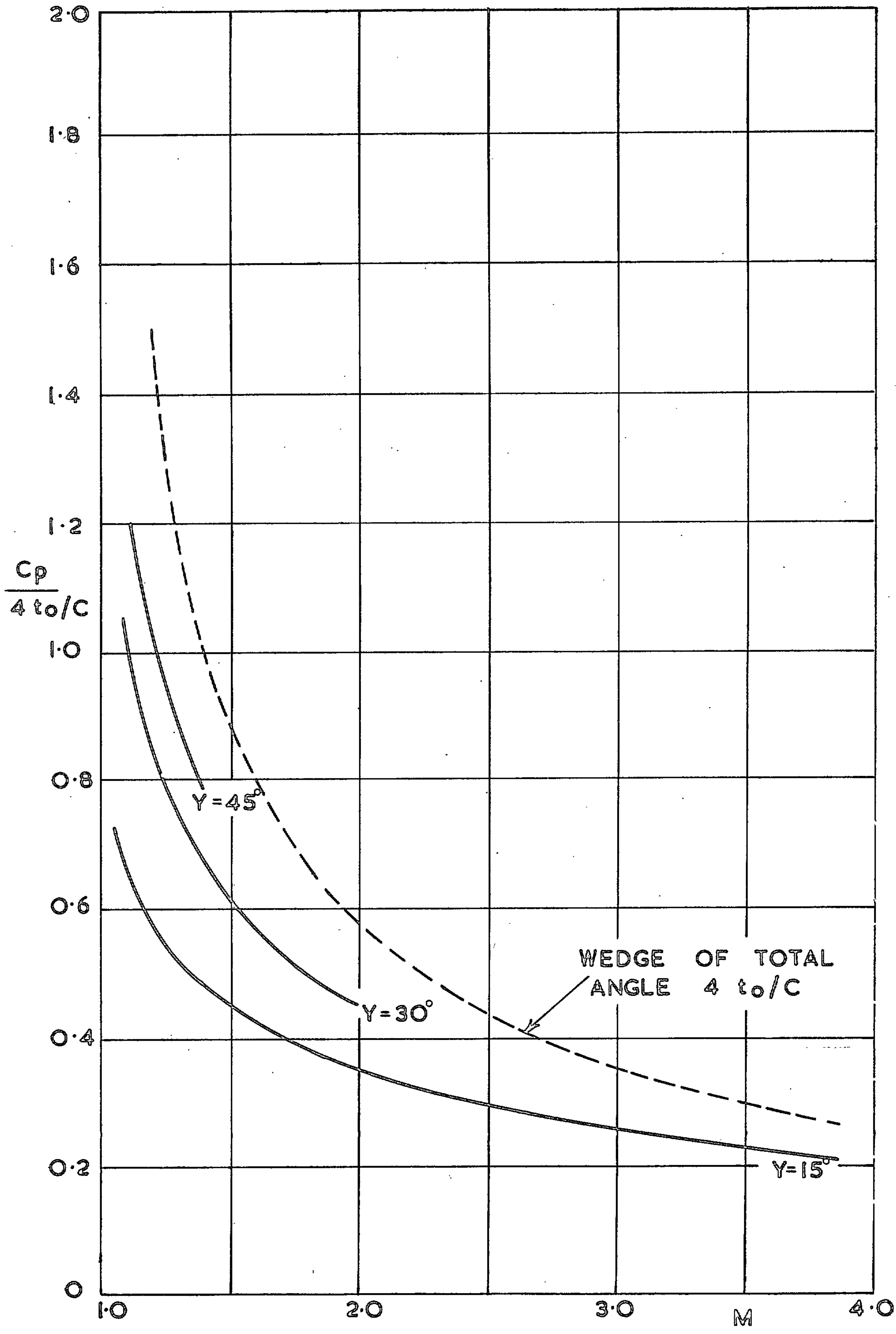

FIG. 5. Pressure on an elliptic cone at zero incidence.

