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On the Statistical Optimisation of Guided-Weapon Systems

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Summary.

The statistical nature of the input to a guided-weapon system (target information and noise) requires that the criterion of weapon performance be itself a statistical quantity. The criterion used in this paper is the mean square miss distance, the mean being taken over a large number of engagements, such that all probable target and noise inputs are encountered, and it is shown that there exists an optimum realisable system for which this mean square miss distance is a minimum.

For the derivation of the optimum system it is necessary to assume that the target and noise inputs, or appropriate functions of these inputs, may be considered to form a stationary (but not necessarily ergodic) ensemble for a short interval prior to collision. Account is taken of the fact that the system must include a missile, with its aerodynamic characteristics and limited available acceleration, and this leads to a number of optimum systems depending on these factors.

The beam-riding system is shown to satisfy the main requirements of the analytical framework, so that this system may be identified with the optimum system. From this identification follows the definition of certain components of the beam-rider, and the optimisation of the latter requires the insertion of electrical networks in the ground tracker or in the missile, or both, depending on the sources of noise.

Explicit formulae are derived for cases in which the noise spectral density is assumed to be constant with frequency, and the target manoeuvres to be such that their lateral accelerations form a stationary ensemble over the necessary interval. The examples given show that a definite improvement results from the use of the optimum system, in that both the miss distance and the acceleration requirement are reduced.

The realisation of networks defined by their transfer functions is discussed in Appendix IV, and examples are given of optimum networks for the beam-riding system. One such example has been the subject of simulator tests, in which it is compared with the 'phase-advance' system.

It is concluded that the missile accelerations required to achieve a given miss distance are considerably less than those hitherto considered necessary, and that the results of the paper warrant a further programme of analysis, simulation and flight trials. Such work might well lead to a significant advance in the efficiency of missile design.

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1. *Introduction.*

1.1. *The Guided-Weapon and Servomechanism Theory.*

The nature of the information available as the input to a Guided-Weapon system plays a dominant role in determining the required characteristics of the system. Such information will be a more or less accurate description of the motion of the target, depending on the quality of the apparatus used to detect and follow the target. That this motion is variable—that it is, to some extent, unpredictable—emphasises the need for a *guided* weapon, a device capable of correcting its error with the aid of up-to-date information concerning the target. In this sense the guided weapon is a servomechanism, a 'closed-loop' system which compares the actual and desired situations and takes appropriate action to minimise the difference between them.

This concept of the guided weapon as a servomechanism has proved a useful one, in that much of the theory of servos is directly applicable to the guided weapon. That this theory is so well developed owes much to the fields of communication engineering and filter theory, with which it is intimately associated.

Recent advances in these latter fields have been concerned with the statistical nature of the message, or signal, and the noise with which it is invariably accompanied. The messages accepted by a telephone-line, for example, differ in detail on each occasion, but nevertheless the ensemble of all possible messages will have properties which characterise the fundamental similarity of messages in general; and it is these statistical properties which influence the design of the system. Noise is also a statistical phenomenon—its precise value on a given occasion is unknown, and its description must be confined to average properties, such as the mean square value.

This communications terminology has been taken over to the field of servomechanisms, and in this context the term noise refers to that part of the input to the system which does not contribute to the desired information. It is clearly desirable to suppress this part of the signal, and this can often be done by filtering, in which use is made of known differences in the frequency content of information and noise; in this situation Wiener[1] has shown that there exists an optimum filter which will separate the message from the noise to the fullest possible extent.

The word optimum in this context requires clarification. The merit of the filter must be judged on the success with which it handles all the possible situations—weighted according to their frequency of occurrence—and not merely on its performance with one or two specific examples of message and noise. Since our knowledge of the latter is confined to their statistical properties, the criterion of merit must itself be statistical in character, describing the 'average' behaviour of the filter. One such criterion is the mean square error existing between the filtered signal and the original message, the mean being taken over all possible messages and noise. In this sense the optimum filter is that network for which the mean square error is a minimum, but other measures of the error are possible, and on occasion preferable.

In view of the fundamental accord which exists between the fields of communications and servomechanisms, it may be expected that the above concepts have significance in the theory of the guided weapon, not only for the various servo-mechanisms included in the system, but also for the system as a whole. In fact, it is found that the statistical theory provides a powerful means not only for assessing the performance of a given system, but for choosing that system having the best performance.

It is the purpose of this paper to develop the theory, taking into account the physical constraints within which the system must operate—limitations imposed, for example, by the aerodynamic behaviour of the missile which forms part of the system: and to derive the best physically realisable system for a given set of conditions and a given criterion—the minimisation of the mean square miss distance.

1.2. *The Target Motion.*

The motion of the target, which provides the useful part of the input to a guided-weapon system, is not entirely random. It is known, for example, that the forward velocity will not undergo rapid variations, and that the lateral acceleration is confined within certain limits, so that although the future position of the target cannot be predicted with certainty from a knowledge of its present and past positions, we can nevertheless eliminate certain positions as unlikely. This leads to the consideration of the statistical properties of a whole set of possible target courses any one of which may provide the input on a particular occasion. Clearly the system should be so designed that it is reasonably effective against any of these targets, with the emphasis on those courses which are most probable.

The mechanism underlying the target motion (i.e. the aircraft, the pilot, the location of objectives in relation to the guided-weapon site, etc.) does not change with time (at least for the period of time over which we are concerned), and it seems reasonable to conclude that the probability distributions which characterize the set of target courses are invariant with respect to time, over a limited region. Such a set is said to be stationary, and it is with the properties of stationary series that the theory of the present paper is mainly concerned.

1.3. *Sources of Error—Noise.*

The information acquired by the guided-weapon system will not be a true record of the target's behaviour: it will include spurious information arising from imperfections in the unit used to detect and follow the target. We are concerned mainly with those systems in which the information is derived from a radar beam which automatically tracks the target, and this type of radar auto-follow is subject to a variety of errors, among which may be cited the so-called 'beam jitter'. This is due to random variations in the radar-reflecting properties of the target, and its effect is to induce random fluctuations in the beam which do not correspond to the true motion of the target. Other errors of a more or less random nature are thermionic noise in the receiver, fading at long range due to lack of transmitted power, and at short ranges the phenomenon of glint—random movements of the point of reflection. In addition errors may be introduced in the tracking servo itself.

For the moment we need not distinguish between these sources of error, and all may be considered as noise. Thus, if $\theta_T(t)$, $\theta(t)$ are measures, in a suitable co-ordinate system, of the true position of the target and the input to the target following system, then

$$\theta(t) = \theta_T(t) + \theta_N(t),$$

and the noise $\theta_N(t)$ is that part of the input which does not contribute information about the target.

It is clear that noise is in the same category as the target motion, in that the future values of the noise cannot be predicted with certainty from a knowledge of past values. If, on the contrary, the noise is known as a function of time, it ceases to have any significance, since it can be removed in effect by subtraction from the input signal. On the other hand, if the target motion is completely pre-determined (from the stand-point of the observer), the noise is again irrelevant: all the required target information is available, and no new information is possible. It is only when both the message and the noise are to some extent unpredictable that the problem of their separation becomes other than trivial.

The underlying processes which generate the noise do not vary from day to day, and it may be expected that the statistical properties derived from the analysis of a large number of records will hold good for similar situations in the future—i.e. the ensemble of noise functions constitutes a stationary series. Any theory which includes the effect of noise must deal with the ensemble rather than with individual functions: the precise statistical quantities required will appear later.

1.4. *The Nature of the Input.*

The two quantities which form the input to the guided-weapon system—target information and noise—are each statistical in character. The uncertainty concerning the target stems from the fact that it is independent of the guided weapon, and is reluctant to supply information leading to its own destruction. This fact means that, as far as the observer is concerned, some element of uncertainty will exist even when all available information is taken into account.

Noise, on the other hand, arises within the system, and is therefore to some extent controllable. The system should clearly be designed to reduce the noise as far as possible, provided that the increased complexity leads to a worthwhile improvement in performance, but again some noise will remain.

Having used all possible target information and having reduced the noise as far as is economically possible, the problem is then so to use the available equipment that the guided-weapon system has the maximum efficiency.

2. *Criteria Governing the Choice of System.*

2.1. *The Mean Square Miss Distance.*

The motion of the missile is governed by the input to the system; symbolically we may write

$$\theta_M(t) = \Gamma\{\theta_T(t) + \theta_N(t)\} = \Gamma\theta(t), \tag{2.1-1}$$

where Γ is an operator (not necessarily linear) which describes the system. Here 'system' connotes the whole chain of events which starts with the motion of the target and ends with a motion of the missile: it therefore includes the aerodynamics and control system of the missile, and the characteristics of the tracking or error-detecting device.

Consider now a large number of attempts against different targets, all of which are engaged at roughly the same range. The target motion and the noise, and therefore the motion of the missile, will in general be different on each occasion. For the rth attempt,

$$\theta_{Mr}(t) = \Gamma\{\theta_{Tr}(t) + \theta_{Nr}(t)\} = \Gamma\theta_r(t), \tag{2.1-2}$$

and if T is the instant of nearest approach of the missile to the target, the difference

$$\theta_{Tr}(T) - \theta_{Mr}(T) \tag{2.1-3}$$

is a measure of the accuracy of the system. If, for example, the θ's refer to angular positions from a datum line, and R is the range of engagement, the miss distance s_r is given by

$$\begin{aligned} s_r &= R[\theta_{Tr}(T) - \theta_{Mr}(T)] \\ &= R[(1-\Gamma)\theta_{Tr}(T) - \Gamma\theta_{Nr}(T)], \end{aligned} \tag{2.1-4}$$

assuming that the operator Γ is associative.

It is assumed in this equation that no *a priori* information is available to distinguish one attack from another, so that the system remains unchanged—i.e. Γ is not a function of r.

The criterion by which the system is judged must take into account the performance against all targets. The criterion used in this paper is the mean square miss distance, the mean being taken over all possible engagements:

$$\langle s_r^2 \rangle = \frac{1}{n} \sum_{r=1}^{n} s_r^2 = \frac{1}{n} \sum_{r=1}^{n} R^2[(1-\Gamma)\theta_{Tr}(T) - \Gamma\theta_{Nr}(T)]^2, \tag{2.1-5}$$

from (2.1-4).

The ultimate object is to obtain the highest lethality, but the relation between miss distance and lethality is a function of the warhead-fuse combination, and not readily expressible. To the extent that this relation can be encompassed by a single number, the most appropriate number appears to be the mean square miss distance. We shall therefore consider the optimum system to be that system for which the mean square miss distance is a minimum. The system is defined by the operator Γ, so that the problem is to select Γ such that $\langle s_r^2 \rangle$ is a minimum.

2.2. *The Admissible Class of Operators.*

The value of the result will clearly depend on the generality allowed in the formulation of the problem, and the population of possible systems from which the optimum is chosen should be as comprehensive as possible; if the problem is restricted to the study of a particular class of operators by arbitrarily excluding all other systems, the value of the result is correspondingly diminished. Nevertheless certain restrictions must be imposed on the operator: in particular, it is necessary to exclude non-linear operators, since no adequate theoretical treatment exists at present. Further, the operator must lead to a physically realisable system, which implies that the operator can only act on past values of the operand. It is also desirable that the resulting system shall have fixed components (at any rate over a sufficiently long interval), and this imposes the further requirement that the operator shall be invariant under translation in time.

From this admissible class of operators, acting only in the past and invariant with time, we wish to select that operator which minimises the mean square miss distance. Even this restricted class includes a wide variety of operators—in electrical terms, any combination of active and passive networks having lumped components can be represented by such an operator.

2.3. *Optimum Considerations.*

The method of optimising the parameters of an arbitrarily chosen system from this class does not ensure that no better system exists; for example, it may be decided from experience and certain criteria that for a particular problem the relation between input and output should be of the form

$$\frac{a_0 + a_1 D + a_2 D^2}{b_0 + b_1 D + b_2 D^2 + b_3 D^3},$$

where $D \equiv d/dt$.

It is then possible to select some or all of the coefficients a_0, b_0, etc., such that the optimum performance is obtained for a chosen criterion. This result applies only to this particular form of operator, and gives no indication of the performance of other possible systems. The same technique can of course be carried out for a number of assumed forms, and the results compared. We might enquire, for example, whether an additional term $b_4 D^4$ might not provide some improvement, and this hypothesis could be tested by repeating the calculation for this configuration; such a procedure however quickly becomes impossible to handle analytically, and still does not ensure that the best system has been investigated.

Where the form of the operator has been chosen for other reasons, or if the system cannot readily be altered, this method is satisfactory, although the criterion adopted should relate to the performance with the type of inputs expected, rather than the response to a step or the sinusoidal response, unless the latter inputs occur frequently. In many cases however there is some latitude in the choice of operational forms; in the guided-weapon case part of the system is fixed by the characteristics of the missile, but considerable freedom exists in choosing other parts of the system, such as the control system and the properties of the target seeker. In order to determine the best characteristics for these components, a more general approach is necessary—a theory which defines the best operator from the widest possible class of operators.

Such a theory, based on linearity, would be of little value if the actual situation were not represented, at least approximately, by a linear system; and it is necessary to show that the mathematical framework will accommodate the more important properties of the system.

3. *Non-linearities in the Guided-Weapon System.*

Non-linearity in a guided weapon may arise in three distinct ways:

(a) The non-linear aerodynamic behaviour of the missile.

(b) Non-linearities resulting from the geometry of the situation.

(c) The effects of saturation in any element of the system.

These three sources are discussed below.

3.1. *Aerodynamic Non-Linearities.*

The motion of the missile is not a linear function of the deflection of the control surfaces. In a fixed-wing vehicle having rear control surfaces, for example, the downwash from the main wings onto the control surfaces results in an effective shift of the centre of pressure with incidence, so that the acceleration produced by a given fin angle is a function of incidence.

The effects of this and other non-linearities can be greatly reduced by the use of negative feedback: this does not of course affect the aerodynamics of the missile, but it does alter the relation between the control signal and the acceleration it produces. The greater the degree of feedback, the more nearly linear does this relation become. The type of feedback used depends on the missile: for the fixed-wing, rear-control-surface vehicle, high acceleration feedback can lead to instability, and additional feedbacks are necessary to overcome this condition.

Other effects—such as those due to the variation of aerodynamic derivatives with changing height and speed—are also reduced by feedback: in effect the missile forms part of a servo system, the performance of which becomes less dependent on the missile as the feedback is increased.

Although the missile system may be made more nearly linear by these methods, it is not immediately obvious that this is beneficial. It will be shown later (Section 7.3) that in fact the best transfer function of the modified missile is unity over the significant frequency band; and although this ideal cannot be attained, it is approached by increasing the feedback, which also reduces the non-linear effects. The application of such feedback is therefore desirable, and at the same time justifies the linear representation.

On these grounds it will be assumed in this paper that the relation between the demanded and achieved missile accelerations, as modified by internal feedback, can be represented by a linear, constant coefficient differential equation: that is

$$f_M(t) = A(D)f_D(t) \tag{3.1-1}$$

where f_M is the lateral acceleration, and $f_D(t)$ a measure of the demanded acceleration. The operator $A(D)$ includes the control-surface actuator characteristics, and those of any instruments—such as accelerometers—from which the feedback is derived. The relation holds over a limited range of $f_D(t)$, as discussed in Section 3.3.

3.2. *Geometrical Non-Linearities.*

3.2.1. *The beam-riding system.*—The influence of the coordinate system can be illustrated by taking the beam-riding system as an example. In this system the angular error $(\theta_B - \theta_M)$ (Fig. 1) existing between the target-tracking beam and the missile is detected by the missile receiver and multiplied by the range r_M, giving a measure of the linear displacement between the missile and the centre of the beam. If the subsequent operations on this signal are denoted by $S(D)$, the demand for acceleration is

$$f_D(t) = S(D)r_M(t)\,[\theta_B(t) - \theta_M(t)]\,, \tag{3.2-1}$$

so that from (3.1-1)

$$f_M = A(D)S(D)r_M(\theta_B - \theta_M)\,. \tag{3.2-2}$$

(In this and subsequent equations the argument t is omitted where there is no ambiguity.)

The operator $S(D)$ includes the receiver characteristics and those of any additional networks included for the purposes of stability, etc.

From Fig. 1,

and

$$\left.\begin{aligned} r_M\ddot{\theta}_M + 2\dot{r}_M\dot{\theta}_M &= f_M \cos(\theta_M - \psi_M) \\ \ddot{r}_M - r_M\dot{\theta}_M{}^2 &= f_M \sin(\theta_M - \psi_M)\,, \end{aligned}\right\} \tag{3.2-3}$$

giving

$$A(D)S(D)r_M(\theta_B - \theta_M) = [(r_M\ddot{\theta}_M + 2\dot{r}_M\dot{\theta}_M)^2 + (\ddot{r}_M - r_M\dot{\theta}_M{}^2)^2]^{1/2} \tag{3.2-4}$$

so that although the system consists of linear elements, the geometry of the situation leads nevertheless to a non-linear equation in θ_M.

3.2.2. *A linear approximation.*—The angle $(\theta_M - \psi_M)$ will normally be small, and may, without great error, be assumed to be zero; the missile axis then lies parallel to the radius vector $\mathbf{r}_M$, and

$$A(D)S(D)r_M(\theta_B - \theta_M) = r_M\ddot{\theta}_M + 2\dot{r}_M\dot{\theta}_M \tag{3.2-5}$$

from (3.2-3) and (3.2-4).

This equation is now linear, but has variable coefficients. The assumption of constant missile range removes the variation, but appears at first sight to be a rather sweeping assumption. As an alternative and closer approximation, consider the range to increase exponentially:

$$r_M = K \exp(c_M t)$$

Then

$$\frac{\dot{r}_M}{r_M} = c_M\,, \qquad \text{a constant.}$$

Equation (3.2-5) then becomes

$$K \exp (c_M t) A(D+c_M) S(D+c_M)(\theta_B - \theta_M) = K \exp (c_M t) D^2 \theta_M + 2Kc_M \exp (c_M t) D\theta_M ,$$

or

$$A(D+c_M) S(D+c_M)(\theta_B - \theta_M) = D(D+2c_M)\theta_M \tag{3.2-6}$$

—a linear equation with constant coefficients.

The actual increase in missile range is more nearly linear than exponential, since $\dot{r}_M$ is practically constant and equal to V_M:

$$r_M = V_M t .$$

If the constants of the exponential approximation are chosen to give the true range and velocity at time t_0, then

$$V_M t_0 = K \exp (c_M t_0), \quad \text{and} \quad V_M = Kc_M \exp (c_M t_0)$$

giving

$$r_M = V_M t_0 \exp (t/t_0 - 1)$$

as the approximation, with $c_M = 1/t_0$.

This function is compared with the true range in Fig. 2, from which it may be seen that the approximation is correct to within 5% in the interval

$$0{\cdot}8\, t_0 < t < 1{\cdot}2\, t_0 .$$

If t_0 is chosen such that $1{\cdot}2\, t_0 = T$, the time of interception reckoned from launch, this interval becomes

$$\frac{2}{3} T < t < T,$$

so that over the final third of the flight the approximation for range is in error by less than 5%.

For the purposes of the theory which follows, the representation need only be correct over an interval immediately prior to interception, provided that this interval is long compared with the time constants of the system, which will not amount to more than a few seconds (Section 7.3). The flight times of interest for a ground-launched vehicle will not normally be less than 20 seconds, and more often 50 to 60 seconds, so that (3.2-6) is valid over a sufficiently long interval. The precise behaviour of the missile before this time is of little consequence, provided that it flies within the beam in a stable manner.

3.2.3. *The overall linear operator with constant coefficients.*—Equation (3.2-6) relates the beam and missile positions: the system is completed by the dependence of the beam position θ_B on the input to the system, which is $\theta_T + \theta_N$. This relationship is governed by the radar tracking system, whose properties we may denote by the operator $T(D)$ acting on the error as measured by the ground radar set (Fig. 3a):

$$\theta_B = T(D)[\theta_T + \theta_N - \theta_B],$$

or

$$\theta_B = \frac{T(D)}{1 + T(D)} (\theta_T + \theta_N) \tag{3.2-7}$$

The operator $T(D)$ may be assumed linear, except for the effects of mechanical hysteresis and friction, which will be discussed later (Section 6.2.2).

The combination of (3.2-6) and (3.2-7) yields

$$\theta_M = \frac{T(D)}{1+T(D)} \frac{A(D+c_M)S(D+c_M)}{A(D+c_M)S(D+c_M)+D(D+2c_M)} (\theta_T+\theta_N) \tag{3.2-8}$$

which may be written as

$$\theta_M = H(D)(\theta_T+\theta_N), \tag{3.2-9}$$

where $H(D)$ is a linear operator with constant coefficients. Thus the whole system can be represented, with the above reservations, by an operator of the class discussed in Section 2.2.

If $\bar{\theta}_M(p)$ denotes the Laplace Transform of $\theta_M(t)$—i.e. if

$$\bar{\theta}_M(p) = \int_0^\infty e^{-pt}\,\theta_M(t)dt,$$

with similar symbols for θ_T, θ_N, we have from (3.2-8) and (3.2-9)

$$\frac{\bar{\theta}_M(p)}{\bar{\theta}_T(p)+\bar{\theta}_N(p)} = \frac{T(p)}{1+T(p)} \frac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M)+p(p+2c_M)} = H(p) \tag{3.2-10}$$

and this is the system transfer function, where $T(p)$, $A(p)$ and $S(p)$ are the transfer functions of the elements defined above. These relations are shown diagrammatically in Figs. 3a, and b. The whole system is equivalent to a filter having the transfer function $H(p)$, although the components of the filter do not correspond to those of the guided-weapon system.

3.3. *Saturation Effects.*

3.3.1. *Limiting.*—The third type of non-linearity is that due to saturation of the various elements composing the system: the linear representation is only valid over the range for which each unit (with feedback if necessary) can be considered linear. In most cases the linear range of the components can be adjusted to meet the requirements without difficulty, but the range of lateral acceleration which the missile can achieve presents a special problem. Here negative feedback is of no assistance, since beyond a certain range a demand for acceleration, augmented by feedback to counter the falling efficiency of the control surfaces, results not in greater acceleration but in disintegration of the missile. It is therefore necessary to limit the demand for acceleration to a level consistent with the structural strength of the vehicle; moreover, this level cannot readily be altered without re-designing the missile.

3.3.2. *Further restrictions on the optimum operator.*—The acceleration limitation cannot be ignored in the analysis: at the same time the analysis is only valid for linear systems. It is however possible to introduce constraints into the theory such that the limiting value is almost never exceeded, in which case the system can again be regarded as linear. This condition imposes a further restriction on the optimum operator—the problem is then to find that physically realisable operator, not demanding an acceleration in excess of a given level, which will minimise the mean square miss distance. The minimum miss distance will of course depend on the available acceleration, as will also the optimum operator with which this minimum is achieved. The optimum operator can be evaluated for various limits, and the resulting minimum miss distances as a function of allowable acceleration furnishes a design criterion for the structural strength of the missile. The absolute minimum is obtained with infinite acceleration, but it will be shown later (Section 7) that the relative gain in accuracy rapidly diminishes as the acceleration is increased.

Although emphasis has been laid on acceleration limiting, it is—for the purposes of analysis—equally necessary that the remaining components (e.g. the receiver) should operate within their linear regimes. It is possible to apply additional constraints to ensure linearity throughout, but this involves specifying much of the system in detail. However, we wish to arrive at the best system by analysis, rather than arbitrarily to specify parts of it in advance: it is therefore preferable to deal only with the most important non-linearity, and then to verify that the realisation of the subsequent operator does not require an unreasonable linear range for any particular component.

4. *The Derivation of the Optimum System without Constraints.*

It has been shown above that the beam-riding system can be characterised by a linear operator $H(D)$ relating input and output. In this section an expression for the mean square miss distance is obtained in terms of the operator and the statistical properties of the target motion and of the noise. The minimisation of this expression leads to an integral equation, the solution of which yields the transfer function of the optimum system. To clarify the discussion the optimum is derived without regard to acceleration limiting, this being deferred to a later section (Section 5).

4.1. *The Minimisation of the Mean Square Miss Distance.*

4.1.1. *The weighting function of the complete system.*—The performance of a system characterised by a linear set of differential equations can be described in a number of ways; since we are concerned with minimising the error at a particular instant (the time of nearest approach), it is convenient to represent the missile position at that time by means of the system weighting function. This is defined as the response of the system to a unit impulse; any input can be specified as a succession of impulses applied at different times, and if the system is linear the resulting output is the sum of the responses of each impulse. It is shown in Appendix I that this leads to

$$\theta_M(t) = \int_0^t h(x)\theta(t-x)dx\,, \tag{4.1-1}$$

where $h(t)$ is the weighting function of the system, and $\theta_M(t)$ the missile position at time t resulting from the input $\theta(t)\{= \theta_T(t) + \theta_N(t)\}$

The conditions implicit in this representation are

(a) the system is linear, so that superposition holds.

(b) It is physically realisable, since it depends only on past values of $\theta(t)$ (the range of x is from 0 to t), but it is not necessarily stable.

(c) The equations describing the system have constant coefficients, since the response is independent of the time at which it is applied.

(d) The system is initially at rest.

These conditions embrace the admissible class of operators discussed in Section 2.2.

The transfer function $H(p)$ is the Laplace Transform of the weighting function $h(t)$ (Appendix I):

$$H(p) = \int_0^\infty h(t)e^{-pt}dt\,. \tag{4.1-2}$$

4.1.2. *The mean square miss distance in terms of the weighting function.*—Consider a large number of attacks against targets $\theta_{T1}, \theta_{T2}, \ldots \theta_{Tn}$, all of which are engaged at the same slant range R, and let T be the time of interception reckoned from launch. Then the miss distance s_r against the rth target is given by

$$s_r = R[\theta_{Tr}(T) - \theta_{Mr}(T)],$$

or

$$\frac{s_r}{R} = \theta_{Tr}(T) - \int_0^T h(x)\theta_r(T-x)dx, \tag{4.1-3}$$

from (4.1-1).

The mean square angular miss distance, taken over n trials, is then

$$\begin{aligned}\sigma^2 &= \frac{1}{n}\sum_{r=1}^{n}\left(\frac{s_r}{R}\right)^2 = \frac{1}{n}\sum_{r=1}^{n}\left[\theta_{Tr}(T) - \int_0^T h(x)\theta_r(T-x)dx\right]^2 \\ &= \langle[\theta_{Tr}(T)]^2\rangle - 2\frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T)\int_0^T h(x)\theta_r(T-x)dx + \\ &\quad + \frac{1}{n}\sum_{r=1}^{n}\int_0^T h(x)\theta_r(T-x)dx\int_0^T h(y)\theta_r(T-y)dy \\ &= \langle[\theta_{Tr}(T)]^2\rangle - 2\int_0^T h(x)\frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T)\theta_{Tr}(T-x)dx + \\ &\quad + \int_0^T h(x)dx\int_0^T h(y)\frac{1}{n}\sum_{r=1}^{n}\theta_r(T-x)\theta_r(T-y)dy.\end{aligned} \tag{4.1-4}$$

Since $\theta_r(t) = \theta_{Tr}(t) + \theta_{Nr}(t)$,

$$\begin{aligned}\frac{1}{n}\sum_{r=1}^{n}\theta_r(T-x)\theta_r(T-y) &= \frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T-x)\theta_{Tr}(T-y) + \frac{1}{n}\sum_{r=1}^{n}\theta_{Nr}(T-x)\theta_{Nr}(T-y) + \\ &\quad + \frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T-x)\theta_{Nr}(T-y) + \frac{1}{n}\sum_{r=1}^{n}\theta_{Nr}(T-x)\theta_{Tr}(T-y)\end{aligned} \tag{4.1-5}$$

If there is no correlation between the target motion and the noise, the last two terms of (4.1-5) are zero. In the absence of any evidence of correlation, it is convenient to make the assumption that none exists; this involves no loss of generality, since if the correlation is known its effect can be included in the analysis.

