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Calculation of Unsteady Generalised Airforces on a Thin Wing Oscillating Harmonically in Subsonic Flow

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1965
PRICE £1 2s. 6d. NET

Calculation of Unsteady Generalised Airforces on a Thin Wing Oscillating Harmonically in Subsonic Flow

By D. E. Davies

Communicated by the Deputy Controller Aircraft (Research and Development),
Ministry of Aviation

*Reports and Memoranda No. 3409**
August, 1963

Summary.

A method of obtaining numerical values of generalised airforces on a thin wing oscillating harmonically in subsonic flow is described. It is assumed that the linearised equations of potential flow are valid. The essence of the method is the approximate solution, by collocation, of the integral equation relating the loading distribution and upwash on the wing and the use of the loading distribution so determined to calculate the generalised forces on the wing and control surfaces at general frequencies of oscillation.

The procedure has been programmed in autocode for the Ferranti Mercury Computer and is available as programme R.A.E.161A. A modification is programme R.A.E.263A for determining the loading distribution over the wing.

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1. *Introduction.*

It is now becoming increasingly the practice to obtain airforces on thin wings by solving approximately the linearised integral equation relating the unknown loading distribution to a known upwash. This practice has the advantage over analytic methods in that wings of more general shape can be considered. Analytical methods are applicable to wings of only a few particular shapes such as the circular wing and the elliptical wing.

When airspeeds were low and wings were straight and of high aspect ratio, the steady lifting force on a wing was determined by assuming that the lift was concentrated on a line lying spanwise across the wing. Even with the advent of higher speeds and the introduction of low-aspect-ratio wings having sweepback this method continued to be used, with the extension that the lift was assumed to be concentrated on several lifting lines.

An important step forward was made by Falkner[1]. Falkner considered the integral equation relating the unknown velocity potential on the wing and in the wake to the upwash and carried out approximate integrations by dividing the wing and wake up by a mesh of lines and assuming simplified behaviour over the resulting lattices.

The final step towards a continuous theory for steady subsonic flow was made by Multhopp[2] and this theory is known as a lifting-surface theory in contrast to the lifting-line theory. The loading distribution is approximated by the first few terms of a series in orthogonal functions and this series is substituted into the integral equation relating the loading distribution to the upwash. The coefficients of the terms in the series for the loading can then be determined by integrating term by term and comparing the values obtained with the values of the known upwash at a set of points on the wing equal in number to the number of terms in the series for the loading. The loading may then be determined and also any required total airforces.

This method was extended[3] to the case of a wing oscillating at low frequency. Further progress was made by Richardson[4] and Acum[5] who extended the method to deal with general frequencies of oscillation.

The method described in this paper is again based on Multhopp's lifting-surface conception and consists of a fusion of ideas presented in the theories of Richardson[4] and Acum[5]. The treatment of the equations is, however, different in detail from that of either of those two theories.

Following the method described in the present paper, a programme has been written in Autocode for the Ferranti Mercury Computer for the calculation of the generalised airforces on a wing capable of oscillation in several rigid or flexible modes at a given frequency in a flow of given main-stream subsonic Mach number. This is available as programme R.A.E.161A. A modification of this programme has been made so that loading distributions in the various modes of oscillation can be determined. This modified programme is available as programme R.A.E.263A.

The theory of Falkner, also, has been extended to deal with oscillating wings and has been used by W. P. Jones[6] and Doris E. Lehrian[7] among others. The integration concepts of Falkner are used by Runyan and Woolston[8], though they use the integral equation for the loading rather than the velocity potential, as Falkner does.

The integral equation relating loading distribution and upwash in oscillatory flow is also the basis of the method of Ref. 9 where the procedure is in some respects similar to that of Multhopp[2]. The loading distribution, however, is not approximated to by the first few terms of a series in orthogonal functions and integrations along the span are performed by dividing the span into several intervals rather than using just one interval.

Stark[10] uses an integral equation based on the integrated velocity potential. One important feature of Stark's method is that the approximation to the integrated velocity potential is not determined by equating the upwash calculated from the integral equation to the known upwash at a set of points but rather by minimising the sum of the squares of the differences between the two values at a larger number of points. This can be expected to be more accurate than merely equating the values at a smaller number of points, but because of the extra numerical work involved a similar feature has not been incorporated into the present method.

The treatment of control surfaces by the lifting-surface theory of Multhopp leads to complications, and as yet has not been carried through. However, a procedure based on equivalent displacements and upwashes was proposed by Falkner in Ref. 11 for a full-span control surface, where each chord was considered as if it were on a two-dimensional wing. Treatment of spanwise effects has since been proposed, and Richardson in Ref. 4 describes how a part-span control surface may be treated by considering chordwise and spanwise effects separately. In the present paper a procedure, again based on equivalent displacements and upwashes, is put forward. With this procedure generalised airforces on a part-span control surface may be obtained without separating the chordwise and spanwise effects. Only some of the generalised airforces on the control surfaces can be obtained with good accuracy but values of the right order of magnitude may be expected for the others.

2. *The Integral Equation Relating the Loading and Upwash on the Wing.*

The wing, which is assumed to be very thin and nearly plane, is immersed in an airstream so that its inclination to the main-stream direction is very small. It is assumed to oscillate with small amplitude about a mean position in either rigid or flexible modes. Accordingly linearised theory is applicable and the wing may be replaced by an oscillating flat plate the mean position of which is parallel to the main-stream direction. Wings in practice are symmetrical about a centre chord so only symmetrically shaped wings will be considered, though in fact the theory is also applicable to non-symmetric wings.

A system of right-handed Cartesian coordinates (x, y, z) is introduced as shown in Fig. 1, which is stationary with respect to the mean position of the flat-plate wing. The origin is taken to be some point on the mean position of the axis of symmetry. The positive x-axis is taken along the main-stream direction, the z-axis normal to the mean position of the flat plate and upwards, and the y-axis is taken mutually at right angles to complete a right-handed system.

The vertical displacement of a point (x, y) on the surface of the wing at time t in a harmonic oscillation with circular frequency ω may be given by

$$Z(x, y, t) = g(x, y)e^{i\omega t} \tag{1}$$

where, as is usual in using complex functions, only the real or the imaginary part represents the pertinent physical quantity.

The boundary condition that the airflow is tangential to the wing surface must be satisfied and this leads to the equation

$$w(x, y) = V\frac{\partial}{\partial x}g(x, y) + i\omega g(x, y) \tag{2}$$

if non-linear terms are neglected. The function $w(x, y)$ in equation (2), called the upwash function, is such that the component of the air velocity in the z-direction at the wing mean plane is $w(x, y)e^{i\omega t}$. V is the velocity of the main stream.

Corresponding to the wing displacement given by equation (1), or the upwash function given by equation (2) there is at the point x, y on the wing surface an upward lifting force per unit area, or loading $l(x, y)e^{i\omega t}$.

Reduced upwash and loading functions are introduced by the equations

$$\alpha(x, y) = \frac{1}{V}w(x, y) \tag{3}$$

$$\lambda(x, y) = \frac{1}{\rho_0 V^2}l(x, y) \tag{4}$$

where ρ_0 is the density of the air in the undisturbed main stream.

Then, as shown in Appendix I, and elsewhere, the integral equation

$$\alpha(x, y) = \frac{1}{4\pi}\iint_S \lambda(x_0, y_0)K(x-x_0, y-y_0)dx_0dy_0 \tag{5}$$

is satisfied, where S represents the area of the wing and the kernel $K(x, y)$ is given by

$$K(x, y) = e^{-i\omega x/V}\left[\int_{(-x+MR)/(1-M^2)}^{\infty} e^{-i\omega u/V}\frac{du}{(u^2+y^2)^{3/2}} + \frac{M(Mx+R)}{R(x^2+y^2)}\exp\left\{-\frac{i\omega}{V}\left(\frac{-x+MR}{1-M^2}\right)\right\}\right] \tag{6}$$

with

$$R = \sqrt{\{x^2 + (1-M^2)y^2\}} \tag{7}$$

and M is the Mach number of the main stream.

If now the modified functions

$$\hat{\alpha}(x, y) = \alpha(x, y)e^{i\omega x/V} \tag{8}$$

$$\hat{\lambda}(x, y) = \lambda(x, y)e^{i\omega x/V} \tag{9}$$

$$\hat{K}(x, y) = K(x, y)e^{i\omega x/V} \tag{10}$$

are introduced into the integral equation (5), it becomes

$$\hat{\alpha}(x, y) = \frac{1}{4\pi}\iint_S \hat{\lambda}(x_0, y_0)\hat{K}(x-x_0, y-y_0)dx_0dy_0\,. \tag{11}$$

Into this integral equation introduce the new variables

$$\left.\begin{aligned} \xi &= \frac{1}{c(y)}[x - x_L(y)] \\ \eta &= \frac{1}{s}y \\ \xi_0 &= \frac{1}{c(y_0)}[x_0 - x_L(y_0)] \\ \eta_0 &= \frac{1}{s}y_0 \end{aligned}\right\} \tag{12}$$

where s is the semi-span of the wing, $c(y)$ is the local chord length and $x_L(y)$ is the coordinate of the leading edge at the spanwise position y as shown in Fig. 1. The integral equation then becomes

$$\bar{\alpha}(\xi, \eta) = \frac{s}{4\pi}\int_{-1}^{+1} c(y_0)d\eta_0 \int_0^1 \bar{\lambda}(\xi_0, \eta_0)\hat{K}(x-x_0, y-y_0)d\xi_0 \tag{13}$$

where

$$\bar{\lambda}(\xi_0, \eta_0) = \hat{\lambda}(x_0, y_0) \tag{14}$$

$$\bar{\alpha}(\xi, \eta) = \hat{\alpha}(x, y)\,. \tag{15}$$

It is convenient to split up the kernel $\hat{K}(x, y)$ into

$$\hat{K}(x, y) = \hat{K}_1(x, y) + \hat{K}_2(x, y) \tag{16}$$

where

$$\begin{aligned} \hat{K}_1(x, y) &= \int_{(-x+MR)/(1-M^2)}^{\infty} e^{-i\omega u/V}\frac{du}{(u^2+y^2)^{3/2}} \\ &= -\frac{\pi}{y^2}\left(\frac{i\omega|y|}{2V}\right)\left[|\mathbf{H}|_{-1}\left(\frac{i\omega|y|}{V}\right) + \frac{2i}{\pi}K_1\left(\frac{\omega|y|}{V}\right) - I_1\left(\frac{\omega|y|}{V}\right)\right] - \\ &\quad - \int_0^{(-x+MR)/(1-M^2)} e^{-i\omega u/V}\frac{du}{(u^2+y^2)^{3/2}} \end{aligned} \tag{17}$$

and

$$\hat{K}_2(x, y) = \frac{M(Mx+R)}{R(x^2+y^2)}\exp\left\{-\frac{i\omega}{V}\left(\frac{-x+MR}{1-M^2}\right)\right\}. \tag{18}$$

In the above I_1 and K_1 are modified Bessel functions of the first and second kinds and first order while $|\mathbf{H}|_{-1}$ is a Struve function in the usual notation (*see* for example Ref. 13).

If the kernel is split up in this manner in the integral equation (13) and an integration by parts carried out on the integral involving the first component $\hat{K}_1(x, y)$ then there results the equation

$$\bar{\alpha}(\xi, \eta) = \frac{s}{4\pi}\int_{-1}^{+1} c(y_0)d\eta_0 \int_{1}^{1} \{\bar{\lambda}(\xi_0, \eta_0)\hat{K}_2(x-x_0, y-y_0) + \\ + c(y_0)\bar{\lambda}^{(1)}(\xi_0, \eta_0)\hat{K}_3(x-x_0, y-y_0)\}d\xi + \\ + \frac{s}{4\pi}\int_{-1}^{+1} c(y_0)\bar{\lambda}^{(1)}(1, \eta_0)\hat{K}_1(x-x_T(y_0), y-y_0)d\eta_0 \quad (19)$$

where

$$\bar{\lambda}^{(1)}(\xi_0, \eta_0) = \int_0^{\xi_0} \bar{\lambda}(v, \eta_0)dv \quad (20)$$

$$\hat{K}_3(x, y) = \frac{1}{R}\frac{(Mx+R)^2}{(x^2+y^2)^2}\exp\left\{-\frac{i\omega}{V}\left(\frac{-x+MR}{1-M^2}\right)\right\} \quad (21)$$

and $x_T(y)$ is the x-coordinate of the trailing edge at the spanwise position y.

The integral equation (19) can be handled more easily than the integral equation (13) for now the rather complicated expression (17) occurs only under a single integral rather than under a double integral.

3. *Approximations to the Loading Function and Location of Loading and Upwash Points.*

The integral equation (19) may be solved numerically for values of the loading function $\bar{\lambda}(\xi_0, \eta_0)$ only at a finite number of points on the wing. Accordingly a set of points, the loading points, at which the values of the loading are to be determined are chosen at the outset and the values of the loadings at these points are regarded as unknowns. The loading function is then represented approximately in terms of its values at the loading points by use of interpolation functions which have the same behaviours as the loading distribution near the edges of the wing.

The upwash distribution on the wing obtained from the integral relation (19) by using the approximations just described for the loading function cannot be made to coincide exactly with the given upwash distribution all over the wing. Coincidence at a number of points, the upwash points, equal in number to the number of loading points can, however, be obtained. In this way a set of simultaneous equations for the values of the loading function at the loading points in terms of the values of the upwash at the upwash points is set up.

The accuracy with which the loading function is determined depends on the number of loading and upwash points taken and also on the choice of their positions over the wing.

In subsonic flow, since the harmonic velocity potential of the flow about the oscillating wing satisfies an elliptic partial differential equation, the loading distribution can be expected to be smooth over the wing, away from any discontinuities in the upwash distribution such as occur at control surface edges, and also away from any discontinuities in slope of the wing edges. This would not be so in supersonic flow where the harmonic velocity potential satisfies a hyperbolic partial differential equation, and, for example, a discontinuity in slope of the wing leading edge gives rise to discontinuities in the loading distribution across a Mach line from the point of discontinuity of slope on the leading edge if the Mach line lies on the surface of the wing.

For a wing without control surfaces in subsonic flow the loading distribution is smooth except in the immediate neighbourhood of any points of discontinuity of slope of its edges, so $\bar{\lambda}(\xi_0, \eta_0)$ may be approximated quite well by a few terms of an expansion in terms of elementary orthogonal functions over the whole wing except in the immediate neighbourhood of these points of discontinuity of slope. The values of total forces on the wing obtained by using this approximation should be little different from the actual values.

In the following theory the positions of the leading and trailing edges of the wing are specified at only a relatively small number of stations along the span and it is assumed that a sufficiently good approximation to the leading and trailing edges is obtained by taking the equations of these edges to be polynomials which give the correct values at the specified stations. This leads to small errors in the neighbourhood of discontinuity of slope in the leading and trailing edges but the overall effect on the total forces is expected to be small.

The loading function $\bar{\lambda}(\xi_0, \eta_0)$ has a singular behaviour like $1/\sqrt{\xi_0}$ near the leading edge and tends to zero like $\sqrt{(1-\xi_0)}$ near the trailing edge. These are the behaviours near the leading and trailing edges of a two-dimensional wing, which must be followed near the leading and trailing edges of a finite wing.

The selection of upwash points along a chord will be made on the basis of two-dimensional steady-flow theory. For a particular finite oscillating wing there may be better selections but the problem of their choice remains. The selection made on the basis of two-dimensional theory should be better than an arbitrary selection.

An approximation to the loading function $\bar{\lambda}(\xi_0, \eta_0)$ along the chord at $\eta = \eta_0$, and which has the correct behaviour at the leading and trailing edges is given by

$$\bar{\lambda}^*(\xi_0, \eta_0) = \left\{\sum_{r=0}^{n-1} a_r(\eta_0)\xi_0^r\right\}\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}. \tag{22}$$

If $\bar{\lambda}^*(\xi_0, \eta_0)$, for a particular value of η_0, represents the loading on a two-dimensional wing lying between $\xi_0 = 0$ and $\xi_0 = 1$ in steady subsonic flow, then the corresponding upwash at any point ξ in (0, 1) may be calculated. If this calculated upwash is equated to the prescribed upwash at each of n points ξ in (0, 1), then there results a system of n simultaneous linear equations which may be solved for the values $a_r(\eta_0)$. The values of the $a_r(\eta_0)$ so obtained will depend on which points have been selected as the n upwash points ξ in (0, 1).

The values of the $a_r(\eta_0)$ for which $\bar{\lambda}^*(\xi_0; \eta_0)$ of equation (22) is the best approximation to $\bar{\lambda}(\xi_0, \eta_0)$ are deemed to be those for which

$$\int_0^1 [\bar{\lambda}(\xi_0, \eta_0) - \bar{\lambda}^*(\xi_0, \eta_0)]^2 \sqrt{\left(\frac{\xi_0}{1-\xi_0}\right)}\, d\xi_0 \tag{23}$$

is a minimum for a given value of η_0. This best set of values of the $a_r(\eta_0)$ cannot be determined exactly since the function $\bar{\lambda}(\xi_0, \eta_0)$ is not known explicitly. However, it is possible to select the n upwash points ξ in (0, 1) so that the values of the $a_r(\eta_0)$ calculated in terms of the two-dimensional steady-state upwashes at these upwash points are, in general, as good approximations to the best set of values of the $a_r(\eta_0)$ as it is possible to get with n points only. The procedure for doing this involves rewriting equation (22) in terms of orthogonal polynomials.

If the set of polynomials $l_r(\xi_0)$ of degree r is defined as an orthogonal set over (0, 1) with respect to $\sqrt{\{(1-\xi_0)/\xi_0\}}$ as weight function, i.e.

$$\int_0^1 l_r(\xi_0) l_s(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}\, d\xi_0 = \delta_{r,s}, \tag{24}$$

where $\delta_{r,s}$ is Kronecker's delta, and the series (22) is written

$$\bar{\lambda}^*(\xi_0, \eta_0) = \left\{ \sum_{r=0}^{n-1} b_r(\eta_0) l_r(\xi_0) \right\} \sqrt{\left(\frac{1-\xi_0}{\xi_0} \right)} \tag{25}$$

then the integral (23) is a minimum when

$$b_r(\eta_0) = \int_0^1 \bar{\lambda}(\xi_0, \eta_0) l_r(\xi_0) d\xi_0, \qquad 0 \leqslant r \leqslant n-1. \tag{26}$$

The $b_r(\eta_0)$ are the coefficients of the first n terms in the infinite expansion of $\bar{\lambda}(\xi_0, \eta_0)$ in terms of the $l_r(\xi_0)$:

$$\bar{\lambda}(\xi_0, \eta_0) = \left\{ \sum_{r=0}^{\infty} b_r(\eta_0) l_r(\xi_0) \right\} \sqrt{\left(\frac{1-\xi_0}{\xi_0} \right)}. \tag{27}$$

Corresponding to the loading distribution

$$l_n(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0} \right)}$$

on the two-dimensional wing in steady subsonic flow let there be an upwash distribution $\alpha_n(\xi)$. Then $\alpha_n(\xi)$ is given by the integral formula

$$\alpha_n(\xi) = \int_0^1 l_n(\xi_0) K(\xi - \xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0} \right)} \, d\xi_0 \tag{28}$$

where $K(\xi)$ is the steady subsonic two-dimensional kernel, and ξ is distance aft of the leading edge. The function $\alpha_n(\xi)$ turns out to be a polynomial of degree n in ξ.