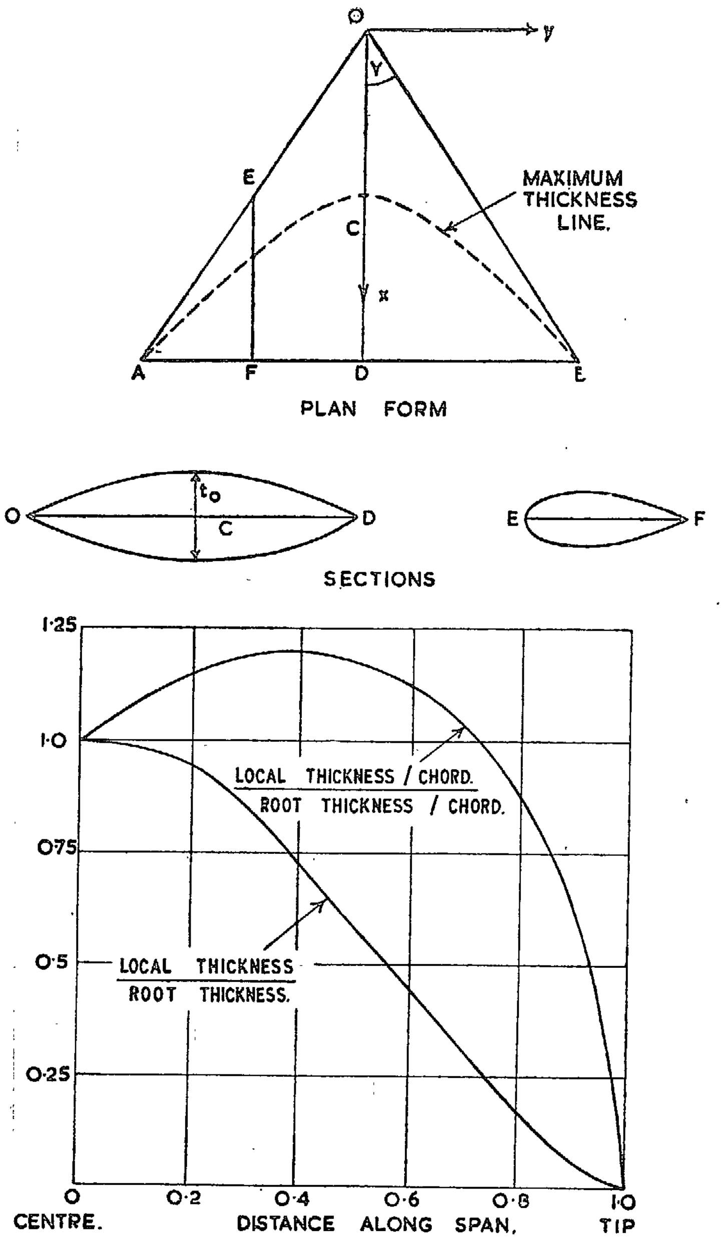

FIG. 6. Shape of wing-like surface.

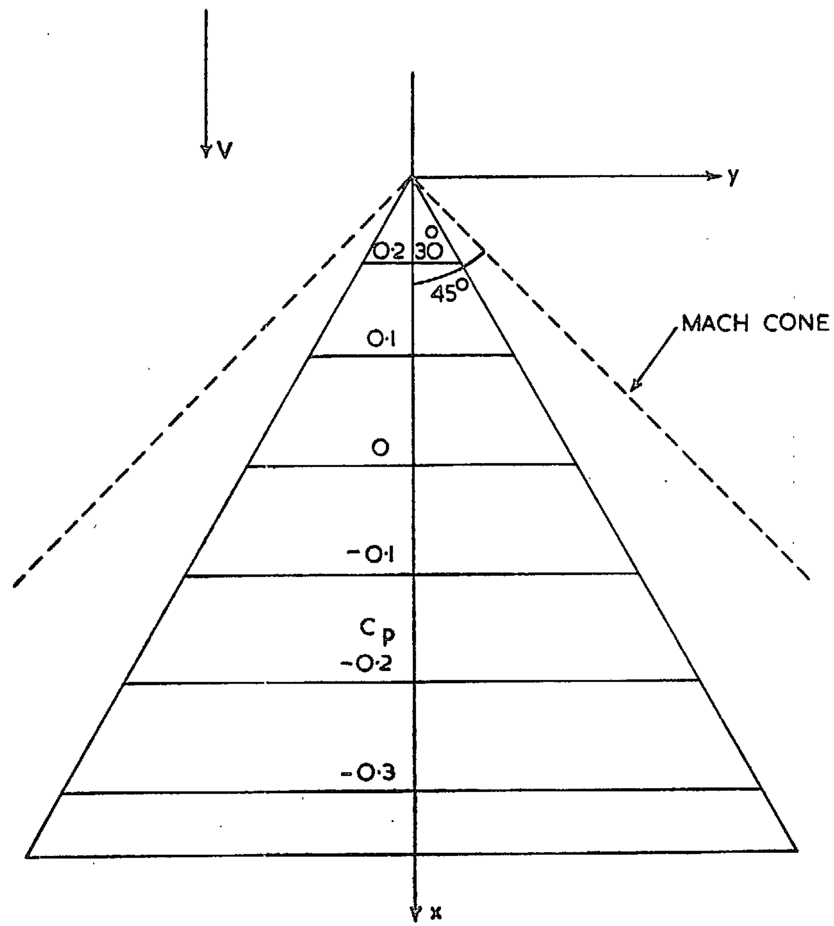

FIG. 7. Pressure distribution on wing for $\gamma = 30$ deg, $M = \sqrt{2}$, $t_0/c = 0{\cdot}10$.