Let

$$\left.\begin{aligned}\varphi_T(x) &= \frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T)\theta_{Tr}(T-x), \\ \chi_T(x, y) &= \frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T-x)\theta_{Tr}(T-y), \\ \text{and}\quad \chi_N(x, y) &= \frac{1}{n}\sum_{r=1}^{n}\theta_{Nr}(T-x)\theta_{Nr}(T-y).\end{aligned}\right\} \tag{4.1-6}$$

Then (4.1-4) becomes, on using (4.1-5) and (4.1-6),

$$\sigma^2 = \varphi_T(0) - 2\int_0^T \varphi_T(x)h(x)dx + \int_0^T h(x)dx\int_0^T [\chi_T(x, y) + \chi_N(x, y)]\,h(y)dy, \tag{4.1-7}$$

which gives the mean square miss distance in terms of $h(x)$ and certain functions of the target motion and of the noise.

4.1.3. *Conditions satisfied by the optimum weighting function.*—Suppose now that the weighting function which minimises σ^2 is $h_0(x)$, and consider a functional variation $\eta(x)$ such that $h(x) = h_0(x) + \epsilon\eta(x)$, where ϵ is a quantity independent of x. Then

$$\sigma^2 = \varphi_T(0) - 2\int_0^T \varphi_T(x)\,[h_0(x) + \epsilon\eta(x)]dx + $$
$$+ \int_0^T [h_0(x) + \epsilon\eta(x)]dx \int_0^T [\chi_T(x, y) + \chi_N(x, y)]\,[h_0(y) + \epsilon\eta(y)]dy\,,$$

from (4.1-7), and σ^2 is now a function of ϵ. The values of ϵ for which σ^2 has stationary values satisfy the equation $d\sigma^2/d\epsilon = 0$—that is

$$\frac{d\sigma^2}{d\epsilon} = -2\int_0^T \varphi_T(x)\eta(x)dx + \int_0^T \eta(x)dx \int_0^T [\chi_T(x, y) + \chi_N(x, y)]\,[h_0(y) + \epsilon\eta(y)]dy + $$
$$+ \int_0^T [h_0(x) + \epsilon\eta(x)]dx \int_0^T [\chi_T(x, y) + \chi_N(x, y)]\eta(y)dy = 0. \qquad (4.1\text{-}8)$$

If σ^2 is to have a stationary value when $h(x) = h_0(x)$, then $\epsilon = 0$ must be a solution of this equation. Hence

$$-2\int_0^T \eta(x)\varphi_T(x)dx + \int_0^T \eta(x)dx \int_0^T [\chi_T(x, y) + \chi_N(x, y)]h_0(y)dy + $$
$$+ \int_0^T h_0(x)dx \int_0^T [\chi_T(x, y) + \chi_N(x, y)]\eta(y)dy = 0.$$

The second and third terms of this expression are equal, as can be seen by changing the order of integration in the third term and then interchanging x and y, since by definition $\chi(x, y) = \chi(y, x)$. Then

$$\int_0^T \eta(x)\left\{\int_0^T [\chi_T(x, y) + \chi_N(x, y)]h_0(y)dy - \varphi_T(x)\right\} dx = 0.$$

This must hold for any $\eta(x)$, with $T \geqslant x \geqslant 0$; the condition is therefore

$$\int_0^T [\chi_T(x, y) + \chi_N(x, y)]h_0(y)dy = \varphi_T(x), \qquad T \geqslant x \geqslant 0\,; \qquad (4.1\text{-}9)$$

this is an integral equation defining the weighting function $h_0(x)$ for which σ^2 has a stationary value. Whether this is a maximum or minimum depends on the sign of $d^2\sigma^2/d\epsilon^2$. From (4.1-8), and using the definitions of χ_T, χ_N of (4.1-6),

$$\frac{d^2\sigma^2}{d\epsilon^2} = 2\int_0^T \eta(x)dx \int_0^T \eta(y)\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T-x)\theta_{Tr}(T-y)dy + $$
$$+ 2\int_0^T \eta(x)dx \int_0^T \eta(y)\frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T-x)\theta_{Nr}(T-y)dy\,,$$
$$= 2\frac{1}{n}\sum_{r=1}^{n}\int_0^T \eta(x)\theta_{Tr}(T-x)dx \int_0^T \eta(y)\theta_{Tr}(T-y)dy + $$
$$+ 2\frac{1}{n}\sum_{r=1}^{n}\int_0^T \eta(x)\theta_{Nr}(T-x)dx \int_0^T \eta(y)\theta_{Nr}(T-y)dy\,,$$
$$= 2\frac{1}{n}\sum_{r=1}^{n}\left[\int_0^T \eta(x)\theta_{Tr}(T-x)dx\right]^2 + 2\frac{1}{n}\sum_{r=1}^{n}\left[\int_0^T \eta(x)\theta_{Nr}(T-x)dx\right]^2,$$

which is non-negative for all x and all $\eta(x)$. The stationary value is therefore a minimum, and the function $h_0(x)$ given by (4.1-9) is the optimum weighting function with which this minimum miss distance is achieved, provided that a unique solution of (4.1-9) exists.

4.2. *The Optimum System when the Ensembles of Target and Noise Angles are Stationary.*

4.2.1. *Ensemble averages.*—The solution of (4.1-9) depends on the properties of $\varphi_T(x)$, $\chi_T(x, y)$ and (x, y). The simplest solution is obtained when the inputs $\theta_{Tr}(T)$ and $\theta_{Nr}(T)$ can be regarded as stationary (though not necessarily ergodic) series, at least over a sufficient interval of time; and this case will be treated first. Returning to the definitions of φ_T, χ_T (4.1-6),

$$\begin{aligned}\chi_T(x, y) &= \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T-x)\theta_{Tr}(T-y) \\ &= \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}[T-y-(x-y)]\theta_{Tr}(T-y).\end{aligned}$$

Assume now that $\chi_T(x, y)$ is invariant for the shift y. Then

$$\begin{aligned}\chi_T(x, y) &= \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}[T-(x-y)]\theta_{Tr}(T) \\ &= \varphi_T(x-y),\end{aligned} \tag{4.2-1}$$

since

$$\varphi_T(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T-x)\theta_{Tr}(T),$$

from (4.1-6).

The assumption of invariance implies that $\varphi_T(x)$ is independent of T—that is, the average product of the target angle at one instant and its value x seconds earlier depends only on the interval x, and not on the particular instant chosen for the evaluation. In other words, the series of target angles is stationary with regard to averages taken over the ensemble, and $\varphi_T(x)$ is the autocorrelation function of the ensemble.

The supposition that the series is stationary does not imply that it is also ergodic: that is, the time average

$$\langle \theta_i(t)\theta_i(t-x)\rangle^t$$

for any one target path θ_i is not necessarily equal to $\varphi_T(x)$ as defined above. As an example of a stationary but non-ergodic series, consider a distribution of quantities a_1, a_2, a_3 . . . , each of which is constant with time. Then

$$\langle a_i^2\rangle^t = a_i^2 \neq \frac{1}{n}\sum_{r=1}^{n} a_r^2,$$

except for a particular choice of i.

The validity of the stationary assumption is further discussed below (Section 4.2.2).

If the noise is also stationary, the same argument leads to

$$\chi_N(x, y) = \varphi_N(x-y), \tag{4.2-2}$$

where $\varphi_N(x)$ is defined as

$$\varphi_N(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T-x)\theta_{Nr}(T), \tag{4.2-3}$$

the autocorrelation function of the noise ensemble.

Equation (4.1-9) can now be rewritten as

$$\left.\begin{aligned} &\int_0^T [\varphi_T(x-y)+\varphi_N(x-y)]h_0(y)dy = \varphi_T(x), \qquad T \geqslant x \geqslant 0, \\ &\text{where} \\ &\varphi_T(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)\theta_{Tr}(T-x), \\ &\text{and} \\ &\varphi_N(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T)\theta_{Nr}(T-x). \end{aligned}\right\} \tag{4.2-4}$$

4.2.2. *The region in which the series are assumed stationary.*—For the general solution of this equation it is convenient to write the upper limit of the integral as infinity instead of T; this is clearly justifiable if $h_0(t)$ is small for $t > T$. The function $h_0(t)$ is a measure of the memory of the system, and if $h_0(T)$ were significantly different from zero, it would imply that the system contained time constants of the order of the time of flight T, so that after this time the system would still be using information concerning the target's position at launch. Although the optimum function $h_0(t)$ is not yet known, it obviously must not contain lags of this order, since the targets can make significant manoeuvres during the interval between launch and strike. We may therefore write

$$h_0(t) = 0, \qquad t > T,$$

and equation (4.2-4) becomes

$$\int_0^\infty [\varphi_T(x-y)+\varphi_N(x-y)]h_0(y)dy = \varphi_T(x), \qquad x \geqslant 0. \tag{4.2-5}$$

The functions χ_T and χ_N were assumed to be invariant for a shift y: since y now runs from zero to infinity, it appears at first sight that the functions should be stationary for this infinite shift. A comparison of (4.1-9) and (4.2-5) however shows that the equivalence of $\chi_T(x, y)$ and $\varphi_T(x-y)$, and of $\chi_N(x, y)$ and $\varphi_N(x-y)$, need extend only over the range of y for which $h_0(y)$ is significantly different from zero, since outside this range the integrand is zero for any χ_T, χ_N. Thus the theory holds provided that the series of target angles and noise can be considered stationary over an interval immediately prior to collision, this interval being long compared with the time constants of the beam-riding system. More precisely, the function

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(t)\theta_{Tr}(t-x)$$

should be independent of t for $T_1 < t \leqslant T$, where T_1 is such that the transient effects of an input applied to the system at time zero have virtually disappeared after a time $T - T_1$—i.e. $h(T-T_1) \doteqdot 0$.

4.2.3. *The solution of equation* (4.2-5).—In Appendix II the solution of (4.2-5) is shown to be

$$H_0(i\omega) = \frac{1}{\{\Phi_T(i\omega)+\Phi_N(i\omega)\}^+}\left[K + \left\{\frac{\Phi_T(i\omega)}{\{\Phi_T(i\omega)+\Phi_N(i\omega)\}^-}\right\}_+\right]. \tag{4.2-6}$$

In this equation,

$$H_0(i\omega) = \int_0^\infty h_0(x)e^{-i\omega x}dx, \tag{4.2-7}$$

$$\Phi_T(i\omega) = \frac{1}{\pi}\int_{-\infty}^\infty \varphi_T(x)e^{-i\omega x}dx, \tag{4.2-8}$$

$$\Phi_N(i\omega) = \frac{1}{\pi}\int_{-\infty}^\infty \varphi_N(x)e^{-i\omega x}dx, \tag{4.2-9}$$

and K is a constant. Also

$$(\Phi_T+\Phi_N)^+(\Phi_T+\Phi_N)^- = \Phi_T + \Phi_N,$$

where $(\Phi_T+\Phi_N)^+$ and $(\Phi_T+\Phi_N)^-$ are functions having all singularities confined to the upper half plane and the lower half plane respectively. Here $\Phi_T + \Phi_N$ is considered as a function of a complex variable ω. If for example

$$\Phi_T + \Phi_N = \frac{1}{1+\omega^2},$$

then

$$(\Phi_T+\Phi_N)^+ = \frac{1}{\omega - i}$$

and

$$(\Phi_T+\Phi_N)^- = \frac{1}{\omega + i}.$$

Similarly, the notation $(F)_+$ and $(F)_-$ indicates that the function F has been expressed as the sum of two functions,

$$F(\omega)_+ + F(\omega)_- = F(\omega),$$

where $F(\omega)_+$ and $F(\omega)_-$ have their poles and zeros in the upper and lower half planes respectively.

$H_0(i\omega)$, the Fourier Transform of the optimum weighting function, is the frequency response function of the system; and its Laplace Transform is $H_0(p)$ (Appendix I).

Φ_T and Φ_N are defined by (4.2-8) and (4.2-9) as the Fourier Transforms of the autocorrelation functions $\varphi_T(x)$ and $\varphi_N(x)$—i.e. Φ_T and Φ_N are the spectral densities of the target angles and noise angles respectively. The spectral density and the autocorrelation function are equivalent ways of defining the same properties; we have used the autocorrelation function to formulate the problem and the spectral density to solve the ensuing equations.

The minimum mean square miss distance in terms of the optimum weighting function can now be found from (4.1-7). As in Section 4.2.2 we may extend to infinity the upper limits of the integrals, at the same time substituting for χ_T, χ_N from (4.2-1) and (4.2-2). This gives

$$\sigma_{\min}{}^2 = \varphi_T(0) - 2\int_0^\infty \varphi_T(x)h_0(x)dx + \int_0^\infty h_0(x)dx\int_0^\infty [\varphi_T(x-y) + \varphi_N(x-y)]\, h_0(y)dy. \tag{4.2-10}$$

Since the solution of (4.2-5) is obtained in terms of the frequency response function $H_0(i\omega)$ {or the transfer function $H_0(p)$} rather than the weighting function, it is convenient to rewrite (4.2-10) in terms of $H_0(i\omega)$, and the spectral densities $\Phi_T(i\omega)$ and $\Phi_N(i\omega)$. It is shown in Appendix II that this transformation leads to

$$\sigma_{\min}{}^2 = \int_0^\infty |1 - H_0(i\omega)|^2\Phi_T(i\omega)d\omega + \int_0^\infty |H_0(i\omega)|^2\Phi_N(i\omega)d\omega, \tag{4.2-11}$$

the first term being the miss due to target motion, and the second that due to noise.

The equation for $H_0(p)$ gives the optimum transfer function of the filter or servomechanism in terms of the spectral densities of the true signal and of the noise. Although the equation has been derived with reference to a guided-weapon system, it is perfectly general, and applies to any system in which it is desired to extract the maximum amount of information from a signal contaminated with noise.

In the guided-weapon case we have considered a large number of trials against different targets with different noise. We have derived a system transfer function for which the r.m.s. miss distance taken over all these trials is less than that for any other linear system. The optimum system $H_0(p)$ depends on the statistical properties of the target motion and of the noise, and not on the individual time functions which occur in each attempt.

4.3. *Non-Stationary Target Motion with Stationary Derivatives.*

4.3.1. *Trends in the target motion.*—The solution (4.2-6) of (4.1-9) has been derived on the assumption that the target angles form a stationary ensemble over an interval before interception. It is often the case that the derivatives of a set of functions constitute a stationary ensemble, while the functions themselves do not. Suppose for example that the target angle $\theta_{Tr}(t)$ consists of a random part $f_r(t)$, together with a trend represented by a polynomial in t:

$$\theta_{Tr}(t) = f_r(t) + \sum_{i=0}^{m-1} a_i t^i, \tag{4.3-1}$$

and suppose further that $f_r(t)$ is stationary with zero mean, having the autocorrelation function $\rho_f(x)$. Then

$$\begin{aligned}\chi_T(x, y) &= \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T-x)\theta_{Tr}(T-y)\\ &= \frac{1}{n}\sum_{r=1}^{n}\left[f_r(T-x) + \sum_{i=0}^{m-1} a_i(T-x)^i\right]\left[f_r(T-y) + \sum_{i=0}^{m-1} a_i(T-y)^i\right]\\ &= \frac{1}{n}\sum_{r=1}^{n} f_r(T-x)f_r(T-y) + \sum_{i=0}^{m-1} a_i(T-x)^i \sum_{i=0}^{m-1} a_i(T-y)^i,\end{aligned}$$

or

$$\chi_T(x, y) = \rho_f(x-y) + \sum_{i=0}^{m-1} a_i(T-x)^i \sum_{i=0}^{m-1} a_i(T-y)^i, \tag{4.3-2}$$

which is a function of T and therefore not stationary. On differentiating (4.3-2) m times with respect to $(T-x)$ and m times with respect to $(T-y)$,

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}^{(m)}(T-x)\theta_{Tr}^{(m)}(T-y) = (-1)^m \rho_f^{(2m)}(x-y), \tag{4.3-3}$$

and this is independent of T. The mth derivatives of the target angles therefore form a stationary set; we may denote its autocorrelation function by $\varphi_{T^{(m)}}(x)$, so that

$$\varphi_{T^{(m)}}(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}^{(m)}(T)\theta_{Tr}^{(m)}(T-x) = (-1)^m \rho_f^{(2m)}(x) \tag{4.3-4}$$

4.3.2. *The solution when the mth derivatives of the target angles form a stationary set.*—Suppose now that the mth derivatives of the target angles can be considered to form a stationary set, while derivatives of order less than m do not. In accordance with experimental evidence, it will be assumed that the noise angle itself is a stationary series, in the sense defined in Section 4.2.2.

Equation (4.1-9) still applies, and can be rewritten as

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T-x)\left[\int_0^\infty \theta_{Tr}(T-y)h_0(y)dy - \theta_{Tr}(T)\right] +$$

$$+\frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T-x)\int_0^\infty \theta_{Nr}(T-y)h_0(y)dy = 0, \qquad x \geqslant 0, \tag{4.3-5}$$

by using the definitions of χ_T, χ_N and φ_T given in (4.1-6). The upper limit of integration has been changed from T to infinity, on the grounds that $h_0(t) = 0$, $t > T$, as discussed in Section 4.2.2.

Now let

$$g(y) = (-1)^m \int\int\int_y^\infty \ldots \int h_0(x)dx, \qquad \text{i.e.} \qquad g^{(m)}(y) = h_0(y),$$

and

$$g^{(i)}(\infty) = 0 \qquad \text{for} \qquad i = 0, 1, 2 \ldots m. \tag{4.3-6}$$

Differentiate (4.3-5) m times with respect to $(T-x)$, and integrate m times by parts, using the above definition of $g(y)$. This leads to

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}^{(m)}(T-x)\left[\int_0^\infty g(y)\theta_{Tr}^{(m)}(T-y)dy - g(0)\theta_{Tr}^{(m-1)}(T) - g^{(1)}(0)\theta_{Tr}^{(m-2)}(T) - \ldots\right.$$

$$\left.- g^{(m-2)}(0)\theta_{Tr}^{(1)}(T) - \{1 + g^{(m-1)}(0)\}\,\theta_{Tr}(T)\right] +$$

$$+\frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}^{(m)}(T-x)\left[\int_0^\infty g(y)\theta_{Nr}^{(m)}(T-y)dy - g(0)\theta_{Nr}^{(m-1)}(T) - \ldots\right.$$

$$\left.- g^{(m-2)}(0)\theta_{Nr}^{(1)}(T) - g^{(m-1)}(0)\theta_{Nr}(T)\right] = 0, \qquad x \geqslant 0, \tag{4.3-7}$$

where (m) denotes the mth derivative with respect to the indicated argument.

We can attach no meaning to terms of the form

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}^{(m)}(T-x)\theta_{Tr}^{(m-i)}(T)$$

unless $i = 0$, since we have assumed that only the mth derivatives of the target angles form a stationary ensemble. It is therefore necessary to eliminate these terms by postulating that the system has no displacement lag, no velocity lag, etc., up to no $(m-1)$th order lag. The necessary conditions (Appendix I) are

$$\int_0^\infty h(x)dx = 1 \qquad \text{for no displacement lag}$$

$$\int_0^\infty h(x)dx = 1 \qquad \text{and} \qquad \int_0^\infty xh(x)dx = 0, \qquad \text{for no velocity lag,}$$

$$\int_0^\infty h(x)dx = 1 \qquad \text{and} \qquad \int_0^\infty x^r h(x)dx = 0, \quad r = 1, 2 \ldots i \text{ for no } i\text{th order lag.}$$

Then

$$\int_0^\infty g^{(m)}(x)dx = 1\,, \quad \text{i.e. } g^{m-1}(0) = -1\,, \quad \text{for no displacement lag.}$$

For no velocity lag,

$$\int_0^\infty xg^{(m)}(x)dx = \left[xg^{(m-1)}(x)\right]_0^\infty - \int_0^\infty g^{(m-1)}(x)dx = 0\,,$$

so that

$$\int_0^\infty g^{(m-1)}(x)dx = 0\,,$$

from (4.3-6), or

$$g^{(m-2)}(0) = 0\,.$$

For no ith order lag,

$$\int_0^\infty x^i g^{(m)}(x)dx = 0\,, \quad \text{i.e.} \quad g^{(m-i-1)}(0) \quad \text{for} \quad i = 1, 2, \ldots m-1\,.$$

Equation (4.3-7) then reduces to

$$\int_0^\infty g(y)\,[\chi_{T^{(m)}}(x, y) + \chi_{N^{(m)}}(x, y)]\,dy + \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T)\theta_{Nr}{}^{(m)}(T-x) = 0\,, x \geqslant 0\,,$$

where

$$\chi_{T^{(m)}}(x, y) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}{}^{(m)}(T-x)\theta_{Tr}{}^{(m)}(T-y)$$

and

$$\chi_{N^{(m)}}(x, y) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}{}^{(m)}(T-x)\theta_{Nr}{}^{(m)}(T-y)\,.$$

We may write this equation as

$$\int_0^\infty g(y)\,[\varphi_{T^{(m)}}(x-y) + \varphi_{N^{(m)}}(x-y)]\,dy + (-1)^m\varphi_N{}^{(m)}(x) = 0\,, x \geqslant 0\,, \tag{4.3-8}$$

where $\varphi_{T^{(m)}}(x)$ is the autocorrelation function of the mth derivatives of the target angles, assumed stationary over the necessary interval. For the noise we have

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T-x)\theta_{Nr}(T-y) = \varphi_N(x-y)\,,$$

so that

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}{}^{(1)}(T-x)\theta_{Nr}(T-y) = -\varphi_N{}^{(1)}(x-y)\,,$$

and

$$\varphi_{N^{(1)}}(x-y) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}{}^{(1)}(T-x)\theta_{Nr}{}^{(1)}(T-y) = -\varphi_N{}^{(2)}(x-y)\,.$$

By successive differentiation,

$$\varphi_{N^{(m)}}(x-y) = (-1)^m\varphi_N{}^{(2m)}(x-y)\,. \tag{4.3-9}$$

This is a particular case of a more general theorem which relates the covariance of two sets of functions $X_r(x)$, $Y_r(y)$ and the covariance of functions derived from $X_r(x)$, $Y_r(y)$ by linear operations. Thus, if

$$\operatorname{cov}\,[X(x); \quad Y(y)] = \rho(x-y),$$

then

$$\frac{1}{n}\sum_{r=1}^{n} X_r(x)\,Y_r(y) = \rho(x-y),$$

so that

$$\frac{1}{n}\sum_{r=1}^{n} F_1(D_x)X_r(x)F_2(D_y)\,Y_r(y) = F_1(D_x)F_2(D_y)\rho(x-y),$$

where $F_1(D_x)$, $F_2(D_y)$ are any two operators.

Thus

$$\operatorname{cov}\,[F_1(D_x)X(x); \quad F_2(D_y)\,Y(y)] = F_1(D_x)F_2(D_y)\operatorname{cov}\,[X(x); \quad Y(y)], \tag{4.3-10}$$

provided that a meaning can be attached to cov $[X(x);\ Y(y)]$. We have assumed that this proviso holds only for the noise and not for the target angles, so that there is no expression corresponding to (4.3-9) for $\varphi_{T^{(m)}}(x)$.

On using (4.3-8) and (4.3-9),

$$\int_0^\infty g(y)\,[\varphi_{T^{(m)}}(x-y) + (-1)^m\varphi_N{}^{(2m)}(x-y)]\,dy + (-1)^m\varphi_N{}^{(m)}(x) = 0\,,\ x \geqslant 0\,, \tag{4.3-11}$$

the solution of which is shown in Appendix II to be

$$H_0 = \frac{1}{\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}} + \Phi_N\right)^+}\left[K + \left\{\frac{\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}}{\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}} + \Phi_N\right)^-}\right\}_+\right] \tag{4.3-12}$$

(The argument $i\omega$ of H_0, $\Phi_{T^{(m)}}$ and Φ_N has been omitted for brevity.) In this expression

$H_0(i\omega)$ = the optimum frequency-response function,

$\Phi_{T^{(m)}}(i\omega)$ = the spectral density of the mth derivatives of the target angles,

$\Phi_N(i\omega)$ = the spectral density of the noise,

and K = a constant, determined by the conditions imposed on $h_0(t)$.

We may note that (4.3-12) could be formally derived from (4.2-6) by writing

$$\Phi_T(i\omega) = \Phi_{T^{(m)}}(i\omega)/\omega^{2m},$$

that is, by ascribing a spectral density to the target angles. This procedure however cannot be justified, since we have seen that the target angles may not form a stationary set, in which case the spectral density will have no meaning. The difficulty has been avoided by postulating that the system shall have no lags up to order $(m-1)$.

The minimum mean square miss distance obtained with the optimum operator can now be found from (4.1-7) and (4.1-9). From these equations

$$\sigma_{\min}{}^2 = \varphi_T(0) - \int_0^\infty \varphi_T(x) h_0(x) dx,$$

$$= \frac{1}{n} \sum_{r=1}^{n} [\theta_{Tr}(T)]^2 - \int_0^\infty h_0(x) \frac{1}{n} \sum_{r=1}^{n} \theta_{Tr}(T) \theta_{Tr}(T-x) dx.$$

If

$$\theta_{Tr}(t) = f_r(t) + \sum_{i=0}^{m-1} a_i t^i,$$

as in equation (4.3-1), then

$$\sigma_{\min}{}^2 = \rho_f(0) + \left(\sum_{i=0}^{m-1} a_i T^i \right)^2 - \int_0^\infty \rho_f(x) h_0(x) dx - \int_0^\infty \sum_{i=0}^{m-1} a_i (T-x)^i h_0(x) dx,$$

or

$$\sigma_{\min}{}^2 = \rho_f(0) - \int_0^\infty \rho_f(x) h_0(x) dx, \tag{4.3-13}$$

since

$$\int_0^\infty h_0(x) dx = 1$$

and

$$\int_0^\infty x^i h_0(x) dx = 0 \qquad \text{for} \qquad i = 1, 2 \ldots m-1.$$

Thus no error arises due to trends in the target motion because the optimum transfer function has no lag up to order $(m-1)$, the order of the polynomial associated with $\theta_{Tr}(t)$.

Now define $S(i\omega)$ as

$$S(i\omega) = \frac{1}{\pi} \int_{-\infty}^\infty \rho_f(x) e^{-i\omega x} dx,$$

so that

$$\rho_f(x) = \frac{1}{2} \int_{-\infty}^\infty S(i\omega) e^{i\omega x} d\omega$$

$$= \frac{1}{2} \int_{-\infty}^\infty S(i\omega) e^{-i\omega x} d\omega,$$

since $\rho_f(x)$ is an even function. On substituting for $\rho_f(x)$ in (4.3-13),

$$\sigma_{\min}{}^2 = \frac{1}{2} \int_{-\infty}^\infty S(i\omega) d\omega - \frac{1}{2} \int_0^\infty h_0(x) dx \int_{-\infty}^\infty S(i\omega) e^{i\omega x} d\omega.$$

Interchanging the order of integration and noting that

$$H_0(i\omega) = \int_0^\infty h_0(x) e^{-i\omega x} dx,$$

$$\sigma_{\min}{}^2 = \frac{1}{2} \int_{-\infty}^\infty [1 - H_0(i\omega)] S(i\omega) d\omega. \tag{4.3-14}$$

But

$$S(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \rho_f(x)e^{-i\omega x}dx = \frac{1}{\pi}\frac{1}{(i\omega)^{2m}}\int_{-\infty}^{\infty} \rho_f^{(2m)}(x)e^{-i\omega x}dx$$

$$= \frac{1}{\pi}\frac{1}{\omega^{2m}}\int_{-\infty}^{\infty} (-1)^m\rho_f^{(2m)}(x)e^{-i\omega x}dx,$$

or

$$S(i\omega) = \frac{1}{\pi}\frac{1}{\omega^{2m}}\int_{-\infty}^{\infty} \varphi_{T^{(m)}}(x)e^{-i\omega x}dx, \qquad \text{from (4.3-4).}$$

$$= \Phi_{T^{(m)}}(i\omega)/\omega^{2m}. \tag{4.3-15}$$

Combining (4.3-14) and (4.3-15) gives

$$\sigma_{\min}{}^2 = \frac{1}{2}\int_{-\infty}^{\infty} [1 - H_0(i\omega)]\frac{\Phi_{T^{(m)}}(i\omega)}{\omega^{2m}}\,d\omega \tag{4.3-16}$$

as the minimum miss distance in terms of the optimum transfer function and the spectral density of the mth derivatives of the target angles.