Then, corresponding to the loading distribution $\bar{\lambda}(\xi_0, \eta_0)$ of equation (27) there is a two-dimensional upwash distribution $u(\xi, \eta_0)$ given by the formula

$$u(\xi, \eta_0) = \sum_{r=0}^{\infty} b_r(\eta_0) \alpha_r(\xi). \tag{29}$$

If equation (29) is written down for n separate points ξ in (0, 1), a set of equations is obtained which may be solved for the $b_r(\eta_0)$, $0 \leqslant r \leqslant n-1$, in terms of the values of the two-dimensional upwash $u(\xi, \eta_0)$ at these points and of the $b_r(\eta_0)$, $r \geqslant n$. Approximate values of the $b_r(\eta_0)$, $0 \leqslant r \leqslant n-1$ are then obtained by neglecting all the $b_r(\eta_0)$, $r \geqslant n$. If, however, the n separate points ξ in (0, 1) are chosen to be the n roots

$$\xi_k^{(w)} \qquad k = 1, 2, \ldots, n, \tag{30}$$

of the polynomial equation

$$\alpha_n(\xi) = 0 \tag{31}$$

then the values of the $b_r(\eta_0)$, $0 \leqslant r \leqslant n-1$ do not depend on the value of $b_n(\eta_0)$. The approximations to the $b_r(\eta_0)$, $0 \leqslant r \leqslant n-1$, will then, in general, be better than those obtainable using the values of $u(\xi, \eta_0)$ at any other selection of n points ξ in (0, 1). The corresponding values of the $a_r(\eta_0)$ are then the values which are to be taken as the approximations to the best set of values of the $a_r(\eta_0)$. It follows that the points (30) are, in general, the best ones to take for the chordwise positions of the upwash points on a two-dimensional wing in steady flow. As mentioned earlier, these points will be taken as the upwash points in the case of a finite oscillating wing. The points are numbered in order from the leading edge.

The functions $\alpha_n(\xi)$ and $l_n(\xi)$ are in a simple relationship, as the following analysis shows:

Since the function $\alpha_r(1-\xi)$ is a polynomial of degree r it may be written as a linear combination of the $l_p(\xi)$ with $p \leqslant r$. It then follows from the relation (24) that

$$\int_0^1 \alpha_r(1-\xi)l_s(\xi)\sqrt{\left(\frac{1-\xi}{\xi}\right)}\,d\xi = 0 \tag{32}$$

when $r < s$. If $r > s$, the integral relation (28) is used, and we obtain

$$\begin{aligned}\int_0^1 \alpha_r(1-\xi)l_s(\xi)\sqrt{\left(\frac{1-\xi}{\xi}\right)}\,d\xi &= \int_0^1 l_s(\xi)\sqrt{\left(\frac{1-\xi}{\xi}\right)}\,d\xi \int_0^1 K(1-\xi-\xi_0)l_r(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}\,d\xi_0 \\ &= \int_0^1 l_r(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}\,d\xi_0 \int_0^1 K(1-\xi-\xi_0)l_s(\xi)\sqrt{\left(\frac{1-\xi}{\xi}\right)}\,d\xi \\ &= \int_0^1 \alpha_s(1-\xi_0)l_r(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}\,d\xi_0 \\ &= 0.\end{aligned} \tag{33}$$

Therefore $\alpha_n(1-\xi)$ must be proportional to $l_n(\xi)$ and the n upwash points are given by

$$\xi_k^{(w)} = 1 - \xi_i^{(l)} \qquad k = 1, 2, \ldots, n \tag{34}$$

where

$$i = n - k + 1 \tag{35}$$

and

$$\xi_i^{(l)} \qquad i = 1, 2, \ldots, n, \tag{36}$$

are the roots, numbered in order of increasing size, of the polynomial equation

$$l_n(\xi) = 0. \tag{37}$$

As is shown in Appendix III, the points $\xi_i^{(l)}$ are given by

$$\xi_i^{(l)} = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2i-1}{2n+1}\pi\right) \qquad i = 1, 2, \ldots, n. \tag{38}$$

The approximate values of $\bar{\lambda}(\xi_0, \eta_0)$ at n points along a chord may be determined from the approximation formula (22). Reciprocally the approximate formula for $\bar{\lambda}(\xi_0, \eta_0)$ may be determined in terms of the approximate values at these n points by the use of interpolation functions having the correct behaviours at the leading and trailing edges. It is very convenient from the point of view of mathematical formulation and numerical computation if these n points are taken to be the n points $\xi_i^{(l)}$ defined above in equation (38). These n points will be called the chordwise loading points.

Corresponding to each point $\xi_i^{(l)}$, an interpolation function $h_i^{(n)}(\xi_0)$ is formed which is unity at the point $\xi_i^{(l)}$ and zero at the other $(n-1)$ loading points, and which is the product of $\sqrt{\{(1-\xi_0)/\xi_0\}}$ with a polynomial of degree $(n-1)$ in ξ_0:

$$h_i^{(n)}(\xi_0) = \frac{l_n(\xi_0)}{(\xi_0 - \xi_i^{(l)})\left[\frac{d}{d\xi_0} l_n(\xi_0)\right]_{\xi_0=\xi_i}} \sqrt{\left(\frac{\xi_i^{(l)}}{1-\xi_i^{(l)}}\right)}\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}. \tag{39}$$

The approximation to the loading along a chord of the wing may then be given as the sum

$$\bar{\lambda}(\xi_0, \eta_0) = \sum_{i=1}^{n} \bar{\lambda}(\xi_i^{(l)}, \eta_0) h_i^{(n)}(\xi_0) \tag{40}$$

where the asterisk has now been dropped from the $\bar{\lambda}$, for it is no longer required if one bears in mind that the quantities denoted by $\bar{\lambda}$ are henceforth approximations to the actual quantities. Formula (40) is exactly equivalent to formula (22).

The loading distribution behaves like $\sqrt{(1-\eta)}$ near the starboard tip of the wing and like $\sqrt{(1+\eta)}$ near the port tip. These are the behaviours near the edges of a very slender rectangular wing.

An approximation to the spanwise function $\bar{\lambda}(\xi_i^{(l)}, \eta_0)$ may then be taken as the product of $\sqrt{(1-\eta_0^2)}$ with a polynomial of degree $(m-1)$ in η_0. Following the procedure for the chordwise variable ξ_0, a set of polynomials $\gamma_r(\eta_0)$ of degree r is defined as an orthogonal set over (0, 1) with $\sqrt{(1-\eta_0^2)}$ as weight function, i.e.

$$\int_0^1 \gamma_r(\eta_0)\gamma_s(\eta_0) \sqrt{(1-\eta_0^2)} d\eta_0 = \delta_{r,s}. \tag{41}$$

To choose the spanwise locations of the upwash points it is observed that the kernel $\hat{K}(x, y)$ in equation (11) behaves like $1/y^2$ near $y = 0$. The spanwise distribution of upwash $w_m(\eta)$ corresponding to the loading distribution $\gamma_m(\eta_0) \sqrt{(1-\eta_0^2)}$, and upon which the choice of upwash points depends is then taken to be

$$w_m(\eta) = \int_{-1}^{+1} \frac{\gamma_m(\eta_0)}{(\eta-\eta_0)^2} \sqrt{(1-\eta_0^2)} d\eta_0 \tag{42}$$

where the singular integral is a principal-value integral which has to be evaluated using Hadamard's 'Finite Part' method of integration. The function $w_m(\eta)$ turns out to be a polynomial of degree m in η. The spanwise locations of the upwash points are then chosen as the m roots

$$\eta_r^{(w)} \qquad r = 1, 2, \ldots, m, \tag{43}$$

of the polynomial equation

$$w_m(\eta) = 0 \tag{44}$$

for reasons similar to the ones for which the chordwise upwash points were chosen. The spanwise points are numbered in order starting from the starboard tip.

The points (43) are given by

$$\eta_r^{(w)} = -\eta_j \qquad r = 1, 2, \ldots, m \tag{45}$$

where

$$j = m - r + 1 \tag{46}$$

and

$$\eta_j \qquad j = 1, 2, \ldots, m \tag{47}$$

are the roots, numbered in order of decreasing η_j from $\eta_0 = 1$ to $\eta_0 = -1$, of the polynomial equation

$$\gamma_m(\eta_0) = 0. \tag{48}$$

The relations (45) and (46) are obtained by an argument similar to the one used in obtaining the relations (34) and (35).

The points η_j are symmetrical with respect to $\eta_0 = 0$, in view of the symmetry of $\sqrt{(1-\eta_0^2)}$ about this point, so that the spanwise locations of the loading and upwash points are the same.

As is shown in Appendix III, the points η_j are given by

$$\eta_j = \cos\left(\frac{j\pi}{m+1}\right), \qquad j = 1, 2, \ldots, m. \tag{49}$$

Again it is convenient if the m points η_j defined in equation (49) are taken as spanwise loading points and the loading given in terms of its values at these points.

Corresponding to each point η_j an interpolation function $g_j^{(m)}(\eta_0)$ is formed which is unity at the point η_j and zero at the other $(m-1)$ spanwise points, and which is the product of $\sqrt{(1-\eta_0^2)}$ with a polynomial of degree $(m-1)$ in η_0:

$$g_j^{(m)}(\eta_0) = \frac{\gamma_m(\eta_0)}{(\eta_0-\eta_j)\left(\dfrac{d}{d\eta_0}\gamma_m(\eta_0)\right)_{\eta_0=\eta_j}} \frac{\sqrt{(1-\eta_0^2)}}{\sqrt{(1-\eta_j^2)}}. \tag{50}$$

The approximation to $\bar{\lambda}(\xi_i^{(l)}, \eta_0)$ is then given by

$$\bar{\lambda}(\xi_i^{(l)}, \eta_0) = \sum_{j=1}^{m} \bar{\lambda}(\xi_i^{(l)}, \eta_j) g_j^{(m)}(\eta_0) \tag{51}$$

so that, from equation (40),

$$\bar{\lambda}(\xi_0, \eta_0) = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\lambda}(\xi_i^{(l)}, \eta_j) h_i^{(n)}(\xi_0) g_j^{(m)}(\eta_0). \tag{52}$$

To end this section formulae for the loading and upwash points on a finite wing are given.

The totality of loading points are given by

$$\left.\begin{aligned} x_{i,j}^{(l)} &= c(y_j)\xi_i^{(l)} + x_L(y_j) \\ y_j &= s\eta_j \end{aligned}\right\} \quad \begin{aligned} i &= 1, 2, \ldots, n \\ j &= 1, 2, \ldots, m \end{aligned} \tag{53}$$

and the totality of upwash points are given by

$$\left.\begin{aligned} x_{k,r}^{(w)} &= c(y_r)\xi_k^{(w)} + x_L(y_r) \\ y_r &= s\eta_r \end{aligned}\right\} \quad \begin{aligned} k &= 1, 2, \ldots, n \\ j &= 1, 2, \ldots, m. \end{aligned} \tag{54}$$

4. *The Numerical Integration.*

Substituting the approximation (40) for $\bar{\lambda}(\xi_0, \eta_0)$ into the form (19) of the integral equation results in

$$\bar{\alpha}(\xi, \eta) = \sum_{i=1}^{n} \int_{-1}^{+1} \frac{\bar{\lambda}(\xi_i^{(l)}, \eta_0)}{(\eta-\eta_0)^2} I_i^{(n)}(\eta, \eta_0, \xi) d\eta_0 \tag{55}$$

where

$$I_i^{(n)}(\eta, \eta_0, \xi) = \frac{s}{4\pi} c(y_0)(\eta-\eta_0)^2 \left[\int_0^1 \{h_i^{(n)}(\xi_0)\hat{K}_2(x-x_0, y-y_0) + \right.$$
$$\left. + c(y_0)h_i^{(1,n)}(\xi_0)\hat{K}_3(x-x_0, y-y_0)\}d\xi_0 + h_i^{(1,n)}(1)\hat{K}_1(x-x_T(y_0), y-y_0)\right] \tag{56}$$

and

$$h_i^{(1,n)}(\xi_0) = \int_0^{\xi_0} h_i^{(n)}(u)du. \tag{57}$$

The function $I_i^{(n)}(\eta, \eta_0, x)$ may be developed into a series of the form

$$I_i^{(n)}(\eta, \eta_0, \xi) = \sum_{s=0}^{\infty} E_{s,i}^{(n)}(\eta, \xi)\,(\eta-\eta_0)^s + (\eta-\eta_0)^2 \log|\eta-\eta_0| \sum_{s=0}^{\infty} F_{s,i}^{(n)}(\eta, \xi)\,(\eta-\eta_0)^s \tag{58}$$

in the neighbourhood of $\eta_0 = \eta$.

The principal value integral in equation (55) could be evaluated approximately by using the interpolation formula

$$\bar{\lambda}(\xi_i^{(l)}, \eta_0) I_i^{(n)}(\eta, \eta_0, \xi) = \sum_{j=1}^{m} \bar{\lambda}(\xi_i^{(l)}, \eta_j) I_i^{(n)}(\eta, \eta_j, \xi) g_j^{(m)}(\eta_0) \tag{59}$$

and integrating each term obtained by putting this series into (55). The accuracy of the value obtained would however be adversely affected by the presence of the logarithmic terms in the expansion (58), and in particular by the lowest-order logarithmic term, especially if the value of m is small. The accuracy can be much improved if the lowest-order logarithmic term is removed from $I_i^{(n)}(\eta, \eta_0, \xi)$ and dealt with separately while the remainder is dealt with by using the interpolation procedure. In order to do this an expression for the function $F_{0,i}^{(n)}(\eta, \xi)$ is required and this is found in Appendix IV to be

$$F_{0,i}^{(n)}(\eta, \xi) = \frac{1}{4\pi} \frac{s}{c(y)} \left\{ -(1-M^2) h_i^{(n)\prime}(\xi) + 2\frac{i\omega}{V} c(y) h_i^{(n)}(\xi) + \frac{\omega^2}{V^2} c^2(y) h_i^{(1,n)}(\xi) \right\}. \tag{60}$$

To evaluate the principal-value integral in equation (55) a procedure similar to that of Mangler and Spencer[14] is carried out.

Put

$$I_i^{(n)}(\eta, \eta_0, \xi) = I_i^{(n)*}(\eta, \eta_0, \xi) + F_{0,i}^{(n)}(\eta, \xi)\,(\eta-\eta_0)^2 \log|\eta-\eta_0| \tag{61}$$

and write the identity

$$\begin{aligned}
&\bar{\lambda}(\xi_i^{(l)}, \eta_0) I_i^{(n)}(\eta, \eta_0, \xi) \\
&= \bar{\lambda}(\xi_i^{(l)}, \eta) \frac{\sqrt{(1-\eta_0^2)}}{\sqrt{(1-\eta^2)}} \{I_i^{(n)}(\eta, \eta_0, \xi) - I_i^{(n)*}(\eta, \eta_0, \xi)\} + \\
&\quad + \left[\bar{\lambda}(\xi_i^{(l)}, \eta_0) I_i^{(n)}(\eta, \eta_0, \xi) - \bar{\lambda}(\xi_i^{(l)}, \eta) \frac{\sqrt{(1-\eta_0^2)}}{\sqrt{(1-\eta^2)}} \{I_i^{(n)}(\eta, \eta_0, \xi) - I_i^{(n)*}(\eta, \eta_0, \xi)\} \right]
\end{aligned} \tag{62}$$

The lowest-order logarithmic singularity is missing in the expression in square brackets so the interpolation process is to be applied to that expression. We then obtain approximately

$$\begin{aligned}
&\bar{\lambda}(\xi_i^{(l)}, \eta_0) I_i^{(n)}(\eta, \eta_0, \xi) \\
&= \bar{\lambda}(\xi_i^{(l)}, \eta) \frac{\sqrt{(1-\eta_0^2)}}{\sqrt{(1-\eta^2)}} F_{0,i}^{(n)}(\eta, \xi)\,(\eta-\eta_0)^2 \log|\eta-\eta_0| + \sum_{j=1}^{m} \Big[\bar{\lambda}(\xi_i^{(l)}, \eta_j) I_i^{(n)}(\eta, \eta_0, \xi) - \\
&\quad - \bar{\lambda}(\xi_i^{(l)}, \eta) \frac{\sqrt{(1-\eta_j^2)}}{\sqrt{(1-\eta^2)}} F_{0,i}^{(n)}(\eta, \xi)\,(\eta-\eta_j)^2 \log|\eta-\eta_j| \Big] g_j^{(m)}(\eta_0).
\end{aligned} \tag{63}$$

Hence, from (55) we obtain the approximate equation

$$\bar{\alpha}(\xi, \eta) = \sum_{i=1}^{n} \frac{\bar{\lambda}(\xi_i^{(l)}, \eta)}{\sqrt{(1-\eta^2)}} F_{0,i}^{(n)}(\eta, \xi) \left[\int_{-1}^{+1} \log|\eta - \eta_0| \sqrt{(1-\eta_0^2)} - \right.$$

$$\left. - \sum_{j=1}^{m} (\eta - \eta_j)^2 \log|\eta - \eta_j| \sqrt{(1-\eta_j^2)} \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0) d\eta_0}{(\eta - \eta_0)^2} \right] +$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\lambda}(\xi_i^{(l)}, \eta_j) I_i^{(n)}(\eta, \eta_0, \xi) \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0) d\eta_0}{(\eta - \eta_0)^2}. \tag{64}$$

If the equation (64) is written down for the *mn* upwash points $(x_{k,r}^{(w)}, y_r)$ on the wing there results the following set of *mn* simultaneous linear equations

$$\bar{\alpha}(\xi_k^{(w)}, \eta_r) = \sum_{i=1}^{n} \frac{\bar{\lambda}(\xi_i^{(l)}, \eta_r)}{\sqrt{(1-\eta_r^2)}} F_{0,i}^{(n)}(\eta_r, \xi_k^{(w)}) \left[\int_{-1}^{+1} \log|\eta_r - \eta_0| \sqrt{(1-\eta_0^2)} d\eta_0 - \right.$$

$$\left. - \sum_{j=1}^{m} (\eta_r - \eta_j)^2 |\log|\eta_r - \eta_j| \sqrt{(1-\eta_j^2)} \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0)}{(\eta_r - \eta_0)^2} d\eta_0 \right] +$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\lambda}(\xi_i^{(l)}, \eta_j) I_i^{(n)}(\eta_r, \eta_j, \xi_k^{(w)}) \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0)}{(\eta_r - \eta_0)^2} d\eta_0$$

$$k = 1, 2, \ldots, n$$

$$r = 1, 2, \ldots, m \tag{65}$$

for the *mn* unknowns $\bar{\lambda}(\xi_i^{(l)}, \eta_j)$.

Integrals required for the evaluation of the coefficients in equations (65) are

$$\int_{-1}^{+1} \log|\eta_r - \eta_0| \sqrt{(1-\eta_0^2)} d\eta_0 = \frac{\pi}{4}(2\eta_r^2 - 1) - \frac{\pi}{2} \log 2 \tag{66}$$

and

$$P_{j,r}^{(m)} = \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0) d\eta_0}{(\eta_r - \eta_0)^2} = \left\{ \begin{array}{ll} -\frac{\pi}{2} \frac{(m+1)}{\sqrt{(1-\eta_j^2)}} & r = j \\ \frac{2\pi}{(m+1)} \frac{\sqrt{(1-\eta_j^2)}}{(\eta_r - \eta_j)^2} & r + j \text{ an odd number} \\ 0 & r + j \text{ an even number and } r \neq j. \end{array} \right\} \tag{67}$$

Multhopp obtained the expressions (67) in Ref. 2, but their derivation is included in Appendix III for completeness.

Also required in equation (65) are the values $I_i^{(n)}(\eta_r, \eta_j, \xi_k^{(w)})$.

If $r \neq j$ values of this quantity may be obtained by performing numerically the integrations appearing on the right-hand side of equation (56). The performing of these integrations is quite straightforward, but care must be exercised when the integrand is changing rapidly with ξ_0 as happens if $(\eta_r - \eta_j)$ is small when ξ_0 is near to $\xi(x_{k,r}^{(w)}, y_j)$. It is suggested that the chord of the wing be divided into a number of intervals over each of which a Gaussian numerical quadrature formula of low order can be used with good accuracy. The lengths of these intervals depend on the value of $(\eta_r - \eta_j)$ and the location of the interval with respect to $x_{k,r}^{(w)}$. These intervals will be short when $(\eta_r - \eta_j)$ is small and when their x-coordinates are near to $x_{k,r}^{(w)}$ and longer otherwise. The singular behaviours of $h_i^{(n)}(\xi_0)$ and $h_i^{(1,n)}(\xi_0)$ near the leading and trailing edges of the wing have to be taken into account in the numerical integration formulae used.