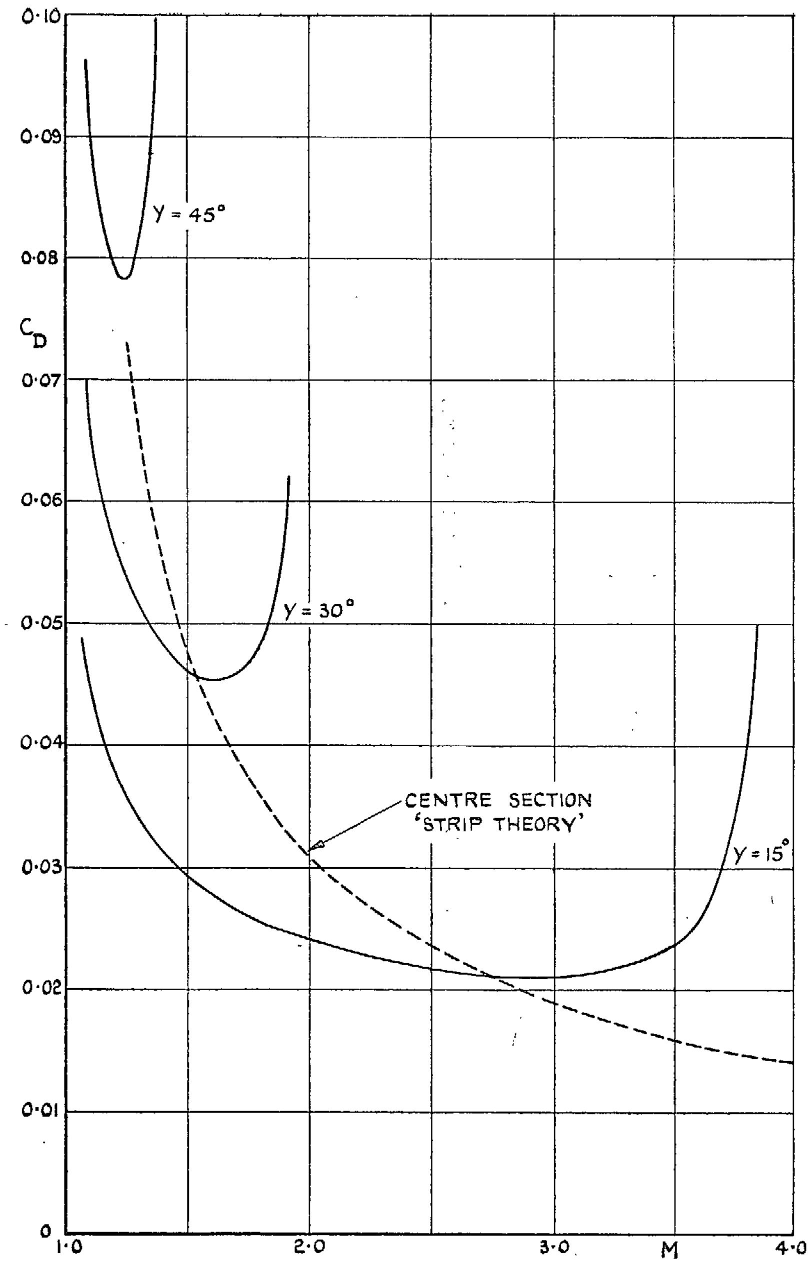

FIG. 8(a). Calculated wing drag $t_0/c = 0{\cdot}10$, C_D based on wing area.