A more useful form for $\sigma_{\min}{}^2$ can be found by using (4.3-2) in (4.1-7); this reduces to

$$\sigma_{\min}{}^2 = \int_0^{\infty} |1 - H_0(i\omega)|^2\frac{\Phi_{T^{(m)}}(i\omega)}{\omega^{2m}}\,d\omega + \int_0^{\infty} |H_0(i\omega)|^2\Phi_N(i\omega)d\omega, \tag{4.3-17}$$

in which the first term is the mean square miss distance due to target motion, and the second that due to the noise alone.

4.4. *Summary of the Theory without Constraints.*

Equations (4.3-12) and (4.3-17) summarise the theory for the cases in which no limit is placed on the lateral acceleration demanded of the missile. (4.3-12) gives the transfer function of the system for which the mean square miss distance against a large number of targets is a minimum, and this minimum miss distance is given by (4.3-17).

In deriving these equations it has been assumed that the functions

$$\varphi_{T^{(m)}}(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{T_r}{}^{(m)}(T)\theta_{T_r}{}^{(m)}(T-x)$$

and

$$\varphi_N(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T)\theta_{Nr}(T-x)$$

are independent of T over a limited region prior to the instant of nearest approach; this means that the ensembles are stationary over this region, so that $\varphi_{T^{(m)}}(x)$ and $\varphi_N(x)$ become autocorrelation functions, and the expression for $H_0(i\omega)$ is given in terms of the Fourier Transforms of these quantities—i.e. their spectral densities.

The equations show that for no noise $H_0(i\omega) = 1$, and $\sigma_{\min}{}^2 = 0$; if the incoming information is a true record of target motion, the missile should follow this input exactly. Such a system could not of course be realised within the framework of a guided-weapon system, even in the absence of noise. It is necessary to introduce further conditions in the formulation of the problem, expressing the fact that the system must contain a missile of given characteristics, with a limited available acceleration. Such constraints are introduced in Section 5.

5. *The Optimum System Subject to Constraints.*

5.1. *The Minimisation Subject to Constraints.*

Suppose that we wish to limit the mean square value of some quantity $\theta_L(t)$ which occurs at a specified point in the system, such that θ_L and θ_M are related by a given linear operator $F(D)$ (Fig. 4). We shall normally choose $F(D)$ such that θ_L is the demanded acceleration, so that $F(D)$ will involve the aerodynamics of the vehicle; this however is not necessary, and in general $F(D)$ may be any operator which serves to define θ_L, the mean square value of which it is desired to restrict. From Fig. 4 we have

$$\theta_L(t) = F(D)\theta_M(t), \tag{5.1-1}$$

so that we wish to determine the optimum $H(D)$ subject to the condition that the mean square of

$$[F(D)\theta_M(t)]_{t=T}$$

shall be limited to a given value.

It is first necessary to express this quantity in terms of θ_T, θ_N and $h(t)$, the weighting function of the system. From (4.1-1),

$$\theta_{Mr}(t) = \int_0^t h(x)\theta_r(t-x)dx \qquad \{\theta_r(t) = \theta_{Tr}(t) + \theta_{Nr}(t)\}.$$

On differentiating,

$$D\theta_{Mr}(t) = \int_0^t h(x)D\theta_r(t-x)dx + h(t)\theta_r(0),$$

so that

$$D\theta_{Mr}(T) = \int_0^T h(x)D\theta_r(T-x)dx,$$

since $h(T) = 0$ (Section 4.2.2).

This may be generalised to give

$$F(D)\theta_{Mr}(T) = \int_0^T h(x)F(D)\theta_r(T-x)dx;$$

thus, if $\sigma_L{}^2$ denotes the mean square value of $\theta_L(T)$, we have

$$\begin{aligned}\sigma_L{}^2 &= \frac{1}{n}\sum_{r=1}^{n} [F(D)\theta_{Mr}(T)]^2 \\ &= \frac{1}{n}\sum_{r=1}^{n} \int_0^T h(x)F(D)\theta_r(T-x)dx \int_0^T h(y)F(D)\theta_r(T-y)dy \end{aligned} \tag{5.1-2}$$

Since $\sigma_L{}^2$ is to have a given constant value, the optimum weighting function which minimises σ^2, the mean square miss distance, must also minimise

$$\sigma^2 + \lambda\sigma_L{}^2,$$

where λ is any constant. But

$$\sigma^2 = \frac{1}{n}\sum_{r=1}^{n}\left[\theta_{Tr}(T) - \int_0^T h(x)\theta_r(T-x)dx\right]^2,$$

so that

$$\begin{aligned}\sigma^2 + \lambda\sigma_L{}^2 = \frac{1}{n}\sum_{r=1}^{n}\Big[&\{\theta_{Tr}(T)\}^2 - 2\,\theta_{Tr}(T)\int_0^T h(x)\theta_r(T-x)dx + \\ &+ \int_0^T h(x)\theta_r(T-x)dx \int_0^T h(y)\theta_r(T-y)dy + \\ &+ \lambda\int_0^T h(x)F(D)\theta_r(T-x)dx \int_0^T h(y)F(D)\theta_r(T-y)dy\Big].\end{aligned}$$

On putting $h(x) = h_0(x) + \epsilon\eta(x)$, and expressing the condition that

$$\left(\frac{d(\sigma^2+\lambda\sigma_L{}^2)}{d\epsilon}\right)_{\epsilon=0} = 0,$$

it is found that the optimum weighting function $h_0(x)$ must satisfy

$$\frac{1}{n}\sum_{r=1}^{n} \theta_r(T-x)\int_0^T h_0(y)\theta_r(T-y)dy + \lambda\frac{1}{n}\sum_{r=1}^{n} F(D)\theta_r(T-x)\int_0^T h_0(y)F(D)\theta_r(T-y)dy$$
$$= \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)\theta_r(T-x), \qquad \text{for} \qquad T \geqslant x \geqslant 0 \tag{5.1-3}$$

Differentiate this equation m times with respect to x, writing

$$D_x = \frac{d}{dx}, \qquad D_y = \frac{d}{dy}.$$

Then

$$\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_r(T-x)\int_0^T h_0(y)\theta_r(T-y)dy +$$
$$+\lambda\frac{1}{n}\sum_{r=1}^{n} D_x{}^m F(-D_x)\theta_r(T-x)\int_0^T h_0(y)F(-D_y)\theta_r(T-y)dy$$
$$= \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)D_x{}^m\theta_r(T-x).$$

Now write $h_0(y) = D_y{}^m g(y)$, so that

$$g(y) = (-1)^m \int\int_y^\infty \dots \int h_0(x)dx.$$

Integrating m times by parts yields

$$\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_r(T-x)\left[\Big\{D_y{}^{m-1}g(y)\theta_r(T-y) - D_y{}^{m-2}g(y)D_y\theta_r(T-y) + \dots\Big\}_0^\infty +\right.$$
$$\left.+(-1)^m\int_0^\infty g(y)D_y{}^m\theta_r(T-y)dy\right] + \lambda\frac{1}{n}\sum_{r=1}^{n} D_x{}^m F(-D_x)\theta_r(T-x) \times$$
$$\times\left[\Big\{D_y{}^{m-1}g(y)F(-D_y)\theta_r(T-y) - D_y{}^{m-2}g(y)D_yF(-D_y)\theta_r(T-y)\dots\Big\}_0^\infty +\right.$$
$$\left.+(-1)^m\int_0^\infty g(y)D_y{}^m F(-D_y)\theta_r(T-y)dy\right] = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)\theta_r(T-x),$$
$$\text{for} \qquad x \geqslant 0, \tag{5.1-4}$$

where the upper limit of integration has been raised from T to infinity.

If, as in Section (4.3.2) the system has no lags up to order $(m-1)$, then

$$D_y{}^{m-1}g(0) = -1, \qquad \text{and} \qquad D_y{}^{m-i-1}g(0) = 0 \qquad \text{for} \qquad i = 1, 2 \dots m-1.$$

Also

$$D_y{}^i g(\infty) = 0 \qquad \text{for} \qquad i = 0, 1, 2 \dots m.$$

With these conditions (5.1-4) reduces to

$$\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_r(T-x)\theta_r(T) + \lambda\left[\frac{1}{n}\sum_{r=1}^{n} D_x{}^m F(-D_x)\theta_r(T-x)F(-D_y)\theta_r(T-y)\right]_{y=0} +$$

$$+ (-1)^m\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_r(T-x)\int_0^\infty g(y)D_y{}^m\theta_r(T-y)dy +$$

$$+ \lambda(-1)^m\frac{1}{n}\sum_{r=1}^{n} D_x{}^m F(-D_x)\theta_r(T-x)\int_0^\infty g(y)D_y{}^m F(-D_y)\theta_r(T-y)dy$$

$$= \frac{1}{n}\sum_{r=1}^{n}\theta_{Tr}(T)D_x{}^m\theta_r(T-x), \qquad x \geqslant 0.$$

But

$$\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_r(T-x)\theta_r(T) = \frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_{Tr}(T-x)\theta_{Tr}(T) + \frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_{Nr}(T-x)\theta_{Nr}(T),$$

and

$$\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_{Tr}(T)\theta_r(T-x) = \frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_{Tr}(T)\theta_{Tr}(T-x),$$

assuming no cross-correlation. Thus

$$\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_{Nr}(T-x)\theta_{Nr}(T) +$$

$$+ (-1)^m\lambda\left[\frac{1}{n}\sum_{r=1}^{n} D_x{}^m F(-D_x)\theta_r(T-x)D_y{}^m F_1(-D_y)\theta_r(T-y)\right]_{y=0} +$$

$$+ (-1)^m\int_0^\infty g(y)\frac{1}{n}\sum_{r=1}^{n} D_x{}^m\theta_r(T-x)D_y{}^m\theta_r(T-y)dy +$$

$$+ \lambda(-1)^m\int_0^\infty g(y)\frac{1}{n}\sum_{r=1}^{n} D_x{}^m F(-D_x)\theta_r(T-x)D_y{}^m F(-D_y)\theta_r(T-y)dy = 0,$$

$$\text{for} \qquad x \geqslant 0, \tag{5.1-5}$$

where

$$F(D_y) = D_y{}^m F_1(D_y) \tag{5.1-6}$$

Equation (5.1-5) may be written as

$$(-1)^m\frac{1}{n}\sum_{r=1}^{n}\theta_{Nr}(T)\theta_{Nr}{}^{(m)}(T-x) +$$

$$+ \lambda(-1)^m\left[F(-D_x)F_1(-D_y)\frac{1}{n}\sum_{r=1}^{n}\theta_r{}^{(m)}(T-x)\theta_r{}^{(m)}(T-y)\right]_{y=0} +$$

$$+ (-1)^m\int_0^\infty g(y)\frac{1}{n}\sum_{r=1}^{n}\theta_r{}^{(m)}(T-x)\theta_r{}^{(m)}(T-y)dy +$$

$$+ \lambda(-1)^m\int_0^\infty g(y)F(-D_x)F(-D_y)\frac{1}{n}\sum_{r=1}^{n}\theta_r{}^{(m)}(T-x)\theta_r{}^{(m)}(T-y)dy = 0,$$

$$x \geqslant 0. \tag{5.1-7}$$

But

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T)\theta_{Nr}^{(m)}(T-x) = (-1)^m\varphi_N^{(m)}(x),$$

and

$$\frac{1}{n}\sum_{r=1}^{n} \theta_r^{(m)}(T-x)\theta_r^{(m)}(T-y) = \varphi_{(m)}(x-y),$$

where

$$\varphi_{(m)}(x) = \varphi_{T^{(m)}}(x) + \varphi_{N^{(m)}}(x),$$

and $\varphi_{T^{(m)}}(x)$, $\varphi_{N^{(m)}}(x)$ denote the autocorrelation functions of the mth derivatives of the target and noise angles, assumed stationary, as in Section 4.3.1. Substituting these relations in (5.1-7), one obtains

$$(-1)^m\varphi_N^{(m)}(x) + \lambda[F(-D_x)F_1(-D_y)\varphi_{(m)}(x-y)]_{y=0} + \\ + \int_0^\infty g(y)[1+\lambda F(-D_x)F(-D_y)]\,\varphi_{(m)}(x-y)dy = 0, \qquad x \geqslant 0. \tag{5.1-8}$$

Equation (5.1-8) is solved in Appendix II; its solution is

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}+\Phi_N\right)^+}\left[K+\left\{\frac{\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}}{(1+\lambda F\bar{F})^-\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}+\Phi_N\right)^-}\right\}_+\right], \tag{5.1-9}$$

where $F = F(i\omega)$, $\bar{F} = F(-i\omega)$.

Equations (4.3-16) and (4.3-17) still apply for the minimum mean square miss distance $\sigma_{\min}^2$; and σ_L^2 may be obtained from (5.1-2) as

$$\sigma_L^2 = \int_0^\infty \frac{|H_0F|^2}{\omega^{2m}}(\Phi_{T^{(m)}}+\Phi_{N^{(m)}})d\omega. \tag{5.1-10}$$

In these equations $F(D)$ is the operator which describes the fixed part of the system, the input to which it is desired to limit to a given mean square value σ_L^2. In the beam-riding case we wish to limit the demanded acceleration, and this is related to θ_M by (3.1-1):

$$f_M = r_M\ddot{\theta}_M + 2\dot{r}_M\dot{\theta}_M = A(D)f_D,$$

or

$$\frac{f_D}{r_M} = \frac{D(D+2c_M)}{A(D+c_M)}\theta_M, \tag{5.1-11}$$

where $\dot{r}/r_M = c_M$, as in Section (3.2.2), and $A(D)$ is the operator relating demanded and achieved accelerations, modified by internal feedbacks. Thus, if $A'(D)$ describes the unmodified aerodynamic behaviour, and $M(D)$ the feedback of lateral acceleration, etc., then

$$A(D) = \frac{A'(D)}{1+M(D)A'(D)}.$$

Comparing (5.1-11) and (5.1-1), it is seen that in order to limit the demanded acceleration we must have

$$F(D) = \frac{D(D+2c_M)}{A(D+c_M)}, \tag{5.1-12}$$

and σ_L^2 is then a measure of the mean square acceleration demanded at the range r_M. If for the moment we assume $c_M = 0$ (i.e. constant range), then

$$F(D) = \frac{D^2}{A(D)},$$

so that

$$F(i\omega) = -\frac{\omega^2}{A(i\omega)};$$

substituting this in (5.1-9) and (5.1-10),

$$H_0 = \frac{1}{\left(1+\frac{\lambda\omega^4}{A\bar{A}}\right)^+ \left(\frac{\Phi_{T(m)}}{\omega^{2m}}+\Phi_N\right)^+} \left[K + \left\{\frac{\frac{\Phi_{T(m)}}{\omega^{2m}}}{\left(1+\frac{\lambda\omega^4}{A\bar{A}}\right)^- \left(\frac{\Phi_{T(m)}}{\omega^{2m}}+\Phi_N\right)^-}\right\}_+\right], \tag{5.1-13}$$

and

$$\sigma_L^2 = \int_0^\infty \left|\frac{H_0}{A}\right|^2 \frac{1}{\omega^{2m-4}} [\Phi_{T(m)} + \Phi_{N(m)}]\, d\omega, \tag{5.1-14}$$

where $A = A(i\omega)$, $\bar{A} = A(-i\omega)$.

(5.1-13) gives the optimum transfer function of a system which includes a missile described by the operator $A(D)$, the mean square demanded acceleration being limited to $r_M^2\sigma_L^2$. In order to find this value it is necessary to determine H_0 from (5.1-13) for a number of trial values of λ, and then to evaluate σ_L^2 from (5.1-14). In this way the value of λ which corresponds to a desired mean square acceleration can be determined.

5.2. *The Maximum and Mean Square Acceleration Demands.*

Suppose now that the *maximum* demand for acceleration must be limited to $f_{\max}$, this being determined by the structural strength of the vehicle. To preserve the validity of the linear analysis, we must ensure that the r.m.s. acceleration demand $r_M\sigma_L$ is such that this limiting acceleration is rarely called for. If the acceleration distribution is assumed to be Gaussian, the chance of the demanded acceleration exceeding the limits is

$$2\frac{1}{r_M\sigma_L\sqrt{(2\pi)}}\int_{f_{\max}}^\infty \exp\left[\frac{-x^2}{2r_M^2\sigma_L^2}\right]dx$$

$$= 1 - \frac{2}{r_M\sigma_L\sqrt{(2\pi)}}\int_0^{f_{\max}} \exp\left[\frac{-x^2}{2r_M^2\sigma_L^2}\right]dx = 1 - \frac{2}{\sqrt{\pi}}\int_0^{f_{\max}/r_M\sigma_L\sqrt{2}} \exp(-y^2)dy$$

$$= 1 - \operatorname{erf}\left[\frac{f_{\max}}{r_M\sigma_L\sqrt{2}}\right]; \tag{5.2-1}$$

if for a given $f_{\max}$—i.e. a given vehicle—σ_L is chosen such that the chance of limiting is small—say 5%—it follows that the system is operating as a virtually linear system, and the analysis holds.

The relation between the miss distance and the available missile acceleration $f_{\max}$ is discussed in Section 7.3, where a number of optimum operators are evaluated for particular forms of $\Phi_{T(m)}$ and Φ_N.

5.3. *The Optimum System when the Target Lateral Accelerations are Stationary.*

5.3.1. *Statistical properties of the target motion.*—The derivation of the optimum operator depends on the function

$$\varphi_{T^{(m)}}(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}^{(m)}(T)\theta_{Tr}^{(m)}(T-x),$$

where the order of the derivative m is such that $\varphi_{T^{(m)}}(x)$ is independent of T over the interval defined in Section 4.2.2. This quantity clearly depends on a large number of factors, such as the type of aircraft, the nature and location of enemy objectives in relation to the weapon site, and the enemy's concept of the most effective avoiding action. In a given tactical situation some of these factors will be known beforehand, leading to a value of $\varphi_{T^{(m)}}(x)$—and therefore to an optimum system—appropriate to that particular situation. The effects of unknown factors must be estimated, and $\varphi_{T^{(m)}}(x)$ will then reflect the average influence of these effects, weighted according to their importance. The 'optimum system' is therefore a subjective concept, depending on the information available to the weapon designer concerning the likely target motion. If for example in a long series of engagements there is no *a priori* information to distinguish one series of attacks from another, there is just one optimum system for the whole series; if, on the other hand, distinguishing features are known to exist, advantage can be taken of this knowledge to construct a number of systems, each of which is the optimum for a particular series of attacks.

A detailed study of target motion has not yet been made; on general grounds however we may note that the lateral acceleration of the target is bounded, and is likely to vary about a mean value of zero, if we suppose that the mean target path is a straight line with random deviations about this path. It therefore appears reasonable to assume that the target lateral accelerations form a stationary ensemble over the necessary interval. If in addition the target path is directed towards the ground radar site—i.e. if the objectives are in this area—the lateral acceleration of the target will also be its acceleration normal to the line of sight:

$$f_T = r_T\ddot{\theta}_T + 2\dot{r}_T\dot{\theta}_T.$$

If we make the same assumption for r_T as for r_M in Section 3.2.2, then

$$\frac{f_T}{r_T} = D(D+2c_T)\theta_T, \qquad \text{where} \qquad c_T = \frac{\dot{r}_T}{r_T}. \tag{5.3-1}$$

Denote by $\psi_T(x)$ the autocorrelation function of f_T/r_T; this may be taken to be the autocorrelation function of f_T, divided by the square of the range r_T, since this varies only slowly over the short range of interest.

Then

$$\begin{aligned}\psi_T(x-y) &= \frac{1}{r_T^2}\frac{1}{n}\sum_{r=1}^{n} f_{Tr}(T-x)f_{Tr}(T-y)\\ &= \frac{1}{n}\sum_{r=1}^{n} -D_x(-D_x+2c_T)\theta_{Tr}(T-x) - D_y(-D_y+2c_T)\theta_{Tr}(T-y),\\ &= \frac{1}{n}\sum_{r=1}^{n} (D_x^2-2c_TD_x)\theta_{Tr}(T-x)\,(D_y^2-2c_TD_y)\theta_{Tr}(T-y),\end{aligned} \tag{5.3-2}$$

from (5.3-1).

5.3.2. *The optimum solution.*—The introduction of $\psi_T(x)$ requires a slight modification to the derivation of equations (5.1-9) and (5.1-10).

If (5.1-3) is multiplied by the operator $(D_x^2 - 2c_T D_x)$, then

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)(D_x^2-2c_TD_x)\theta_r(T-x)$$

$$= \frac{1}{n}\sum_{r=1}^{n} (D_x^2-2c_TD_x)\theta_r(T-x)\int_0^\infty h_0(y)\theta_r(T-y)dy +$$

$$+\lambda\frac{1}{n}\sum_{r=1}^{n} (D_x^2-2c_TD_x)F(-D_x)\theta_r(T-x)\int_0^\infty h_0(y)F(-D_y)\theta_r(T-y)dy\,,\ x \geqslant 0\,. \quad (5.3\text{-}3)$$

Now let

$$h_0(y) = (D_y^2+2c_TD_y)g(y)\,. \quad (5.3\text{-}4)$$

Then

$$\int_0^\infty g^{(2)}(y)\theta_r(T-y)dy = \Big[g^{(1)}(y)\theta_r(T-y) - g(y)D_y\theta_r(T-y)\Big]_0^\infty + \int_0^\infty g(y)D_y^2\theta_r(T-y)dy\,,$$

and

$$\int_0^\infty g^{(1)}(y)\theta_r(T-y)dy = \Big[g(y)\theta_r(T-y)\Big]_0^\infty - \int_0^\infty g(y)D_y\theta_r(T-y)dy\,.$$

Hence

$$\int_0^\infty h_0(y)\theta_r(T-y)dy = \int_0^\infty \theta_r(T-y)(D_y^2+2c_TD_y)g(y)dy$$

$$= \Big[\{g^{(1)}(y) + 2c_Tg(y)\}\,\theta_r(T-y) - g(y)D_y\theta_r(T-y)\Big]_0^\infty +$$

$$+\int_0^\infty g(y)(D_y^2-2c_TD_y)\theta_r(T-y)dy\,.$$

For no displacement lag,

$$\int_0^\infty (D_y^2+2c_TD_y)g(y)dy = 1\,, \qquad \text{or} \qquad (D_y+2c_T)g(0) = -1 \quad (5.3\text{-}5)$$

Thus

$$\int_0^\infty h_0(y)\theta_r(T-y)dy = \theta_r(T) + g(0)\left[D_y\theta_r(T-y)\right]_{y=0} + \int_0^\infty g(y)(D_y^2-2c_TD_y)\theta_r(T-y)dy\,.$$

Substituting this last expression in (5.3-3) gives

$$\frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)(D_x^2-2c_TD_x)\theta_r(T-x) = \frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)\theta_r(T-x)\theta_r(T) +$$

$$+\frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)\theta_r(T-x)g(0)\left[D_y\theta_r(T-y)\right]_{y=0} +$$

$$+\lambda\left[\frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)F(-D_x)\theta_r(T-x)F(-D_y)\theta_r(T-y)\right]_{y=0} +$$

$$+\lambda\left[\frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)F(-D_x)\theta_r(T-x)g(0)D_yF(-D_y)\theta_r(T-y)\right]_{y=0} +$$

$$+\frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)\theta_r(T-x)\int_0^\infty g(y)(D_y^2-2c_TD_y)\theta_r(T-y)dy +$$

$$+\lambda\frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)F(-D_x)\theta_r(T-x)\int_0^\infty g(y)(D_y^2-2c_TD_y)F(-D_y)\theta_r(T-y)dy\,,$$

$$x \geqslant 0\,. \quad (5.3\text{-}6)$$

Now put

$$\psi(x-y) = \frac{1}{n}\sum_{r=1}^{n} (D_x^2-2c_T D_x)\theta_r(T-x)\,(D_y^2-2c_T D_y)\theta_r(T-y)\,, \tag{5.3-7}$$

and let

$$F(D_y) = (D_y^2+2c_T D_y)F_2(D_y)\,. \tag{5.3-8}$$

Then (5.3-6) becomes

$$\begin{aligned} 0 = {} & \frac{1}{n}\sum_{r=1}^{n} (D_x^2-2c_T D_x)\theta_N(T-x)\theta_N(T) + \lambda\,[F(-D_x)F_2(-D_y)\psi(x-y)]_{y=0} + \\ & + \int_0^\infty g(y)\,[1+\lambda F(-D_x)F(-D_y)]\,\psi(x-y)dy + \\ & + g(0)\left[(1+\lambda F(-D_x)F(-D_y))\frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_T D_x)\theta_r(T-x)D_y\theta_r(T-y)\right]_{y=0}, \\ & \qquad x \geqslant 0\,, \end{aligned} \tag{5.3-9}$$

the solution of which is given in Appendix II as

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+\left(\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T^2)}+\Phi_N\right)^+}\left[K+\left\{\frac{\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T^2)}}{(1+\lambda F\bar{F})^-\left(\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T^2)}+\Phi_N\right)^-}\right\}_+\right] \tag{5.3-10}$$

where Ψ_T is the spectral density of the target lateral acceleration, divided by r_T^2:

$$\Psi_T(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \psi_T(x)e^{-i\omega x}dx\,,$$

$\psi_T(x)$ being the autocorrelation function defined in (5.3-2).

If we wish to limit the mean square demanded acceleration, we again have

$$F(D) = \frac{D(D+2c_M)}{A(D+c_M)}$$

as in (5.1-12); and σ_L^2 is then given by

$$\sigma_L^2 = \int_0^\infty \frac{\omega^2+4c_M^2}{\omega^2+4c_T^2}\left|\frac{H_0(i\omega)}{A(i\omega+c_M)}\right|^2 [\Psi_T(i\omega)+\omega^2(\omega^2+4c_T^2)\Phi_N(i\omega)]\,d\omega\,, \tag{5.3-11}$$

which reduces to (5.1-14) for $c_M = c_T = 0$ and $m = 2$, for then $\Psi_T(i\omega) = \Phi_{T(m)}(i\omega)$.

Finally, the minimum mean square miss distance, obtained from (4.1-4) and (5.3-2), is given by

$$\sigma_{\min}^2 = \int_0^\infty |1-H_0(i\omega)|^2\frac{\Psi_T(i\omega)}{\omega^2(\omega^2+4c_T^2)}\,d\omega + \int_0^\infty |H_0(i\omega)|^2\Phi_N(i\omega)d\omega\,. \tag{5.3-12}$$

6. *The Optimum Transfer Function in Relation to the Beam-riding System.*

6.1. *Necessary Conditions—the Choice of Networks.*

We have seen (Section 3.2.3) that with certain approximations the overall transfer function of the beam-riding system is

$$\frac{T(p)}{1+T(p)} \; \frac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M)+p^2+2c_M p}\,,$$

where $A(p)$ is the transfer function of the modified aerodynamics relating demanded and achieved acceleration, $S(p)$ that of the missile receiver and any additional networks, and $T(p)$ the open-loop transfer function of the target-tracking system.