If $r = j$ the integrands in equation (56) are singular and non-integrable so that the value of the quantity can only be obtained by taking the limit as $\eta_0 \to \eta$. It is shown in Appendix V that this process leads to

$$I_i^{(n)}(\eta, \eta, \xi) = \frac{1}{2\pi} \frac{c(y)}{s} h_i^{(1,n)}(\xi). \tag{68}$$

The values of $F_{0,i}^{(n)}(\eta_r, \xi_k^{(w)})$ are obtained straightforwardly from equation (60).

All the coefficients of $\bar{\lambda}(\xi_i^{(l)}, \eta_j)$ in equations (65) may be determined and then the set may be solved.

5. *Matrix Formulation of the Equations.*

In what follows it will be assumed that the number of spanwise points m is an even number. Then there is no spanwise point on the centre section and the set of simultaneous linear equations (65) may be written as the matrix equation

$$\begin{bmatrix} \bar{\alpha}^+ \\ \bar{\alpha}^- \end{bmatrix} = \begin{bmatrix} \Lambda^{++} & \Lambda^{+-} \\ \Lambda^{-+} & \Lambda^{--} \end{bmatrix} \begin{bmatrix} \bar{\lambda}^+ \\ \bar{\lambda}^- \end{bmatrix} \tag{69}$$

where the elements are submatrices defined below.

$[\bar{\alpha}^+]$ is a column matrix of $\frac{1}{2}mn$ elements with the element

$$\bar{\alpha}(\xi_k^{(w)}, \eta_r) \qquad \begin{aligned} k &= 1, 2, \ldots, n \\ r &= 1, 2, \ldots, m/2 \end{aligned} \tag{70}$$

in the $n(m/2-r) + k$'th row.

$[\bar{\alpha}^-]$ is a column matrix of $\frac{1}{2}mn$ elements with the element

$$\bar{\alpha}(\xi_k^{(w)}, \eta_{m/2+r}) \qquad \begin{aligned} k &= 1, 2, \ldots, n \\ r &= 1, 2, \ldots, m/2 \end{aligned} \tag{71}$$

in the $n(r-1) + k$'th row.

$[\bar{\lambda}^+]$ is a column matrix of $\frac{1}{2}mn$ elements with the element

$$\bar{\lambda}(\xi_i^{(l)}, \eta_j) \qquad \begin{aligned} i &= 1, 2, \ldots, n \\ j &= 1, 2, \ldots, m/2 \end{aligned} \tag{72}$$

in the $n(m/2-j) + i$'th row.

$[\bar{\lambda}^-]$ is a column matrix of $\frac{1}{2}mn$ elements with the element

$$\bar{\lambda}(\xi_i^{(l)}, \eta_{m/2+j}) \qquad \begin{aligned} i &= 1, 2, \ldots, n \\ j &= 1, 2, \ldots, m/2 \end{aligned} \tag{73}$$

in the $n(j-1) + i$'th row.

$[\Lambda^{++}]$ is a square matrix of order $\frac{1}{2}mn \times \frac{1}{2}mn$ with the element

$$I_i^{(n)}(\eta_r, \eta_j, \xi_k^{(w)}) \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0)}{(\eta_r - \eta_j)^2} d\eta_0 +$$

$$+ \delta_{j,r} F_{0,i}^{(n)}(\eta_r, \xi_k^{(w)}) \frac{1}{\sqrt{(1-\eta_r^2)}} \left[\left[\int_{-1}^{+1} \log |\eta_r - \eta_0| \sqrt{(1-\eta_0^2)} d\eta_0 - \right.\right.$$

$$\left. - \sum_{p=1}^{m} (\eta_r - \eta_p)^2 \log |\eta_r - \eta_p| \sqrt{(1-\eta_p^2)} \int_{-1}^{+1} \frac{g_p^{(m)}(\eta_0)}{(\eta_r - \eta_0)^2} d\eta_0 \right]$$

$$i = 1, 2, \ldots, n; \qquad k = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, m/2; \qquad r = 1, 2, \ldots, m/2 \tag{74}$$

in the $n(m/2-r) + k$'th row and $n(m/2-j) + i$'th column.

$[\Lambda^{+-}]$ is a square matrix of order $\frac{1}{2}mn \times \frac{1}{2}mn$ with the element

$$I_i^{(n)}(\eta_r, \eta_{m/2+j}, \xi_k^{(w)}) \int_{-1}^{+1} \frac{g_{m/2+j}^{(m)}(\eta_0)}{(\eta_r - \eta_0)^2} d\eta_0$$

$$i = 1, 2, \ldots, n; \qquad k = 1, 2, \ldots, n$$

$$j = 1, 2, \ldots, m/2; \qquad r = 1, 2, \ldots m/2 \tag{75}$$

in the $n(m/2-r) + k$'th row and $n(j-1) + i$'th column.

Since the wing is assumed to be symmetric about the centre chord the remaining submatrices may be defined by

$$\Lambda^{-+} = \Lambda^{+-} \tag{76}$$

and

$$\Lambda^{--} = \Lambda^{++}. \tag{77}$$

It will be noticed that the + sign is associated with the starboard side of the wing and the − sign with the port side.

The arrangement of elements in the above matrices corresponds with counting the points from the leading to the trailing edge along the chords at each spanwise section in turn, starting with the spanwise section nearest the wing centre line and proceeding towards the tip. This applies to both starboard and port sides of the wing.

Let the elements in the column matrices in equation (69) be written as the sum of symmetric and antisymmetric components by

$$\begin{bmatrix} \bar{\alpha}^+ \\ \bar{\alpha}^- \end{bmatrix} = \begin{bmatrix} \bar{\alpha}^s \\ \bar{\alpha}^s \end{bmatrix} + \begin{bmatrix} \bar{\alpha}^a \\ -\bar{\alpha}^a \end{bmatrix}, \quad \begin{bmatrix} \bar{\lambda}^+ \\ \bar{\lambda}^- \end{bmatrix} = \begin{bmatrix} \bar{\lambda}^s \\ \bar{\lambda}^s \end{bmatrix} + \begin{bmatrix} \bar{\lambda}^a \\ -\bar{\lambda}^a \end{bmatrix}, \tag{78}$$

where

$$[\bar{\alpha}^s] = \tfrac{1}{2}[\bar{\alpha}^+ + \bar{\alpha}^-], \quad \bar{\lambda}^s = \tfrac{1}{2}[\bar{\lambda}^+ + \bar{\lambda}^-]$$

$$[\bar{\alpha}^a] = \tfrac{1}{2}[\bar{\alpha}^+ - \bar{\alpha}^-], \quad \bar{\lambda}^a = \tfrac{1}{2}[\bar{\lambda}^+ - \bar{\lambda}^-]. \tag{79}$$

Then from (69) in virtue of relations (76) and (77),

$$\begin{bmatrix} \bar{\alpha}^s \\ \bar{\alpha}^s \end{bmatrix} = \begin{bmatrix} \Lambda^{++} & \Lambda^{+-} \\ \Lambda^{-+} & \Lambda^{--} \end{bmatrix} \begin{bmatrix} \bar{\lambda}^s \\ \bar{\lambda}^s \end{bmatrix}, \quad \begin{bmatrix} \bar{\alpha}^a \\ -\bar{\alpha}^a \end{bmatrix} = \begin{bmatrix} \Lambda^{++} & \Lambda^{+-} \\ \Lambda^{-+} & \Lambda^{--} \end{bmatrix} \begin{bmatrix} \bar{\lambda}^a \\ -\bar{\lambda}^a \end{bmatrix} \tag{80}$$

which may be replaced by

$$[\bar{\alpha}^s] = \left[\frac{\Lambda^{++} + \Lambda^{+-}}{2} \right] [2\bar{\lambda}^s], \; [\bar{\alpha}^a] = \left[\frac{\Lambda^{++} - \Lambda^{+-}}{2} \right] [2\bar{\lambda}^a]. \tag{81}$$

The set of mn equations in mn unknowns has therefore been reduced to two sets of $\frac{1}{2}mn$ equations in $\frac{1}{2}mn$ unknowns corresponding respectively to symmetric and antisymmetric oscillations.

The equations (81) may be solved for the matrices $[2\bar{\lambda}^s]$ or $[2\bar{\lambda}^a]$. Then $\bar{\lambda}(\xi_0, \eta_0)$ may be obtained from equation (52) and so the reduced loading $\lambda(x_0, y_0)$ may be obtained by using equations (9) and (14). Programme R.A.E.263A has been constructed to determine values of $\lambda(x_0, y_0)$ at points (x_0, y_0) on the wing when it is oscillating in given symmetric or antisymmetric modes at a given frequency in a main-stream flow of given Mach number.

6. *Modes of Oscillation and Associated Generalised Forces.*

If the wing surface may be displaced in a linear combination of a number (k say) of independent modes of displacement then the vertical displacement $Z(x, y, t)$ of the point (x, y) on the wing at time t in any vibration involving these modes only can be given by the formula

$$Z(x, y, t) = l \sum_{p=1}^{k} f_p(x, y) b_p(t) \tag{82}$$

where l is a typical dimension of the wing, the $f_p(x, y)$, $p = 1, 2, \ldots, k$, define the shapes of the k independent modes and the $b_p(t)$, $p = 1, 2, \ldots, k$, are independent generalised coordinates which are functions of the time.

If at time t the wing surface undergoes an incremental virtual displacement

$$\delta Z(x, y) = l \sum_{p=1}^{k} f_p(x, y) \delta b_p \tag{83}$$

where the δb_p, $p = 1, 2, \ldots, k$ are incrementally small and arbitrary, then the virtual work done by the airforces on the wing at time t in this virtual displacement is given by

$$\begin{aligned} \delta W &= \iint_S L(x_0, y_0, t) \delta Z(x_0, y_0) dx_0 dy_0 \\ &= \sum_{p=1}^{k} \delta b_p l \iint_S L(x_0, y_0, t) f_p(x_0, y_0) dx_0 dy_0 \end{aligned} \tag{84}$$

where $L(x, y, t)$ is the upward lifting force per unit area, or loading function, at a point (x, y) on the wing at time t when the wing is vibrating according to equation (82).

In a small incremental virtual displacement, however, the virtual work is given by

$$\sum_{p=1}^{k} Q_p(t) \delta b_p \tag{85}$$

where $Q_p(t)$ is the generalised aerodynamic force in the p'th mode at time t.

Comparing (82) and (83), noting that the δb_p are arbitrary, leads to

$$Q_p(t) = l \iint_S L(x_0, y_0, t) f_p(x_0, y_0) dx_0 dy_0 . \tag{86}$$

When the wing is oscillating harmonically with circular frequency ω the generalised coordinate $b_p(t)$ is written

$$b_p(t) = b_{p0} e^{i\omega t} . \tag{87}$$

Since the governing equations of the flow about the wing are linearised the principle of superposition holds and the loading may be given by

$$L(x, y, t) = \rho V^2 \sum_{q=1}^{k} \lambda_q(x, y) b_{q0} e^{i\omega t} \tag{88}$$

where

$$\rho V^2 \lambda_q(x, y) e^{i\omega t} \tag{89}$$

is the loading distribution on the wing corresponding to the harmonic oscillation

$$Z_q(x, y, t) = l f_q(x, y) e^{i\omega t}. \tag{90}$$

If the expression (88) for $L(x, y, t)$ is substituted into (86) there results

$$\begin{aligned} Q_p(t) &= \rho V^2 l \sum_{q=1}^{k} b_{q0} e^{i\omega t} \iint_S f_p(x_0, y_0) \lambda_q(x_0, y_0) dx_0 dy_0 \\ &= \rho V^2 s l^2 \sum_{q=1}^{k} Q_{p,q} b_{q0} e^{i\omega t} \end{aligned} \tag{91}$$

where

$$Q_{p,q} = \frac{1}{sl} \iint_S f_p(x_0, y_0) \lambda_q(x_0, y_0) e^{i\omega t} dx_0 dy_0 \tag{92}$$

The quantity $Q_{p,q}$ is a generalised aerodynamic force coefficient and is a dimensionless complex number. For similar wings oscillating in similar modes p and q, from dimensional considerations one can see that it depends only on the Mach number M of the flow and the frequency parameter ν, where

$$\nu = \frac{\omega l}{V}. \tag{93}$$

In flutter theory it is often convenient to write

$$Q_{p,q} = Q_{p,q}' + i\nu Q_{p,q}'' \tag{94}$$

where $Q_{p,q}'$ and $Q_{p,q}''$ are real numbers.

By making the transformation of variables from (x_0, y_0) to (ξ_0, η_0) as defined in equation (12) in the integration variables of (92) there results

$$\begin{aligned} Q_{p,q} &= \int_{-1}^{+1} \frac{c(y_0)}{l} d\eta_0 \int_0^1 f_p(x_0, y_0) \lambda_q(x_0, y_0) d\xi_0 \\ &= \int_{-1}^{+1} \frac{c(y_0)}{l} d\eta_0 \int_0^1 f_p(x_0, y_0) e^{-i\omega x_0/V} \bar{\lambda}_q(\xi_0, \eta_0) d\xi_0. \end{aligned} \tag{95}$$

Then, using the expression (52) with the suffix q attached to the λ's this becomes

$$Q_{p,q} = \sum_{i=1}^{n} \sum_{j=1}^{m} \bar{\lambda}_q(\xi_i^{(l)}, \eta_j) \int_{-1}^{+1} \frac{c(y_0)}{l} g_j^{(m)}(\eta_0) d\eta_0 \int_0^1 h_i^{(m)}(\xi_0) e^{-i\omega x_0/V} f_p(x_0, y_0) d\xi_0. \tag{96}$$

It will be assumed that an adequate approximation to

$$c(y_0) e^{-i\omega x_0/V} f_p(x_0, y_0) \tag{97}$$

over the whole wing may be given by a double polynomial of degree not greater than the n'th in ξ_0 and not greater than the m'th in η_0. This approximation may not be so good near points of discontinuity of slope of leading and trailing edges of the wing, as obtains for example at the wing apex, but this is expected to be only a local effect in subsonic flow, and is equivalent to modifying the wing contour so that there are no such discontinuities. The values of $(\omega x_0/V)$ must also not be too large anywhere on the wing, its greatest permissible magnitude being determined mainly by the number of chordwise points. If large values of $(\omega x_0/V)$ occur then oscillations in the function $e^{-i\omega x_0/V}$ become important and this would need special treatment.

In Appendix III it is shown that if $a(\xi_0)$ is a polynomial of degree not greater than the n'th in ξ_0 and if $e(\eta_0)$ is a polynomial of degree not greater than the m'th in η_0, then

$$\int_0^1 a(\xi_0)h_i^{(n)}(\xi_0)d\xi_0 = a(\xi_i^{(l)})H_i^{(n)} \tag{98}$$

and

$$\int_{-1}^{+1} e(\eta_0)g_j^{(m)}(\eta_0)d\eta_0 = e(\eta_j)G_j^{(m)} \tag{99}$$

where

$$H_i^{(n)} = \int_0^1 h_i^{(n)}(\xi_0)d\xi_0 \tag{100}$$

and

$$G_j^{(m)} = \int_{-1}^{+1} g_j^{(m)}(\eta_0)d\eta_0 . \tag{101}$$

So equation (96) may be replaced by

$$Q_{p,q} = \sum_{i=1}^{n}\sum_{j=1}^{m} H_i^{(n)}G_j^{(m)}\frac{c(y_j)}{l}\exp\left\{-\frac{i\omega}{V}x_{i,j}^{(l)}\right\}f_p(x_{i,j}^{(l)}, y_j)\bar{\lambda}_q(\xi_i^{(l)}, \eta_j) \tag{102}$$

or, in matrix form

$$Q_{p,q} = [f_p^+, f_p^-]\begin{bmatrix} B & O \\ O & B\end{bmatrix}\begin{bmatrix}\bar{\lambda}_q^+ \\ \bar{\lambda}_q^-\end{bmatrix} \tag{103}$$

for a symmetric wing. The submatrices appearing as elements in equation (103) are defined below:

$[f_p^+]$ is a row matrix of $\frac{1}{2}mn$ elements with the element

$$f_p(x_{i,j}^{(l)}, y_j) \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m/2 \end{matrix} \tag{104}$$

in the $n(m/2-j)+i$'th column.

$[f_p^-]$ is a row matrix of $\frac{1}{2}mn$ elements, with the element

$$f_p(x_{i,\,m/2+j}^{(l)}, y_{m/2+j}) \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m/2 \end{matrix} \tag{105}$$

in the $n(j-1)+i$'th element.

$[B]$ is a diagonal matrix of order $\frac{1}{2}mn \times \frac{1}{2}mn$ with the element

$$\frac{c(y_j)}{l}H_i^{(n)}G_j^{(m)}\exp\left\{-\frac{i\omega}{V}x_{i,j}^{(l)}\right\} \qquad \begin{matrix} i = 1, 2, \ldots, n \\ j = 1, 2, \ldots, m/2 \end{matrix} \tag{106}$$

in the $n(m/2-j)+i$'th row and column.

The column matrices $\bar{\lambda}_q^+$ and $\bar{\lambda}_q^-$ are defined by (72) and (73) only now with the addition of the suffix q.

The equation (103) may be replaced by

$$Q_{p,q} = [f_p^s]\,[B]\,[2\bar{\lambda}_q^s] + [f_p^a]\,[B]\,[2\bar{\lambda}_q^a] \tag{107}$$

where

$$[f_p^s] = \tfrac{1}{2}[f_p^+ + f_p^-]\,,\ [f_p^a] = \tfrac{1}{2}\,[f_p^+ - f_q^-] \tag{108}$$

$$[\bar{\lambda}_q^s] = \tfrac{1}{2}\,[\bar{\lambda}_q^+ + \bar{\lambda}_q^-]\,,\ [\bar{\lambda}_q^a] = \tfrac{1}{2}\,[\bar{\lambda}_q^+ - \bar{\lambda}_q^-] \tag{109}$$

are respectively symmetric and antisymmetric components of the row and column matrices appearing in equation (103)

The matrices $[2\bar{\lambda}_q^s]$ and $[2\bar{\lambda}_q^a]$ in equation (107) are obtained by solving equations (81) with suffices q added to the $\bar{\lambda}$'s and the $\bar{\alpha}$'s, and on using these solutions in (107) there results

$$[Q_{p,q}] = [f_p^s]\,[B]\left[\frac{\Lambda^{++} + \Lambda^{+-}}{2}\right]^{-1}[\bar{\alpha}_q^s] + [f_p^a]\,[B]\left[\frac{\Lambda^{++} - \Lambda^{+-}}{2}\right]^{-1}[\bar{\alpha}_q^a] \tag{110}$$

where $\bar{\alpha}_q$ is obtained from equations (2), (3), (8) and (15) on putting

$$g(x, y) = lf_q(x, y) \tag{111}$$

and the suffix q is added to the α's in these equations.

So

$$\alpha_q(x, y) = l\frac{\partial}{\partial x}f_q(x, y) + ivf_q(x, y)\,. \tag{112}$$

Now define column matrices $[\alpha_q^+]$ and $[\alpha_q^-]$ as follows:

$[\alpha_q^+]$ is a column matrix of $\frac{1}{2}mn$ elements with the element

$$\alpha_q(x_{k,r}^{(w)}, y_r) \qquad \begin{array}{l} k = 1, 2, \ldots, n \\ r = 1, 2, \ldots, m/2 \end{array} \tag{113}$$

in the $n(m/2-r) + k$'th row.

$[\alpha_q^-]$ is a column matrix of $\frac{1}{2}mn$ elements with the element

$$\alpha_q(x_{k,m/2+r}^{(w)}, y_r) \qquad \begin{array}{l} k = 1, 2, \ldots, n \\ r = 1, 2, \ldots, m/2 \end{array} \tag{114}$$

in the $n(r-1) + k$'th row.

As before, define

$$[\alpha_q^s] = \tfrac{1}{2}\,[\alpha_q^+ + \alpha_q^-]\ [\alpha_q^a] = \tfrac{1}{2}\,[\alpha_q^+ - \alpha_q^-] \tag{115}$$

as the symmetric- and antisymmetric-component column matrices.