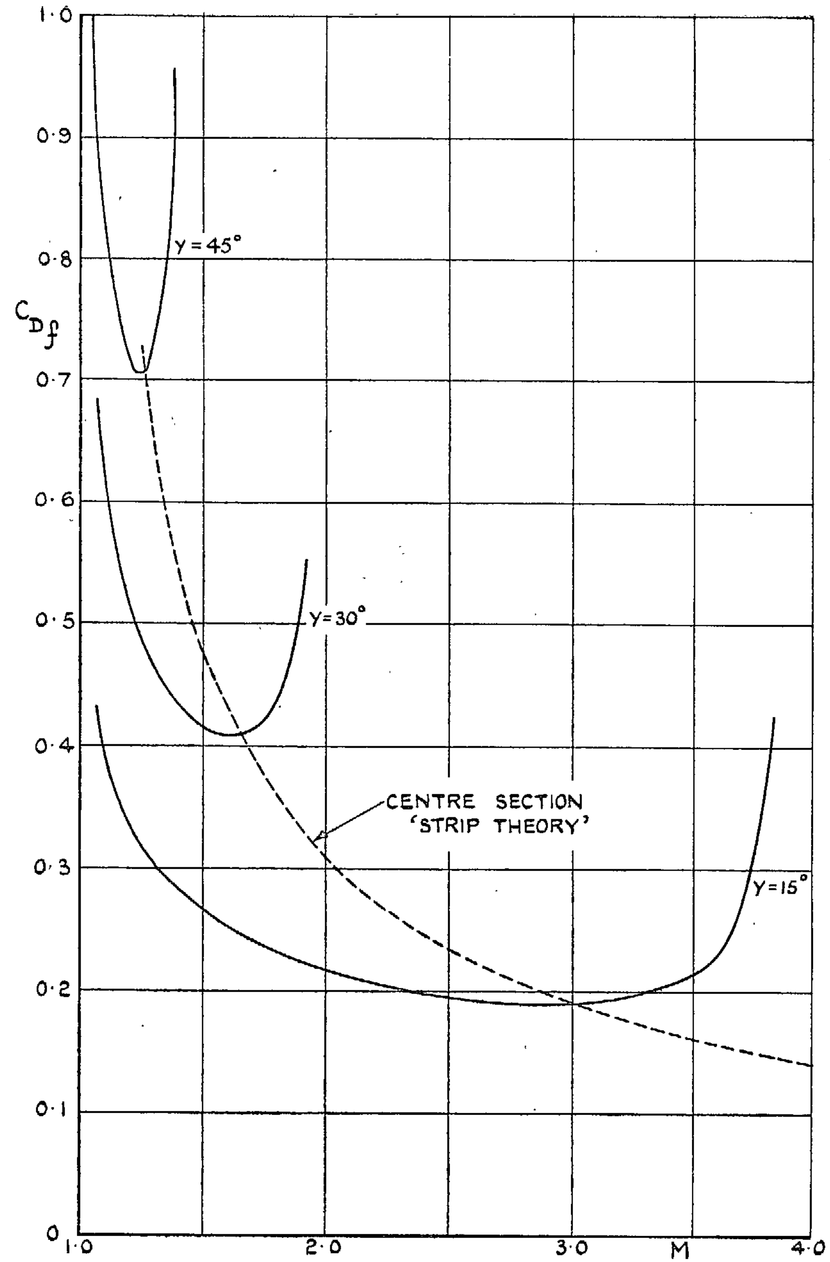

FIG. 8(b). Calculated wing drag $t_0/c = 0{\cdot}10$, C_{Df} based on frontal area.

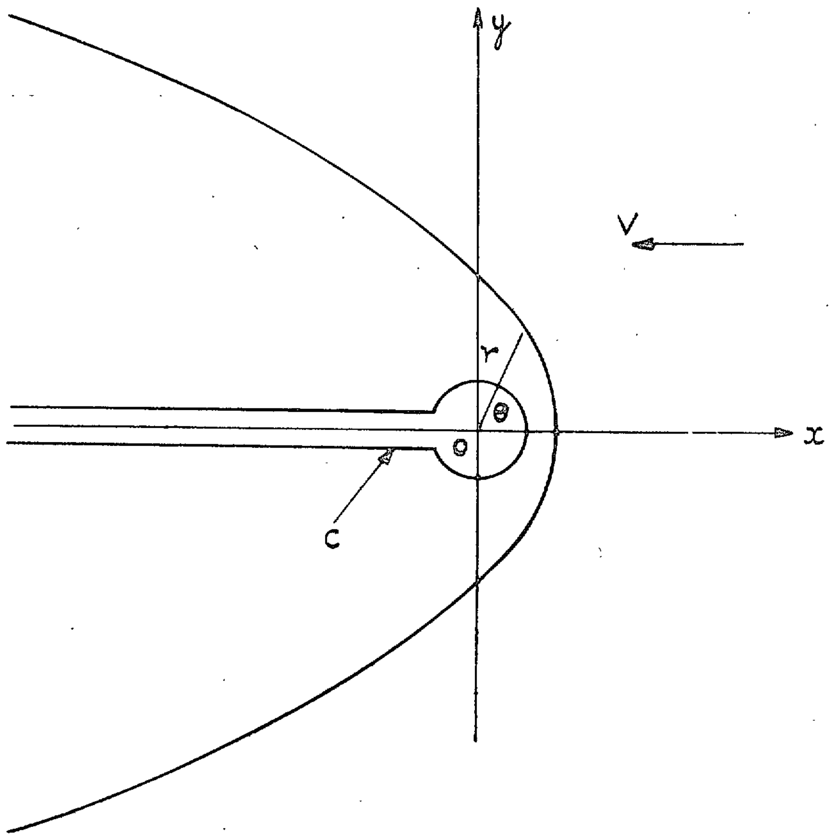

FIG. 9. Calculation of leading-edge force.

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