We have shown that in fact the optimum transfer function is $H_0(p)$, so that $A(p)$, $S(p)$ and $T(p)$ must satisfy

$$\frac{T(p)}{1 + T(p)} \frac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M) + p^2 + 2c_M p} = H_0(p). \tag{6.1-1}$$

where $H_0(i\omega)$ can be evaluated in terms of $\Phi_{T(m)}(i\omega)$, $\Phi_N(i\omega)$, $A(i\omega)$ and σ_L^2, the limit imposed on the mean square demanded acceleration. In (6.1-1) we must regard $A(p)$ as fixed, since $H_0(p)$ is the optimum transfer function for a system which includes a missile whose transfer function is $A(p)$. The realisation of the optimum then rests on the choice of the functions $T(p)$ and $S(p)$; equation (6.1-1) does not define each separately, but only the relation between them, so that the same overall system can be obtained in a number of different ways—i.e. a given system may be optimised by a correction network in the tracker or the missile, or both.

If the missile-borne equipment $S(p)$ is chosen independently, it is necessary to adjust the transfer function of the tracker to satisfy (6.1-1), provided that such a procedure is permissible (*see* Section 6.2).

Rearrangement of (6.1-1) yields

$$T(p) = \frac{H_0(p)}{\dfrac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M) + p^2 + 2c_M p} - H_0(p)}, \tag{6.1-2}$$

which gives the required transfer function of the tracker.

This equation shows that as far as minimising the mean square miss distance is concerned, there is no particular requirement for the response of the missile system to beam movements: from other considerations however it is desirable that the missile loop should form a stable and reasonably well-behaved system, so that the missile can recover from disturbances which may arise in the early stages—gathering and initial dispersion. The precise nature of the response is immaterial, provided that $T(p)$ is adjusted to satisfy (6.1-2). The conditions imposed on $H_0(p)$ are such that it is stable and physically realisable, so that for any choice of $S(p)$ for which the missile loop is stable, $T(p)$ is also stable and physically realisable.

Suppose now that a tracking system having the open-loop transfer function $Y(p)$ exists, and we wish to convert this system to the optimum arrangement. This requires the addition of a network $N(p)$ to the tracker, such that $N(p)\ Y(p) = T(p)$, or

$$N(p) = \frac{1}{Y(p)} \frac{H_0(p)}{\dfrac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M) + p^2 + 2c_M p} - H_0(p)}. \tag{6.1-3}$$

This equation is valid if the incoming noise $\theta_N(t)$ is independent of the transfer function of the tracking servo: the treatment when this is not the case is given in Section 6.2 below.

6.2. *Sources of Noise.*

In the derivation of the optimum operator the effects of noise have been represented by the function $\theta_N(t)$ applied as an input to the whole system. The fact that there may be several sources of noise acting at different points in the system does not invalidate this representation, since the effects of these noise sources can be referred to the input.

6.2.1. *Linear noise sources.*—Suppose that the radar tracking system consists of a number of elements denoted by the operators $Y_1(D)$, $Y_2(D) \ldots Y_n(D)$ (Fig. 5), and that noise enters the system from various sources at the points indicated in Fig. 5a. Assuming that the system is linear,

$$Y_1(D)Y_2(D) \ldots Y_n(D)(\theta_T - \theta_B + \theta_{N1}) + Y_2(D)Y_3(D) \ldots Y_n(D)\theta_{N2} + \ldots + Y_n(D)\theta_{Nn} = \theta_B,$$

or

$$\left[\theta_T - \theta_B + \theta_{N1} + \frac{\theta_{N2}}{Y_1(D)} + \frac{\theta_{N3}}{Y_1(D)Y_2(D)} + \ldots + \frac{\theta_{Nn}}{Y_1(D)Y_2(D) \ldots Y_{n-1}(D)}\right] Y(D) = \theta_B,$$

where

$$Y(D) = Y_1(D)Y_2(D) \ldots Y_n(D).$$

Thus

$$\theta_B = \frac{Y(D)}{1+Y(D)}\left[\theta_T + \theta_{N1} + \frac{\theta_{N2}}{Y_1(D)} + \frac{\theta_{N3}}{Y_1(D)Y_2(D)} + \ldots + \frac{\theta_{Nn}}{Y_1(D)Y_2(D) \ldots Y_{n-1}(D)}\right]$$

$$= \frac{Y(D)}{1+Y(D)}(\theta_T + \theta_N).$$

This is equivalent to a system having one source of noise θ_N at the input Fig. 5b, where

$$\theta_N = \theta_{N1} + \frac{\theta_{N2}}{Y_1(D)} + \frac{\theta_{N3}}{Y_1(D)Y_2(D)} + \ldots + \frac{\theta_{Nn}}{Y_1(D)Y_2(D) \ldots Y_{n-1}(D)} \qquad (6.2\text{-}1)$$

If the spectral densities of the various noise functions are denoted by Φ_{N1}, $\Phi_{N2} \ldots \Phi_{Nn}$, then from (6.2-1)

$$\Phi_N(i\omega) = \Phi_{N1}(i\omega) + \frac{\Phi_{N2}(i\omega)}{|Y_1(i\omega)|^2} + \frac{\Phi_{N3}(i\omega)}{|Y_1(i\omega)Y_2(i\omega)|^2} + \ldots, \qquad (6.2\text{-}2)$$

where Φ_N is the spectral density to be used in the expressions for the optimum operator. It will be noted that Φ_N, and therefore H_0, is now a function of the elements Y_1, Y_2 etc., so that any modification to the tracking system for the purpose of realising the optimum system must leave these elements unchanged—otherwise H_0 is no longer the optimum transfer function. Further, the correcting network $N(D)$ must be introduced beyond the point at which the last noise source appears—in Fig. 5a, after the element Y_n. For then

$$\theta_B = \frac{Y(D)N(D)}{1+Y(D)N(D)}\left(\theta_T + \theta_{N1} + \frac{\theta_{N2}}{Y_1(D)} + \ldots + \frac{\theta_{Nn}}{Y_1(D)Y_2(D)Y_{n-1}(D)}\right),$$

and (6.1-3) gives the correcting network $N(p)$.

If $N(p)$ is introduced at some intermediate point—after Y_1, say, in Fig. 5, we have

$$\theta_B = \frac{Y(D)N(D)}{1+Y(D)N(D)}\left(\theta_T + \theta_{N1} + \frac{\theta_{N2}}{Y_1(D)N(D)} + \ldots + \frac{\theta_{Nn}}{Y_1(D)Y_2(D) \ldots Y_{n-1}(D)N(D)}\right),$$

so that the equivalent noise input θ_N is now different from that assumed in the evaluation of $H_0(p)$, and the equations are invalid.

Thus for distributed noise sources the optimum system can still be derived by modifying the tracking servo system, provided that the elements $Y_1(D)$, etc. remain unaltered, and the correcting network $N(p)$ is placed so as to act on all the sources of noise. Since Φ_N depends on θ_{N1}, $\theta_{N2} \ldots$, and on $Y_1(D)$, $Y_2(D)$, etc., the optimum thus obtained applies only to a particular system whose tracking servo includes these elements and noise sources: a servo having different components would lead to a different optimum system.

It is evident that the same treatment can be applied to sources of noise within the missile, in that they may be expressed as an equivalent noise to the input of the complete system. In this case the optimising network must be placed in the missile, beyond the sources of noise.

If all the noise sources θ_{N2}, $\theta_{N3} \ldots \theta_{Nn}$—i.e. noise arising in the servo itself—could be removed, Φ_N would depend only on θ_{N1}, and $H_0(p)$ would therefore be independent of the servo system, and would apparently give an absolute optimum. However, the function $\theta_{N1}(t)$, which may be described as the primary radar noise, itself depends on the characteristics of the tracker—e.g. the transmitted power and size of dish—so that the optimum applies to that particular set of radar characteristics which gives rise to the noise function $\theta_{N1}(t)$.

6.2.2. *Non-linear noise.*—It has been assumed above that the sources of servo noise are such that the resultant behaviour of the system is still linear. A servo system however will nearly always contain non-linear sources of error, such as those due to hysteresis and static friction. In a well designed system the effects of such errors should be small, but they may nevertheless form a not insignificant proportion of the total noise. We may approximate to this situation by postulating a hypothetical linear servo, with a certain noise input spectral density, such that the output noise spectral density of this servo is the same as that observed with the actual system. When backlash and static friction are dominant, the equivalent noise input and the target input will be correlated; this causes no theoretical inconvenience, although it may be difficult to ascertain the degree of correlation from the data available.

Thus the distribution of noise sources, linear or non-linear, serves to define an equivalent noise spectral density Φ_N, from which the optimum transfer function $H_0(p)$ is derived as before, and equation (6.1-1) applies. If the noise arises from non-linear effects, however, it is not permissible to realise the optimum system by a correcting network in the tracker, since the alteration in the noise output thus obtained would not be as predicted for a linear system; the efficacy of an additional filter is likely to be much less against backlash noise than against linear noise. It is therefore necessary to leave the whole tracking system unchanged, and to realise the optimum transfer function by adjusting $S(p)$ to satisfy (6.1-1). On rearranging this equation,

$$S(p+c_M) = \frac{H_0(p)\,[1 + T(p)]\,[p^2 + 2c_M p]}{A(p+c_M)\,[T(p) - H_0(p)(1+T(p)]}, \tag{6.2-3}$$

and here $T(p)$ is the transfer function of the tracking system with which the equivalent noise spectral density Φ_N was obtained. Again, H_0 is a restricted optimum, giving the best system that can be achieved with this particular target-tracking system.

It is probable that in a practical system the noise contributions from non-linear sources will be small enough to be regarded as emanating from linear sources, at least for small variations of the servo elements; so that a correcting network inserted in the servo may be deemed to have the same effect as on a linear system, provided that the alteration is small. In this case the optimum system would be achieved by modifying both $T(p)$ and $S(p)$ to satisfy (6.1-1).

7. *Optimum Transfer Functions for Particular Forms of Statistical Target and Noise Inputs.*

7.1. *Data and Assumptions Used in the Examples.*

In the following sections a number of examples of optimum beam-riding systems is given (for motion in one plane only), together with the r.m.s. miss distances attained, and the r.m.s. achieved and demanded accelerations. The data and conditions to which these examples refer are summarised below.

7.1.1. *The autocorrelation function and spectral density of the target acceleration.*—It is shown in Appendix III that with certain assumptions {discussed in Section (5.3.1)} the autocorrelation function of the target acceleration normal to the sight line, divided by r_T, is of the form

$$\psi_T(x) = \sigma_T^2 e^{-\beta|x|}. \tag{7.1-1}$$

This function would result if the lateral acceleration of the target were as illustrated in Fig. 6. The acceleration is changed abruptly at random intervals, determined by a Poisson distribution of mean length $1/\beta$—i.e. the chance of finding an interval of duration y is $\beta e^{-\beta y}$. The value of the acceleration varies randomly from interval to interval, with a distribution such that the r.m.s. acceleration is $r_T \sigma_T$. The length of the interval and the acceleration level during the interval are assumed to be uncorrelated.

The spectral density associated with (7.1-1) is

$$\Psi_T = \frac{1}{\pi} \int_{-\infty}^{\infty} \psi_T(x) e^{-i\omega x} dx = \sigma_T^2 \frac{2}{\pi} \frac{\beta}{\beta^2 + \omega^2}. \tag{7.1-2}$$

In the calculations which follow $\beta = 0 \cdot 1$ rad/sec, and $\sigma_T^2 = 12 \cdot 5 \times 10^{-8}$ rad^2 sec^{-4}. Thus the mean duration of the steps in Fig. 6 is 10 sec, and the target r.m.s. acceleration is $r_T \sigma_T$, which is $1g$ at a range of 30 000 yards.

7.1.2. *The spectral density of the noise.*—It has been assumed throughout that the noise angles form a stationary ensemble; experimental evidence supports this view and suggests that at medium and long ranges the noise spectral density is constant over the frequency band of interest—i.e. the bandwidth of the complete system. We may therefore write

$$\Phi_N(i\omega) = k^2, \tag{7.1-3}$$

and the fact that this is in error for high frequencies does not invalidate the results, provided that the representation holds over the necessary frequency band.

The values of the noise spectral density used in the examples are

$$\Phi_N(i\omega) = k^2 = 4 \times 10^{-8} \text{ rad}^2/\text{rad/sec}, \; 0 \cdot 5 \times 10^{-8} \text{ rad}^2/\text{rad/sec},$$

and zero.

The first two figures embrace the range of noise levels which may be expected in a typical tracking system; the actual value of k^2 depends on a number of factors, such as the transmitted power, dish size, scanning system and type of target being tracked, but it is probable that the figure of $0 \cdot 5 \times 10^{-8}$ rad^2/rad/sec is near the lower limit which can be achieved in a practical system. $k^2 = 0$ corresponds of course to a perfect tracking system.

7.1.3. *The missile.*—The aerodynamic behaviour of the missile was represented in Section (3.1) by the operator $A(D)$, where

$$f_M(t) = A(D) f_D(t), \tag{3.1-1}$$

where $A(D)$ includes the effects of internal feedbacks introduced to modify the response. We shall take $A(D)$ to be of the form

$$A(D) = \frac{1}{(1+DT)^2}, \tag{7.1-4}$$

i.e. the transfer function relating the demanded and achieved accelerations is

$$\frac{1}{(1+pT)^2},$$

so that the modified weathercock mode is critically damped with a frequency of $^1/T$ rad/sec. This simplification of the situation is unnecessary as far as the derivation of H_0 is concerned: the full aerodynamics and feedback terms can be included in $A(D)$ (provided that the feedbacks are sufficient to allow a linear representation), but the evaluation of H_0 is then more laborious. It is shown later however that provided the missile weathercock response is fairly rapid, the actual form of the response is of little consequence.

Three values of T are used: 1 sec, 0·1 sec and zero, corresponding to weathercock frequencies of 1 rad/sec, 10 rad/sec and infinity. In the latter case $A(D) = 1$, representing an ideal missile with no time lags.

7.1.4. *The conditions of engagement.*—The missile and target velocities are taken to be 2000 and 1000 ft/sec respectively, the target being intercepted at a slant range of 100 000 feet. This gives a time of flight of approximately 50 sec.

7.2. *Explicit formulae for optimum transfer functions and miss distances*

The appropriate formula when the spectral density of the target acceleration is given is that of (5.3-10):

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+ \left(\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T^2)} + \Phi_N\right)^+} \left[K + \left\{ \frac{\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T^2)}}{(1+\lambda F\bar{F})^- \left(\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T^2)} + \Phi_N\right)^-} \right\}_+ \right]; \quad (5.3\text{-}10)$$

The optimum operators are derived with various limits imposed on the r.m.s. demanded acceleration. This requires (Section 5.2.1) that

$$F(D) = \frac{D(D+2c_M)}{A(D+c_M)}, \quad (5.2\text{-}12)$$

so that from (7.1-4)

$$F(D) = D(D+2c_M)\,[1 + (D+c_M)T]^2,$$

or

$$F\bar{F} = F(i\omega)F(-i\omega) = \omega^2(\omega^2+4c_M^2)\,[(1+c_M T)^2 + \omega^2 T^2]^2. \quad (7.2\text{-}1)$$

The mean square demanded acceleration is then $R^2\,\sigma_L^2$, where R is the range at interception and

$$\sigma_L^2 = \int_0^\infty \frac{\omega^2 + 4c_M^2}{\omega^2 + 4c_T^2} \left| \frac{H_0(i\omega)}{A(i\omega + c_M)} \right|^2 [\Psi_T(i\omega) + \omega^2(\omega^2+4c_T^2)\Phi_N(i\omega)]\,d\omega, \quad (5.3\text{-}11)$$

and the minimum mean square miss distance is $R^2\sigma_{\min}^2$ where

$$\sigma_{\min}^2 = \int_0^\infty |1 - H_0(i\omega)|^2 \frac{\Psi_T(i\omega)}{\omega^2(\omega^2+4c_T^2)}\,d\omega + \int_0^\infty |H_0(i\omega)|^2\Phi_N(i\omega)\,d\omega. \quad (5.3\text{-}12)$$

It has been found that when Ψ_T, Φ_N and $A(D)$ have the forms given above (Section 7.1), the presence of the terms c_M, c_T has a negligible effect on the optimum transfer function $H_0(p)$. With missile and target velocities of 2000 and 1000 ft/sec, and an interception range of 100 000 feet, we have $c_M = 0{\cdot}02$ sec^{-1} and $c_T = 0{\cdot}01$ sec^{-1}; and operators evaluated with these values are practically identical with those for $c_M = c_T = 0$. This remains true for medium ranges, but for short ranges ($<$ 20 000 feet) it is necessary to include the correct values of c_M and c_T—unless, of course, the velocities are correspondingly reduced.

In what follows we may write $c_M = c_T = 0$, in order to ease the labour of computation, with the proviso that the results apply for ranges greater than about 20 000 feet. The assumption is that

$\dot{r}_M = \dot{r}_T = 0$, and therefore implies a constant range; it can only be justified by demonstrating that the results obtained do not differ appreciably from those obtained with a closer approximation to the true situation. This condition is satisfied for the cases under discussion.

On putting $c_M = c_T = 0$, and substituting for Ψ_T, Φ_N, $A(i\omega)$ and $F(i\omega)F(-i\omega)$ from (7.1-2), (7.1-3), (7.1-4) and (7.2-1), equation (5.3-10) becomes

$$H_0 = \frac{1}{\{1+\lambda\omega^4(1+\omega^2T^2)^2\}^+\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)}+k^2\right)^+} \times \left[K + \left\{\frac{\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)}}{(1+\lambda\omega^4(1+\omega^2T^2)^2)^-\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)}+k^2\right)^-}\right\}_+\right]. \tag{7.2-2}$$

Equations (5.3-12) and (5.3-11) become

$$\sigma_{\min}^2 = \int_0^\infty |1-H_0(i\omega)|^2 \frac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)}\,d\omega + \int_0^\infty k^2|H_0(i\omega)|^2 d\omega\,, \tag{7.2-3}$$

and

$$\sigma_L^2 = \int_0^\infty \omega^4(1+\omega^2T^2)^2|H_0(i\omega)|^2\left[\frac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)}+k^2\right]d\omega\,. \tag{7.2-4}$$

We shall also be interested in the r.m.s. acceleration *achieved* by the missile; since

$$f_M = A(D)f_D = \frac{1}{(1+DT)^2}f_D\,,$$

we have from (7.2-4)

$$\sigma_M^2 = \int_0^\infty \omega^4|H_0(i\omega)|^2\left[\frac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)}+k^2\right]d\omega\,, \tag{7.2-5}$$

where $R^2\sigma_M^2 = \langle f_M^2\rangle$, the mean square achieved acceleration.

Equations (7.2-2) to (7.2-5) are the formulae used in evaluating the optimum operators given below.

Before dealing with the general case, it is instructive to consider a number of special cases in which either k^2, λ or T is taken to be zero.

7.2.1. *Zero noise and no constraint.*—Consider first the case when there is no noise and no limit imposed on the acceleration demanded of the missile, i.e. $k^2 = \lambda = 0$. Then from (7.2-2),

$$H_0(i\omega) = \frac{K}{\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)}\right)^+} + 1\,.$$

The best value of K is clearly zero, for then $H_0(i\omega) = 1$ and $\sigma_{\min}^2 = 0$, from (7.2-3). Also

$$\sigma_L^2 = \int_0^\infty (1+\omega^2T^2)^2\frac{2\beta\sigma_T^2}{\pi\beta(\omega^2+\beta^2)}\,d\omega\,.$$

This integral does not converge, so that $\sigma_L^2 \to \infty$.

The achieved acceleration is given by (7.2-5):

$$\sigma_M^2 = \int_0^\infty \frac{2\beta\sigma_T^2}{\pi(\beta^2+\omega^2)}\,d\omega = \sigma_T^2$$

—i.e. the mean square acceleration achieved by the missile is equal to the mean square target acceleration.

Thus for $k^2 = \lambda = 0$ the optimum transfer function is unity, giving zero mean square miss distance but requiring infinite demanded acceleration. This of course cannot be realised, since in practice the missile control surfaces would be heavily saturated.

7.2.2. *Unlimited missile acceleration in the presence of noise.*—In this case $\lambda = 0$ but $k^2 \neq 0$. We then have from (7.2-2)

$$H_0(i\omega) = \frac{1}{\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)} + k^2\right)^+}\left[K + \left\{\frac{\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)}}{\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^-}\right\}_+\right]. \tag{7.2-6}$$

But

$$\frac{2\beta\sigma_T^2}{\pi\omega^4(\beta^2+\omega^2)} + k^2 = k^2\,\frac{\omega^6 + \beta^2\omega^4 + \dfrac{2\beta\sigma_T^2}{\pi k^2}}{\omega^4(\beta^2+\omega^2)}.$$

The factor ω^4 in the denominator of this expression gives a quadruple pole at the origin, and it is not obvious how the poles should be divided between the upper and lower half planes. To resolve this difficulty write ω^4 as $(\omega^2+\epsilon^2)^2$, with ϵ positive.

Then

$$k^2\,\frac{\omega^6 + \beta^2\omega^4 + \dfrac{2\beta\sigma_T^2}{\pi k^2}}{(\omega^2+\epsilon^2)^2(\omega^2+\beta^2)} = k^2\,\frac{\omega^6 + \beta^2\omega^4 + \dfrac{2\beta\sigma_T^2}{\pi k^2}}{(\omega+i\epsilon)^2(\omega-i\epsilon)^2(\omega+i\beta)(\omega-i\beta)}. \tag{7.2-7}$$

Now write the factors of $\omega^6 + \beta^2\omega^4 + 2\beta\sigma_T^2/\pi k^2$ as

$$\omega^6 + \beta^2\omega^4 + \frac{2\beta\sigma_T^2}{\pi k^2} \equiv (\omega+ia)(\omega-ia)(\omega+c+id)(\omega+c-id)(\omega-c+id)(\omega-c-id). \tag{7.2-8}$$

Then from (7.2-7) and (7.2-8)

$$\left(k^2\,\frac{\omega^6 + \beta^2\omega^4 + \dfrac{2\beta\sigma_T^2}{\pi k^2}}{(\omega^2+\epsilon^2)^2(\omega^2+\beta^2)}\right)^+ = k^2\,\frac{(\omega-ia)(\omega+c-id)(\omega-c-id)}{(\omega-i\epsilon)^2(\omega-i\beta)}, \tag{7.2-9}$$

and

$$\left(k^2\,\frac{\omega^6 + \beta^2\omega^4 + \dfrac{2\beta\sigma_T^2}{\pi k^2}}{(\omega^2+\epsilon^2)^2(\omega^2+\beta^2)}\right)^- = \frac{(\omega+ia)(\omega+c+id)(\omega-c+id)}{(\omega+i\epsilon)^2(\omega+i\beta)}, \tag{7.2-10}$$

since each expression has all its poles and zeros confined to the upper half plane (U.H.P.) and lower half plane (L.H.P.) respectively. From (7.2-10),

$$\left\{\frac{\dfrac{2\beta\sigma_T^2}{\pi(\omega^2+\epsilon^2)^2(\omega^2+\beta^2)}}{\left(k^2\,\dfrac{\omega^6 + \beta^2\omega^4 + \dfrac{2\beta\sigma_T^2}{\pi k^2}}{(\omega^2+\epsilon^2)^2(\omega^2+\beta^2)}\right)^-}\right\} = \frac{2\beta\sigma_T^2}{\pi}\,\frac{(\omega+i\epsilon)^2(\omega+i\beta)}{(\omega^2+\epsilon^2)^2(\omega^2+\beta^2)(\omega+ia)(\omega+c+id)(\omega-c+id)}$$

$$= \frac{2\beta\sigma_T^2}{\pi}\,\frac{1}{(\omega-i\epsilon)^2(\omega-i\beta)(\omega+ia)(\omega+c+id)(\omega-c+id)}. \tag{7.2-11}$$

It is now necessary to express (7.2-11) as the sum of two functions, each of which has its singularities confined to one half plane. To do this (7.2-11) is expressed as partial fractions:

$$\left\{\ \right\} \equiv \frac{2\beta\sigma_T^2}{\pi}\left[\frac{A}{(\omega-i\epsilon)}+\frac{B}{(\omega-i\epsilon)^2}+\frac{C}{\omega-i\beta}+\frac{D+E\omega+F\omega^2}{(\omega+ia)(\omega+c+id)(\omega-c+id)}\right]. \tag{7.2-12}$$

The first three terms have poles in the U.H.P., and the fourth has its poles in the L.H.P. We may now discard the ϵ—its purpose was to show how the poles of $1/\omega^4$ must be distributed between the upper and lower half planes.

Thus

$$\left\{\ \right\}_+ = \frac{2\beta\sigma_T^2}{\pi}\left[\frac{A}{\omega}+\frac{B}{\omega^2}+\frac{C}{\omega-i\beta}\right], \tag{7.2-13}$$

where the constants A, B and C are obtained from the identity (7.2-12). They are

$$\left.\begin{aligned} A &= i\,\frac{(a-\beta)(c^2+d^2)-2a\beta d}{a^2\beta^2(c^2+d^2)^2}, \\ B &= -\frac{1}{a\beta(c^2+d^2)}, \\ \text{and}\qquad C &= -i\,\frac{1}{\beta^2(\beta+a)[c^2+(\beta+d)^2]}. \end{aligned}\right\} \tag{7.2-14}$$

On using (7.2-13) and (7.2-10) (with $\epsilon = 0$) in (7.2-6), we obtain

$$\begin{aligned} H_0(i\omega) &= \frac{\omega^2(\omega-i\beta)}{k^2(\omega-ia)(\omega+c-id)(\omega-c-id)}\left[K+\frac{2\beta\sigma_T^2}{\pi}\left(\frac{A}{\omega}+\frac{B}{\omega^2}+\frac{C}{\omega-i\beta}\right)\right] \\ &= \frac{2\beta\sigma_T^2}{\pi k^2}\,\frac{K'\omega^2(\omega-i\beta)+A\omega(\omega-i\beta)+B(\omega-i\beta)+C\omega^2}{(\omega-ia)(\omega+c-id)(\omega-c-id)}, \end{aligned}$$

where $K' = \pi K/2\beta\sigma_T^2$.

This expression is of order zero in ω, so that the mean square miss distance due to noise is infinite, since the noise integral

$$\int_0^\infty k^2|H_0(i\omega)|^2 d\omega$$

does not converge. To avoid this we must write $K' = 0$, so that

$$\begin{aligned} H_0(i\omega) &= \frac{2\beta\sigma_T^2}{\pi k^2}\,\frac{(A+C)\omega^2+(B-iA\beta)\omega-iB\beta}{(\omega-ia)(\omega^2-2id\omega-c^2-d^2)} \\ &= \frac{2\beta\sigma_T^2}{\pi k^2}\,\frac{-i(A+C)\omega^2-(B-iA\beta)i\omega-B\beta}{(i\omega+a)(c^2+d^2+2di\omega-\omega^2)}. \end{aligned}$$

This is the optimum frequency-response function $H_0(i\omega)$; since it is stable, its transfer function is $H_0(p)$:

$$H_0(p) = \frac{2\beta\sigma_T^2}{\pi k^2}\,\frac{i(A+C)p^2-(B-iA\beta)p-B\beta}{(p+a)(p^2+2dp+c^2+d^2)}.$$

On noting from (7.2-8) that

$$\frac{2\beta\sigma_T^2}{\pi k^2} = a^2(c^2+d^2)^2,$$

and substituting for A, B, and C from (7.2-14), we have finally

$$H_0(p) = \frac{A'p^2 + (c^2+d^2+2ad)p + a(c^2+d^2)}{(p+a)(p^2+2dp+c^2+d^2)} \tag{7.2-15}$$

where

$$A' = \frac{[2a\beta d + (\beta-a)(c^2+d^2)][(\beta+a)(c^2+(\beta+d)^2)] + a^2(c^2+d^2)^2}{\beta^2(\beta+a)(c^2+(\beta+d)^2)},$$

and this gives the optimum transfer function in terms of the roots of

$$\omega^6 + \beta^2\omega^4 + \frac{2\beta\sigma_T{}^2}{\pi k^2} = 0.$$

Equation (7.2-15) shows that $H_0(p)$ has no displacement lag and no velocity lag, in agreement with the conditions imposed in the derivation of the general equation (5.3-10).