Then for a symmetric wing

$$[\bar{\alpha}_q^s] = [D]\,[\alpha_q^s]\,,\ [\bar{\alpha}_q^a] = [D]\,[\alpha_q^a] \tag{116}$$

where

D is a diagonal matrix of order $\frac{1}{2}mn \times \frac{1}{2}mn$ with the element

$$\exp\left\{\frac{i\omega}{V}\,x_{k,r}^{(w)}\right\} \qquad \begin{array}{l} k = 1, 2, \ldots, n \\ r = 1, 2, \ldots, m/2 \end{array} \tag{117}$$

in the $n(m/2-r) + k$'th row and column.

(91474)

The expression for $[Q_{p,q}]$ may now be written

$$[Q_{p,q}] = [f_p^s]\,[B]\left[\frac{\Lambda^{++} + \Lambda^{+-}}{2}\right]^{-1}[D]\,[\alpha_q^s] + [f_p^a]\,[B]\left[\frac{\Lambda^{++} - \Lambda^{+-}}{2}\right]^{-1}[D]\,[\alpha_q^a]. \qquad (118)$$

In flutter theory the modes of oscillation $f_p(x, y)$ are either purely symmetric or purely antisymmetric. Hence only one of the matrices $[f_p^s]$ and $[f_p^a]$ is non-null and also only one of the matrices $[\alpha_q^s]$ and $[\alpha_q^a]$ is non-null. So if p and q refer to modes which are not both symmetric or not both antisymmetric then

$$[Q_{p,q}] = 0. \qquad (119)$$

If p and q both refer to symmetric modes, then

$$Q_{p,q} = [f_p]\,[B]\left[\frac{\Lambda^{++} + \Lambda^{+-}}{2}\right]^{-1}[D]\,[\alpha_q] \qquad (120)$$

while if p and q both refer to antisymmetric modes, then

$$Q_{p,q} = [f_p]\,[B]\left[\frac{\Lambda^{++} - \Lambda^{+-}}{2}\right]^{-1}[D]\,[\alpha_q] \qquad (121)$$

where in both (120) and (121)

$$[f_p] = [f_p^+] \qquad (122)$$

and

$$[\alpha_q] = [\alpha_q^+]. \qquad (123)$$

The row matrix $[f_p]$ and the column matrix $[\alpha_q]$ are made up of numbers associated with the displacement and upwash on the starboard half of the wing in both symmetric and antisymmetric oscillations.

Equations (120) and (121) as they stand determine just one of the possible k^2 generalised airforce coefficients $Q_{p,q}$.

If the rows $[f_p]$, $p = 1, 2, \ldots, k$ are arranged consecutively beneath each other to form a matrix $[f]$ of order $k \times mn$ and if the columns $[\alpha_q]$ are arranged consecutively alongside each other to form a matrix $[\alpha]$ of order $mn \times k$ then for symmetric modes of oscillation

$$[Q] = [f]\,[B]\left[\frac{\Lambda^{++} + \Lambda^{+-}}{2}\right]^{-1}[D]\,[\alpha] \qquad (124)$$

and for antisymmetric modes of oscillation

$$[Q] = [f]\,[B]\left[\frac{\Lambda^{++} - \Lambda^{+-}}{2}\right]^{-1}[D]\,[\alpha] \qquad (125)$$

where $[Q]$ is a square matrix of order $k \times k$ with the element $Q_{p,q}$ in the p'th row and q'th column.

The matrices obtained from the products

$$[B]\left[\frac{\Lambda^{++} + \Lambda^{+-}}{2}\right]^{-1}[D] \qquad (126)$$

and

$$[B]\left[\frac{\Lambda^{++} - \Lambda^{+-}}{2}\right]^{-1}[D] \qquad (127)$$

may be respectively called the symmetric and antisymmetric influence matrices for the wing oscillating at a given frequency parameter in a flow of given main-stream Mach number.

A programme in autocode for the Ferranti Mercury Computer has been constructed which can be used to determine the matrix Q of generalised airforce coefficients given by equations (124) or (125). Details of wing geometry, number of chordwise and spanwise points, frequency parameter and Mach number of the main stream as well as the matrices $[f]$ and $[\alpha]$ are required as data by this programme.

The programme is available as programme R.A.E. 161A.

7. *Examples.*

The first example taken will be that of a symmetric tapered wing as shown in Fig. 2. The typical dimension l of the wing will be taken to be the root chord $c_0 = 2$. The wing will be assumed immersed in a subsonic flow with free-stream Mach number $M = 0 \cdot 9$ and to be oscillating with frequency parameter $\nu = 0 \cdot 6$ in one of two rigid modes of oscillation defined by

$$f_1(x, y) = 1 \tag{128}$$

and

$$f_2(x, y) = \frac{x}{l}. \tag{129}$$

The number of spanwise points across the whole span of the wing is taken to be $m = 4$ and the number of loading and of upwash points along a chord is taken to be $n = 2$.

The x-coordinates of the loading and upwash points on the starboard half of the wing are found by using the formulae (53) and (54). They are:

$$\begin{array}{ll} x_{1,1}^{(l)} = 0 \cdot 6715 & x_{1,2}^{(l)} = 0 \cdot 3745 \\ x_{2,1}^{(l)} = 1 \cdot 1255 & x_{2,2}^{(l)} = 1 \cdot 2389 \end{array} \tag{130}$$

and

$$\begin{array}{ll} x_{1,1}^{(w)} = 0 \cdot 8745 & x_{1,2}^{(w)} = 0 \cdot 7611 \\ x_{2,1}^{(w)} = 1 \cdot 3285 & x_{2,2}^{(w)} = 1 \cdot 6255. \end{array} \tag{131}$$

The matrices Λ^{++} and Λ^{+-} whose elements are defined by (74) and (75) are found to be

$$\Lambda^{++} = \begin{bmatrix} -0 \cdot 3026 & -0 \cdot 0091 & +0 \cdot 0204 & +0 \cdot 0001 \\ -0 \cdot 2848 & -0 \cdot 5001 & +0 \cdot 0441 & +0 \cdot 0602 \\ +0 \cdot 1121 & +0 \cdot 0159 & -0 \cdot 2543 & +0 \cdot 0117 \\ +0 \cdot 1367 & +0 \cdot 1081 & -0 \cdot 2356 & -0 \cdot 4223 \end{bmatrix} + i \begin{bmatrix} -0 \cdot 0298 & -0 \cdot 1021 & -0 \cdot 0196 & -0 \cdot 0201 \\ +0 \cdot 0057 & -0 \cdot 0780 & -0 \cdot 0100 & -0 \cdot 0250 \\ -0 \cdot 0493 & -0 \cdot 0659 & -0 \cdot 0147 & -0 \cdot 0504 \\ -0 \cdot 0321 & -0 \cdot 0881 & +0 \cdot 0028 & -0 \cdot 0385 \end{bmatrix} \tag{132}$$

$$\Lambda^{+-} = \begin{bmatrix} +0 \cdot 0559 & -0 \cdot 0098 & 0 & 0 \\ +0 \cdot 0879 & +0 \cdot 0920 & 0 & 0 \\ 0 & 0 & -0 \cdot 0020 & -0 \cdot 0032 \\ 0 & 0 & -0 \cdot 0010 & -0 \cdot 0032 \end{bmatrix} + i \begin{bmatrix} -0 \cdot 0432 & -0 \cdot 0394 & 0 & 0 \\ -0 \cdot 0225 & -0 \cdot 0663 & 0 & 0 \\ 0 & 0 & -0 \cdot 0015 & -0 \cdot 0001 \\ 0 & 0 & -0 \cdot 0028 & -0 \cdot 0025 \end{bmatrix}. \tag{133}$$

The modes of oscillation are symmetric modes, so the symmetric influence matrix defined by (126) is required. This is

$$[B]\left[\frac{\Lambda^{++}+\Lambda^{+-}}{2}\right]^{-1}[D]$$

$$= \begin{bmatrix} +1\cdot507 & -0\cdot102 & +0\cdot132 & -0\cdot057 \\ -0\cdot881 & +1\cdot451 & +0\cdot013 & +0\cdot208 \\ +0\cdot212 & -0\cdot039 & +0\cdot443 & -0\cdot006 \\ +0\cdot016 & +0\cdot109 & -0\cdot371 & +0\cdot429 \end{bmatrix} + i \begin{bmatrix} -0\cdot008 & -0\cdot526 & -0\cdot069 & -0\cdot142 \\ +0\cdot469 & -0\cdot080 & +0\cdot085 & -0\cdot088 \\ -0\cdot074 & -0\cdot149 & +0\cdot026 & -0\cdot077 \\ +0\cdot070 & -0\cdot057 & +0\cdot053 & +0\cdot013 \end{bmatrix} \quad (134)$$

To obtain the generalised airforce coefficients, matrices $[f]$ of the displacement at the loading points and $[\alpha]$ of the upwash at the upwash points are used. These are

$$[f] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0\cdot1873 & 0\cdot6195 & 0\cdot3358 & 0\cdot5627 \end{bmatrix} \quad (135)$$

and

$$[\alpha] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} + 0\cdot6i \begin{bmatrix} 1 & 0\cdot3806 \\ 1 & 0\cdot8127 \\ 1 & 0\cdot4373 \\ 1 & 0\cdot6642 \end{bmatrix} \quad (136)$$

The matrix of the generalised airforce coefficients obtained using equation (24) is then

$$[Q] = \begin{bmatrix} -0\cdot336 & -3\cdot450 \\ +0\cdot031 & -1\cdot137 \end{bmatrix} + 0\cdot6i \begin{bmatrix} -3\cdot064 & -1\cdot022 \\ -1\cdot075 & -0\cdot995 \end{bmatrix} \quad (137)$$

The values of the elements in (137) cannot be expected to be really good approximations to the generalised airforce coefficients since the number of spanwise and chordwise points is rather small. A better matrix of values is obtained by using $m = 8$ and $n = 2$. The matrix is then

$$[Q] = \begin{bmatrix} -0\cdot331 & -3\cdot446 \\ +0\cdot042 & -1\cdot145 \end{bmatrix} + 0\cdot6i \begin{bmatrix} -3\cdot068 & -1\cdot061 \\ -1\cdot087 & -1\cdot060 \end{bmatrix} \quad (138)$$

and these values are not far removed from the values (137).

Intermediate results have been given in the above example to illustrate some of the matrices obtained in the procedure. In the next example only the final result will be given.

The second and final example is that of a circular wing oscillating at very low frequency in incompressible flow. The reason for choosing this example is that van Spiegel[15] has also obtained values for airforces on a circular wing oscillating at very low frequency in incompressible flow. van Spiegel's method[15] is completely different from that of the present paper in that he solves directly the differential equation governing the flow about the wing, and so it is of interest to compare the two sets of results obtained.

The quantities $Q_{p,q}'$ and $Q_{p,q}''$ defined in equation (94) are now for the limiting case of vanishingly small frequency parameter.

The typical dimension l of the wing will be taken to be the radius of the circle.

First, the symmetrical modes of oscillation defined by

$$f_1(x, y) = 1 \tag{139}$$

$$f_2(x, y) = \frac{x}{l} \tag{140}$$

$$f_3(x, y) = \frac{x^2}{l^2} \tag{141}$$

$$f_4(x, y) = \frac{y^2}{l^2} \tag{142}$$

are considered.

The values of the quantities $Q_{p,q}'$ and $Q_{p,q}''$ obtained from the method of the present paper will be given with the corresponding values obtained from van Spiegel's method[15] written in brackets beneath them.

The values obtained from the method of the present paper were obtained by Woodcock with $m = 12$ spanwise points and $n = 4$ chordwise points using programme R.A.E.161A and are reported in full by him in Ref. 16 together with a collection of other results he obtained using this programme.

Van Spiegel[15] did not determine $Q_{p,q}'$ and $Q_{p,q}''$ for all combinations of $p = 1, 2, 3, 4$ and $q = 1, 2, 3, 4$. Here only those values which were also determined by van Spiegel are given. They are

$$\left.\begin{array}{llll} Q_{1,1}' = 0 & Q_{1,2}' = -2\cdot820 & Q_{1,3}' = -2\cdot963 & Q_{1,4}' = 0 \\ \quad (0) & \quad (-2\cdot812) & \quad (-2\cdot931) & \quad (0) \\ Q_{2,1}' = 0 & Q_{2,2}' = 1\cdot485 & Q_{2,3}' = -1\cdot398 & Q_{2,4}' = 0 \\ \quad (0) & \quad (1\cdot465) & \quad (-1\cdot379) & \quad (0) \end{array}\right\} \tag{143}$$

$$\left.\begin{array}{llll} Q_{1,1}'' = -2\cdot820 & Q_{1,2}'' = -3\cdot822 & Q_{1,3}'' = -0\cdot822 & Q_{1,4}'' = -0\cdot698 \\ \quad (-2\cdot812) & \quad (-3\cdot766) & \quad (-0\cdot809) & \quad (-0\cdot695) \\ Q_{2,1}'' = 1\cdot485 & Q_{2,2}'' = -0\cdot885 & Q_{2,3}'' = -0\cdot945 & Q_{2,4}'' = 0\cdot309 \\ \quad (1\cdot465) & \quad (-0\cdot847) & \quad (-0\cdot935) & \quad (0\cdot302) \end{array}\right\} \tag{144}$$

Secondly the antisymmetrical modes of oscillation defined by

$$f_5(x, y) = \frac{y}{l} \tag{145}$$

$$f_6(x, y) = \frac{xy}{l^2} \tag{146}$$

are considered. The values of $Q_{p,q}'$ and $Q_{p,q}''$ in this case are

$$\left.\begin{array}{llll} Q_{5,5}' = 0 & Q_{5,6}' = -0\cdot386 & Q_{6,5}' = 0 & Q_{6,6}' = 0\cdot186 \\ \quad (0) & \quad (-0\cdot385) & \quad (0) & \quad (0\cdot181) \\ Q_{5,5}'' = -0\cdot386 & Q_{5,6}'' = -0\cdot550 & Q_{6,5}'' = 0\cdot186 & Q_{6,6}'' = -0\cdot100 \\ \quad (-0\cdot385) & \quad (-0\cdot527) & \quad (0\cdot181) & \quad (-0\cdot102) \end{array}\right\} \tag{147}$$

The two sets of values are seen to be in good agreement in all cases.

The values given in van Spiegel's report[15] are those given here divided by π. The value of $Q_{5,6}''$ given by van Spiegel (in a different notation in his paper) is incorrect. Woodcock[16] discovered a wrong sign in formula (3.4.13), page 102 of Ref. 15, and when this sign was corrected the above result was obtained. The values $Q_{6,5}'$, $Q_{6,5}''$, $Q_{6,6}'$ and $Q_{6,6}''$ are not given at all by van Spiegel but their values have been obtained by Woodcock from intermediate results given by van Spiegel.

8. *The Treatment of Control Surfaces.*

If there are control surfaces on the wing then modes of displacement are possible in which the displacement function $f_p(x, y)$ of equation (82) is not smooth across all the inboard edges of the control surfaces (i.e. edges of the control surface which are not outside edges of the planform). Such modes of displacement will be called control-surface modes. If the mode q is a control-surface mode then $\alpha_q(x, y)$ as given by equation (112) is not smooth near the inboard edges of the control surfaces and this in turn leads to a corresponding reduced loading distribution $\lambda_q(x, y)$ with singularities at these inboard edges. An approximation of the form (52) for $\bar{\lambda}_q(\xi_0, \eta_0)$ is then not valid and it should be replaced by an approximation which takes into account the singularities at the inboard edges of the control surfaces. The subsequent procedure would be considerably more complicated than that which has been described already in the present paper, and as far as the writer is aware this has not been carried through to a successful conclusion. For a control-surface mode $f_p(x, y)$ the approximation to (97) by a low-order polynomial would not be permissible and the evaluation of the integrals in (95) would have to be carried out by a different procedure from that described in Section 6.

The procedure described below for obtaining the generalised airforce coefficients $Q_{p,q}$ is valid only when p and q do not both refer to control-surface modes at the same time. This procedure is related to that of Richardson[4].

Even though the procedure is not justified when both p and q refer to control-surface modes it may be used to provide estimates of the coefficients $Q_{p,q}$ in these cases until a proper procedure is available.

If the mode q is not a control-surface mode, then it may be expected that the corresponding reduced loading function $\lambda_q(x_0, y_0)$ can be represented with good accuracy by the double series of a finite number of terms

$$\lambda_q(x_0, y_0) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_q(x_{i,j}^{(l)}, y_j) h_i^{(n)}(\xi_0) g_j^{(m)}(\eta_0) . \tag{148}$$

This equation is quite similar to equation (52). The difference is that a factor $e^{i\omega x/V}$ was introduced by equation (9) to modify $\lambda(x, y)$ so as to simplify the integral equation, whereas here no such factor has been introduced to modify the reduced loading in equation (148) since the analysis in simpler without it.

A form of equation completely equivalent to (148) is

$$\lambda_q(x_0, y_0) = \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} a_{rs} l_r(\xi_0) \gamma_s(\eta_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} \sqrt{(1-\eta_0^2)} \tag{149}$$

where the coefficients a_{rs} are constants. The generalised force coefficient to be evaluated is

$$\begin{aligned} Q_{p,q} &= \frac{1}{sl} \int\int_S f_p(x_0, y_0) \lambda_q(x_0, y_0) dx_0 dy_0 \\ &= \int_{-1}^{+1} \frac{c(y_0)}{l} d\eta_0 \int_0^1 f_p(x_0, y_0) \lambda_q(x_0, y_0) d\xi_0 \end{aligned} \tag{150}$$

and if the series (149) is substituted into this it becomes

$$Q_{p,q} = \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} a_{rs} b_{rs} \tag{151}$$

where

$$b_{rs} = \int_{-1}^{+1} \frac{c(y_0)}{l} \gamma_s(\eta_0) \sqrt{(1-\eta_0^2)} d\eta_0 \int_0^1 l_r(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} f_p(x_0, y_0) d\xi_0 . \tag{152}$$

The constants b_{rs} can be determined, even when p refers to a control-surface mode, by numerical integration procedures. The generalised airforce coefficient $Q_{p,q}$ can then be determined from equation (151) provided the mode q is not a control-surface mode.

It is possible to write the equation for the coefficient $Q_{p,q}$ in the form of equation (102) provided the values of $f_p(x_{i,j}^{(l)}, y_j)$ in that equation are replaced by equivalent values as described below.

Define the equivalent function $\bar{f}_p^{(e)}(\xi_0, \eta_0)$ by

$$\bar{f}_p^{(e)}(\xi_0, \eta_0) = \frac{l}{c(y_0)} \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} b_{rs} l_r(\xi_0) \gamma_s(\eta_0) . \tag{153}$$

Then using the orthogonal properties (24) and (41), we obtain

$$b_{rs} = \int_{-1}^{+1} \frac{c(y_0)}{l} \gamma_s(\eta_0) \sqrt{(1-\eta_0^2)} d\eta_0 \int_0^1 \bar{f}_p^{(e)}(\xi_0, \eta_0) l_r(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} d\xi_0 \tag{154}$$

so that

$$Q_{p,q} = \sum_{r=0}^{n=1} \sum_{s=0}^{m-1} a_{rs} \int_{-1}^{+1} \frac{c(y_0)}{l} \gamma_s(\eta_0) \sqrt{(1-\eta_0^2)} d\eta_0 \int_0^1 \bar{f}_p^{(e)}(\xi_0, \eta_0) l_r(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} d\xi_0$$

$$= \int_{-1}^{+1} \frac{c(y_0)}{l} d\eta_0 \int_0^1 \bar{f}_p^{(e)}(\xi_0, \eta_0) \lambda_q(x_0, y_0) d\xi_0 \tag{155}$$

and then using (148)

$$Q_{p,q} = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_q(x_{i,j}^{(l)}, y_j) \int_{-1}^{+1} \frac{c(y_0)}{l} g_j(\eta_0) d\eta_0 \int_0^1 \bar{f}_p^{(e)}(\xi_0, \eta_0) h_i^{(n)}(\xi_0) d\xi_0 . \tag{156}$$

Now from (153) we see that

$$\frac{c(y_0)}{l} \bar{f}_p^{(e)}(\xi_0, \eta_0) \tag{157}$$

is a double polynomial of the n'th degree in ξ_0 and the m'th degree in η_0, so using the properties (98) and (99) we can replace equation (156) by

$$Q_{p,q} = \sum_{i=1}^{n} \sum_{j=1}^{m} H_i^{(n)} G_j^{(m)} \frac{c(y_j)}{l} \bar{f}_p^{(e)}(\xi_i^{(l)}, \eta_j) \lambda_q(\xi_{i,j}^{(l)}, y_j)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} H_i^{(n)} G_j^{(m)} \frac{c(y_j)}{l} \exp\left(-\frac{i\omega}{V} x_{i,j}^{(l)}\right) \bar{f}_p^{(e)}(\xi_i^{(l)}, \eta_j) \bar{\lambda}_q(\xi_i^{(l)}, \eta_j) \tag{158}$$

which is analagous to equation (102), with the values $f_p(x_{i,j}^{(l)}, y_j)$ replaced by the equivalent values $\bar{f}_p^{(e)}(\xi_i^{(l)}, \eta_j)$.