The mean square miss distance achieved with this optimum function can now be obtained on using (7.2-15) in (7.2-3). This integral converges, since $H_0(p)$ is of order -1 in p, so that $H_0(i\omega)^2$ is $O(-2)$ in ω. The integrals of (7.2-4) and (7.2-5) however do not converge, and both the demanded and achieved accelerations are infinite.

On evaluating (7.2-15) for $k^2 = 4 \times 10^{-8}$ rad^2/rad/sec, with the other constants as given in Section (7.1), one obtains

$$H_0(p) = \frac{0{\cdot}446 + 1{\cdot}169\,p + 1{\cdot}432\,p^2}{0{\cdot}446 + 1{\cdot}169\,p + 1{\cdot}532\,p^2 + p^3},$$

giving an acceleration lag of $(1{\cdot}532 - 1{\cdot}432)/0{\cdot}446 = 0{\cdot}22$ sec^2. For this case the r.m.s. miss distance is 0·42 mils, or 42 feet at a range of 100 000 feet. That part of the miss distance due to target motion (the first integral of (7.2-3) is 18 feet, while that due to the noise alone (the second integral) is 38·5 feet.

The results for $k^2 = 0{\cdot}5 \times 10^{-8}$ rad^2/rad/sec and $k^2 = 0$ are given in Table 1. The system is still not practicable, since it requires infinite acceleration from the missile.

TABLE 1

Miss Distances, etc., with no Constraints

k^2 (rad^2/rad/sec)	$H_0(p)$	Accn. Lag $a_0{}^2$ (sec^2)	$R\sigma_1$ ft	$R\sigma_2$ ft	$R\sigma_{\min}$ ft	$R\sigma_M$ g's
0	1	0	0	0	0	$R\,\sigma_T$
$0{\cdot}5 \times 10^{-8}$	$\dfrac{1{\cdot}26 + 2{\cdot}34p + 2{\cdot}08p^2}{(1{\cdot}08 + p)(1{\cdot}17 + 1{\cdot}08p + p^2)}$	0·07	16	7	18	∞
4×10^{-8}	$\dfrac{0{\cdot}45 + 1{\cdot}17p + 1{\cdot}43p^2}{(0{\cdot}77 + p)(0{\cdot}58 + 0{\cdot}77p + p^2)}$	0·22	38·5	18	42	∞

In Table 1

	R	the range at interception = 100 000 ft
	a_0^2	the acceleration lag of the overall system $H_0(p)$
	$R\sigma_{\min}$	the minimum r.m.s. miss distance
	$R\sigma_1$	the r.m.s. miss distance due to noise
and	$R\sigma_2$	the r.m.s. miss distance due to target motion, so that

$$\sigma_1^2 + \sigma_2^2 = \sigma_{\min}^2 ;$$

$R\sigma_M$ the r.m.s. missile acceleration.

7.2.3. *Limited achieved missile acceleration and zero noise.*—Suppose now that we impose a limit on the achieved acceleration. Since

$$f_M = D(D+2c_M)\theta_M ,$$

we have from (5.2-1) that

$$F(D) = D(D+2c_M), \quad \text{or} \quad F(D) = D^2 \quad \text{if} \quad c_M = 0 .$$

Thus $F(i\omega)F(-i\omega) = \omega^4$, so that in equation (7.2-2) the term $\lambda\omega^4(1+\omega^2T^2)^2$ must be replaced by $\lambda\omega^4$. The same effect is achieved by putting $T = 0$: for then the missile is perfect, and the demanded and achieved accelerations are equal.

With $k^2 = 0$ (7.2-2) reduces to

$$H_0(i\omega) = \frac{1}{(1+\lambda\omega^4)^+S_T^+}\left[K + \left\{\frac{S_T}{(1+\lambda\omega^4)^-S_T^-}\right\}_+\right], \tag{7.2-16}$$

where

$$S_T = \frac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)}$$

Thus

$$(1+\lambda\omega^4)S_T = \frac{2\beta\sigma_T^2}{\pi\mu^4}\,\frac{\omega^4+\mu^4}{\omega^4(\omega^2+\beta^2)},$$

where $\mu^4 = \dfrac{1}{\lambda}$.

Then

$$(1+\lambda\omega^4)^+S_T^+ = \frac{2\beta\sigma_T^2}{\pi\mu^4}\,\frac{\left(\omega - \mu\dfrac{i+1}{\sqrt{2}}\right)\left(\omega - \mu\dfrac{i-1}{\sqrt{2}}\right)}{\omega^2(\omega-i\beta)}, \tag{7.2-17}$$

and

$$(1+\lambda\omega^4)^-S_T^- = \frac{\left(\omega + \mu\dfrac{i+1}{\sqrt{2}}\right)\left(\omega + \mu\dfrac{i-1}{\sqrt{2}}\right)}{\omega^2(\omega+i\beta)} . \tag{7.2-18}$$

{We may deduce from Section (7.2.2) how the poles of $1/\omega^4$ are distributed, so that it is unnecessary to write $(\omega^2+\epsilon^2)^2$ for ω^4.}

From (7.2-18)

$$\frac{S_T}{(1+\lambda\omega^4)^- S_T^-} = \frac{2\beta\sigma_T^2}{\pi} \frac{\omega^2(\omega+i\beta)}{\omega^4(\omega^2+\beta^2)\left(\omega+\mu\frac{i+1}{\sqrt{2}}\right)\left(\omega+\mu\frac{i-1}{\sqrt{2}}\right)},$$

$$= \frac{2\beta\sigma_T^2}{\pi} \frac{1}{\omega^2(\omega-i\beta)\left(\omega+\mu\frac{i+1}{\sqrt{2}}\right)\left(\omega+\mu\frac{i-1}{\sqrt{2}}\right)},$$

$$\equiv \frac{2\beta\sigma_T^2}{\pi}\left[\frac{A}{\omega}+\frac{B}{\omega^2}+\frac{C}{\omega-i\beta}+\frac{D+E\omega}{\left(\omega+\mu\frac{i+1}{\sqrt{2}}\right)\left(\omega+\mu\frac{i-1}{\sqrt{2}}\right)}\right]. \quad (7.2\text{-}19)$$

Therefore

$$\left\{\frac{S_T}{(1+\lambda\omega^4)^- S_T^-}\right\}_+ = \frac{2\beta\sigma_T^2}{\pi}\left[\frac{A}{\omega}+\frac{B}{\omega^2}+\frac{C}{\omega-i\beta}\right], \quad (7.2\text{-}20)$$

the constants A, B and C being found from the identity (7.2-19). Using (7.2-20) and (7.2-17) in (7.2-16), we have

$$H_0(i\omega) = \frac{\pi\mu^4}{2\beta\sigma_T^2} \frac{\omega^2(\omega-i\beta)}{\left(\omega-\mu\frac{i+1}{\sqrt{2}}\right)\left(\omega-\mu\frac{i-1}{\sqrt{2}}\right)}\left[K+\left(\frac{2\beta\sigma_T^2}{\pi}\frac{A}{\omega}+\frac{B}{\omega^2}+\frac{C}{\omega-i\beta}\right)\right]$$

$$= \mu^4 \frac{[K'\omega^2(\omega-i\beta)+A\omega(\omega-i\beta)+B(\omega-i\beta)+C\omega^2]}{\left(\omega-\mu\frac{i+1}{\sqrt{2}}\right)\left(\omega-\mu\frac{i-1}{\sqrt{2}}\right)}.$$

The integral for the r.m.s. achieved acceleration (7.2-5) does not converge unless $K' = 0$, so that

$$H_0(i\omega) = \mu^4 \frac{(A+C)\omega^2 - i\omega(iB+\beta A) - i\beta B}{\omega^2-\sqrt{2}\,\mu i\omega-\mu^2}.$$

On evaluating the constants A, B and C from (7.2-19), this reduces to

$$H_0(i\omega) = \frac{\mu^2+\sqrt{2}\,\mu i\omega - \dfrac{\mu^2+\sqrt{2}\,\mu\beta}{\mu^2+\sqrt{2}\,\mu\beta+\beta^2}\,\omega^2}{\mu^2+\sqrt{2}\,\mu i\omega-\omega^2},$$

or

$$H_0(p) = \frac{\mu^2+\sqrt{2}\,\mu p + \dfrac{\mu^2+\sqrt{2}\,\mu\beta}{\mu^2+\sqrt{2}\,\mu\beta+\beta^2}\,p^2}{\mu^2+\sqrt{2}\,\mu p+p^2}. \quad (7.2\text{-}21)$$

The transfer function is seen to represent a stable system, with no displacement or velocity lag, and an acceleration lag of

$$\frac{\beta^2}{\mu^2(\mu^2+\sqrt{2}\,\mu\beta+\beta^2)}.$$

In this equation μ is any positive constant, to each value of which corresponds a particular r.m.s. achieved acceleration. The mean square miss distance and mean square acceleration can be evaluated by using (7.2-21) in (7.2-3) and (7.2-5): the results are

$$\sigma_{\min}^2 = \sigma_T^2 \frac{\beta^4(\sqrt{2}\,\mu^3 - \mu^2\beta + \beta^3)}{\sqrt{2}\,\mu^3(\mu^4+\beta^4)(\mu^2+\sqrt{2}\,\mu\beta+\beta^2)^2} \tag{7.2-22}$$

and

$$\sigma_M^2 = \sigma_T^2 \frac{\mu\left[\mu^7 + 2\sqrt{2}\,\mu^6\beta + 4\mu^5\beta^2 + 2\sqrt{2}\,\mu^4\beta^3 + \frac{5\sqrt{2}}{2}\,\mu^2\beta^5 + 4\mu\beta^6 + \frac{3\sqrt{2}}{2}\,\beta^7\right]}{(\mu^4+\beta^4)(\mu^2+\beta^2+\sqrt{2}\,\mu\beta)^2} \tag{7.2-23}$$

The results are tabulated in Fig. 7 for various values of μ, with $\beta = 0{\cdot}1$ rad/sec, and $\sigma_{\min}/\sigma_T$ is plotted as a function of the ratio of missile and target r.m.s. accelerations σ_M/σ_T. To each value of the missile r.m.s. acceleration σ_M there corresponds an optimum transfer function of the form (7.2-21). The ordinates of Fig. 7 give a measure of the r.m.s. miss distance achieved with these various operators, and their acceleration lags are also plotted.

In particular, $\mu = \infty$ corresponds to no acceleration limit: in this case $H_0(p) = 1$, $\sigma_{\min}^2 = 0$ and $\sigma_M^2 \to \sigma_T^2$ as in Section (7.2.1). At the other extreme $\mu = 0$, $H_0(p) = 0$, $\sigma_{\min}^2 \to \infty$ and $\sigma_M^2 = 0$. It will be seen from Fig. 7 that the miss distance in the absence of noise is critically dependent on the available missile acceleration as a proportion of the target acceleration.

Equation (7.2-21) shows that even when there is no noise present, the optimum operator is not unity if account is taken of the limited available acceleration of the missile. It will be noted that both the form of the optimum function and its parameters are completely specified when the target spectral density and the allowable achieved acceleration are given.

7.2.4. *Limited achieved acceleration in the presence of noise.*—The appropriate equation for this case is obtained by putting $T = 0$ in (7.2-2). This gives

$$H_0(i\omega) = \frac{1}{(1+\lambda\omega^4)^+\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^+}\left[K + \left\{\frac{\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)}}{(1+\lambda\omega^4)^-\left(\dfrac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^-}\right\}_+\right].$$

The treatment of this expression is similar to that of (7.2-6), with the extra factor $(1+\lambda\omega^4)$. Following the method given in Section 7.2.2, one obtains

$$H_0(p) = \frac{a\mu^2(c^2+d^2) + [\mu^2(c^2+d^2+2ad) + a\mu\sqrt{2}(c^2+d^2)]p + Cp^2}{(p+a)(p^2+2dp+c^2+d^2)(p^2+\mu\sqrt{2}\,p+\mu^2)}, \tag{7.2-24}$$

where $\pm\, ia$, $\pm\,(c+id)$ and $\pm\,(c-id)$ are the roots of

$$\omega^6 + \beta^2\omega^4 + \frac{2\beta\sigma_T^2}{\pi k^2} = 0,$$

as in Section 7.2.2;

$$C = \frac{a^2\mu^4(c^2+d^2)^2}{\beta^2(a+\beta)[c^2+(\beta+d)^2](\beta^2+\mu^2+\sqrt{2}\,\mu\beta)} - \frac{a\mu^2(c^2+d^2) - \beta[(c^2+d^2)(\mu^2+\sqrt{2}\,a\mu) + 2a\mu^2 d]}{\beta^2}$$

and μ is a constant which determines the r.m.s. achieved acceleration.

Comparing (7.2-24) with (7.2-15), it is seen that the effect of limiting the mean square achieved acceleration is to introduce a further quadratic term in the denominator, as well as modifying the numerator coefficients. $H_0(p)$ is then of order (-3) in p, so that $|H_0(i\omega)|^2$ is $0(-6)$ in ω, and the integrals (7.2-3) and (7.2-5) converge, giving finite $\sigma_{\min}$ and σ_M. The integral (7.2-4) however does not converge, showing that the demanded acceleration is still infinite.

The constants of (7.2-24) have been evaluated for $k^2 = 0\cdot 5 \times 10^{-8}$ and 4×10^{-8} rad²/rad/sec, with various values of μ. The results are given in Figs. 8 and 9, which also show the r.m.s. miss distances obtained with various limits on the achieved r.m.s. acceleration. Similar results for $T = 0$ were obtained by Keats in Ref. 2.

7.2.5. *Limited demanded acceleration in the presence of noise.*—The results of the previous section refer to a system containing a missile of limited available acceleration, but which is otherwise perfect, in that it responds immediately to a demand for acceleration. We now study the more realistic case in which this response is of the form

$$\frac{1}{(1+pT)^2}, \quad \text{with} \quad T \neq 0,$$

and in which the demand for acceleration is limited to some finite r.m.s. value $R\sigma_2$; this of course implies a limit also on the achieved acceleration.

The optimum system is then given by (7.2-2), which may be evaluated as follows.

We have

$$(1+\lambda F\bar{F}) = 1 + \lambda\omega^4(1+\omega^2T^2)^2 = \frac{1}{\mu^4}(\mu^4+\omega^4+2T^2\omega^6+T^4\omega^8), \quad \text{where} \quad \frac{1}{\mu^4} = \lambda.$$

$$= \frac{T^4}{\mu^4}\left(\omega^8 + \frac{2}{T^2}\omega^6 + \frac{1}{T^4}\omega^4 + \frac{\mu^4}{T^4}\right). \tag{7.2-25}$$

In order to find $(1+\lambda F\bar{F})^+$ and $(1+\lambda F\bar{F})^-$, we require the factors of (7.2-25). Let

$$\omega^8 + \frac{2}{T^2}\omega^6 + \frac{1}{T^4}\omega^4 + \frac{\mu^4}{T^4} \equiv (\omega+l_1+im_1)(\omega-l_1+im_1)(\omega+l_2+im_2)(\omega-l_2+im_2) \times$$
$$\times (\omega+l_1-im_1)(\omega-l_1-im_1)(\omega+l_2-im_2)(\omega-l_2-im_2). \tag{7.2-26}$$

Then

$$(1+\lambda F\bar{F})^+ = \frac{T^4}{\mu^4}(\omega+l_1-im_1)(\omega-l_1-im_1)(\omega+l_2-im_2)(\omega-l_2-im_2) \tag{7.2-27}$$

and

$$(1+\lambda F\bar{F})^- = (\omega+l_1+im_1)(\omega-l_1+im_1)(\omega+l_2+im_2)(\omega-l_2+im_2). \tag{7.2-28}$$

The functions

$$\left(\frac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^+ \quad \text{and} \quad \left(\frac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^-$$

have already been obtained in Section 7.2.2: they are ($\epsilon \to 0$)

$$\left(\frac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^+ = k^2\frac{(\omega-ia)(\omega+c-id)(\omega-c-id)}{\omega^2(\omega-i\beta)} \tag{7.2-9}$$

and

$$\left(\frac{2\beta\sigma_T^2}{\pi\omega^4(\omega^2+\beta^2)} + k^2\right)^- = \frac{(\omega+ia)(\omega+c+id)(\omega-c+id)}{\omega^2(\omega+i\beta)}, \tag{7.2-10}$$

where a, c and d are defined by the identity (7.2-8).

From (7.2-28) and (7.2-10),

$$\left\{\frac{\dfrac{2\beta\sigma_T{}^2}{\pi\omega^4(\omega^2+\beta^2)}}{(1+\lambda\omega^4(1+\omega^2T^2)^2)^-\left(\dfrac{2\beta\sigma_T{}^2}{\pi\omega^4(\omega^2+\beta^2)}+k^2\right)^-}\right\}$$

$$=\frac{2\beta\sigma_T{}^2}{\pi}\frac{\omega^2(\omega+i\beta)}{\omega^4(\omega^2+\beta^2)(\omega+ia)(\omega+c+id)(\omega-c+id)(\omega\pm l_{12}+im_{12})},$$

where for brevity we have used the notation

$$(\omega\pm l_{12}+im_{12})\equiv(\omega+l_1+im_1)(\omega-l_1+im_1)(\omega+l_2+im_2)(\omega-l_2+im_2).$$

Thus

$$\left\{\ \right\}=\frac{2\beta\sigma_T{}^2}{\pi}\frac{1}{\omega^2(\omega-i\beta)(\omega+ia)(\omega+c+id)(\omega-c+id)(\omega\pm l_{12}+im_{12})}$$

$$\equiv\frac{2\beta\sigma_T{}^2}{\pi}\left[\frac{D}{\omega}+\frac{E}{\omega^2}+\frac{F}{\omega-i\beta}+\frac{G\omega^6+H\omega^5+\ldots+L\omega+M}{(\omega+ia)(\omega+c+id)(\omega-c+id)(\omega\pm l_{12}+im_{12})}\right] \tag{7.2-29}$$

where $D, E, F \ldots M$ are constants determined by this identity.

The first three terms have poles in the U.H.P. (the first two are actually $(\omega-i\epsilon)$, $(\omega-i\epsilon)^2$), and the last term has poles in the L.H.P. Therefore

$$\left\{\ \right\}_+=\frac{2\beta\sigma_T{}^2}{\pi}\left[\frac{D}{\omega}+\frac{E}{\omega^2}+\frac{F}{\omega-i\beta}\right]$$

$$=\frac{2\beta\sigma_T{}^2}{\pi}\frac{D\omega(\omega-i\beta)+E(\omega-i\beta)+F\omega^2}{\omega^2(\omega-i\beta)},$$

so that

$$K+\left\{\ \right\}_+=\frac{K\omega^2(\omega-i\beta)+\dfrac{2\beta\sigma_T{}^2}{\pi}[(D+F)\omega^2+(E-i\beta D)\omega-i\beta E]}{\omega^2(\omega-i\beta)}. \tag{7.2-30}$$

On substituting from (7.2-27), and (7.2-9) and (7.2-30) in (7.2-2), we have

$$H_0(i\omega)=\frac{\mu^4}{T^4k^2}\frac{K\omega^2(\omega-i\beta)+\dfrac{2\beta\sigma_T{}^2}{\pi}[(D+F)\omega^2+(E-i\beta D)\omega-i\beta E]}{(\omega-ia)(\omega+c-id)(\omega-c-id)(\omega\pm l_{12}-im_{12})}.$$

This expression is $0(-4)$ in ω, so that $|H_0(i\omega)|^2$ is $0(-8)$, and the integral (7.2-4) does not converge unless $K=0$. Hence

$$H_0(i\omega)=\frac{2\beta\sigma_T{}^2\mu^4}{\pi k^2T^4}\frac{(D+F)\omega^2+(E-i\beta D)\omega-i\beta E}{(\omega-ia)(\omega+c-id)(\omega-c-id)(\omega\pm l_{12}-im_{12})}.$$

On evaluating the constants D, E and F from the identity (7.2-29), and writing p for $i\omega$, we have finally

$$H_0(p) = \frac{A + Bp + Cp^2}{(p+a)(p^2+2dp+c^2+d^2)(p^2+2m_1p+l_1^2+m_1^2)(p^2+2m_2p+l_2^2+m_2^2)}, \tag{7.2-31}$$

where

$$A = a(c^2+d^2)(l_1^2+m_1^2)(l_2^2+m_2^2),$$

$$B = (c^2+d^2+2ad)(l_1^2+m_1^2)(l_2^2+m_2^2) + 2a(c^2+d^2)[m_1(l_2^2+m_2^2) + m_2(l_1^2+m_1^2)]$$

and

$$C = \frac{a^2(c^2+d^2)^2(l_1^2+m_1^2)^2(l_2^2+m_2^2)^2}{\beta^2(a+\beta)[c^2+(\beta+d)^2][l_1^2+(\beta+m_1)^2][l_2^2+(\beta+m_2)^2]} - \frac{A-\beta B}{\beta^2}.$$

The positive constants a, c, d, and l_1, l_2, m_1, m_2 are defined by the identities (7.2-8) and (7.2-26) respectively; they depend on the roots of polynomials, and it is for this reason that $H_0(p)$ cannot be expressed more directly in terms of the primary constants.

The constant coefficient of the denominator of (7.2-31) is equal to A, and the coefficient of p in the denominator is equal to B, so that the system has no displacement or velocity lag. We may write (7.2-31) as

$$H_0(p) = \frac{A + Bp + Cp^2}{A + Bp + C'p^2 + D'p^3 + E'p^4 + F'p^5 + G'p^6 + p^7}, \tag{7.2-32}$$

and its acceleration lag is

$$\frac{C'-C}{A}. \tag{7.2-33}$$

The minimum r.m.s. miss distance, and the r.m.s. demanded and achieved accelerations, can now be found by using the optimum transfer function (7.2-31) in (7.2-3), (7.2-4) and (7.2-5). Since $H_0(i\omega)$ is $0(-5)$, $|H_0(i\omega)|^2$ is $0(-10)$, so that all the integrals converge, giving finite mean square values for these quantities.

The $H_0(p)$ of equation (7.2-31) applies to a system which contains a missile whose modified weathercock transfer function is

$$\frac{1}{(1+pT)^2},$$

the demanded acceleration being limited to a desired r.m.s. value (depending on μ). The noise spectral density k^2 is assumed to be constant over the frequency range covered by (7.2-31), and the target lateral acceleration spectral density is of the form

$$\sigma_T^2 \frac{2}{\pi} \frac{\beta}{\omega^2+\beta^2}.$$

With this data (7.2-31) gives the transfer function for which the mean square miss distance is less than for any other linear system.

7.3. *Numerical Examples.*

7.3.1. *Comments on Figs.* 10 *to* 13.—The examples which have been evaluated, and which are shown in Figs. 10 to 13, are all for $\beta = 0{\cdot}1$ rad/sec and $\sigma_T^2 = 12{\cdot}5 \times 10^{-8}$ rad^2 sec^{-4}. Two values of the weathercock time constant T—$0{\cdot}1$ sec and 1 sec—have been taken, each with noise levels of $0{\cdot}5 \times 10^{-8}$ and 4×10^{-8} rad^2/rad/sec. The special cases for $T = 0$ and $k^2 = 0$ have already been discussed.

Figs. 10b, c to 13 b, c show the variation of the minimum r.m.s. miss distance (at $R = 100\,000$ feet) with the achieved missile acceleration, and the relation between the r.m.s. achieved and demanded accelerations. The curvature of the latter graphs has no connection with aerodynamic or other non-linearities: it is due to the fact that each point on the curves is associated with a particular optimum transfer function, which is different for different points. This means that the spectral density of the demanded acceleration is varying with different operators, and therefore equal r.m.s. demanded accelerations for the same missile result in different r.m.s. achieved accelerations. For any one system the plot of demanded vs. achieved acceleration is of course a straight line up to the limit imposed by structural considerations, assuming that the missile has been linearised by heavy degenerative feedback.

Some of the optimum transfer functions corresponding to certain points of the miss distance and acceleration curves of Figs. 10b, c to 13b, c are given in Figs. 10a to 13a. It will be seen that all the miss-distance curves have a pronounced knee: it is clearly advantageous to choose a system such that the operating point is slightly to the right of this knee: for a system requiring less acceleration gives a greatly increased miss distance, while little improvement in miss distance accrues from employing greater acceleration, particularly in view of the increased drag with which this is associated.

A comparison of Fig. 7 and Figs. 8 to 13 shows that the effect of the noise is to raise the level of the miss distance, as well as making the latter less critically dependent on the achieved missile acceleration. It is also clear from these figures that the best system is obtained with $T = 0$—i.e. perfect aerodynamics; however, the curves for $T = 0 \cdot 1$ sec are already close to this ideal.

The optimum transfer functions have been derived in terms of angular quantities, so that the results for miss distances, etc., for any range are obtained by multiplying the relevant quantities by the range in question. There are two reservations to be made, however; the first is that the time of flight must be sufficiently long for the approximations $h_0(T) = 0$ (Section 4.2.2) and $c_M = c_T = 0$ (Section 7.2) to be valid. For given missile and target velocities this implies a minimum range, below which the results are in error.

The second reservation concerns the behaviour of the target: its r.m.s. lateral acceleration f_T is likely to be independent of its range from the weapon site, whereas in the analysis it has been assumed that the quantity σ_T is constant, where $R\sigma_T = f_T$. For a smaller range, therefore, f_T is smaller if σ_T is kept constant, and conversely.

We may conclude that the results apply over a limited range of ranges; and that for greater or smaller ranges it is necessary to re-evaluate the optimum operator for a different choice of σ_T, such that f_T remains constant. In the present examples we have

$$\sigma_T = 12 \cdot 5 \times 10^{-8} \text{ rad}^2 \text{ sec}^{-4},$$

so that for $R = 100\,000$ feet, $f_T = 1 \cdot 1g$. We may assume that the results hold for ranges of, say, 80 000 to 120 000 feet, which implies a variation of f_T between $0 \cdot 9g$ and $1 \cdot 3g$ r.m.s.—a not unlikely occurrence.

7.3.2. *The optimum system and miss distance for a given missile acceleration limit.*—As an example, consider a missile having a modified weathercock frequency of 10 rad/sec. ($T = 0 \cdot 1$ sec), and so designed that the acceleration must be limited to a maximum of $10g$ (briefly, a '$10g$ missile'). Since we have assumed critical damping of the weathercock mode, it follows that the demanded acceleration must be limited to $10g$. Suppose further that the noise spectral density is $k^2 = 4 \times 10^{-8}$ rad^2/rad/sec.

In order for the analysis to remain valid, the r.m.s. demanded acceleration must be such that the $10g$ limits are never (or rarely) brought into operation. If a gaussian distribution is assumed for the demanded acceleration, the proportion of time for which the demand exceeds the limits is given by

$$q = 1 - \operatorname{erf}\left[\frac{f_{\max}}{R\sigma_L\sqrt{2}}\right], \tag{5.2-1}$$

where $R\sigma_L$ is the r.m.s. demanded acceleration and $\pm f_{\max}$ the level at which the demand is limited—in the present case $f_{\max} = 10g$. This curve is shown in Fig. 14, from which it is seen that for

$$\frac{R\sigma_L}{f_{\max}} = 0\cdot51,$$

the demand is on the limits for 5% of the time.

We may suppose for the moment that the difference between limiting for 5% of the time and the absence of limits is sufficiently small not to affect the linear analysis. The choice of 5% is however arbitrary, and requires further investigation.

For the $10g$ missile and 5% limiting, we have

$$R\sigma_L = 5\cdot1g;$$

the nearest operator which has been evaluated is that given for $\mu^4 = 10$ (Fig. 11a): it is

$$H_0(p) = \frac{141 + 509\,p + 827\,p^2}{(0\cdot766+p)(0\cdot582+0\cdot776\,p+p^2)(100\cdot2+20\cdot01\,p+p^2)(3\cdot16+2\cdot47\,p+p^2)},$$

with which the r.m.s. demanded acceleration $R\sigma_L$ is $4\cdot24g$. The actual operator which gives $R\sigma_L = 5\cdot1g$ can of course be found more exactly by evaluating (7.2-31) for values of μ between 10 and 50—for $\mu^4 = 50$, $R\sigma_L = 6\cdot57g$ (Fig. 11a).

From Fig. 11c, the r.m.s. achieved acceleration with this optimum system ($R\sigma_L = 5\cdot1g$) is $4\cdot1g$, and Fig. 11b gives the miss distance as 67 feet r.m.s.

Thus this optimum system will give an r.m.s. miss of 67 feet, and the demanded acceleration will exceed $10g$ for only 5% of the time.

If we wish to reduce this limiting to 1%, we have from Fig. 14 that

$$R\sigma_L = 4g \quad \text{for} \quad f_{\max} = 10g.$$

The corresponding transfer function is then very nearly that given for $\mu^4 = 10$ (Fig. 11a). The achieved acceleration is $3\cdot4g$ r.m.s. (Fig. 11c); and from Fig. 11b the r.m.s. miss distance is 73 feet. The insertion of $10g$ limits will clearly have a negligible effect on this assessment, since they will only be in operation for 1% of the time.

It may be concluded that the r.m.s. miss distance for a $10g$ missile need not be more than 73 ft (under the stated conditions of noise, etc.), and probably nearer 67 ft. As mentioned above, the degree of saturation for which the linear analysis gives markedly optimistic results remains to be determined: it is estimated however that at least a 5% saturation can be allowed before the difference becomes appreciable.

7.3.3. *The missile acceleration required for a given r.m.s. miss distance.*—As a further example, suppose that we wish to achieve a miss distance of 50 ft r.m.s. with a missile having a weathercock frequency of 1 rad/sec ($T = 1$ sec), in the presence of a noise spectral density of

$$k^2 = 0\cdot5 \times 10^{-8}\ \text{rad}^2/\text{rad/sec}.$$

From Fig. 12b, the optimum system requires an achieved acceleration of $2 \cdot 6g$, and the demanded acceleration (Fig. 12c) is then $16g$. For 5% saturation this would require acceleration limits of $32g$—a quite impractical figure. This result is due to the low weathercock frequency, giving rise to a large difference between the r.m.s. demanded and achieved accelerations.

The remedy lies in increasing the weathercock frequency; suppose that additional feedback is applied so that it is raised to 10 rad/sec ($T = 0 \cdot 1$ sec). The appropriate curves are then those of Fig. 10. For a miss distance of 50 feet, an achieved acceleration of $2 \cdot 2g$ r.m.s. is required (Fig. 10b). Fig. 10c gives the demanded acceleration as $2 \cdot 5g$, so that if the demand is limited to $4 \cdot 9g$ maximum (Fig. 14), the limits will only be reached for 5% of the time. Under these conditions, therefore, a $5g$ missile is necessary to give an r.m.s. miss distance of 50 feet.

The appropriate optimum transfer function lies between those listed in Fig. 10a for $\mu^4 = 4$ and $\mu^4 = 10$. Again, the precise optimum can be evaluated from (7.2-31).

For a 30 ft r.m.s. miss distance, the same procedure shows that a missile capable of $9 \cdot 2g$ maximum is required—probably a worthwhile improvement. However, a further reduction in miss distance is only achieved at the expense of greatly increased acceleration requirements, as is evident from Fig. 10b; the benefits of a rather smaller miss distance are outweighed by the adverse factors invoked by higher accelerations.

Figs. 8 to 13 also show the merit of a moderately high weathercock frequency. The improvement shown by increasing this frequency from 1 to 10 rad/sec is considerable, but the results for infinite frequency ($T = 0$) are not greatly different from those for 10 rad/sec. Thus, for the target spectral density chosen in these examples, the weathercock frequency should be rather more than 10 rad/sec—say 2 to 3 c/sec—but there is little advantage to be gained from higher frequencies.

8. *The Adjustable Components of the Optimum System.*

In Section (6.1) it was shown that in order to achieve the optimum system for a given set of conditions, the various components of the system must satisfy the equation

$$H_0(p) = \frac{T(p)}{1 + T(p)} \frac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M) + p^2 + 2c_M p}. \tag{6.1-1}$$

In this equation $A(p)$ is fixed {it has been used in the derivation of $H_0(p)$}, but we are at liberty to choose either $T(p)$ or $S(p)$ to satisfy (6.1-1). Examples of both are given below for some of the operators evaluated in Section 7.

8.1. *The Optimum Tracking System when the Missile Control System is Given.*

8.1.1. *The form of the optimum tracker.*—The appropriate equation when $S(p)$ is given is

$$T(p) = \frac{H_0(p)}{\dfrac{A(p+c_M)S(p+c_M)}{A(p+c_M)S(p+c_M) + p^2 + 2c_M p} - H_0(p)}. \tag{6.1-2}$$

Since the optimum transfer functions have been evaluated with $c_M = c_T = 0$, we must make the same approximation in the derivation of $T(p)$. (There is of course no difficulty in retaining the correct values of c_M and c_T, but the work becomes rather more laborious.) Thus

$$T(p) = \frac{H_0(p)}{\dfrac{A(p)S(p)}{A(p)S(p) + p^2} - H_0(p)}. \tag{8.1-1}$$

As an example, suppose that the missile control system consists of a phase-advance network; this type of control system has received much attention, and is one way of stabilising a zero velocity lag system. In this case we have

$$S(p) = K\frac{1 + p\tau}{1 + np\tau}, \tag{8.1-2}$$

where K is the stiffness of the missile system, τ is the phase-advance time constant, and n a constant (< 1). Any other time constant which is present, of necessity or by design, can be included in $S(p)$, but for simplicity we shall assume that (8.1-2) gives the main term.

We have already assumed (Section 7.1.3) that the missile weathercock characteristic is such that

$$A(p) = \frac{1}{(1+pT)^2}; \tag{7.1-4}$$

On substituting for $H_0(p)$, $S(p)$ and $A(p)$ {equations (7.2-32), (8.1-2) and (7.1-4)} in (8.1-1), one obtains

$$T(p) = \frac{(A+Bp+Cp^2)[K(1+p\tau)+p^2(1+pT)^2(1+np\tau)]}{p^2[K(1+p\tau)(C'-C+D'p+E'p^2+F'p^3+G'p^4+p^5)-(1+pT)^2(1+np\tau)(A+Bp+Cp^2)]} \quad \ldots (8.1\text{-}3)$$

which is the open-loop transfer function of the optimum tracking system, the constants A, B, C, etc., being defined by (7.2-31) and (7.2-32); they are determined by the target acceleration and noise spectral densities, and the permissible missile acceleration, as in Section 7.

Equation (8.1-3) may be written as

$$T(p) = \frac{f(p)}{p^2}, \tag{8.1-4}$$

and the presence of the double pole at the origin indicates that the tracker has no velocity lag. If we write a_0^2, a_R^2 and a_M^2 as the acceleration lags of the overall system, the tracker and the missile system respectively, then

$$\frac{1}{a_R^2} = f(0) = \frac{AK}{K(C'-C)-A} = \frac{1}{\dfrac{C'-C}{A} - \dfrac{1}{K}}$$

or

$$\frac{1}{a_R^2} = \frac{1}{a_0^2 - a_M^2}, \tag{8.1-5}$$

since $a_M^2 = 1/K$, where K is the stiffness of the missile loop, and

$$a_0^2 = \frac{C'-C}{A} \tag{7.2-33}$$

Thus, from (8.1-5) $a_0^2 = a_R^2 + a_M^2$, as required.

The simplest possible servo system having zero displacement and velocity lags has the transfer function of the form

$$\frac{k_1 + k_2 p}{k_1 + k_2 p + p^2},$$

the open-loop transfer function being

$$\frac{k_1 + k_2 p}{p^2}.$$

The function $F(p)$ (8.1-4) is therefore to be compared with the simple phase advance $(k_1 + k_2p)$.

Neither (k_1+k_2p) nor $f(p)$ are physically realisable in the exact sense: they are both $0(1)$ in p, implying an infinite response for infinite frequency. However, a network can readily be found which has the required characteristics over the bandwidth of the system, and this is all that is necessary. The realisation of networks such as $f(p)$ is taken up in Appendix IV.

8.1.2. *A numerical example.*—As an example of the derivation of the optimum tracking system, consider the case in which

$$k^2 = 0\cdot 5 \times 10^{-8}\ \mathrm{rad}^2/\mathrm{rad/sec},$$

and

$$\Psi_T = \sigma_T^{\,2}\frac{2}{\pi}\frac{\beta}{\omega^2+\beta^2},$$

with

$$\sigma_T^{\,2} = 12\cdot 5 \times 10^{-8}\ \mathrm{rad}^2\ \mathrm{sec}^{-4}$$

and

$$\beta = 0\cdot 1\ \mathrm{rad/sec},$$

as before, the target being intercepted at a range of 100 000 feet. Suppose also that the missile has been modified by feedback so that the weathercock mode is critically damped with a frequency of 10 rad/sec—i.e.

$$A(p) = \frac{1}{(1+0\cdot 1p)^2}.$$

The relevant curves for the optimum system are those of Fig. 10. As mentioned earlier, it is preferable to work just to the right of the knee of Fig. 10b. A suitable operator is that for $\mu^4 = 10$ (Fig. 10a). It is

$$H_0(p) = \frac{399 + 1130p + 1472p^2}{399 + 1130p + 1603p^2 + 1374p^3 + 695p^4 + 204p^5 + 24\cdot 7p^6 + p^7} \qquad (8.1\text{-}6)$$

for which the r.m.s. demanded and achieved accelerations are $2\cdot 80g$ and $2\cdot 41g$ respectively. From Fig. 10b, the r.m.s. miss distance is 38 feet. If we again take 5% saturation as the allowable limit, Fig. 14 shows that

$$f_{\max} = \frac{2\cdot 80}{0\cdot 51}g = 5\cdot 5g,$$

so that a missile with a $5\cdot 5g$ limit would be suitable. For a $5g$ missile the saturation would be $7\cdot 5$% (Fig. 14), which is probably still sufficiently small to admit the linear analysis.

From (8.1-6) and (7.2-32), the constants A, B etc. are

$$\begin{array}{llr} A = 399 & D' = 1374 & \\ B = 1130 & E' = 695 & \\ C = 1472 & F' = 204 & (8.1\text{-}7) \\ C' = 1603 & G' = 24\cdot 7 & \end{array}$$

The optimum tracking system (8.1-3) now depends on the choice of the missile control system. It should be emphasised that, within limits, this choice is arbitrary as far as the realisation of the optimum system is concerned: different missile control systems merely alter the transfer function of the optimum tracking system, the overall transfer function remaining unchanged.

Suppose that the missile control system chosen is

$$\frac{K(1+p\tau)}{(1+np\tau)},$$

as discussed above. The choice of the constants K, τ and n is of course also arbitrary; however, it is necessary to make the missile response to beam motion reasonably well damped and fairly fast, in order to deal with initial transients. The overall acceleration lag of the selected optimum system (8.1-6) is $a_0{}^2 = 0\cdot33\ \text{sec}^2$, so that

$$a_R{}^2 + a_M{}^2 = 0\cdot33\ \text{sec}^2.$$

Thus the acceleration lag of the missile system must not exceed $0\cdot33\ \text{sec}^2$—otherwise $a_R{}^2$ will be negative, implying an unstable tracker. This imposes a lower limit on the missile stiffness $K = 1/a_M{}^2$.

A reasonable set of parameters for the missile loop is as follows:

$$K = 5\ \text{sec}^{-2}\ (a_M{}^2 = 0\cdot2\ \text{sec}^2)\ \text{i.e. a stiffness of } 6\cdot4\ \text{ft}/g$$

$$\tau = 0\cdot65\ \text{sec},$$

and

$$n = 1/20.$$

The values of τ and n are such as to give approximately half-critical damping for the response of the missile to beam movement. The natural frequency of this mode is

$$\sqrt{K} = 2\cdot2\ \text{rad/sec}.$$

With these values and those of (8.1-7), the transfer function of the optimum tracker (8.1-3) becomes

$$T(p) = \frac{(399+1130p+1472p^2)\,[5(1+0\cdot65p) + p^2(1+0\cdot1p)^2(1+0\cdot0325p)]}{p^2\,[5(1+0\cdot65p)(131+1374p+695p^2+204p^3+24\cdot7p^4+p^5) - (1+0\cdot1p)^2(1+0\cdot0325p)(399+1130p+1472p^2)]}$$

or

$$T(p) = \frac{7\cdot75(1+3\cdot98p+5\cdot72p^2+3\cdot01p^3+0\cdot87p^4+0\cdot18p^5+0\cdot013p^6+0\cdot00025p^7)}{p^2\,[1 + 23\cdot7p + 24\cdot2p^2 + 11\cdot4p^3 + 2\cdot97p^4 + 0\cdot34p^5 + 0\cdot012p^6]}.$$

The acceleration lag of this function is $a_R{}^2 = 0\cdot13\ \text{sec}^2$, i.e. the loop gain of the servo is $7\cdot75\ \text{sec}^{-2}$. The amplitude response of the optimum tracker—that is,

$$\left|\frac{T(i\omega)}{1 + T(i\omega)}\right|$$

—is plotted in Fig. 15. For comparison the response of the simple tracker whose forward transfer function is

$$\frac{1 + 2\delta a_R p}{a_R{}^2 p^2}$$

is also plotted for the same value of $a_R{}^2$ ($0\cdot13\ \text{sec}^2$), and for various values of δ, the damping ratio.

It will be seen that it is not possible to approximate to the optimum system by varying the parameters of the simple system. In this latter system the undamped natural frequency is given by

$$\omega_0 = \frac{1}{a_R},$$

but this simple relation does not hold for the optimum system; in fact the main effect of the function $f(p)$ is to reduce the noise bandwidth of the tracker without the accompanying increase in acceleration lag which occurs in the simple system. In this particular case the noise bandwidth of the optimum system is

$$\int_0^\infty \left|\frac{T(i\omega)}{1+T(i\omega)}\right|^2 d\omega = 5\cdot 3 \text{ rad/sec},$$

while for the simple system it is

$$\int_0^\infty \left|\frac{1+2\delta a_R i\omega}{1+2\delta a_R i\omega - a_R^2\omega^2}\right|^2 d\omega = \frac{\pi}{a_R}\left(\delta + \frac{1}{4\delta}\right),$$

which has a minimum value of π/a_R when $\delta = 0\cdot 5$. With $a_R^2 = 0\cdot 13$ sec^2 this is $8\cdot 8$ rad/sec; and the resonant frequency is more than an octave above that of the optimum system. To obtain the same resonant frequency it would be necessary to increase the acceleration lag four-fold, resulting in a greatly increased miss distance.

The optimum transfer function $T(p)$ is such that the closed-loop system is stable, since this condition was inserted in the derivation of the optimum system: the stability is however conditional —i.e. the system is only stable for a limited variation of the loop gain, instability being reached if the gain is greater or less than the permissible range of values. This contrasts with the normal kind of stability, in which there is only an upper limit to the loop gain, all lower values giving a stable system.

The networks of the following Section (8.2) are also of the conditionally stable type, and the phenomenon is examined in more detail in that section.

8.2. *The Optimum Missile System for a Given Tracker.*

In the preceding section the optimum overall transfer function was realised by arbitrarily defining the missile control system and deducing the required properties of the ground tracker. As discussed in Section 6.2, this procedure is only permissible when the noise arises from linear sources external to the system. If the noise is due wholly or in part to servo noise, the insertion of the optimising filter will not have the desired effect, and it is then necessary to provide a correcting network in the missile rather than in the tracker, using (6.2-3). A case in which this treatment is required will now be considered.

8.2.1. *The transfer function of the optimum missile system.*—Suppose that observations with a radar set of the type

$$T(p) = \frac{1+2\delta a_R p}{a_R^2 p^2} \qquad (8.2\text{-}1)$$

have shown that the beam jitter has an r.m.s. value of 1 mil for $\delta = 1$ and $a_R = 0\cdot 157$ sec, with a spectral density consistent with the representation of white noise applied at the input to the servo: that is, the spectral density of the beam jitter is

$$k^2 \left|\frac{1+2\delta a_R i\omega}{1+2\delta a_R i\omega - a_R^2\omega^2}\right|^2.$$

The integral of this expression over ω gives the mean square value: it is

$$\sigma_B^2 = \frac{\pi}{a_R}\left(\delta + \frac{1}{4\delta}\right) k^2,$$

and for $\sigma_B = 1$ mil, $\delta = 1$ and $a_R = 0\cdot 157$ sec, we have

$$k^2 = 4 \times 10^{-8} \text{ rad}^2/\text{rad/sec}.$$

This then is the appropriate value of k^2 to be used in the derivation of the optimum transfer function. It will also be assumed that the beam jitter cannot be reduced by varying the parameters δ and a_R—i.e. it is due to causes which are not affected by these parameters.

The missile weathercock frequency will again be taken as 10 rad/sec, and for $k^2 = 4 \times 10^{-8}$ rad^2/rad/sec the appropriate curves are those of Fig. 11. For a $5g$ missile the relevant operator is that for $\mu^4 = 1$ (Fig. 11a):

$$H_0(p) = \frac{44\cdot 6 + 189p + 351p^2}{(0\cdot 77+p)(0\cdot 58+0\cdot 77p+p^2)(100+20p+p^2)(1+1\cdot 41p+p^2)} \qquad (8.2\text{-}2)$$

This transfer function gives a miss distance of 93 ft r.m.s., requiring achieved and demanded accelerations of $2\cdot 33g$ and $2\cdot 56g$ (Figs. 11b and c). Thus the chance of limiting in a $5g$ missile is 5% (Fig. 14). Fig. 11b shows that the miss distance could be substantially reduced by using higher accelerations—e.g. a $12g$ missile would give a miss of 60 ft r.m.s.; however, the $5g$ missile has been chosen with a view to possible flight trials in a test vehicle (the R.T.V.1), which is limited to this figure.

Since the tracking system cannot be modified, the correct overall transfer function must be obtained by modifying the missile control system $S(p)$. From (6.2-3)

$$S(p) = \frac{H_0(p)[1 + T(p)]p^2}{A(p)[T(p) - H_0(p)(1+T(p))]}, \qquad (8.2\text{-}3)$$

for $c_M = c_T = 0$. In this case,

$$A(p) = \frac{1}{(1+0\cdot 1p)^2}$$

and

$$T(p) = \frac{1 + 0\cdot 314p}{0\cdot 025p^2},$$

since $\delta = 1$ and $a_R = 1\cdot 57$, from (8.2-1). Thus

$$S(p) = \frac{p^2(1+0\cdot 1p)^2 H_0(p)}{\dfrac{1 + 314p}{1 + 0\cdot 314p + 0\cdot 025p^2} - H_0(p)}, \qquad (8.2\text{-}4)$$

where $H_0(p)$ is given by (8.2-2). The combination of (8.2-2) and (8.2-4) yields

$$S(p) = \frac{1}{1\cdot 04}\frac{(1+0\cdot 1p)^2(1+0\cdot 157p)^2(1+4\cdot 23p+7\cdot 87p^2)}{(1+0\cdot 095p)(1+0\cdot 106p)(1+10\cdot 31p+2\cdot 90p^2)(1+0\cdot 60p+0\cdot 233p^2)}.$$

The product $(1+0\cdot 095p)$ $(1+0\cdot 106p)$ is very nearly $(1+0\cdot 1p)^2$, so that we may write

$$S(p) = \frac{1}{1\cdot 04}\frac{(1+0\cdot 314p+0\cdot 025p^2)(1+4\cdot 23p+7\cdot 87p^2)}{(1+10\cdot 31p+2\cdot 90p^2)(1+0\cdot 60p+0\cdot 233p^2)}. \qquad (8.2\text{-}5)$$

This is the required transfer function of the missile control system, and may be regarded as replacing the simple network

$$\frac{K(1+p\tau)}{(1+np\tau)}$$

of the previous example (Section 8.1).

From (8.2-5), the missile loop acceleration lag $a_M{}^2 = 1\cdot 04$ sec^2, which is equivalent to a stiffness of $33\cdot 5$ ft/g. This agrees with the fact that the overall acceleration lag is $a_0{}^2 = 1\cdot 07$ sec^2 (Fig. 11a, $\mu^4 = 1$), and we have chosen $a_R{}^2$ as $0\cdot 025$ sec^2. Thus most of the lag is in the missile system, resulting in a rather slow response to beam movement. In practice, it would be preferable to allocate the total lag more equally between the tracker and the missile systems, but the figures have been chosen to give a large beam jitter, so that the system can be tested with the equipment at present available for flight trials.

If, for the same conditions, a missile having a greater limiting acceleration is available, we may choose $H_0(p)$ to give a smaller miss distance. The optimum system for $\mu^4 = 50$ (Fig. 11a), for example, gives a miss distance of 63 ft r.m.s., with achieved and demanded accelerations of $5 \cdot 04g$ and $6 \cdot 57g$; and a $13g$ missile would be required for 5% saturation. For this system the necessary missile control function is

$$S(p) = \frac{1}{0 \cdot 491} \frac{(1+0 \cdot 314p+0 \cdot 025p^2)(1+3 \cdot 34p+5 \cdot 05p^2)}{(1+10 \cdot 63p+2 \cdot 64p^2)(1+0 \cdot 33p+0 \cdot 078p^2)}, \tag{8.2-6}$$

with the tracker as in equation (8.2-1).

In this case $a_M{}^2 = 0 \cdot 491$ sec^2—a stiffness of $15 \cdot 8$ ft/g.

8.2.2. *Physical realisability.*—Equations (8.2-5) and (8.2-6) are of the same form. The factor

$$\frac{(1+0 \cdot 314p+0 \cdot 025p^2)}{(1+10 \cdot 31p+2 \cdot 90p^2)}$$

of (8.2-5), and the corresponding factor of (8.2-6), have two real zeros and two real poles, and can be realised by passive networks consisting of capacitances and resistances. The remaining factors contain a pair of complex zeros and a pair of complex poles; their realisation requires active networks, since the use of inductance is impracticable at the low frequencies involved. The realisation of such functions is treated in Appendix IV.

8.2.3. *Conditional stability.*—The equation relating missile acceleration and the beam-to-missile error is

$$A(D)S(D)\,[\theta_B - \theta_M] = D^2\theta_M, \tag{3.2-6}$$

since we have assumed $c_M = 0$. Thus

$$\theta_M = \frac{A(D)S(D)}{A(D)S(D) + D^2}\,\theta_B,$$

so that the transfer function relating beam to missile motion is

$$\frac{A(p)S(p)}{A(p)S(p) + p^2}.$$

For the example of the previous Section (8.2-1), we have

$$A(p) = \frac{1}{(1+0 \cdot 1p)^2},$$

and $S(p)$ is given by (8.2-5).

The open-loop transfer function is $\{A(p)S(p)\}/p^2$, and the frequency response of this function is shown in Fig. 16. Inspection of this diagram shows that the missile system is conditionally stable, in that a decrease as well as an increase in gain leads to instability. The phase margin is 19°, and the gain margins are—9 db and 6 db. Thus to maintain stability the loop gain must not be allowed to fall below its nominal value by more than a factor of 3: it must also not increase by more than a factor of 2. These are fairly wide limits, giving a permissible stiffness range of 15 ft/g to 100 ft/g, as far as stability is concerned. The stiffness must clearly be controlled much more accurately than this, if the optimum performance of the system is to be achieved.

It will be noted from Fig. 16 that the phase margin attains its maximum value at precisely the point where it is required—when the loop gain is 0 db; in this way the system is stabilised with the minimum increase in noise bandwidth. The Nyquist diagrams of Fig. 17 illustrate the effect of stabilising the missile system by a phase-advance network and by the network $S(p)$ of (8.2-5).

With no stabilising network (Fig. 17a) the missile loop is clearly unstable; the addition of the phase-advance network

$$\frac{1 + p\tau}{1 + np\tau}$$

gives stability, at the expense of increasing the response to higher frequencies (Fig. 17b). The optimum system (Fig. 17c) achieves (conditional) stability by providing a local distortion of Fig. 17a in the neighbourhood of the critical point $(-1, i0)$, and decreases the response for higher frequencies. This may also be seen by comparing the high-frequency response of $S(p)$ and of the network

$$\frac{K(1+p\tau)}{(1+np\tau)}.$$

The gain of the latter for high frequencies is approximately K/n; for the same stiffness $K = 1/1\cdot 04$ sec^{-2}, and for stability n is necessarily in the region 1/10 to 1/20. The optimum network, on the other hand, has an *attenuation* of approximately 3 for these frequencies $\left[\lim_{p\to\infty} S(p).\right]$

8.2.4. *The missile response to beam motion.*—The response of the missile to a step of the beam, with the optimum network of (8.2-5), is shown in Fig. 18. The recovery is rather slow, accompanied by a large overshoot: this is a consequence of the rather artificial conditions which have been chosen to give a large beam jitter with a $5g$ missile. Normally a larger share of the overall acceleration lag would be incorporated in the tracker; alternatively, the missile acceleration could be increased without altering the tracker constants, leading to a faster response of the missile—this is the case for the network of (8.2-6), for which a $13g$ missile is required for 5% saturation.

9. *A Comparison of the Optimum System and the Phase-Advance System.*

Having derived the optimum systems for a number of cases, it is useful to estimate whether the increased efficiency of the optimised system is sufficient to warrant the inclusion of additional networks. That the optimum system does in fact lead to a considerable improvement may be illustrated by comparison with the phase-advance system, which has received extensive theoretical and simulator treatment. The system referred to has a missile stabilising network of the form

$$\frac{K(1+p\tau)}{(1+np\tau)},$$

associated with a radar tracker whose open-loop transfer function is

$$\frac{1 + 2\delta a_R p}{a_R^2 p^2},$$

as in Section 8. It is shown in Ref. 2 that this arrangement achieves its maximum efficiency when the various parameters are such that the demand for acceleration frequently exceeds the limits; in which case the system is non-linear, and analysis is only possible for simple target motions—such as a constant angular acceleration of the line of sight.

As a specific example, suppose that the following conditions obtain:

Noise spectral density	$k^2 = 0 \cdot 5 \times 10^{-8}$ rad²/rad/sec
Interception range	100 000 feet
Target manoeuvre	$2g$ normal to beam
Missile weathercock response	$\dfrac{1}{(1+0 \cdot 1p)^2}$.

Using the methods of Ref. 2, the parameters K, a_R, τ, etc., can be found such that the r.m.s. miss distance against the $2g$ target is a minimum. The results for $5g$ and $10g$ missiles are given in Table 2.