Now from (153) and (152) we have

$$\bar{f}_p^{(e)}(\xi_i^{(l)}, \eta_j) = \frac{l}{c(y_j)} \sum_{r=0}^{n-1} \sum_{s=0}^{m-1} l_r(\xi_i^{(l)}) \gamma_s(\eta_j) \int_{-1}^{+1} \frac{c(y_0)}{l} \gamma_s(\eta_0) \sqrt{(1-\eta_0^2)} d\eta_0 \times$$

$$\times \int_0^1 f_p(x_0, y_0) l_r(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} d\xi_0$$

$$= \frac{l}{c(y_j)} \int_{-1}^{+1} \frac{c(y_0)}{l} \sum_{s=0}^{m-1} \gamma_s(\eta_j) \gamma_s(\eta_0) \sqrt{(1-\eta_0^2)} d\eta_0 \times$$

$$\times \int_0^1 f_p(x_0, y_0) \sum_{r=0}^{n-1} l_r(\xi_i^{(l)}) l_r(\xi_0) \sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} d\xi_0$$

$$= \frac{1}{H_i^{(n)} G_j^{(m)}} \frac{l}{c(y_j)} \int_{-1}^{+1} \frac{c(y_0)}{l} g_j^{(m)}(\eta_0) d\eta_0 \int_0^1 f_p(x_0, y_0) h_i^{(n)}(\xi_0) d\xi_0 \qquad (159)$$

on using results from Appendix III. The double integral in (159) is to be evaluated numerically, taking into account the discontinuities of $f_p(x_0, y_0)$ and its derivatives at the inboard edges of the control surfaces. The equivalent values $\bar{f}_p^{(e)}(\xi_i^{(l)}, \eta_j)$ are then obtained from (159) rather than from (153) and (152).

If the function

$$c(y_0) f_p(x_0, y_0) \qquad (160)$$

can be represented with good accuracy by a polynomial of degree not greater than n in ξ_0 and not greater than m in η_0 then using the properties (98) and (99) in (159) leads to

$$\bar{f}_p^{(e)}(\xi_i^{(l)}, \eta_j) = f_p(x_{i,j}^{(l)}, \eta_j) \qquad (161)$$

so that in this case the equivalent values are equal to the actual values of the displacement function at the loading points.

If the mode q is a control-surface mode then the reduced upwash distribution $\alpha_q(x, y)$ of equation (112) is not smooth at the inboard edges of the control surface and it cannot be expected that equation (148) will hold with good accuracy. The value of the generalised airforce coefficient $Q_{p,q}$ can, however, still be obtained with good accuracy provided that the mode p is not a control-surface mode. This is justified with the aid of the reverse-flow theorem[17].

If the reduced loading function $\lambda_q(x, y)$ in the direct flow corresponds to the reduced upwash function $\alpha_q(x, y)$ and if the reduced loading function $\zeta_p(x, y)$ in the reverse flow corresponds to the reduced upwash function $f_p(x, y)$, then according to the reverse-flow theorem

$$Q_{p,q} = \iint_S f_p(x_0, y_0) \lambda_q(x_0, y_0) dx_0 dy_0 = \iint_S \alpha_q(x_0, y_0) \zeta_p(x_0, y_0) dx_0 dy_0 . \qquad (162)$$

Since $f_p(x, y)$ is now assumed to be smooth, the reduced loading $\zeta_p(x, y)$ can be represented by a series of the form (148) with the difference that the singularities at the leading and trailing edges of the wing are reversed. This series is

$$\zeta_p(x_0, y_0) = \sum_{k=1}^{n} \sum_{r=1}^{m} \zeta_p(x_{k,r}^{(w)}, y_j) \, h_i^{(n)}(1-\xi_0) g_j^{(m)}(\eta_0) \qquad (163)$$

for according to (34) the loading points in the reverse flow become the upwash points in the forward flow.

The integral on the right of equation (162) may then be replaced by the integral

$$Q_{p,q} = \iint_S \bar{\alpha}_q^{(e)}(\xi_0, \eta_0)\zeta_p(x_0, y_0)dx_0dy_0 \tag{164}$$

where $\bar{\alpha}_q^{(e)}(\xi_0, \eta_0)$ is an equivalent upwash function obtained by repeating the process by which $\bar{f}_p^{(e)}(\xi_0, \eta_0)$ of equation (153) was obtained, only with ξ_0 replaced by $1 - \xi_0$.

Applying the reverse-flow theorem to (164) then leads to

$$Q_{p,q} = \iint_S \lambda_q^{(e)}(x_0, y_0)f_p(x_0, y_0)dx_0dy_0 \tag{165}$$

where $\lambda_q^{(e)}(x_0, y_0)$ is the loading corresponding to the equivalent upwash function $\bar{\alpha}_q^{(e)}(\xi_0, \eta_0)$ in the direct flow.

To obtain $\lambda_q^{(e)}(x_0, y_0)$ by the procedures of the present paper the values of $\bar{\alpha}_q(\xi_k^{(w)}, \eta_r)$ are required on the starboard half of the wing. These are obtained by analogy from equation (159) on replacing ξ_0 and $\xi_i^{(l)}$ by $1 - \xi_0$ and $1 - \xi_i^{(l)}$ respectively and then taking

$$\xi_k^{(w)} = 1 - \xi_i^{(l)} \tag{166}$$

where, from (35)

$$k = n - i + 1. \tag{167}$$

These values are given by

$$\bar{\alpha}_q^{(e)}(\xi_k^{(w)}, \eta_r) = \frac{1}{H_{n-k+1}^{(n)}G_r^{(m)}} \frac{l}{c(y_r)} \int_{-1}^{+1} \frac{c(y_0)}{l} g_r^{(m)}(\eta_0)d\eta_0 \int_0^1 \alpha_q(x_0, y_0)h_{n-k+1}^{(n)}(1-\xi_0)d\xi_0 . \tag{168}$$

The double integral in (168) is to be evaluated numerically, taking into account the discontinuities of $\alpha_q(x_0, y_0)$ and its derivatives at the inboard edges of the control surfaces, and then the equivalent values $\bar{\alpha}_q^{(e)}(\xi_k^{(w)}, \eta_r)$ are determined.

If the function

$$c(y_0)\alpha_q(x_0, y_0) \tag{169}$$

can be represented with good accuracy by a polynomial of degree not greater than n in ξ_0 and not greater than m in η_0 then using the properties (98) and (99) in (168) leads to

$$\bar{\alpha}_q^{(e)}(\xi_k^{(w)}, \eta_r) = \alpha_q(x_{k,r}^{(w)}, y_r) \tag{170}$$

so that in this case the equivalent values are equal to the actual values of the upwash at the upwash points.

If the leading edge of a control surface does not coincide with the hinge then the displacement function corresponding to control rotation will have a finite step jump at the leading edge and consequently the upwash will have the behaviour of a dirac delta function at the control leading edge. The integral (168) can still be evaluated and no difficulty occurs.

Hence $Q_{p,q}$ can be determined when at most one of p and q refers to a control-surface mode provided equivalent values of displacement or upwash are used in equation (102).

9. *Conclusions.*

A lifting-surface theory has been described for determining generalised airforces on a flat-plate wing oscillating harmonically in a subsonic stream. The calculations are long but straightforward and are best carried out on an electronic digital computer. For this purpose programme R.A.E.161A has been constructed. Also programme R.A.E.263A has been constructed for obtaining loading distributions on a wing.

Only two examples have been given. The first is put in to illustrate some of the numbers occurring during the course of a calculation and is concerned with a symmetric tapered wing. The second example gives results for the generalised airforces on a circular wing, and these results are seen to compare favourably with results obtained by a completely different method by van Spiegel[15].

No other examples are given. Systematic application of programme R.A.E.161A to a selection of wings of different shapes using different numbers of chordwise and spanwise points and with different frequency parameters and flow Mach numbers would provide results which should indicate how many chordwise and spanwise points are to be taken in the case of any particular wing to obtain reliable results. The results of Woodcock[16] obtained using programme R.A.E.161A are a selection of such results.

SYMBOLS

a		Speed of sound in the undisturbed main stream
$b_p(t)$		Generalised coordinate
B		Matrix, the elements of which are defined in equation (106)
$c(y)$		Wing chord at spanwise station y
D		Matrix whose elements are defined in equation (117)
$E_{s,i}^{(n)}(\eta, \xi)$, $F_{s,i}^{(n)}(\eta, \xi)$		Defined in equation (58)
$f_p(x, y)$		Shape of p'th mode of oscillation
$\bar{f}_p^{(e)}(\xi_0, \eta_0)$		A function equivalent to $f_p(x_0, y_0)$ and defined by equation (153)
$g(x, y)$		Mode shape, defined in equation (1)
$g_j^{(m)}(\eta_0)$		Interpolation functions, defined in equation (50), $j = 1, 2, \ldots, m$
$G_j^{(m)}$	=	$\int_{-1}^{+1} g_j^{(m)}(\eta_0) d\eta_0$
$h_i^{(n)}(\xi_0)$		Interpolation functions, defined in equation (39), $i = 1, 2, \ldots, n$
$h_i^{(1,n)}(\xi_0)$		Defined in equation (57), $i = 1, 2, \ldots, n$
$H_i^{(n)}$	=	$\int_0^1 h_i^{(n)}(\xi_0) d\xi_0$
$I_i^{(n)}(\eta, \eta_0, \xi)$		Defined in equation (56)
$K(x, y)$		Kernel function defined in equation (6)
$\hat{K}(x, y)$		Modified kernel function, defined in equation (10)
$\hat{K}_1(x, y)$, $\hat{K}_2(x, y)$		Constituents of $\hat{K}_1(x, y)$ defined by equations (17) and (18)
$\hat{K}_3(x, y)$		Defined in equation (21)
l		Typical dimension of the wing
$l(x, y)$		Loading function
$l_s(\xi_0)$		Polynomial of degree r satisfying equation (24)
$L(x, y, t)$		Upward lifting force per unit area at time t at a point (x, y) on the wing
m		Number of spanwise points, taken to be an even number for most of this paper
M	=	V/a, Mach number of the main stream

$\bar{\alpha}^+$, etc.		Matrices defined from equation (70) onwards
$\gamma_r(\eta_0)$		Polynomial of degree r satisfying equation (41)
η_j		Spanwise points, defined in equation (49), $j = 1, 2, \ldots, m$
$\lambda(x, y)$		Reduced loading function, defined in equation (4)
$\hat{\lambda}(x, y)$		Modified reduced loading function, defined in equation (9)
$\bar{\lambda}(\xi_0, \eta_0)$	=	$\hat{\lambda}(x_0, y_0)$
$\lambda_q(x, y)$		Reduced loading function due to oscillation in mode q
$\bar{\lambda}^+$, etc.		Matrices defined from equation (70) onwards
Λ^{++}, etc.		Submatrix elements appearing in equation (69)
μ		Strength of doublet distribution
ν	=	$\dfrac{\omega l}{V}$, frequency parameter
ξ, η, ξ_0, η_0		Transformed variables, defined in equation (12)
$\xi_i^{(l)}$		Chordwise loading points, defined in equation (38), $i = 1, 2, \ldots, n$
$\xi_k^{(w)}$		Chordwise upwash points, defined in equations (34) and (35), $k = 1, 2, \ldots, n$
ρ_0		Free-stream density
ϕ		Velocity potential
ω		Circular frequency

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APPENDIX I

Derivation of the Integral Equation

Besides the system (x, y, z) of right-handed Cartesian coordinates introduced in Section 2 of the main text, another system (X, Y, Z) of right-handed Cartesian coordinates is introduced which is stationary with respect to the main-stream flow and which coincides with the system (x, y, z) at time $t = 0$. Then at time t the following relationships exist between the coordinates of a point in the two systems

$$\left.\begin{aligned} x &= X + Vt \\ y &= Y \\ z &= Z. \end{aligned}\right\} \tag{171}$$

If the flow of air about the wing is assumed to be irrotational then a velocity potential ϕ exists such that the velocity $\mathbf{q}$ of a fluid particle relative to the (X, Y, Z) coordinate system is given by

$$\mathbf{q} = \mathbf{i}\frac{\partial\phi}{\partial X} + \mathbf{j}\frac{\partial\phi}{\partial Y} + \mathbf{k}\frac{\partial\phi}{\partial Z} \tag{172}$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors directed along the X, Y, and Z axes respectively.

With the usual assumptions of linearised theory it is found from Euler's equation of motion of inviscid flow, the continuity equation, and the adiabatic equation of state that ϕ satisfies the wave equation

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2}\right)\phi = \frac{1}{a^2}\frac{\partial^2\phi}{\partial t^2}. \tag{173}$$

The airflow must be tangential to the surface of the wing and this leads to a boundary condition. Within the accuracy of linearised theory it is permissible to apply this condition at the mean position of the wing in the plane $Z = 0$ rather than on its surfaces. The condition may then be written

$$\left(\frac{\partial\phi}{\partial Z}\right)_{Z=0} = w(x, y, t) \tag{174}$$

over the area of the wing in its mean position, where

$$w(x, y, t) = \left(V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) Z(x, y, t) \tag{175}$$

and $Z(x, y, t)$ is the vertical displacement of the point x, y on the wing at time t.

The function

$$\phi_1(X, Y, Z, t) = -\frac{1}{4\pi}\frac{\partial}{\partial Z}\left\{\frac{\mu(t-r/a)}{r}\right\} \tag{176}$$

where

$$r = \sqrt{\{(X-X_0)^2 + (Y-Y_0)^2 + (Z-Z_0)^2\}} \tag{177}$$

and $\mu(t)$ is an arbitrary differentiable function, satisfies the wave equation (173) and corresponds to the potential about a doublet of strength $\mu(t)$ at time t placed at the fixed point (X_0, Y_0, Z_0) and orientated in the positive direction of Z.

If there is a doublet layer on the $Z_0 = 0$ plane then the potential of the flow about this layer, by the principle of superposition, is

$$\phi(X, Y, Z, t) = -\frac{1}{4\pi}\iint_{Z_0=0\text{ plane}}\left[\frac{\partial}{\partial Z}\left\{\frac{\mu(X_0, Y_0, t-r/a)}{r}\right\}\right]_{Z_0=0} dX_0 dY_0. \tag{178}$$

As is usual with doublet layers, there is a discontinuity of potential across the layer. In Appendix II it is proved that the discontinuity in potential across a layer at any point on it is of amount equal to the strength of the layer at that point, so that

$$\phi(X_0, Y_0, +0, t) - \phi(X_0, Y_0, -0, t) = \mu(X_0, Y_0, t). \tag{179}$$

In linearised theory the wake shed by the wing from its trailing edge is plane and parallel to the main-stream flow. The wing and its wake will be replaced by a doublet sheet and the strength of the sheet will be adjusted so that the boundary condition (175) is satisfied on the wing and so that no loading is sustained by the wake. The flow about this system will then be exactly the same as that about the wing and its wake and the airforces on the doublet sheet will be equal to those on the wing.

The linearised Bernoulli equation is

$$\frac{\partial\phi}{\partial t} = -\frac{(p-p_0)}{\rho_0} \tag{180}$$

where p is the pressure at a point in the flow, p_0 is the free-stream pressure and ρ_0 is the free-stream density.

Let

$$\begin{aligned} L(X_0, Y_0, t) &= p(X_0, Y_0, -0, t) - p(X_0, Y_0, +0, t) \\ &= \rho_0 \frac{\partial}{\partial t}[\phi(X_0, Y_0, +0, t) - \phi(X_0, Y_0, -0, t)] \\ &= \rho_0 \frac{\partial}{\partial t}[\mu(X_0, Y_0, t)] \end{aligned} \tag{181}$$

be the upward force per unit area, or loading, on the wing.

The equations of the leading and trailing edges of the wing are respectively

$$x = x_L(y), \qquad z = 0; \tag{182}$$

and

$$x = x_T(y), \qquad z = 0. \tag{183}$$

Since the two systems of axes coincide at time $t = 0$, the point $(X_0, Y_0, 0)$ is on the leading edge of the wing at time

$$t_0 = \frac{x_L(Y_0) - X_0}{V} \tag{184}$$

provided

$$|Y_0| < s. \tag{185}$$

Before time t_0 the strength of the doublet layer at the point $(X_0, Y_0, 0)$ is zero for the wing has not yet reached it.

Integrating equation (181) and making use of this last observation leads to

$$\mu(X_0, Y_0, t) = \frac{1}{\rho_0}\int_{\{x_L(Y_0)-X_0\}/V}^{t} L(X_0, Y_0, u)du. \tag{186}$$

On making the change of variables

$$\chi_0 = X_0 + Vu \tag{187}$$

in the integral in (186) we obtain

$$\mu(X_0, Y_0, t) = \frac{1}{\rho_0 V}\int_{x_L(Y_0)}^{X_0+Vt} L\left\{X_0, Y_0, \frac{\chi_0 - X_0}{V}\right\} d\chi_0. \tag{188}$$

The velocity potential, due the presence of the doublet layer, is then according to equation (178)

$$\phi = -\frac{1}{4\pi\rho_0 V}\frac{\partial}{\partial Z}\iint_{\text{wing and wake}} dX_0 dY_0 \left[\frac{1}{r}\int_{x_L(Y_0)}^{X_0+V(t-r/a)} L\left\{X_0, Y_0, \frac{\chi_0 - X_0}{V}\right\} d\chi_0\right]_{Z_0=0}. \tag{189}$$

If

$$l(x_0, y_0, t) = L(X_0, Y_0, t) \tag{190}$$

where x_0, y_0 in the (x, y, z) coordinate system corresponds to X_0, Y_0 in the (X, Y, Z) coordinate system, then $l(x_0, y_0, t)$ is the loading distribution on the wing and wake as a function of the coordinates fixed relative to the mean position of the wing, and it is non zero only on the wing.

Since

$$\begin{aligned} L\left\{X_0, Y_0, \frac{\chi_0 - X_0}{V}\right\} &= l\left\{X_0 + V\left(\frac{\chi_0 - X_0}{V}\right), Y_0, \frac{\chi_0 - X_0}{V}\right\} \\ &= l\left\{\chi_0, Y_0, \frac{\chi_0 - x_0}{V} + t\right\} \end{aligned} \tag{191}$$

the expression (189) for ϕ becomes

$$\phi(x, y, z, t) = -\frac{1}{4\pi\rho_0 V}\frac{\partial}{\partial z}\int_{-s}^{+s} dy_0 \int_{x_L(y_0)}^{\infty}\frac{dx_0}{[r]}\int_{x_L(y_0)}^{x_0 - M[r]} l\left\{\chi_0, y_0, \frac{\chi_0 - x_0}{V} + t\right\} d\chi_0 \tag{192}$$

where

$$[r] = \sqrt{\{(x-x_0)^2 + (y-y_0)^2 + z^2\}} \tag{193}$$

$$M = \frac{V}{a}. \tag{194}$$

The order of integration of the inner two integrals in (192) is to be changed. Relevant areas of integration of the corresponding double integral are shown shaded in Figs. 3, 4 and 5 for the three cases (i) $M < 1$, (ii) $M > 1$ and (iii) $M = 1$.