The appropriate curves for the comparable optimum system are those of Fig. 10. If the achieved acceleration is taken to be $2 \cdot 5g$, the r.m.s. miss distance is 35 ft (Fig. 10b). This gives a demanded acceleration of $3g$ (Fig. 10c), so that for a $5g$ missile the demand would be on the limits for about 10% of the time (Fig. 14). These values are of course for the statistical ensemble of target motions for which the system is optimised, but since the system is linear its performance against any other target is readily deduced. For a target having a constant acceleration of f_T normal to the beam, the mean miss distance is simply $f_T\, a_0^2$, where a_0^2 is the acceleration lag of the complete system.

The scatter due to noise alone is given in Fig. 10b together with a_0^2. The r.m.s. achieved and demanded accelerations against the $2g$ target are slightly different from those quoted above, because the latter include the acceleration due to the target ensemble motion as well as that due to noise.

Table 2 compares the results of the two systems.

TABLE 2

Comparison of the Optimum and Phase-advance Systems for a Target Accelerating at 2g

Limiting missile accn.	r.m.s. demand accn. (before limits)	% of time on limits	r.m.s. achieved accn.	Mean miss dist.	Dis-persion	r.m.s. miss dist.	Accn. lag (sec²)	Type of system
$5g$	$3 \cdot 2g$	12%	$2 \cdot 7g$	19 ft	22 ft	29 ft	0·3	Optimum System
$5g$	$8 \cdot 2g$	58%	$3 \cdot 2g$	47 ft	20 ft	51 ft	0·51	Phase-advance system with opti-
$10g$	$10 \cdot 9g$	37%	$6 \cdot 0g$	32 ft	21 ft	38 ft	0·4	mum parameters for a $2g$ target.

The results for the optimum system have been obtained on the assumption that the 12% saturation can be neglected, whereas for the phase-advance system the effect of saturation has been included.

It is evident that the optimum system is considerably better than the phase-advance method, in that it results in a smaller miss distance with smaller accelerations; even with a $10g$ missile the phase-advance system gives a greater miss distance than does a $5g$ missile with the optimum arrangement, while the r.m.s. acceleration required for the former system is nearly twice as great. It will be noted from Table 2 that while the manoeuvre miss distance in the optimum system is simply $f_T\, a_0^2$, for the phase-advance system it is greater than this quantity: this results from the interaction of the limits and noise signals[4].

It should be emphasised that the optimum system above is an optimum with respect to an ensemble of targets having the autocorrelation function

$$\psi_T(x) = e^{-\beta|x|},$$

and not for targets having a constant acceleration; the performance of such a system against the constant acceleration target is nevertheless superior to that of the phase-advance system. It is possible to derive an optimum operator for the constant-target-acceleration case, but for this it is necessary to revert to the integral equation (4.1-9), since the assumptions inherent in the Fourier Transform solution of this equation are no longer apposite.

10. *Simulator Tests with Optimum and Phase-advance Systems.*

10.1. *The Optimum System Used in the Simulator.*

In quoting numerical results for a typical system, a number of approximations and simplifications have been made. The simplifications have been introduced to ease the labour of computation, while the approximations were necessary to render the problem analytically tractable. The extent to which such approximations are likely to affect the conclusions reached can be usefully assessed by solving particular cases with the aid of a simulator, which can readily provide a closer representation of the true situation than is possible within a mathematical framework limited by considerations of linearity, etc. The greater flexibility of the simulator is of course offset by the fact that only particular cases can be studied: its proper role therefore is to verify and amplify the theoretical work, or, as in the present case, to assess the influence of factors which have been neglected or idealised in the analysis.

Although an exhaustive simulator study has not been undertaken, a preliminary examination of a particular optimum system has been carried out. The example used in the simulator work is that of Section 8.2, where it was assumed that the tracker could not be modified, and that it was therefore necessary to place the optimising network in the missile control system.

The choice of this example was influenced by the facilities at present available for simulator tests and flight trials: these reside with the test vehicle R.T.V.1 (limited to $5g$), and a radar tracking system (the S.C.R. 584) in which the noise level is greater than may be expected with later systems. The combination of large noise and low acceleration limits results in a heavily filtered system (cf. the response of the missile to a beam step, Fig. 18), and in fairly large miss distances. Although this situation is unlikely to arise in an actual weapon—the noise, for example, will certainly be less—it nevertheless serves our present purpose, which is to assess the validity of the theory when neglected factors are taken into account, and to compare the optimum system with other possible systems.

The appropriate network for the missile is the $S(p)$ of (8.2-5):

$$S(p) = \frac{1}{1{\cdot}04}\,\frac{(1+0{\cdot}314p+0{\cdot}025p^2)(1+4{\cdot}23p+7{\cdot}87p^2)}{(1+10{\cdot}31p+2{\cdot}90p^2)(1+0{\cdot}60p+0{\cdot}233p^2)}\,.$$

The realisation of such a transfer function is given in Appendix IV, and Fig. 23 shows the actual circuit used in the simulation. It will be recalled (Section 8.2.3) that this network leads to a conditionally stable system for the missile loop, with the characteristics given in Figs. 16 and 17.

10.2 *Additional Factors Included in the Simulation.*

The main factors neglected in the analysis but included in the simulator are:

(a) Non-linearity of the aerodynamics. The simulation of the R.T.V.1 aerodynamics includes non-linear effects such as downwash, and since only a moderate degree of rate gyro and accelerometer feedback is at present applied, the resulting weathercock mode is only approximately linear; for example, the stiffness varies between zero incidence and the incidence necessary for $5g$ by a factor of $1{\cdot}4$. In the theory the stiffness is of course considered constant with incidence.

(b) The transfer function of the weathercock mode of a fixed-wing vehicle with rear control surfaces is of the form

$$P(p)/Q(p),$$

where both P and Q are quadratic in p. The coefficients of p and p^2 in $P(p)$ are negative, because of the rear control, and the system behaves as a non-minimum phase network. It is this fact which limits the amount of acceleration feedback which can be applied—it is necessary to use an accelerometer displaced from the centre of gravity, or some equivalent arrangement, to avoid instability.

The coefficients of p and p^2 in $P(p)$ are normally small, and they have been discarded in the theory (for the examples evaluated), where we have used the representation

$$\frac{1}{(1+pT)^2}$$

for the modified weathercock mode. In the simulator however these terms are present.

(c) As derived in the analysis, the function $S(p)$ relates the demanded acceleration $f_D(t)$ to the error from the beam:

$$f_D(t) = S(D) r_M(t)\{\theta_B(t) - \theta_M(t)\}. \tag{3.2-1}$$

Thus $S(p)$ should properly include the transfer function of the receiver as well as the network introduced for control purposes. If this last network is given the transfer function $S(p)$, it implies that the receiver is perfect, or nearly so. In the simulator tests the correcting network $S(p)$ was introduced in addition to the actual receiver and a simulated control-surface actuator.

The differences between the simulator arrangement and the theory noted in (b) and (c) arise from simplifications introduced to ease calculation, and do not represent limitations of the theory; item (a) however is a necessary approximation for the analysis.

10.3. *Results for the Optimum and Phase-Advance Systems.*

Since the R.T.V.1 control system is normally of the phase-advance type, it is convenient to compare this system with the optimum arrangement.

The simulation of the two systems is indicated in Fig. 19. For the optimum system the network $S(p)$ (Fig. 23) replaces the phase-advance network of the normal R.T.V.1 system. At first sight the optimum network appears more complicated,* but in fact the phase-advance requires additional

* *See* footnote on p. 65 to Section 12.2.

frequencies lie well above that of the correcting network $S(p)$, their effect will be negligible. It must be borne in mind however that in our present example the complete missile system is rather sluggish, for the reasons given in Section 10.1, so that the response times demanded of receiver and actuator are correspondingly less severe. In a more practical situation—e.g. less noise—the optimum stiffness is likely to be higher, in which case it may be necessary to take account of these components in designing the correcting network.

11. *Conclusions.*

11.1. Given certain information about the target and the noise, and the characteristics of the missile (its weathercock response and structural strength) we have shown that a practical system can be devised for which the r.m.s. miss distance is less than that for any other linear system. In the derivation it has been assumed that the target and noise functions

$$\varphi_{T^{(m)}}(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{T_r}{}^{(m)}(T)\theta_{T_r}{}^{(m)}(T-x)$$

and

$$\varphi_N(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T)\theta_{Nr}(T-x)$$

are independent of T, the time of strike, over an interval immediately prior to engagement. This assumption leads to an optimum transfer function for the system, which may be identified with the beam-riding system, since we have shown that the latter can be regarded as linear in the sense required for the analysis. This leads to the definition of certain components of the beam-riding system for optimum performance—components which, depending on the sources of noise, may be associated with either the tracker or the missile, or both. The conditions imposed in the derivation of the optimum operator ensure that the components required are physically realisable within the framework of a beam-riding system.

It is interesting to note that with a perfect missile (i.e. no time lags and no acceleration limit), the best transfer function of the system is not unity, but of the form

$$\frac{a + bp + cp^2}{a + bp + c'p^2 + d'p^3};$$

and that even with a practical missile the r.m.s. miss distance with the optimum system may be less than the r.m.s. error in the beam estimate of the target's position. This is the case for the examples discussed in Section 8.2.1.

In the cases evaluated the optimisation leads to a conditionally stable loop either for the missile or for the tracker, depending on where the correcting network is placed: this is a consequence of the assumption that the target lateral acceleration is stationary. It has been shown that for a case in which the correction is applied in the missile, the range of stiffness over which stability is maintained is such that the conditional stability is of academic interest rather than an obstacle to the practical design.

11.2. From the numerical results given in Section 7.3—with the assumptions regarding the target motion, noise, etc., of that section—we may draw the following conclusions:

(a) The optimum system achieves a smaller miss distance, and at the same time requires less acceleration, than the comparable phase-advance system. Since the induced drag is proportional to the mean square lateral acceleration, the latter point is of significance, in that for a given performance the demand on the missile and on its propulsion unit are less stringent. In Table 2, for example, the phase-advance system gives a five-fold increase in induced drag, to achieve an r.m.s. miss distance which is 9 feet greater than the optimum system against a '$2g$ target'.

(b) Figs. 10 to 13 show that when the optimum system is used there exists a fairly well-defined achieved acceleration, beyond which the increase in accuracy is more than offset by the increased fuel consumption, etc., which the higher accelerations involve. The optimum point depends on the targets, the noise, and the overall logistics of the weapon defence system. In Fig. 10, which covers the set of conditions likely to be realised ($k^2 = 0 \cdot 5 \times 10^{-8}$ rad²/rad/sec, $T = 0 \cdot 1$ sec), the optimum r.m.s. acceleration appears to be in the region of $3\frac{1}{2}g$, where the r.m.s. miss distance is 30 feet at a range of 100 000 feet. (For the higher noise, the figure is about $4\frac{1}{2}g$.) Under these conditions the demand for acceleration in a $10g$ missile would only reach this limit for 5% of the time, and it is probable that the acceleration limits could be reduced without materially affecting the performance (*see* Section 12.2).

It is noteworthy that these accelerations are considerably less than those previously deemed necessary; and that the results are achieved by a more efficient use of the available information, requiring electrical components at a low power level rather than the costly provision of missile acceleration greatly in excess of possible target acceleration.

(c) The modified weathercock frequency of the missile should be in the region of 2 to 3 c/sec; lower values react adversely on the miss distance obtained with a given acceleration, while higher values provide little improvement.

(d) The numerical results quoted are all for a particular target spectral density, and this subject has not yet been fully investigated; the form used, however, even if shown to be inadequate, leads to a system which has a better performance against constant acceleration targets than the phase-advance system. The example quoted in (b) above for example would have a miss distance of 30 ft r.m.s. against a $3g$ target at a range of 100 000 feet.

Although the present theory leads to the best practical linear system, it may be argued that a non-linear system may exist which has a better performance. This cannot be denied and the search for such a system should continue: meanwhile full use should be made of a system which appears to have some advantages over those at present employed.

12. *Further Work.*

It is considered that the preliminary results of the above analysis warrant a further investigation, directed along the following lines:

12.1. *Analysis.*

This would include a more detailed study of possible target motion, with a view to making more direct use of preliminary target data—such as height, course and speed—furnished by the radar search systems; such specific information eliminates some of the uncertainties regarding the target's subsequent behaviour, thus easing the task of the guided weapon. This implies a different optimum system for each target, which would be possible if the alterations could be confined to the tracker.

For a given system, there will be an optimum target spectral density which will render the system as ineffective as possible; this should not be overlooked as a probable type of target behaviour.

The present paper is concerned mostly with beam-riding. The application of the theory to proportional navigation is less tractable, in that it is essentially a variable-coefficient system; it is possible that a similar approach would yield an optimum arrangement.

12.2. *Simulation.*

There are a number of problems for which the simulator approach is more expedient than analysis:

(a) It is necessary to establish the degree of saturation which can be tolerated before the theoretically predicted performance becomes markedly optimistic. With a missile of a given structural strength, it may be preferable to choose an optimum system for which the saturation level is, say, 25%, rather than the optimum for a lower acceleration which is more nearly linear. Reference to Figs. 10 to 13 shows that the steep rise in miss distance for the smaller accelerations may more than offset the loss in accuracy due to saturation. The simulator programme would therefore consist of the evaluation of a series of optimum systems for given acceleration limits.

(b) In the simulation so far attempted (Section 10) the optimum filter has been realised accurately. Further tests are necessary to determine to what extent the networks can be simplified without sacrifice of performance, and the permissible tolerances on the components.*

12.3. *Flight Trials.*

(a) The facilities at present available are such that the jitter of the radar beam is rather greater than may be expected in a weapon system. However, it is possible to design an optimum system for these conditions (Section 8.2) and to test its performance in the presence of jitter. If this is in agreement with the theory, there is no reason to suppose that an optimum system designed for less jitter would be less predictable.

(b) The overall acceleration lag of the optimum system when the jitter level is low ($k^2 = 0 \cdot 5 \times 10^{-8}$ rad²/rad/sec) is rather small (Fig. 10b), and since there is a limit to the permissible loop gain of the tracker, it follows that a stiff missile control system (of up to 5 ft/g) may be necessary. It is not essential that the flight trials to test such arrangements be carried out in the presence of jitter: since the systems are sensibly linear, a few comparatively simple experiments with stationary and moving beams would suffice to give a fairly accurate assessment of the performance of a weapon.

13. *Acknowledgments.*

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* Since this paper was written Mr. H. Lewis has shown that a good approximation to the required transfer functions can be realised by the use of passive networks, involving capacitance and resistance only. This results in a considerable simplification, and such networks are now being tested on the simulator.

LIST OF PRINCIPAL SYMBOLS

$A(D)$	Operator relating demanded and achieved missile lateral accelerations
a_0^2	Overall acceleration lag of the optimum beam-riding system
a_M^2	Acceleration lag of the missile system
a_R^2	Acceleration lag of the ground-radar tracking system

$$\left.\begin{aligned} c_M &= \dot{r}_M/r_M \\ c_T &= \dot{r}_T/r_T \end{aligned}\right\}\text{, considered constant}$$

$F(D)$	Operator defining θ_L, the mean square value of which it is desired to limit, in terms of θ_M: $$\theta_L(t) = F(D)\theta_M(t)$$
f_M	Achieved missile acceleration
f_D	Demanded missile acceleration
$f_{\max}$	Maximum permissible acceleration which can be demanded of the missile
f_T	Target acceleration perpendicular to the line of sight
$H(D)$	Operator defining the complete beam-riding system
$H_0(D)$	Optimum operator
$h(t)$	Weighting function of the complete system
$h_0(t)$	Optimum weighting function
R	Range at which interception occurs
$r_M(t)$	Missile range at time t
$r_T(t)$	Target range at time t
$S(D)$	Operator relating the missile-to-beam error and the demanded missile acceleration: $$f_D(t) = S(D)r_M(t)[\theta_B(t) - \theta_M(t)]$$
s_r	Miss distance for the rth attempt
T	Time, reckoned from launch, at which interception occurs. A time constant in the missile aerodynamic response
$T(D)$	Operator defining the response of the ground radar tracking system: $$\theta_B(t) = T(D)[\theta_T(t) + \theta_N(t) - \theta_B(t)]$$

$\theta_T(t)$ Angle which the line of sight to the target makes with a datum line through the ground radar set

$\theta_B(t)$ Angle which the centre-line of the beam makes with this datum

$\theta_M(t)$ Angle which the line of sight to the missile makes with this datum

$\theta_N(t)$ 'Noise' angle, i.e. the difference between the true angular error $(\theta_T - \theta_B)$ and the error as given by the ground radar receiver

$\theta_L(t)$ Quantity in the missile system whose mean square value it is desired to restrict

Additional suffix r refers to the values of $\theta_T(t)$, $\theta_B(t)$, etc., during the rth attempt

σ r.m.s. angular miss distance taken over a large number of attempts:

$$\sigma^2 = \frac{1}{R^2}\frac{1}{n}\sum_{r=1}^{n} s_r^2$$

$\sigma_{\min}$ Minimum r.m.s. angular miss distance when the optimum operator $H_0(D)$ is used

σ_1, σ_2 r.m.s. angular miss distances due to the noise and to target manoeuvre respectively—i.e.

$$\sigma_{\min}^2 = \sigma_1^2 + \sigma_2^2$$

σ_L r.m.s. value of the quantity it is desired to limit—e.g. the demanded missile acceleration:

$$\sigma_L^2 = R^2\frac{1}{n}\sum_{r=1}^{n} [\theta_{Lr}(T)]^2$$

σ_M r.m.s. missile achieved acceleration, divided by the range R

σ_T r.m.s. target acceleration normal to the line of sight, divided by the range R

$\varphi_{T^{(m)}}(x)$ Autocorrelation function of the mth derivatives of the target angles:

$$\varphi_{T^{(m)}}(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}{}^{(m)}(T)\theta_{Tr}{}^{(m)}(T-x)$$

$\varphi_{N^{(m)}}(x)$ Autocorrelation function of the mth derivatives of the noise angles

$$\varphi_{N^{(m)}}(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}{}^{(m)}(T)\theta_{Nr}{}^{(m)}(T-x).$$

$\psi_T(x)$ Autocorrelation function of the target acceleration normal to the line of sight, divided by R^2

$\left.\begin{array}{r}\Phi_{T^{(m)}}(i\omega),\\ \Phi_{N^{(m)}}(i\omega)\\ \Psi_T(i\omega)\end{array}\right\}$ Fourier Transforms of $\varphi_{T^{(m)}}(x)$, $\varphi_{N^{(m)}}(x)$ and $\psi_T(x)$,

REFERENCES

No.	*Author*	*Title, etc.*
1	N. Wiener	*Extrapolation, interpolation and smoothing of stationary time series.* Wiley.
2	E. G. C. Burt	A theoretical treatment of noise and non-linearities in a beam-riding system. A.R.C. R. & M. 3417. November, 1951.
3	James, Nichols and Phillips ..	*Theory of servomechanisms.* Radiation Laboratory Series, McGraw-Hill.
4	L. B. C. Cunningham and W. R. B. Hynd.	Random processes in air warfare. Supplement to the *J.R.Statist. Soc.*, Vol. 8, No. 1. 1946.
5	Lars. N. Enequist	On some integral equations arising in the study of noise in linear systems. Symposium on Noise Reduction. Los Angeles, Cal. February, 1949.

APPENDIX I

Some Properties of Linear Differential Equations

I.1. *The Weighting Function.*

The general nth order linear differential equation may be written as

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots . + a_1(t)\frac{dy}{dt} + a_0(t)y = f(t) \tag{I-1}$$

where $a_{n-1}, \ldots, a_1(t)$, $a_0(t)$ and $f(t)$ are arbitrary functions of time. If the complementary function of this equation is known, the complete solution may readily be found.

Let the complementary function be

$$y = c_1 y_1 + c_2 y_2 + \ldots . + c_n y_n . \tag{I-2}$$

In this $y_1, y_2, \ldots, y_n$ are the n independent solutions of the homogeneous equation corresponding to (I-1), i.e. of

$$\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \ldots . + a_1(t)\frac{dy}{dt} + a_0(t)y = 0, \tag{I-3}$$

and $c_1, c_2, \ldots, c_n$ are arbitrary constants which may be determined from the initial conditions of the problem.

Consider

$$y = v_1 y_1 + v_2 y_2 + \ldots . + v_n y_n ; \tag{I-4}$$

where $v_1, v_2, \ldots, v_n$ are functions of t. Then

$$\dot{y} = v_1 \dot{y}_1 + v_2 \dot{y}_2 + \ldots . + v_n \dot{y}_n ,$$

provided that

$$\dot{v}_1 y_1 + \dot{v}_2 y_2 + \ldots . + \dot{v}_n y_n = 0.$$

Also

$$\ddot{y} = v_1 \ddot{y}_1 + v_2 \ddot{y}_2 + \ldots . + v_n \ddot{y}_n ,$$

provided that

$$\dot{v}_1 \dot{y}_1 + \dot{v}_2 \dot{y}_2 + \ldots . + \dot{v}_n \dot{y}_n = 0.$$

Continuing in this way up to the $(n-1)$th derivative, we have

$$y^{(n-1)} = v_1 y_1^{(n-1)} + v_2 y_2^{(n-1)} + \ldots . + v_n y_n^{(n-2)},$$

provided that

$$\dot{v}_1 y_1^{(n-2)} + \dot{v}_2 y_2^{(n-2)} + \dot{v}_n y_n^{(n-2)} = 0.$$

Thus $(n-1)$ linear homogeneous relations have been established between $\dot{v}_1, \dot{v}_2, \ldots, \dot{v}_n$. Also

$$y^{(n)} = v_1 y_1^{(n)} + v_2 y_2^{(n)} + \ldots . + v_n y_n^{(n)} + \dot{v}_1 y_1^{(n-1)} + \dot{v}_2 y_2^{(n-1)} + \ldots . + \dot{v}_n y_n^{(n-1)}.$$

If now we substitute y as given by (I-4) in equation (I-1), the condition that y be a solution of (I-1) leads to the further relation

$$\dot{v}_1 y_1^{(n-1)} + \dot{v}_2 y_2^{(n-1)} + \ldots . + \dot{v}_n y_n^{(n-1)} = f(t).$$

This provides the nth relation between the derivatives, and the system of equations

$$\begin{array}{l}
\dot{v}_1 y_1 + \dot{v}_2 y_2 + \dots + \dot{v}_r y_r + \dots + \dot{v}_n y_n = 0 \\
\dot{v}_1 \dot{y}_1 + \dot{v}_2 \dot{y}_2 + \dots + \dot{v}_r \dot{y}_r + \dots + \dot{v}_n \dot{y}_n = 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\dot{v}_1 y_1^{(n-2)} + \dot{v}_2 y_2^{(n-2)} + \dots + \dot{v}_r y_r^{(n-2)} + \dots + \dot{v}_n y_n^{(n-2)} = 0 \\
\dot{v}_1 y_1^{(n-1)} + \dot{v}_2 y_2^{(n-1)} + \dots + \dot{v}_r y_r^{(n-1)} + \dots + \dot{v}_n y_n^{(n-1)} = f(t)
\end{array}$$

may be solved to give $\dot{v}_1, \dots, \dot{v}_n$. Define

$$W = \begin{vmatrix} y_1 & y_2 & \dots\dots & y_n \\ \dot{y}_1 & \dot{y}_2 & \dots\dots & \dot{y}_n \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots\dots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots\dots & y_n^{(n-1)} \end{vmatrix};$$

then

$$\dot{v}_r = f(t)\,\frac{A_{nr}}{W}, \qquad r = 1, 2, \dots, n,$$

where A_{nr} is the cofactor of $y_r^{(n-1)}$ in the determinant W. Hence

$$v_r = \int_0^t f(x)\,\frac{A_{nr}(x)}{W(x)}\,dx,$$

$$y = v_1 y_1 + \dots + v_n y_n = \sum v_r y_r$$

$$= \sum \int_0^t f(x)\,\frac{1}{W(x)}\,y_r(t) A_{nr}(x)\,dx$$

$$= \int_0^t f(x)\,\frac{1}{W(x)} \begin{vmatrix} y_1(x) & y_2(x) & \dots\dots & y_n(x) \\ \dot{y}_1(x) & \dot{y}_2(x) & \dots\dots & \dot{y}_n(x) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots\dots & y_n^{(n-2)}(x) \\ y_1(t) & y_2(t) & \dots\dots & y_n(t) \end{vmatrix} dx$$

$$= \int_0^t f(x)\,\frac{(-1)^{n-1}}{W(x)} \begin{vmatrix} y_1(t) & y_2(t) & \dots\dots & y_n(t) \\ y_1(x) & y_2(x) & \dots\dots & y_n(x) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \dots\dots & y_n^{(n-2)}(x). \end{vmatrix} dx.$$

Hence the general solution of equation (I-1) may be written as

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n + \int_0^t f(x) \frac{(-1)^{n-1}}{W(x)} \begin{vmatrix} y_1(t) & y_2(t) & \ldots\ldots & y_n(t) \\ y_1(x) & y_2(x) & \ldots\ldots & y_n(x) \\ \dot{y}_1(x) & \dot{y}_2(x) & \ldots\ldots & \dot{y}_n(x) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \ldots\ldots & y_n^{(n-2)}(x) \end{vmatrix} dx, \tag{I-5}$$

where W, termed the Wronskian, is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \ldots\ldots & y_n(x) \\ \dot{y}_1(x) & \dot{y}_2(x) & \ldots\ldots & \dot{y}_n(x) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \ldots\ldots & y_n^{(n-2)}(x) \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \ldots\ldots & y_n^{(n-1)}(x) \end{vmatrix}. \tag{I-6}$$

The expression (I-5) is the complete solution of (I-1) since it involves n arbitrary constants. The integral term is of course the particular integral. It has been assumed in writing the integral term in (I-5) that the process described by the differential equation (I-1) starts at time t equal to zero. If the system starts from rest at $t = 0$, i.e. if $y, \dot{y}, \ldots, y^{(n-1)}$ are all zero, then $c_1, c_2, \ldots, c_n$ are all zero and the solution reduces to

$$y = \int_0^t f(x) \frac{(-1)^{n-1}}{W(x)} \begin{vmatrix} y_1(t) & y_2(t) & \ldots\ldots & y_n(t) \\ y_1(x) & y_2(x) & \ldots\ldots & y_n(x) \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \ldots\ldots & y_n^{(n-2)}(x) \end{vmatrix} dx. \tag{I-7}$$

The expression (I-7) may be simplified as follows. Since $y_1, y_2, \ldots, y_n$ are separately solutions of equation (I-3), we have the following n relations

$$\begin{aligned} y_1^{(n)} + a_{n-1} y_1^{(n-1)} + \ldots\ldots + a_1 \dot{y}_1 + a_0 y_1 &= 0 \\ y_2^{(n)} + a_{n-1} y_2^{(n-1)} + \ldots\ldots + a_1 \dot{y}_2 + a_0 y_2 &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ y_n^{(n)} + a_{n-1} y_n^{(n-1)} + \ldots\ldots + a_1 \dot{y}_n + a_0 y_n &= 0. \end{aligned}$$

From this set of n equations we may eliminate $a_0, a_1, \ldots\ldots, a_{n-2}$ and this leads to the simple equation

$$\frac{dW}{dt} + a_{n-1}(t)W = 0, \tag{I-8}$$

since

$$\frac{dW}{dt} = \begin{vmatrix} y_1(t) & y_2(t) & \cdots\cdots & y_n(t) \\ \dot{y}_1(t) & \dot{y}_2(t) & \cdots\cdots & \dot{y}_n(t) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \cdots\cdots & y_n^{(n-2)}(t) \\ y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots\cdots & y_n^{(n)}(t). \end{vmatrix}$$

Then from (I-8) we have

$$W = W_0 \exp\left(-\int^t a_{n-1}(t)dt\right), \tag{I-9}$$

where W_0 is some constant. Thus (I-7) becomes

$$y = \frac{(-1)^{n-1}}{W_0} \int_0^t f(x) \exp\left(\int^x a_{n-1}(t)dt\right) \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \\ \dot{y}_1(x) & \dot{y}_2(x) & \cdots & \dot{y}_n(x) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x). \end{vmatrix} dx. \tag{I-10}$$

This is a perfectly general result, and does not depend on the form of the functions $a_0(t)$, $a_1(t) \ldots,$ $a_{n-1}(t)$.