The curved boundary of the areas has the equation

$$\begin{aligned} \chi_0 &= x_0 - M[r] \\ &= x_0 - M\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + z^2\}}. \end{aligned} \tag{195}$$

To invert the order of integration the points of intersection of this curve with the straight line

$$\chi_0 = \text{const.} \tag{196}$$

must be obtained. These can be obtained by solving equation (195) for x_0 in terms of χ_0. The result is

$$x_0 = x + \frac{1}{(1-M^2)}\{\chi_0 - x \pm M\sqrt{[(\chi_0 - x)^2 + (1-M^2)\{(y-y_0)^2 + z^2\}]}\}. \tag{197}$$

Since, according to (195)

$$x_0 - \chi_0 > 0\,, \tag{198}$$

only the positive sign of the square root in (197) is applicable when $M < 1$, and so the boundary curve is intersected in only one point by the straight line (196).

If $M > 1$, then from (195)

$$\begin{aligned} x - \chi_0 &= (x - x_0) + M\sqrt{\{(x-x_0)^2 + (y-y_0)^2 + z^2\}} \\ &> 0 \end{aligned} \tag{199}$$

and so both signs are applicable in (197) provided

$$x - \chi_0 > \sqrt{[(M^2-1)\{(y-y_0)^2 + z^2\}]} \tag{200}$$

while there in no intersection at all otherwise.

If $M \to 1+0$ the second point of intersection moves to infinity and there is then a point of intersection only if

$$x - X_0 > 0\,. \tag{201}$$

If, for the present, the case $M < 1$ is considered, then changing the order of the inner two integrals in (192) yields

$$\begin{aligned} \phi(x, y, z, t) = &-\frac{1}{4\pi\rho_0 V}\frac{\partial}{\partial z}\int_{-s}^{+s} dy_0 \int_{x_L(y_0)}^{\infty} d\chi_0 \times \\ &\times \int_{x+\{1/(1-M^2)\}\{(\chi_0-x)+M\sqrt{[(\chi_0-x)^2+(1-M^2)\{(y-y_0)^2+z^2\}]}\}}^{\infty} l\left\{\chi_0, y_0, \frac{\chi_0 - x_0}{V} + t\right\}\frac{dx_0}{[r]} \end{aligned} \tag{202}$$

$$\begin{aligned} \phi(x, y, z, t) = &-\frac{1}{4\pi\rho_0 V}\frac{\partial}{\partial z}\int_{-s}^{+s} dy_0 \int_{x_L(y_0)}^{\infty} d\chi_0 \times \\ &\times \int_{\{M/(1-M^2)\}\{M(\chi_0-x)+\sqrt{[(\chi_0-x)^2+(1-M^2)\{(y-y_0)^2+z^2\}]}\}}^{\infty} l\left\{\chi_0, y_0, t - \frac{\sigma}{V}\right\}\frac{d\sigma}{\sqrt{\{(\sigma - x+\chi_0)^2 + (y-y_0)^2 + z^2\}}} \end{aligned} \tag{203}$$

If the wing is oscillating harmonically, then we may write

$$l(x, y, t) = l(x, y)e^{i\omega t} \tag{204}$$

where only the real or imaginary part of a complex function represents the physical quantity. So, using the fact that

$$l(x, y, t) = 0 \tag{205}$$

for

$$x > x_T(y)\,, \tag{206}$$

i.e. beyond the trailing edge, the expression for the potential becomes

$$\begin{aligned} \phi(x, y, z, t) = &-\frac{e^{i\omega t}}{4\pi\rho_0 V}\frac{\partial}{\partial z}\int_{-s}^{+s} dy_0 \int_{x_L(y_0)}^{x_T(y_0)} l(\chi_0, y_0) d\chi_0 \times \\ &\times \int_{\{M/(1-M^2)\}\{M(\chi_0-x)+\sqrt{[(\chi_0-x)^2+(1-M^2)\{(y-y_0)^2+z^2\}]}\}}^{\infty} e^{-i\omega\sigma/V}\frac{d\sigma}{\sqrt{\{(\sigma - x+\chi_0)^2 + (y-y_0)^2 + z^2\}}} \end{aligned} \tag{207}$$

The upwash due to the doublet layer is then

$$w(x, y, t) = \left(\frac{\partial \phi}{\partial z}\right)_{z=0}$$

$$= -\frac{e^{i\omega t}}{4\pi\rho_0 V} \lim_{z\to 0} \frac{\partial^2}{\partial z^2} \int_{-s}^{+s} dy_0 \int_{x_L(y_0)}^{x_T(y_0)} l(\chi_0, y_0) d\chi_0 \times$$

$$\times \int_{\{M/(1-M^2)\}\{M(\chi_0 - x) + \sqrt{[(\chi_0-x)^2 + (1-M^2)\{(y-y_0)^2+z^2\}]}\}}^{\infty} e^{-i\omega\sigma/V} \frac{d\sigma}{\sqrt{\{(\sigma - x + \chi_0)^2 + (y-y_0)^2 + z^2\}}} \tag{208}$$

Let

$$K(x, y) = \lim_{z\to 0} -\frac{\partial^2}{\partial z^2} \int_{\{M/(1-M^2)\}(-Mx+R_1)}^{\infty} e^{-i\omega\sigma/V} \frac{d\sigma}{\sqrt{\{(\sigma - x)^2 + y^2 + z^2\}}}$$

$$= \lim_{z\to 0} -e^{-i\omega x/V} \frac{\partial^2}{\partial z^2} \int_{(-x+MR_1)/(1-M^2)}^{\infty} e^{-i\omega u/V} \frac{du}{\sqrt{(u^2+y^2+z^2)}}$$

$$= \lim_{z\to 0} -e^{-i\omega x/V} \frac{\partial}{\partial z} \times$$

$$\times \left[-\int_{(-x+MR_1)/(1-M^2)}^{\infty} e^{-i\omega u/V} \frac{z\,du}{(u^2+y^2+z^2)^{3/2}} - \frac{Mz}{R_1} \frac{\exp\left\{-\frac{i\omega}{V}\left(\frac{-x+MR_1}{1-M^2}\right)\right\}}{\sqrt{\left\{\left(\frac{-x+MR_1}{1-M^2}\right)^2 + y^2 + z^2\right\}}} \right]$$

$$= \lim_{z\to 0} -e^{-i\omega x/V} \frac{\partial}{\partial z} \times$$

$$\times \left[-\int_{(-x+MR_1)/(1-M^2)}^{\infty} e^{-i\omega u/V} \frac{z\,du}{(u^2+y^2+z^2)^{3/2}} - \frac{M(Mx+R_1)z}{R_1(x^2+y^2+z^2)} \exp\left\{-\frac{i\omega}{V}\left(\frac{-x+MR_1}{1-M^2}\right)\right\} \right]$$

$$= e^{-i\omega x/V} \times$$

$$\times \left[\int_{(-x+MR)/(1-M^2)}^{\infty} e^{-i\omega u/V} \frac{du}{(u^2+y^2)^{3/2}} + \frac{M(Mx+R)}{R(x^2+y^2)} \exp\left\{-\frac{i\omega}{V}\left(\frac{-x+MR}{1-M^2}\right)\right\} \right] \tag{209}$$

where

$$R_1 = \sqrt{\{x^2 + (1-M^2)(y^2+z^2)\}} \tag{210}$$

$$R = \sqrt{\{x^2 + (1-M^2)y^2\}}. \tag{211}$$

If then the limit and differentiation are taken under the first two integral signs in (208) we obtain the equation

$$w(x, y) = \frac{1}{4\pi\rho_0 V} \iint_S l(x_0, y_0) K(x - x_0, y - y_0) dx_0 dy_0 \tag{212}$$

where X_0 has been replaced by x_0 as it is only a variable of integration, S is the wing area and $w(x, y)$ is defined by

$$w(x, y, t) = w(x, y) e^{i\omega t}. \tag{213}$$

The kernel $K(x, y)$ of the integral has a non-integrable singularity at $x = 0$, $y = 0$ and this is to be dealt with using Hadamard's 'Finite Part' method of integration.

Using equations (3) and (4) of the main text in (212) then leads to the form of integral equation given in equation (5).

If $M > 1$, the infinite upper limit in equation (202) must be replaced by

$$x - \frac{1}{(M^2-1)} \{(\chi_0 - x) - M\sqrt{[(\chi_0 - x)^2 - (M^2-1)\{(y-y_0)^2 + z^2\}]}\} \tag{214}$$

when

$$x - \chi_0 > \sqrt{[(M^2-1)\{(y-y_0)^2 + z^2\}]}, \tag{215}$$

and the whole inner integral must be replaced by zero otherwise.

The procedure following equation (202) is then followed, and it again leads to an integral equation of the form (212) with the kernel $K(x, y)$ given by

$$K(x,y) = \begin{cases} e^{-i\omega x/V}\left[\int_{(x-MR)/(M^2-1)}^{(x+MR)/(M^2-1)} e^{-i\omega u/V} \frac{du}{(u^2+y^2)^{3/2}} + \frac{M(Mx-R)}{R(x^2+y^2)} \exp\left\{-\frac{i\omega}{V}\left(\frac{x+MR}{M^2-1}\right)\right\} + \right. \\ \left. \qquad + \frac{M(Mx+R)}{R(x^2+y^2)} \exp\left\{-\frac{i\omega}{V}\left(\frac{x-MR}{M^2-1}\right)\right\}\right] & x > \sqrt{(M^2-1)}|y| \\ 0 & x < \sqrt{(M^2-1)}|y|. \end{cases} \tag{216}$$

If $M = 1$, the lower limit in equation (202) must be replaced by

$$\lim_{M\to 1-0} x + \frac{1}{(1-M^2)}\{(\chi_0 - x) + M\sqrt{[(\chi_0 - x)^2 + (1-M^2)\{(y-y_0)^2 + z^2\}]}\}$$

$$= \begin{cases} x - \frac{1}{2}(x-\chi_0) + \frac{1}{2}\frac{(y-y_0)^2 + z^2}{(x-\chi_0)}, & x_0 - \chi_0 > 0 \\ \infty, & x_0 - \chi_0 < 0. \end{cases} \tag{217}$$

The same procedure again leads to an integral equation of the form (212) with the kernel $K(x, y)$ given by

$$K(x,y) = \begin{cases} e^{-i\omega x/V}\left[\int_{-x/2+y^2/2x}^{\infty} e^{-i\omega u/V} \frac{du}{(u^2+y^2)^{3/2}} + \frac{2}{x^2+y^2}\exp\left\{\frac{i\omega}{2V}\left(x - \frac{y^2}{2x}\right)\right\}\right] & x > 0 \\ 0 & x < 0. \end{cases} \tag{218}$$

The kernel (218) would be obtained by proceeding to the limit $M = 1 - 0$ in (209) or to the limit $M = 1 + 0$ in (216).

If the change of variables

$$\lambda = \frac{\omega}{a}[-u + \sqrt{(u^2+y^2)}] \tag{219}$$

is made in the integral on the right-hand side of equation (218) then the following form of the kernel function $K(x, y)$ is obtained for the case $M = 1$

$$K(x,y) = \begin{cases} e^{-i\omega x/a}\left[\frac{2}{y^2}\exp\left\{\frac{i\omega}{2a}\left(x - \frac{y^2}{x}\right)\right\} - \frac{i}{y^2}\int_0^{\omega x/a} \exp\left\{\frac{i}{2}\left(\lambda - \frac{\omega^2 y^2}{a^2\lambda}\right)\right\} d\lambda\right] & x > 0 \\ 0 & x < 0. \end{cases} \tag{220}$$

APPENDIX II

Discontinuity across a Doublet Layer

We have to consider the behaviour of

$$\phi(X, Y, Z, t) = -\frac{1}{4\pi}\iint_{Z_0=0 \text{ plane}} \left[\frac{\partial}{\partial Z}\left\{\frac{u(X_0, Y_0, t-r/a)}{r}\right\}\right]_{Z_0=0} dX_0 dY_0 \tag{221}$$

near the plane $Z = 0$, where

$$r = \sqrt{\{(X-X_0)^2 + (Y-Y_0)^2 + (Z-Z_0)^2\}}. \tag{222}$$

The doublet strength $\mu(X_0, Y_0, t)$ is assumed to be non-zero only in a strip of finite width $-s \leqslant Y_0 \leqslant s$, for in general the integral (221) is not convergent.

The integrand in the integral (221) becomes infinite at $X_0 = X$, $Y_0 = Y$ when $Z \to 0$. However for any point X_0, Y_0 outside a finite neighbourhood of the point (X, Y) in the $Z_0 = 0$ plane the integrand remains finite as $Z \to 0$, and so the value of the integral over the area outside this finite neighbourhood is continuous in Z at $Z = 0$.

The integral

$$I(X, Y, Z, t) = -\frac{1}{4\pi}\iint_C \left[\frac{\partial}{\partial Z}\left\{\frac{\mu(X_0, Y_0, t-r/a)}{r}\right\}\right]_{Z_0=0} dX_0 dY_0 \tag{223}$$

is now considered. The point (X, Y) is taken within the doublet strip, and the finite neighbourhood C of the point (X, Y) is taken to be a circle radius R and centre (X, Y) lying entirely within the doublet strip.

Write

$$I_1(X, Y, Z, t) = -\frac{1}{4\pi}\iint_C \left[\frac{\partial}{\partial z}\left\{\frac{\mu(X, Y, t-r/a)}{r}\right\}\right]_{Z_0=0} dX_0 dY_0 \tag{224}$$

$$I_2(X, Y, Z, t) = -\frac{1}{4\pi}\iint_C \left[\frac{\partial}{\partial Z}\left\{\frac{\mu(X_0, Y_0, t-r/a)}{r} - \frac{\mu(X, Y, t-r/a)}{r}\right\}\right]_{Z_0=0} dX_0 dY_0 \tag{225}$$

then

$$I(X, Y, Z, t) = I_1(X, Y, Z, t) + I_2(X, Y, Z, t). \tag{226}$$

Introduce polar coordinates (σ, θ) by means of the equations

$$\left.\begin{aligned} X_0 - X &= \sigma\cos\theta \\ Y_0 - Y &= \sigma\cos\theta \end{aligned}\right\} \tag{227}$$

so that

$$[r]_{Z_0=0} = \sqrt{(\sigma^2 + Z^2)}. \tag{228}$$

Then

$$I_1(X, Y, Z, t) = -\frac{1}{4\pi} Z \int_0^{2\pi} d\theta \int_0^R \left[\frac{\partial}{\partial\sigma}\left\{\frac{\mu(X, Y, t-r/a)}{r}\right\}\right]_{Z_0=0} d\sigma$$

$$= -\frac{1}{2} Z \left\{\frac{\mu\left\{X, Y, t - \frac{\sqrt{(R^2+Z^2)}}{a}\right\}}{\sqrt{(R^2+Z^2)}} - \frac{\mu\left(X, Y, t - \frac{|z|}{a}\right)}{|z|}\right\}. \tag{229}$$

Thus, on making $Z \to 0$ from above and below, the equation

$$I_1(X, Y, +0, t) - I_1(X, Y, -0, t) = \mu(X, Y, t) \tag{230}$$

is obtained.

Also

$$I_2(X, Y, Z, t) = \frac{1}{4\pi} Z \int_0^{2\pi} d\theta \int_0^R \left[\frac{\mu(X_0, Y_0, t-r/a)}{r^3} - \frac{\mu(X, Y, t-r/a)}{r^3}\right]_{Z_0=0} \sigma\, d\sigma +$$

$$+ \frac{1}{4\pi} Z \int_0^{2\pi} d\theta \int_0^R \left[\frac{\mu_t(X_0, Y_0, t-r/a)}{ar^2} - \frac{\mu_t(X, Y, t-r/a)}{ar^2}\right]_{Z_0=0} \sigma\, d\sigma. \tag{231}$$

It will be assumed that $\mu(X_0, Y_0, t)$ satisfies the uniform Lipschitz condition

$$|\mu(X_0, Y_0, t) - \mu(X, Y, t)| \leqslant B\sigma \tag{232}$$

inside the circle C, where B is a constant.

The Lipschitz condition is certainly satisfied by any function $\mu(X_0, Y_0, t)$ whose derivatives with respect to X_0, Y_0 are uniformly bounded. The assumption that these derivatives exist everywhere is, however, avoided by taking the Lipschitz condition (232).

It will also be assumed that $\mu_t(X_0, Y_0, t)$ is bounded inside the circle C, i.e.

$$|\mu_t(X_0, Y_0, t)| \leqslant D$$

within and on C, where D is a constant.

So

$$\left|\int_0^{2\pi} d\theta \int_0^R \left[\frac{\mu(X_0, Y_0, t-r/a)}{r^3} - \frac{\mu(X, Y, t-r/a)}{r^3}\right]_{Z_0=0} \sigma\, d\sigma\right|$$

$$\leqslant 2\pi B \int_0^R \frac{\sigma^2 d\sigma}{(\sigma^2+Z^2)^{3/2}}$$

$$= 2\pi B\left[\log\left\{\frac{\sqrt{(R^2+Z^2)}+R}{|Z|}\right\} - \frac{R}{\sqrt{(R^2+Z^2)}} + 1\right] \tag{233}$$

and

$$\left|\int_0^{2\pi} d\theta \int_0^R \left[\frac{\mu_t(X_0, Y_0, t-r/a)}{ar^2} - \frac{\mu_t(X, Y, t-r/a)}{ar^2}\right]_{Z_0=0} \sigma\, d\sigma\right|$$

$$\leqslant \frac{4\pi D}{a} \int_0^R \frac{\sigma\, d\sigma}{(\sigma^2+Z^2)}$$

$$= \frac{2\pi D}{a} \log\left\{\frac{\sqrt{(R^2+Z^2)}}{|Z|}\right\}. \tag{234}$$

It follows, therefore, that

$$\lim_{Z\to 0} I_2(X, Y, Z, t) = 0 \tag{235}$$

so that

$$\phi(X, Y, +0, t) - \phi(X, Y, -0, t) = \mu(X, Y, t) \tag{236}$$

which is the required discontinuity relation.

If the point $(X, Y, 0)$ is outside the doublet strip all the integrands are finite and ϕ is continuous across $Z = 0$.

If the point $(X, Y, 0)$ is on the boundary of the doublet strip, then a point within the strip is first considered and the limit taken as the point on the boundary is approached.

APPENDIX III

Properties of Orthogonal Polynomials

If, in the integral relation (24) of the main text, the change of variables

$$\xi_0 = \frac{1}{2}(1-\cos\theta_0) \tag{237}$$

is made in the integrand, the relation becomes

$$\int_0^\pi l_r\left(\frac{1-\cos\theta_0}{2}\right) l_s\left(\frac{1-\cos\theta_0}{2}\right)\cos^2\frac{\theta_0}{2}\,d\theta_0 = \delta_{r,s}. \tag{238}$$

It follows that

$$l_r\left(\frac{1-\cos\theta_0}{2}\right) = \sqrt{\left(\frac{2}{\pi}\right)}\,\frac{\cos\left(r+\frac{1}{2}\right)\theta_0}{\cos\frac{1}{2}\theta_0} \tag{239}$$

for this is a polynomial of degree r in the original variable ξ_0, and (238) is satisfied.

There are just n values of $\cos\theta_0$ at which the expression (239) is zero when $r = n$ and these are

$$\cos\theta_0 = \cos\left(\frac{2i-1}{2n+1}\right)\pi \qquad i = 1, 2, \ldots, n \tag{240}$$

so that the n chordwise loading points are given by

$$\xi_i^{(l)} = \frac{1}{2} - \frac{1}{2}\cos\theta_i \qquad i = 1, 2, \ldots, n \tag{241}$$

where

$$\theta_i = \frac{2i-1}{2n+1}\pi \qquad i = 1, 2, \ldots, n. \tag{242}$$

The interpolation function defined in equation (39) of the main text may be given by the equivalent formula

$$h_i^{(n)}\left(\frac{1-\cos\theta_0}{2}\right) = \frac{\cos\left(n+\frac{1}{2}\right)\theta_0}{\sin\frac{1}{2}\theta_0}\,\frac{1}{(\cos\theta_0-\cos\theta_i)}\lim_{\theta_0\to\theta_i}\left[\frac{(\cos\theta_0-\cos\theta_i)\sin\frac{1}{2}\theta_0}{\cos\left(n+\frac{1}{2}\right)\theta_0}\right]$$

$$= \frac{2(-1)^{i+1}}{(2n+1)}\sin\theta_i\sin\frac{1}{2}\theta_i\,\frac{\cos\left(n+\frac{1}{2}\right)\theta_0}{\sin\frac{1}{2}\theta_0}\,\frac{1}{(\cos\theta_0-\cos\theta_i)}. \tag{243}$$

Then

$$H_i^{(n)} = \int_0^1 h_i^{(n)}(\xi_0)d\xi_0 = \frac{1}{2}\int_0^\pi h_i^{(n)}\left(\frac{1-\cos\theta_0}{2}\right)\sin\theta_0 d\theta_0$$

$$= 2\frac{(-1)^{i+1}}{(2n+1)}\sin\theta_i\sin\frac{1}{2}\theta_i\int_0^\pi \frac{\cos\left(n+\frac{1}{2}\right)\theta_0\cos\frac{1}{2}\theta_0}{(\cos\theta_0-\cos\theta_i)}d\theta_0$$

$$= \frac{(-1)^{i+1}}{(2n+1)}\sin\theta_i\sin\frac{1}{2}\theta_i\int_0^\pi \frac{\cos(n+1)\theta_0+\cos n\theta_0}{\cos\theta-\cos\theta_i}d\theta_0$$

$$= \frac{(-1)^{i+1}\pi}{(2n+1)}\sin\theta_i\sin\frac{1}{2}\theta_i\left[\frac{\sin(n+1)\theta_i}{\sin\theta_i}+\frac{\sin n\theta_i}{\sin\theta_i}\right]$$

$$= \frac{(-1)^{i+1}2\pi}{(2n+1)}\sin\frac{1}{2}\theta_i\sin\left(n+\frac{1}{2}\right)\theta_i\cos\frac{1}{2}\theta_i$$

$$= \frac{\pi}{(2n+1)}\sin\theta_i. \tag{244}$$

Now suppose $p_n(\xi_0)$ is a polynomial of degree not greater than the n'th in ξ_0, then

$$p_n(\xi_0) = q_n(\xi_0)(\xi_0-\xi_i^{(l)})+p_n(\xi_i^{(l)}) \tag{245}$$

where $q_n(\xi_0)$ is a polynomial of degree not greater than the $(n-1)$ st. So, using the definition of $h_i^{(n)}(\xi_0)$ given in equation (39) of the main text, we get

$$\int_0^1 p_n(\xi_0)h_i^{(n)}(\xi_0)d\xi_0 = p_n(\xi_i^{(l)})H_i^{(n)}+\int_0^1 q_n(\xi_0)(\xi_0-\xi_i^{(l)})h_i^{(n)}(\xi_0)d\xi_0$$

$$= p_n(\xi_i^{(l)})H_i^{(n)}+\frac{1}{l_n'(\xi_i^{(l)})\sqrt{\left(\frac{1-\xi_i^{(l)}}{\xi_i^{(l)}}\right)}}\int_0^1 q_n(\xi_0)l_n(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}d\xi_0$$

$$= p_n(\xi_i^{(l)})H_i^{(n)} \tag{246}$$

since

$$\int_0^1 q_n(\xi_0)l_n(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)}d\xi_0 = 0 \tag{247}$$

for $q_n(\xi_0)$, being a polynomial of at most degree $n-1$, can be expressed as a linear combination of polynomials $l_r(\xi_0)$ with $r \leqslant n-1$ and then relation (24) applied to each term gives the result.

Now $h_i^{(n)}(\xi_0)$ can be given by the sum

$$h_i^{(n)}(\xi_0) = \sum_{r=0}^{n-1} d_r l_r(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} \tag{248}$$

where the coefficients d_r are found by multiplying equation (248) by $l_s(\xi)$, integrating between 0 and 1, and applying the relation (24) and (246). Doing this leads to

$$h_i^{(n)}(\xi_0) = H_i^{(n)}\sum_{r=0}^{n-1} l_r(\xi_i^{(l)})l_r(\xi_0)\sqrt{\left(\frac{1-\xi_0}{\xi_0}\right)} \tag{249}$$

or

$$h_i^{(n)}\left(\frac{1-\cos\theta_0}{2}\right) = \frac{4}{(2n+1)}\frac{\sin\frac{1}{2}\theta_i}{\sin\frac{1}{2}\theta_0}\sum_{r=0}^{n-1}\cos(r+\tfrac{1}{2})\theta_i\cos(r+\tfrac{1}{2})\theta_0. \tag{250}$$

If in the integral relation (41) of the main text the change of variables

$$\eta_0 = \cos \phi_0 \tag{251}$$

is made in the integrand, it becomes

$$\int_0^\pi \gamma_r(\cos \phi_0)\gamma_s(\cos \phi_0) \sin^2 \phi_0 d\phi_0 = \delta_{r,s}. \tag{252}$$

It follows that

$$\gamma_r(\cos \phi_0) = \sqrt{\left(\frac{2}{\pi}\right)} \frac{\sin [(r+1)\theta_0]}{\sin \phi_0} \tag{253}$$

for this is a polynomial of degree r in the original variable η_0, and (252) is satisfied. There are just m values of $\cos \phi_0$ at which the expression (253) is zero when $r = m$, and these are

$$\cos \phi_0 = \cos \left(\frac{j\pi}{m+1}\right) \qquad j = 1, 2, \ldots, m \tag{254}$$

so that the m spanwise points are given by

$$\eta_j = \cos \phi_j \qquad j = 1, 2, \ldots, m \tag{255}$$

where

$$\phi_j = \frac{j\pi}{m+1} \qquad j = 1, 2, \ldots, m. \tag{256}$$

The interpolation function defined in equation (50) of the main text may be given by the equivalent formula

$$\begin{aligned} g_j^{(m)}(\cos \phi_0) &= \frac{\sin (m+1)\phi_0}{\cos \phi_0 - \cos \phi_j} \lim_{\phi_0 \to \phi_j} \left(\frac{\cos \phi_0 - \cos \phi_j}{\sin (m+1)\phi_0}\right) \\ &= \frac{(-1)^{j+1}}{(m+1)} \frac{\sin (m+1)\phi_0 \sin \phi_j}{(\cos \phi_0 - \cos \phi_j)}. \end{aligned} \tag{257}$$

Then

$$\begin{aligned} G_j^{(m)} &= \int_{-1}^{+1} g_j^{(m)}(\eta_0)d\eta_0 = \int_0^\pi g_j^{(m)}(\cos \phi_0) \sin \phi_0 d\phi_0 \\ &= (-1)^{j+1} \frac{\sin \phi_j}{(m+1)} \int_0^\pi \frac{\sin (m+1)\phi_0 \sin \phi_0}{(\cos \phi_0 - \cos \phi_j)} d\phi_0 \\ &= (-1)^{j+1} \frac{1}{2} \frac{\sin \phi_j}{(m+1)} \int_0^\pi \frac{\cos (m\phi_0) - \cos \{(m+2)\phi_0\}}{(\cos \phi_0 - \cos \phi_j)} d\phi_0 \\ &= (-1)^{j+1} \frac{\pi}{2} \frac{\sin \phi_j}{(m+1)} \left[\frac{\sin (m\phi_j)}{\sin \phi_j} - \frac{\sin \{(m+2)\phi_j\}}{\sin \phi_j}\right] \\ &= (-1)^j \frac{\pi}{(m+1)} \cos \{(m+1)\phi_j\} \sin \phi_j \\ &= \frac{\pi}{(m+1)} \sin \phi_j. \end{aligned} \tag{258}$$

If $r_m(\eta_0)$ is a polynomial of degree not greater than the m'th in η_0, then it follows in exactly the same manner as that used in proving equation (246) that

$$\int_{-1}^{+1} r_m(\eta_0) g_j^{(m)}(\eta_0) d\eta_0 = r_m(\eta_j) G_j^{(m)} \tag{259}$$

and in exactly the same manner as that used in proving equation (249) that

$$g_j^{(m)}(\eta_0) = G_j^{(m)} \sum_{s=0}^{m-1} \gamma_s(\eta_j)\gamma_s(\eta_0)\sqrt{(1-\eta_0^2)} \tag{260}$$

or

$$g_j^{(m)}(\cos\phi_0) = \frac{2}{(m+1)} \sum_{s=0}^{m-1} \sin(s+1)\phi_j \sin(s+1)\phi_0 . \tag{261}$$

Define

$$P_{j,r}^{(m)} = \int_{-1}^{+1} \frac{g_j^{(m)}(\eta_0)}{(\eta_0-\eta_r)^2}\, d\eta_0 . \tag{262}$$

Then using the expression (261) this becomes

$$P_{j,r}^{(m)} = \frac{2}{(m+1)} \sum_{s=0}^{m-1} \sin(s+1)\phi_j \int_0^{\pi} \frac{\sin(s+1)\phi_0 \sin\phi_0}{(\cos\phi_0-\cos\phi_r)^2}\, d\phi_0 . \tag{263}$$

Now

$$\begin{aligned}
\int_0^{\pi} \frac{\sin(s+1)\phi_0 \sin\phi_0}{(\cos\phi_0-\cos\phi_r)^2}\, d\phi_0 &= \int_0^{\pi} \sin(s+1)\phi_0 \frac{d}{d\phi_0}\left[\frac{1}{\cos\phi_0-\cos\phi_r}\right] d\phi_0 \\
&= -(s+1)\int_0^{\pi} \frac{\cos(s+1)\phi_0}{(\cos\phi_0-\cos\phi_r)}\, d\phi_0 \\
&= -\pi(s+1)\frac{\sin(s+1)\phi_r}{\sin\phi_r}
\end{aligned} \tag{264}$$

where the integrals occurring are principal-value integrals.

Hence

$$\begin{aligned}
P_{j,r}^{(m)} &= -\frac{2\pi}{(m+1)}\frac{1}{\sin\phi_r} \sum_{s=0}^{m-1} (s+1)\sin(s+1)\phi_j \sin(s+1)\phi_r \\
&= -\frac{\pi}{(m+1)}\frac{1}{\sin\phi_r} \sum_{s=0}^{m-1} (s+1)[\cos(s+1)(\phi_j-\phi_r) - \cos(s+1)(\phi_j+\phi_r)] .
\end{aligned} \tag{265}$$

Then

$$\begin{aligned}
P_{j,j}^{(m)} &= -\frac{\pi}{(m+1)}\frac{1}{\sin\phi_j}\left[\frac{1}{2}m(m+1) + \frac{1-(m+1)\cos 2m\phi_j + m\cos 2(m+1)\phi_j}{2(1-\cos 2\phi_j)}\right] \\
&= -\frac{\pi}{(m+1)}\frac{1}{\sin\phi_j}\left[\frac{1}{2}m(m+1) + \frac{1}{2}(m+1)\right] \\
&= -\frac{\pi}{2}\frac{(m+1)}{\sqrt{(1-\eta_j^2)}} .
\end{aligned} \tag{266}$$

Also, for $r \neq j$

$$\begin{aligned}
P_{j,r}^{(m)} = -\frac{\pi}{(m+1)}\frac{1}{\sin\phi_r}\Bigg[&\frac{-1+(m+1)\cos m(\phi_j-\phi_r) - m\cos(m+1)(\phi_j-\phi_r)}{2\{1-\cos(\phi_j-\phi_j)\}} + \\
&+\frac{+1-(m+1)\cos m(\phi_j+\phi_r) + m\cos(m+1)(\phi_j+\phi_r)}{2\{1-\cos(\phi_j+\phi_r)\}}\Bigg] .
\end{aligned} \tag{267}$$

Using the relations

$$\cos(m+1)(\phi_j \pm \phi_r) = (-1)^{j+r} \tag{268}$$

$$\cos\{m(\phi_j \pm \phi_r)\} = (-1)^{j+r}\cos(\phi_j \pm \phi_r) \tag{269}$$

and

$$[1-\cos(\phi_j-\phi_r)]\,[1-\cos(\phi_j+\phi_r)] = (\cos\phi_j-\cos\phi_r)^2 \tag{270}$$

in (267) we obtain

$$\begin{aligned}
P_{j,r}^{(m)} &= -\frac{\pi}{(m+1)}\frac{1}{\sin\phi_r}\left[\frac{-1+1(-1)^{j+r}(m+1)\cos(\phi_j-\phi_r)-m(-1)^{j+r}}{2\{1-\cos(\phi_j-\phi_r)\}}+\right.\\
&\quad\left.+\frac{+1-(-1)^{j+r}(m+1)\cos(\phi_j+\phi_r)+m(-1)^{j+r}}{2\{1-\cos(\phi_j+\phi_r)\}}\right]\\
&= -\frac{\pi}{(m+1)}\frac{1}{\sin\phi_r}\frac{[\cos(\phi_j+\phi_r)-\cos(\phi_j-\phi_r)]\,[1-(-1)^{j+r}]}{2(\cos\phi_j-\cos\phi_r)^2}\\
&= \frac{2\pi}{(m+1)}\frac{\sin\phi_j}{(\cos\phi_j-\cos\phi_r)^2}\left[\frac{1-(-1)^{j+r}}{2}\right].
\end{aligned} \tag{271}$$

Hence

$$P_{j,r}^{(m)} = \begin{cases} 0 & \text{if } (j+r) \text{ is even and } j \neq r \\[2mm] \dfrac{2\pi\sqrt{(1-\eta_j^2)}}{(m-1)(\eta_r-\eta_j)^2} & \text{if } (j+r) \text{ is odd} \\[2mm] -\dfrac{\pi}{2}\dfrac{(m+1)}{\sqrt{(1-\eta_j^2)}} & \text{if } j = r. \end{cases} \tag{272}$$

APPENDIX IV

The Lowest-Order Logarithmic Singularity

Instead of equation (56) we may write the equivalent equation

$$I_i^{(n)}(\eta, \eta_0, \xi) = \frac{s}{4\pi} c(y_0)(\eta-\eta_0)^2 \int_0^1 h_i^{(n)}(\xi_0)[\hat{K}_1(x-x_0, y-y_0) + \hat{K}_2(x-x_0, y-y_0)]d\xi_0 \tag{273}$$

which comes from the form (13) of the integral equation on using relation (16).

Now, by integrating by parts, we obtain

$$\begin{aligned}
&\int_0^1 h_i^{(n)}(\xi_0)\hat{K}_1(x-x_0, y-y_0)d\xi_0 \\
&= \left[\int_{\xi(x,y_0)}^{\xi_0} h_i^{(n)}(u)du\hat{K}_1(x-x_0, y-y_0)\right]_0^1 - \int_0^1 \left\{\int_{\xi(x,y_0)}^{\xi_0} h_i^{(n)}(u)du\right\} \frac{\partial \hat{K}_1}{\partial \xi_0}(x-x_0, y-y_0)d\xi_0 \\
&= \hat{K}_1(x-x_T(y_0), y-y_0)\int_{\xi(x,y_0)}^1 h_i^{(n)}(u)du + \hat{K}_1(x-x_L(y_0), y-y_0)\int_0^{\xi(x,y_0)} h_i^{(n)}(u)du - \\
&\quad - \int_0^1 \frac{\partial \hat{K}_1}{\partial \xi_0}(x-x_0, y-y_0)d\xi_0 \int_{\xi(x,y_0)}^{\xi_0} h_i^{(n)}(u)du \\
&= \hat{K}_1(x-x_L(y_0), y-y_0)\int_0^{\xi(x,y_0)} h_i^{(n)}(u)du + \hat{K}_1(x-x_T(y_0), y-y_0)\int_{\xi(x,y_0)}^1 h_i^{(n)}(u)du + \\
&\quad + c(y_0)\int_0^1 \hat{K}_3(x-x_0, y-y_0)d\xi_0 \int_{\xi(x,y_0)}^{\xi_0} h_i^{(n)}(u)du
\end{aligned} \tag{274}$$

where

$$\xi(x, y_0) = \frac{1}{c(y_0)}[x - x_L(y_0)]. \tag{275}$$

Hence

$$\begin{aligned}
&I_i^{(n)}(\eta, \eta_0, \xi) \\
&= \frac{s}{4\pi} c(y_0)(\eta-\eta_0)^2 \left\{\hat{K}_1(x-x_L(y_0), y-y_0)\int_0^{\xi(x,y_0)} h_i^{(n)}(u)du + \right. \\
&\quad + \hat{K}_1(x-x_T(y_0), y-y_0)\int_{\xi(x,y_0)}^1 h_i^{(n)}(u)du + \\
&\quad \left. + \int_0^1 h_i^{(n)}(\xi_0)\hat{K}_2(x-x_0, y-y_0)d\xi_0 + c(y_0)\int_0^1 \hat{K}_3(x-x_0, y-y_0)d\xi_0 \int_{\xi(x,y_0)}^{\xi_0} h_i^{(n)}(u)du. \right\}
\end{aligned} \tag{276}$$

The various terms in (276) are now considered.

In

$$\hat{K}_1(x, y) = \int_{(-x+MR)/(1-M^2)}^{\infty} e^{(-i\omega/V)u} \frac{du}{(u^2+y^2)^{3/2}}, \tag{277}$$

if $x < 0$ the lower limit of integration is positive and the integrand therefore remains integrable throughout the range of integration as $y \to 0$. So a logarithmic singularity does not occur in this case.

If, however, $x > 0$, the lower limit of integration is negative when y is sufficiently small since $M < 1$, and the integrand does not then remain integrable in the neighbourhood of $u = 0$ as $y \to 0$. This case must therefore be investigated further and to do this the range of integration is split up so that

$$\hat{K}_1(x, y) = \int_{(-x+MR)/(1-M^2)}^{-\delta} + \int_{-\delta}^{+\delta} + \int_{\delta}^{\infty} \frac{e^{(-i\omega/V)u}}{(u^2+y^2)^{3/2}} du \tag{278}$$

where δ is any positive number. Only the integral over the interval $(-\delta, \delta)$ contributes to the logarithmic singularity for this is the only interval over which the integrand does not remain integrable as $y \to 0$. The integral over this interval may then be written

$$\int_{-\delta}^{\delta} \frac{e^{(-i\omega/V)u}}{(u^2+y^2)^{3/2}}\, du = \frac{1}{y^2}\int_{-\delta/|y|}^{\delta/|y|} \frac{e^{(-i\omega/V)|y|v}}{(v^2+1)^{3/2}}\, dv$$

$$= \frac{2}{y^2}\int_{0}^{\delta/|y|} \frac{\cos\left(\dfrac{\omega}{V}yv\right)}{(v^2+1)^{3/2}}\, dv$$

$$= \frac{2}{y^2}\sum_{s=0}^{\infty} \frac{(-1)^s}{(2s)!}\left(\frac{\omega y}{V}\right)^{2s} \int_{0}^{\delta/|y|} \frac{v^{2s}}{(v^2+1)^{3/2}}\, dv. \qquad (279)$$

If y is so small that

$$\delta/|y| > 1 \qquad (280)$$

then

$$\int_{0}^{\delta/|y|} \frac{v^{2s}}{(v^2+1)^{3/2}}\, dv = \int_{0}^{1} \frac{v^{2s}}{(v^2+1)^{3/2}}\, dv + \int_{1}^{\delta/|y|} v^{2s-3}\left(1+\frac{1}{v^2}\right)^{-3/2} dv$$

$$= \int_{0}^{1} \frac{v^{2s}}{(v^2+1)^{3/2}}\, dv + \int_{1}^{\delta/|y|} v^{2s-3}\left[1 + \binom{-3/2}{1}\frac{1}{v^2} + \binom{-3/2}{2}\frac{1}{v^4} + \dots\right] dv$$

$$= \int_{1}^{\delta/|y|} \binom{-3/2}{s-1}\frac{dv}{v} + \text{terms not containing } \log|y| \text{ if } s \neq 0$$

$$= \binom{-3/2}{s-1}\log(\delta/|y|) + \text{terms not containing } \log|y| \text{ if } s \neq 0. \qquad (281)$$

If $s = 0$ there are no terms at all involving logarithims.

The contribution from (279) to the lowest-order logarithmic singularity comes from the $s = 1$ term only of the summation and hence

$$\int_{-\delta}^{\delta} \frac{e^{(-i\omega/V)u}}{(u^2+y^2)^{3/2}}\, du = \left(\frac{\omega}{V}\right)^2 \log|y| + \text{higher-order logarithmic terms} + \text{other terms}. \qquad (282)$$

It follows that the contribution of

$$\frac{s}{4\pi}c(y_0)(\eta-\eta_0)^2\left\{\hat{K}_1(x-x_L(y_0), y-y_0)\int_0^{\xi(x,y_0)} h_i^{(n)}(u)du + \right.$$

$$\left. + \hat{K}_1(x-x_T(y_0), y-y_0)\int_{\xi(x,y_0)}^{1} h_i^{(n)}(u)du\right\} \qquad (283)$$

to the lowest-order logarithmic singularity is given by

$$\frac{1}{4\pi}\frac{s}{c(y_0)}\left(\frac{\omega c(y_0)}{V}\right)^2(\eta-\eta_0)^2\log|\eta-\eta_0|\int_0^{\xi(x,y_0)} h_i^{(n)}(u)du, \qquad (284)$$

since $x-x_T(y_0) < 0$ and so no logarithmic singularity arises from $\hat{K}_1(x-x_T(y_0), y-y_0)$.

Next the term

$$\frac{s}{4\pi}c(y_0)(\eta-\eta_0)^2\int_0^1 h_i^{(n)}(\xi_0)\hat{K}_2(x-x_0, y-y_0)d\xi_0 \qquad (285)$$

is considered.

Now

$$\int_0^1 h_i^{(n)}(\xi_0)\hat{K}_2(x-x_0\,,\,y-y_0)d\xi_0$$

$$= \int_0^1 h_i^{(n)}(\xi_0)\hat{K}_2[c(y_0)\,(\xi(x,y_0)-\xi_0)\,,\,s(\eta-\eta_0)]d\xi_0$$

$$= \int_{\xi(x,y_0)-1}^{\xi(x,y_0)} h_i^{(n)}(\xi(x,y_0)-\xi_0)\hat{K}_2[c(y_0)\xi_0\,,\,s(\eta-\eta_0)]d\xi_0$$

$$= \int_{\xi(x,y_0)-1}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\xi(x,y_0)} h_i^{(n)}(\xi(x,y_0)-\xi_0)\hat{K}_2[c(y_0)\xi_0\,,\,s(\eta-\eta_0)]d\xi_0 \tag{286}$$

where δ is any positive number such that

$$\delta < \xi(x, y_0) \tag{287}$$

and

$$\delta < 1 - \xi(x, y_0)\,. \tag{288}$$

Only the middle interval can give rise to a logarithmic singularity for this is the only interval inside which the integrand does not remain integrable as $\eta_0 \to \eta$.

Using the convergent Taylor expansion

$$h_i^{(n)}\{\xi(x,y_0)-\xi_0\} = h_i^{(n)}\{\xi(x,y_0)\} - \xi_0 h_i^{(n)\prime}\{\xi(x,y_0)\} + \frac{1}{2}\,\xi_0^2 h_i^{(n)\prime\prime}\{\xi(x,y_0)\} + \ldots \tag{289}$$

in the integral over $(-\delta,\delta)$ we obtain

$$\int_{-\delta}^{\delta} h_i^{(n)}(\xi(x,y_0)-\xi_0)\hat{K}_2[c(y_0)\xi_0, s(\eta-\eta_0)]d\xi_0$$

$$= \int_{-\delta}^{\delta} h_i^{(n)}(\xi(x,y_0)-\xi_0)\,\frac{M[Mc(y_0)\xi_0 + \sqrt{\{c^2(y_0)\xi_0^2 + (1-M^2)s^2(\eta-\eta_0)^2\}}]}{\sqrt{\{c^2(y_0)\xi_0^2 + (1-M^2)s^2(\eta-\eta_0)^2\}}\,[c^2(y_0)\xi_0^2 + s^2(\eta-\eta_0)^2]} \times$$

$$\times \exp\left\{-\left(\frac{i\omega}{V}\right)\frac{[-c(y_0)\xi_0 + M\sqrt{\{c^2(y_0)\xi_0^2 + (1-M^2)s^2(\eta-\eta_0)^2\}}]}{1-M^2}\right\} d\xi_0$$

$$= \frac{1}{sc(y_0)|\eta-\eta_0|}\int_{-c(y_0)\delta/(s|\eta-\eta_0|)}^{c(y_0)\delta/(s|\eta-\eta_0|)} h_i^{(n)}\left\{\xi(x,y_0) - \frac{s|\eta-\eta_0|v_0}{c(y_0)}\right\}\frac{M\{Mv_0 + \sqrt{(v_0^2+1-M^2)}\}}{(1+v_0^2)\sqrt{(v_0^2+1-M^2)}} \times$$

$$\times \exp\left[\left(\frac{i\omega}{V}\right)\frac{s|\eta-\eta_0|}{(1-M^2)}\{v_0 - M\sqrt{(v_0^2+1-M^2)}\}\right] dv_0$$

$$= \frac{1}{sc(y_0)|\eta-\eta_0|}\int_{-c(y_0)\delta/(s|\eta-\eta_0|)}^{c(y_0)\delta/(s|\eta-\eta_0|)}\left[h_i^{(n)}\{\xi(x,y_0)\} - \frac{s|\eta-\eta_0|}{c(y_0)}\,v_0 h_i^{(n)\prime}\{\xi(x,y_0)\} + \ldots\right] \times$$

$$\times \frac{M\{Mv_0 + \sqrt{(v_0^2+1-M^2)}\}}{(1+v_0^2)\sqrt{(v_0^2+1-M^2)}}\left\{1 + \left(\frac{i\omega}{V}\right)\frac{s|\eta-\eta_0|}{1-M^2}\,[v_0 - M\sqrt{(v_0^2+1-M^2)}] + \ldots\right\} dv_0$$

$$= \frac{2}{sc(y_0)|\eta-\eta_0|}\int_0^{c(y_0)\delta/(s|\eta-\eta_0|)}\left[\left\{\frac{M}{1+v_0^2} - \left(\frac{i\omega}{V}\right)\frac{s|\eta-\eta_0|M^2}{(1+v_0^2)\sqrt{(v_0^2+1-M^2)}} + \ldots\right\} h_i^{(n)}\{\xi(x,y_0)\} +\right.$$

$$+ \left\{-\frac{s|\eta-\eta_0|M^2v_0^2}{c(y_0)\,(1+v_0^2)\sqrt{(v_0^2+1-M^2)}} - \left(\frac{i\omega}{V}\right)\frac{s^2(\eta-\eta_0)^2}{c(y_0)}\,\frac{Mv_0^2}{1+v_0^2} +\right.$$

$$\left.\left. + \ldots\right\} h_i^{(n)\prime}\{\xi(x,y_0)\} + \ldots\right] dv_0 \tag{290}$$

where the omitted terms do not contribute to the lowest-order logarithmic singularity.

The lowest-order logarithmic term in (290) is obtained by assuming that $c(y_0)\delta/(s|\eta-\eta_0|)$ is very large compared with unity, expanding the integrand near the upper limit of the integral in ascending powers of $1/v_0$, and retaining only terms in $1/v_0$. On doing this we obtain

$$\int_{-\delta}^{+\delta} h_i^{(n)}(\xi(x, y_0)-\xi_0)\hat{K}_2[c(y_0)\xi_0, s|\eta-\eta_0|]d\xi_0$$

$$= \frac{2M^2}{c^2(y_0)} h_i^{(n)\prime}\{\xi(x, y_0)\} \log|\eta-\eta_0| +$$

$$+ \text{ higher order logarithmic terms } +$$

$$+ \text{ other terms}. \tag{291}$$

The contribution of

$$\frac{s}{4\pi} c(y_0)(\eta-\eta_0)^2 \int_0^1 h_i^{(n)}(\xi_0)\hat{K}_2(x-x_0, y-y_0)d\xi_0 \tag{292}$$

to the lowest-order logarithmic singularity is therefore given by

$$\frac{M^2}{2\pi}\frac{s}{c(y_0)} h_i^{(n)\prime}\{\xi(x, y_0)\}(\eta-\eta_0)^2 \log|\eta-\eta_0|. \tag{293}$$

Finally

$$\frac{s}{4\pi} c^2(y_0)(\eta-\eta_0)\int_0^1 \hat{K}_3(x-x_0, y-y_0)d\xi_0 \int_{\xi(x, y_0)}^{\xi_0} h_i^{(n)}(u)du \tag{294}$$

is considered.

Now

$$\int_0^1 \hat{K}_3(x-x_0, y-y_0)d\xi_0 \int_{\xi(x, y_0)}^{\xi_0} h_i^{(n)}(u)du$$

$$= \int_{\xi(x, y_0)-1}^{\xi(x, y_0)} \hat{K}_3\{c(y_0)\xi_0, s(\eta-\eta_0)\}d\xi_0 \int_{\xi(x, y_0)}^{\xi(x, y_0)-\xi_0} h_i^{(n)}(u)du$$

$$= \int_{\xi(x, y_0)-1}^{-\delta} + \int_{-\delta}^{+\delta} + \int_{\delta}^{\xi(x, y_0)} K_3\{c(y_0)\xi_0, s(\eta-\eta_0)\}d\xi_0 \int_{\xi(x, y_0)}^{\xi(x, y_0)-\xi_0} h_i^{(n)}(u)du \tag{295}$$

where δ is any positive number, such that

$$\delta < \xi(x, y_0) \tag{296}$$

and

$$\delta < 1 - \xi(x, y_0). \tag{297}$$

Only the middle interval can give rise to a logarithmic singularity for this is the only interval inside which the integrand does not remain integrable as $\eta_0 \to \eta$.

Using the convergent Taylor expansion

$$\int_{\xi(x, y_0)}^{\xi(x, y_0)-\xi_0} h_i^{(n)}(u)du = -\xi_0 h_i^{(n)}\{\xi(x, y_0)\} + \frac{1}{2}\xi_0^2 h_i^{(n)\prime}\{\xi(x, y_0)\} + \dots \tag{298}$$

we obtain

$$\int_{-\delta}^{\delta} \hat{K}_3\{c(y_0)\xi_0,\, s(\eta-\eta_0)\}d\xi_0 \int_{\xi(x,\,y_0)}^{\xi(x,\,y_0)-\xi_0} h_i^{(n)}(u)du$$

$$= \int_{-\delta}^{\delta} \left\{\int_{\xi(x,\,y_0)}^{\xi(x,\,y_0)-\xi_0} h_i^{(n)}(u)du\right\} \frac{[Mc(y_0)\xi_0 + \sqrt{\{c^2(y_0)\xi_0^2 + (1-M^2)s^2(\eta-\eta_0)^2\}}]^2}{\sqrt{\{c^2(y_0)\xi_0^2 + (1-M^2)s^2(\eta-\eta_0)^2\}}\,[c^2(y_0)\xi_0^2 + s^2(\eta-\eta_0)^2]} \times$$

$$\times \exp\left\{-\left(\frac{i\omega}{V}\right)\frac{[-c(y_0)\xi_0 + M\sqrt{\{c^2(y_0)\xi_0^2 + (1-M^2)s^2(\eta-\eta_0)^2\}}]}{1-M^2}\right\} d\xi_0$$

$$= \frac{2}{sc(y_0)(\eta-\eta_0)^2}\int_0^{c(y_0)\delta/(s|\eta-\eta_0|)} \left[\left\{-\frac{s|\eta-\eta_0|}{c(y_0)}\frac{2Mv_0^2}{(1+v_0^2)^2} - \left(\frac{i\omega}{V}\right)\frac{s^2(\eta-\eta_0)^2}{c(y_0)} \times\right.\right.$$

$$\left.\times \frac{[v_0^4 + v_0^2(1-2M^2)]}{(1+v_0^2)^2\sqrt{\{v_0^2 + (1+M^2)\}}} + \ldots\right\} h_i^{(n)}\{\xi(x,y_0)\} +$$

$$+\left\{\frac{1}{2}\frac{s^2(\eta-\eta_0)^2}{c^2(y_0)}\frac{v_0^2(M^2v_0^2+v_0^2+1-M^2)}{(1+v_0^2)^2\sqrt{(v_0^2+1-M^2)}} + \frac{1}{2}\left(\frac{i\omega}{V}\right)\frac{s^2(\eta-\eta_0)^3}{c^2(y_0)} \times\right.$$

$$\left.\left.\times \frac{v_0^2(Mv_0^2-M)}{(1+v_0^2)^2} + \ldots\right\} h_i^{(n)\prime}\{\xi(x,y_0)\} + \ldots\right] dv_0 \tag{299}$$

$$= -\frac{2}{s^2c(y_0)(\eta-\eta_0)^2}\left\{-\left(\frac{i\omega}{V}\right)\frac{s^2(\eta-\eta_0)^2}{c(y_0)} h_i^{(n)}\{\xi(x,y_0)\} +\right.$$

$$\left.+\frac{1}{2}\frac{s^2(\eta-\eta_0)^2}{c^2(y_0)}(M^2+1)h_i^{(n)\prime}\{\xi(x,y_0)\}\right\}\log|\eta-\eta_0| +$$

+ higher order logarithmic terms +

+ other terms. (300)

The contribution of

$$\frac{s}{4\pi}c^2(y_0)(\eta-\eta_0)^2\int_0^1 \hat{K}_3(x-x_0,\, y-y_0)d\xi_0\int_{\xi(x,\,y_0)}^{\xi_0} h_i^{(n)}(u)du \tag{301}$$

to the lowest order logarithmic singularity is therefore given by

$$\frac{1}{4\pi}\frac{s}{c(y_0)}(\eta-\eta_0)^2\log|\eta-\eta_0|\left[2\left(\frac{i\omega}{V}\right)c(y_0)h_i^{(n)}\{\xi(x,y_0)\} - (1+M^2)h_i^{(n)\prime}\{\xi(x,y_0)\}\right]. \tag{302}$$

Then, by adding together the contributions (284), (293) and (302) and expanding $c(y_0)$ and $\xi(x, y_0)$ as Taylor series about $y_0 = y$ we obtain

$$F_{0,i}^{(m)}(\eta,\xi) = \frac{1}{4\pi}\frac{s}{c(y)}\left[-(1-M^2)h_i^{(n)\prime}\{\xi(x,y)\} + 2\left(\frac{i\omega}{V}\right)c(y)h_i^{(n)}\{\xi(x,y)\} +\right.$$

$$\left.+\frac{\omega^2}{V^2}c^2(y)\int_0^{\xi(x,\,y)} h_i^{(n)}(u)du\right]. \tag{303}$$

APPENDIX V

Determination of a Limit

It is required to determine

$$\lim_{\eta_0\to\eta} I_i^{(n)}(\eta, \eta_0, \xi)$$

$$= \lim_{\eta_0\to\eta} \frac{s}{4\pi} c(y_0)(\eta-\eta_0)^2 \int_0^1 h_i^{(n)}(\xi_0)[\hat{K}_1(x-x_0, y-y_0) + \hat{K}_2(x-x_0, y-y_0)]d\xi_0 \tag{304}$$

where the form (273) has been used for $I_i^{(n)}(\eta, \eta_0, \xi)$.

Using (17) we obtain

$$\lim_{\eta\to 0} \eta^2 \hat{K}_1(x, y) = \lim_{\eta\to 0} \eta^2 \int_{(-x+MR)/(1-M^2)}^{\infty} e^{-i\omega u/V} \frac{du}{(u^2+y^2)^{3/2}}$$

$$= \lim_{\eta\to 0} \frac{\eta^2}{y^2} \int_{(1/|y|)\{(-x+MR)/(1-M^2)\}}^{\infty} e^{(-i\omega/V)|y|v} \frac{dv}{(v^2+1)^{3/2}}.$$

Now

$$\lim_{\eta\to 0} \frac{1}{|y|}\left(\frac{-x+MR}{1-M^2}\right) = \begin{cases} -\infty & \text{if } x > 0 \\ +\infty & \text{if } x < 0. \end{cases} \tag{305}$$

Hence if $x > 0$

$$\lim_{\eta\to 0} \eta^2 \hat{K}_1(x, y) = \frac{1}{s^2} \int_{-\infty}^{\infty} \frac{dv}{(v^2+1)^{3/2}}$$

$$= \frac{2}{s^2} \tag{306}$$

and if $x < 0$

$$\lim_{\eta\to 0} \eta^2 \hat{K}_1(x, y) = \frac{1}{s^2} \int_{\infty}^{\infty} \frac{dv}{(v^2+1)^{3/2}}$$

$$= 0. \tag{307}$$

From (18) we see that $\hat{K}_2(x, y)$ is finite as $y \to 0$ for both $x > 0$ and $x < 0$, so

$$\lim_{\eta\to 0} \eta^2 \hat{K}_2(x, y) = 0. \tag{308}$$

Hence using the limit relations (306), (307) and (308) in (304) we obtain

$$\lim_{\eta_0\to\eta} I_i^{(n)}(\eta, \eta_0, \xi) = \frac{1}{2\pi} \frac{c(y_0)}{s} \int_0^{\xi} h_i^{(n)}(\xi) d\xi. \tag{309}$$

The point $x = 0$ has not been considered in the limit relations (306), (307) and (308), since the expressions are indeterminate in the neighbourhood of $x = 0$. This does not affect the result (309) however, since the expressions concerned are never infinite and the point $x = 0$ corresponds to an isolated point in the integral in (309).

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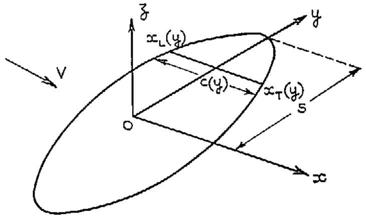

FIG. 1. The wing and a coordinate system.

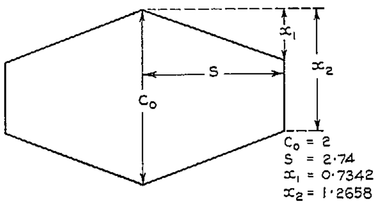

FIG. 2. Planform of a symmetric tapered wing.

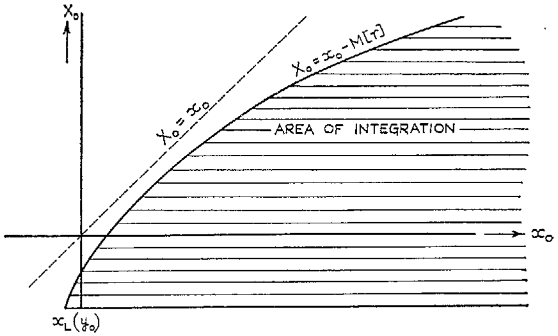

FIG. 3. Area of integration when M < 1.

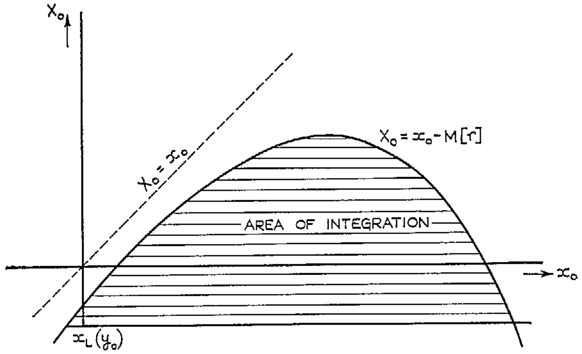

FIG. 4. Area of integration when M > 1.

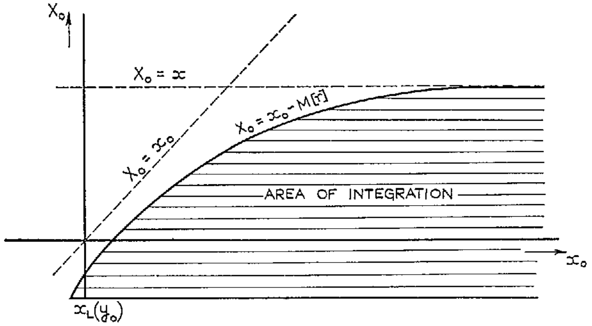

FIG. 5. Area of integration when M = 1.

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Printed and published by
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S.O. Code No. 23-3409