Now let us suppose that $a_0(t)$, $a_1(t)$, $\ldots$, $a_{n-1}(t)$ are all constants—i.e. the original equation (I-1) is now the general nth order linear differential equation with constant coefficients. In this case the functions $y_1(t), \ldots\ y_n(t)$ are all of the form

$$y_r(t) = \exp(\alpha_r t) \qquad r = 1, 2, \ldots, n$$

where $\alpha_1, \ldots, \alpha_n$ are roots of the characteristic equation

$$\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0 \equiv (\alpha-\alpha_1)(\alpha-\alpha_2)\ldots(\alpha-\alpha_n) = 0. \tag{I-11}$$

We have excluded the possibility of equal roots, but the analysis may be readily extended to include this. We note that

$$y_r^{(s)}(t) = (\alpha_r)^s y_r(t),$$

and from (I-11) we also have

$$\alpha_1 + \alpha_2 \ldots . + \alpha_n = \sum_1^n \alpha_r = -a_{n-1}. \tag{I-12}$$

Thus (I-10) becomes

$$y = \frac{(-1)^{n-1}}{W_0} \int_0^t f(x) \exp\left(-x \sum_1^n \alpha_r\right) \begin{vmatrix} \exp(\alpha_1 t) & \exp(\alpha_2 t) & \ldots . & \exp(\alpha_n t) \\ \exp(\alpha_1 x) & \exp(\alpha_2 x) & \ldots . & \exp(\alpha_n x) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{n-1}\exp(\alpha_1 x) & \alpha_2^{n-2}\exp(\alpha_2 x) & \ldots . & \alpha_n^{n-2}\exp(\alpha_n x) \end{vmatrix} dx,$$

$$= \frac{(-1)^{n-1}}{W_0} \int_0^t f(x) \begin{vmatrix} \exp\{\alpha_1(t-x)\} & \exp\{\alpha_2(t-x)\} & \ldots . & \exp\{\alpha_n(t-x)\} \\ 1 & 1 & \ldots . & 1 \\ \alpha_1 & \alpha_2 & \ldots . & \alpha_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{(n-2)} & \alpha_2^{(n-2)} & \ldots . & \alpha_n^{(n-2)} \end{vmatrix} dx. \tag{I-13}$$

The constant W_0 is readily found: for from (I-9) and (I-12),

$$W_0 = W(t) \exp\left(t \sum_1^n \alpha_r\right);$$

hence

$$W_0 = W(0) = \begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ \alpha_1 & \alpha_2 & \ldots\ldots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \ldots\ldots & \alpha_n^{n-1} \end{vmatrix}, \tag{I-14}$$

using equation (I-6) which defined $W(x)$.

[*continued overleaf*

Thus equation (I-13) may be written as

$$y = \int_0^t f(x)h(t-x)dx, \tag{I-15}$$

$$= \int_0^t h(x)f(t-x)dx, \tag{4.1-1}$$

where

$$h(x) \equiv (-1)^{n-1} \frac{\begin{vmatrix} \exp(\alpha_1 x) & \exp(\alpha_2 x) & \cdots\cdots & \exp(\alpha_n x) \\ 1 & 1 & \cdots\cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots\cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{n-2} & \alpha_2^{n-2} & \cdots\cdots & \alpha_n^{n-2} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots\cdots & \alpha_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \cdots\cdots & \alpha_n^{n-1} \end{vmatrix}}. \tag{I-16}$$

The function $h(x)$ is usually termed the weighting function,[9] and it will be noted from (I-15) that

$$y = h(t)$$

is the response of the system if the input, $f(x)$, is a delta function.

It should be pointed out that it is only possible to write the response of the system in the form

$$y = \int_0^t f(x)h(t-x)dx,$$

if the functions $y_1, y_2, \ldots, y_n$ are exponentials, i.e. if the differential equation of which it is the solution has constant coefficients. For the general case in which the coefficients are not constants it is possible to write {from (I-10)}

$$y = \int_0^t f(x)h(t, x)dx$$

where $h(t, x)$ is given by

$$h(t, x) = \frac{(-1)^{n-1}}{W_0} \exp\left(\int^x a_{n-1}(t)dt\right) \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1(x) & y_2(x) & \cdots & y_n(x) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-2)}(x) & y_2^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \end{vmatrix}.$$

This function may again be termed the weighting function, but it will be noted that in general t and x do not necessarily occur only in the combination $t - x$.

For the constant coefficient system (4.1-1) may be written as

$$y(t) = \int_0^\infty h(x)f(t-x)dx - \int_t^\infty h(x)f(t-x)dx,$$

and if the system is stable $h(t)$ vanishes for sufficiently large t, so that

$$y(t) = \int_0^\infty h(x)f(t-x), dx, \quad \text{for } t \text{ large.} \tag{I-17}$$

Thus equation (I-17) holds for a stable system if the interval between the application of the input and the time t is large enough for the transient effects to have decayed. (I-17) therefore gives the steady state response, while (4.1-1) gives the complete solution (assuming that the system starts from rest).

I.2. *The Convolution Integral.*

If two systems, designated by weighting functions $h_1(t)$ and $h_2(t)$, are arranged such that the output of the first provides the input to the second, the overall response to a unit impulse applied to the first will be

$$\int_0^t h_2(x)h_1(t-x)dx,$$

from (4.1-1). Thus if the weighting function of the complete system is $h(t)$ we have

$$h(t) = \int_0^t h_2(x)h_1(t-x)dx = \int_0^t h_1(x)h_2(t-x)dx.$$

I.3. *Conditions for No Displacement Lag, etc., in Terms of the Weighting Function.*

If a system is to have no displacement lag, the output in response to a constant input must in the steady state assume the same constant value. The steady-state response to a unit step input is, from (I-17),

$$y(t) = \int_0^\infty h(x)dx,$$

so that

$$\int_0^\infty h(x)dx = 1,$$

for no displacement lag.

The response to an input $f(t) = t$ is

$$y(t) = \int_0^\infty h(x)(t-x)dx = t\int_0^\infty h(x)dx - \int_0^\infty xh(x)dx$$

$$= t - \int_0^\infty xh(x)dx,$$

and this must equal the input t if there is to be no velocity (or 1st order) lag. Thus

$$\int_0^\infty h(x)dx = 1 \quad \text{and} \quad \int_0^\infty xh(x)dx = 0$$

are the necessary conditions.

In general, for no lags up to the ith order, an input t^i must in the steady state yield an output t^i, so that

$$t^i = \int_0^\infty h(x)(t-x)^i dx$$
$$= t^i \int_0^\infty h(x)dx - \ldots + (-1)^i \int_0^\infty x^i h(x)dx,$$

giving

$$\int_0^\infty h(x)dx = 1, \quad \text{and} \quad \int_0^\infty x^r h(x)dx = 0, \quad r = 1, 2, \ldots, i$$

as the conditions for no ith order lag.

I.4. *The Transfer Function.*

I.4.1. The general linear differential equation may be written as

$$H_1(D)y(t) = H_2(D)f(t),$$

or

$$y(t) = H(D)f(t), \tag{I-18}$$

where $f(t)$ is the driving function and $y(t)$ the response, $H_1(D)$, $H_2(D)$ are polynomials in D and $H(D) = H_2(D)/H_1(D)$. On defining $Y(p)$, $F(p)$ as

$$Y(p) = \int_0^\infty y(t)e^{-pt}dt \quad \text{and} \quad F(p) = \int_0^\infty f(t)e^{-pt}dt,$$

i.e. the Laplace Transforms of $y(t)$, $f(t)$, we have from (I-18)

$$Y(p) = H(p)F(p), \quad \text{for} \quad f(t) = 0, \quad t < 0,$$

and $H(p)$ is the transfer function relating the Laplace transforms of input and output. From the inverse transform

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Y(p)e^{pt}dp$$

we have

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(p)F(p)e^{pt}dp, \tag{I-19}$$

where c is such that all the poles are to the left of the path of integration. This ensures that $y(t) = 0$, $t < 0$.

I.4.2. Equation (I-19) gives the general solution for the driving function $f(t)$. For an input $\delta(t)$, $F(p) = 1$, and we have defined the response of the system to a δ-function as $h(t)$, the weighting function.

Thus, from (I-19)

$$h(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(p)e^{pt}dp, \tag{I-20}$$

so that by inversion

$$H(p) = \int_0^\infty h(t)e^{-pt}dt, \tag{I-21}$$

i.e. the transfer function is the Laplace Transform of the weighting function. This applies to stable or unstable systems.

I.5. *The Frequency-Response Function.*

If the system is stable, the integral

$$\int_0^\infty h(t)e^{-i\omega t}dt$$

converges, in which case

$$H(i\omega) = \int_0^\infty h(t)e^{-i\omega t}dt, \tag{I-22}$$

from (I-21).

The steady-state response to an input $\sin \omega t$ is, from (I-17)

$$\begin{aligned}
y(t) &= \int_0^\infty h(x) \sin \omega(t-x)dx \\
&= \frac{1}{2i}\left[\int_0^\infty h(x)e^{i\omega(t-x)}dx - \int_0^\infty h(x)e^{-i\omega(t-x)}dx\right] \\
&= \frac{1}{2i}e^{i\omega t}\int_0^\infty h(x)e^{-i\omega x}dx - \frac{1}{2i}e^{-i\omega t}\int_0^\infty h(x)e^{i\omega x}dx \\
&= \frac{1}{2i}e^{i\omega t}H(i\omega) - \frac{1}{2i}e^{-i\omega t}H(-i\omega), \qquad \text{from (I-22)} \\
&= (A^2+B^2)^{1/2}\sin\left(\omega t + \tan^{-1}\frac{B}{A}\right), \qquad \text{where} \quad H(i\omega) = A + iB \\
&= |H(i\omega)| \sin\{\omega t + \arg H(i\omega)\},
\end{aligned}$$

so that the $H(i\omega)$ defined by (I-22) is in fact the frequency-response function.

If the system is stable, the transfer function $H(p)$ will have no poles in the right half of the p-plane, so that the path of integration in (I-20) may be taken along the imaginary axis. Thus

$$h(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} H(p)e^{pt}dp.$$

On changing the variable of integration to $i\omega$, we have

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\omega)e^{i\omega t}d\omega. \tag{I-23}$$

I.6. *Summary.*

The operator $H(D)$, the transfer function $H(p)$, the weighting function $h(t)$, and (if the system is stable) the frequency response function $H(i\omega)$ all serve to define a linear system with constant coefficients. The solution $y(t)$ for a driving function $f(t)$ may be found either as

$$y(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} H(p)F(p)e^{pt}dp$$

or as

$$y(t) = \int_0^t h(t)f(t-x)dx,$$

and

$$H(p) = \int_0^\infty h(t)e^{-pt}dt.$$

If the system is stable: its steady state response is

$$y(t) = \int_0^\infty h(x)f(t-x)dx,$$

in which case

$$H(i\omega) = \int_0^\infty h(t)e^{-i\omega t}dt.$$

In the derivation of the optimum system and its subsequent realisation, use has been made of all these functions.

APPENDIX II

The Solutions of the Integral Equations for the Optimum Weighting Functions

II.1. *Introduction.*

The integral equations derived for the optimum weighting functions under various conditions can be reduced basically to the form

$$\int_0^\infty K(x-y)h(y)dy = f(x), \qquad x \geqslant 0,$$

where $h(y)$ is the unknown function, and the kernel $K(x-y)$ is symmetrical in x and y. Its solution has been discussed by several authors[1, 4, 5]; the treatment allows a ready extension to the cases involving constraints.

II.2. *The solution of* (4.2-5).

The equation may be written as

$$\int_0^\infty \varphi(x-y)h_0(y)dy - \varphi_T(x) = 0, \qquad x \geqslant 0, \tag{4.2-5}$$

where

$$\varphi(x) = \varphi_T(x) + \varphi_N(x).$$

The fact that the equation need only hold for $x \geqslant 0$ prevents a direct solution by means of a Fourier Transform: we may not write

$$\int_{-\infty}^\infty e^{-i\omega x}dx \left[\int_0^\infty \varphi(x-y)h_0(y)dy - \varphi_T(x)\right] = 0,$$

since this assumes that (4.2-5) holds for all x. If the range of x in the integration is restricted to $x \geqslant 0$, however, the equation

$$\int_0^\infty e^{-i\omega x}dx \left[\int_0^\infty \varphi(x-y)h_0(y)dy - \varphi_T(x)\right] = 0 \tag{II-1}$$

is valid, but does not lead to a solution except for special forms of $\varphi(x-y)$. For, on interchanging the order of integration of (II-1),

$$\int_0^\infty h_0(y)dy \int_0^\infty \varphi(x-y)e^{-i\omega x}dx - \int_0^\infty \varphi_T(x)e^{-i\omega x}dx = 0$$

or

$$\int_0^\infty h_0(y)e^{-i\omega y}dy \int_0^\infty \varphi(x-y)e^{-i\omega(x-y)}dx - \int_0^\infty \varphi_T(x)e^{-i\omega x}dx = 0,$$

giving

$$\int_0^\infty h_0(y)e^{-i\omega y}dy \int_{-y}^\infty \varphi(x)e^{-i\omega x}dx - \int_0^\infty \varphi_T(x)e^{-i\omega x}dx = 0,$$

and this is not soluble for a general $\varphi(x)$, since the integral

$$\int_{-y}^\infty \varphi(x)e^{-i\omega x}dx$$

is a function of y.

We may however make use of the Fourier Transform in the following way. Returning to (4.2-5), let

$$\int_0^\infty \varphi(x-y)h_0(y)dy - \varphi_T(x) = l(x). \tag{II-2}$$

Then the condition to be fulfilled is $l(x) = 0,\ x \geqslant 0$. Now define $L(i\omega)$ as

$$L(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} l(x)e^{-i\omega x}dx, \tag{II-3}$$

so that

$$l(x) = \frac{1}{2}\int_{-\infty}^{\infty} L(i\omega)e^{i\omega x}d\omega. \tag{II-4}$$

Then from (II-2) and (II-3),

$$L(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\omega x}dx \int_0^\infty \varphi(x-y)h_0(y)dy - \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_T(x)e^{-i\omega x}dx.$$

Interchanging the order of integration,

$$L(i\omega) = \int_0^\infty h_0(y)e^{-i\omega y}dy \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(x-y)e^{-i\omega(x-y)}dx - \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_T(x)e^{-i\omega x}dx,$$

provided that these integrals exist—i.e. provided that the system is stable.

Thus

$$L(i\omega) = \int_0^\infty h_0(y)e^{-i\omega y}dy \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(x)e^{-i\omega x}dx - \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_T(x)e^{-i\omega x}dx.$$

On defining $\Phi(i\omega)$, $\Phi_T(i\omega)$ as

$$\Phi(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi(x)e^{-i\omega x}dx \tag{II-5}$$

and

$$\Phi_T(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_T(x)e^{-i\omega x}dx, \tag{II-6}$$

and noting from (I-22) that

$$H_0(i\omega) = \int_0^\infty h_0(y)e^{-i\omega y}dy,$$

we have

$$L(i\omega) = H_0(i\omega)\Phi(i\omega) - \Phi_T(i\omega). \tag{II-7}$$

We now deduce the properties of $L(i\omega)$ corresponding to the condition that $l(x) = 0,\ x \geqslant 0$. The function $l(x)$ may be expressed as a contour integral with respect to a complex variable z, of which ω is the real part. Thus

$$l(x) = \frac{1}{2}\int_{-\infty}^{\infty} L(i\omega)e^{i\omega x}d\omega = \frac{1}{2}\int_{c_1} L(iz)e^{izx}dz - \frac{1}{2}\int_{c_2} L(iz)e^{izx}dz$$

where c_1 is a closed contour consisting of the real axis and the infinite semi-circle in the upper half of the z-plane, and c_2 the open contour along the infinite semi-circle. By Jordan's Lemma, the second integral vanishes for $x > 0$, so that

$$l(x) = \frac{1}{2}\int_{c_1} L(iz)e^{izx}dz, \qquad x > 0.$$

But $l(x) = 0$, $x > 0$, so that the contour c_1 encloses no poles. Thus $L(i\omega)$ has no poles in the upper half of the ω-plane, ω being regarded as complex.

For a stable system, we also have the condition that $H_0(p)$ has no poles in the right half p-plane, so that $H_0(i\omega)$ has no poles in the lower half ω-plane.

Returning to equation (II-7), let

$$\Phi(i\omega) = \Phi^+(i\omega)\,\Phi^-(i\omega), \tag{II-8}$$

where $\Phi^+(i\omega)$ has all its poles and zeros in the U.H.P., while those of $\Phi^-(i\omega)$ are confined to the L.H.P. If, for example

$$\Phi(i\omega) = \frac{1}{1+\omega^2} = \frac{1}{(\omega+i)}\frac{1}{(\omega-i)},$$

then

$$\Phi^+(i\omega) = \frac{1}{\omega - i}, \qquad \Phi^-(i\omega) = \frac{1}{\omega + i}.$$

Using (II-8) and (II-7),

$$\frac{L}{\Phi^-} = H_0\Phi^+ - \frac{\Phi_T}{\Phi^-} \tag{II-9}$$

(The argument $i\omega$ has been omitted for brevity)

Now let

$$\frac{\Phi_T}{\Phi^-} = \left(\frac{\Phi_T}{\Phi^-}\right)_+ + \left(\frac{\Phi_T}{\Phi^-}\right)_-, \tag{II-10}$$

where $(\Phi_T/\Phi^-)_+$ has all its poles and zeros in the U.H.P., and $(\Phi_T/\Phi^-)_-$ in the L.H.P.

Substituting (II-10) in (II-9),

$$\frac{L}{\Phi^-} + \left(\frac{\Phi_T}{\Phi^-}\right)_- = H_0\Phi^+ - \left(\frac{\Phi_T}{\Phi^-}\right)_+.$$

Since L has no poles in the U.H.P., the left-hand side has no poles in the U.H.P., so that

$$H_0\Phi^+ - \left(\frac{\Phi_T}{\Phi^-}\right)_+$$

has no poles in the U.H.P. But this quantity has no poles in the L.H.P., since H_0 has no poles in L.H.P. A function which has no poles anywhere must be either a constant or zero, so that

$$H_0\Phi^+ - \left(\frac{\Phi_T}{\Phi^-}\right)_+ = K,$$

or

$$H_0 = \frac{1}{\Phi^+}\left[K + \left(\frac{\Phi_T}{\Phi^-}\right)_+\right].$$

Replacing Φ by $\Phi_T + \Phi_N$, we have finally

$$H_0 = \frac{1}{(\Phi_T+\Phi_N)^+}\left[K + \left\{\frac{\Phi_T}{(\Phi_T+\Phi_N)^-}\right\}_+\right] \tag{4.2-6}$$

which is the desired solution.

$H_0(i\omega)$ is the frequency response function of the optimum system—i.e. $H_0(p)$ is the optimum transfer function; Φ_T, Φ_N are the spectral densities of the target and noise angles θ_T and θ_N, defined as the Fourier Transforms of the respective autocorrelation functions:

$$\Phi_{T,N} = \frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_{T,N}(x)e^{-i\omega x}dx$$

where

$$\varphi_T(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Tr}(T)\theta_{Tr}(T-x),$$

and

$$\varphi_N(x) = \frac{1}{n}\sum_{r=1}^{n} \theta_{Nr}(T)\theta_{Nr}(T-x).$$

The constant K is determined by the restrictions imposed on $H_0(i\omega)$.

The mean square miss distance given by the optimum system is given by (4.2-10):

$$\sigma_{\min}^2 = \varphi_T(0) - 2\int_0^{\infty} \varphi_T(x)h_0(x)dx + \int_0^{\infty} h_0(x)dx\int_0^{\infty} \varphi(x-y)h_0(y)dy.$$

But

$$\varphi_T(x) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T(i\omega)e^{i\omega x}d\omega = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T(i\omega)e^{-i\omega x}d\omega \;\{\text{since } \varphi_T(x) = \varphi_T(-x)\}$$

and

$$\varphi(x) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi(i\omega)e^{i\omega x}d\omega,$$

from (II-5) and (II-6). Thus

$$\sigma_{\min}^2 = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T d\omega - \frac{1}{2}\int_0^{\infty} h_0(x)dx\int_{-\infty}^{\infty} \Phi_T e^{i\omega x}d\omega - \frac{1}{2}\int_0^{\infty} h_0(x)dx\int_{-\infty}^{\infty} \Phi_T e^{-i\omega x}d\omega +$$

$$+\frac{1}{2}\int_0^{\infty} h_0(x)dx\int_0^{\infty} h_0(y)dy\int_{-\infty}^{\infty} \Phi e^{i\omega(x-y)}d\omega,$$

or

$$\sigma_{\min}^2 = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T d\omega - \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T d\omega\int_0^{\infty} h_0(x)e^{i\omega x}dx - \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T d\omega\int_0^{\infty} h_0(x)e^{-i\omega x}dx +$$

$$+\frac{1}{2}\int_0^{\infty} h_0(x)dx\int_{-\infty}^{\infty} \Phi e^{i\omega x}d\omega\int_0^{\infty} h_0(y)e^{-i\omega y}dy.$$

On using

$$H_0(i\omega) = \int_0^{\infty} h_0(x)e^{-i\omega x}dx,$$

we have

$$\sigma_{\min}^2 = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_T d\omega - \frac{1}{2}\int_{-\infty}^{\infty} \overline{H}_0\Phi_T d\omega - \frac{1}{2}\int_{-\infty}^{\infty} H_0\Phi_T d\omega + \frac{1}{2}\int_{-\infty}^{\infty} H_0\overline{H}_0\Phi d\omega,$$

where

$$\overline{H}_0 = H_0(-i\omega).$$

But

$$\Phi = \Phi_T + \Phi_N,$$

so that

$$\sigma_{\min}{}^2 = \frac{1}{2}\int_{-\infty}^{\infty} (1-\overline{H}_0)(1-H_0)\Phi_T d\omega + \frac{1}{2}\int_{-\infty}^{\infty} H_0\overline{H}_0\Phi_N d\omega$$

or

$$\sigma_{\min}{}^2 = \int_0^{\infty} |1 - H_0|^2\Phi_T d\omega + \int_0^{\infty} |H_0|^2\Phi_N d\omega \qquad (4.2\text{-}11)$$

II.3. *The Solution of* (4.3-11).

The equation

$$\int_0^{\infty} g(y)\left[\varphi_{T^{(m)}}(x-y) + (-1)^m\varphi_N{}^{(2m)}(x-y)\right] dy - (-1)^{m+1}\varphi_N{}^{(m)}(x) = 0, \quad x \geqslant 0 \qquad (4.3\text{-}11)$$

has the same form as (4.2-5), and the solution may be written down from (4.2-6):

$$G(i\omega) = \frac{1}{(\Phi_{T^{(m)}} + S_1)^+}\left[K + \left\{\frac{S_2}{(\Phi_{T^{(m)}} + S_1)^-}\right\}_+\right], \qquad (\text{II-11})$$

where

$$G(i\omega) = \int_0^{\infty} g(x)e^{-i\omega x}dx \qquad (\text{II-12})$$

$$S_1(i\omega) = (-1)^m\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_N{}^{(2m)}(x)e^{-i\omega x}dx$$

and

$$S_2(i\omega) = (-1)^{m+1}\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_N{}^{(m)}(x)e^{-i\omega x}dx.$$

Integrating these last two expressions by parts,

$$S_1(i\omega) = (-1)^m\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_N{}^{(2m)}(x)e^{-i\omega x}dx = \omega^{2m}\Phi_N \qquad (\text{II-13})$$

and

$$S_2(i\omega) = (-1)^{m+1}\frac{1}{\pi}\int_{-\infty}^{\infty} \varphi_N{}^{(m)}e^{-i\omega x}dx = (-1)^{m+1}(i\omega)^m\Phi_N, \qquad (\text{II-14})$$

so that by substitution in (II-11)

$$G(i\omega) = \frac{1}{(\Phi_{T^{(m)}} + \omega^{2m}\Phi_N)^+}\left[K + \left\{\frac{(-1)^{m+1}(i\omega)^m\Phi_N}{(\Phi_{T^{(m)}} + \omega^{2m}\Phi_N)^-}\right\}_+\right]. \qquad (\text{II-15})$$

To obtain the equation for H_0, we have, from (4.3-6)

$$h_0(x) = g^{(m)}(x),$$

so that

$$H_0 = \int_0^{\infty} h_0(x)e^{-i\omega x}dx = \int_0^{\infty} g^{(m)}(x)e^{-i\omega x}dx$$

$$= \left[g^{(m-1)}(x) + i\omega g^{(m-2)}(x) + \ldots + (i\omega)^{(m-1)}g(x)\right]_0^{\infty} + (i\omega)^m\int_0^{\infty} g(x)e^{-i\omega x}dx,$$

or

$$H_0 = 1 + (i\omega)^m G, \qquad (\text{II-16})$$

since, from Section 4.3.3,

$$g^{(m-1)}(0) = -1$$

$$g^{(m-i-1)}(0) = 0, \qquad i = 1, 2, \ldots m-1$$

and

$$g^{(i)}(\infty) = 0, \qquad i = 0, 1 \ldots m.$$

Equations (II-15) and (II-16) determine H_0.

Alternatively, the solution may be obtained in a different form by transforming the left-hand side of (4.3-11). Denoting the transform by $L(i\omega)$,

$$L(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-i\omega x}dx \int_{0}^{\infty} [\varphi_{T^{(m)}}(x-y) + (-1)^m \varphi_N{}^{(2m)}(x-y)]\, g(y)dy +$$

$$+ (-1)^m \int_{-\infty}^{\infty} \varphi_N{}^{(m)}(x) e^{-i\omega x}dx,$$

or

$$L = G\Phi_{T^{(m)}} + \omega^{2m} G\Phi_N + (-1)^m (i\omega)^m \Phi_N :$$

using (II-12, 13 and 14). Then from (II-16)

$$L = \frac{H_0 - 1}{(i\omega)^m}[\Phi_{T^{(m)}} + \omega^{2m}\Phi_N] + (-1)^m (i\omega)^m \Phi_N,$$

which reduces to

$$L = \frac{1}{(i\omega)^m}[H_0(\Phi_{T^{(m)}} + \omega^{2m}\Phi_N) - \Phi_{T^{(m)}}] \tag{II-17}$$

Now write $i\omega$ as $i\omega + \epsilon$, where ϵ is small and positive. Then

$$L = \frac{1}{(i\omega+\epsilon)^m}[H_0(\Phi_{T^{(m)}} + (\omega^2+\epsilon^2)^m \Phi_N) - \Phi_{T^{(m)}}].$$

On dividing through by $(\epsilon - i\omega)^m$,

$$\frac{L}{(\epsilon - i\omega)^m} = H_0\left(\frac{\Phi_{T^{(m)}}}{(\omega^2+\epsilon^2)^m} + \Phi_N\right) - \frac{\Phi_{T^{(m)}}}{(\omega^2+\epsilon^2)^m},$$

and the left-hand side has no poles in the U.H.P.

Writing

$$L_1 = \frac{L}{(\epsilon - i\omega)^2},$$

$$\Phi_1 = \frac{\Phi_{T^{(m)}}}{(\omega^2+\epsilon^2)^m} + \Phi_N, \qquad \text{and} \qquad \Phi_2 = \frac{\Phi_{T^{(m)}}}{(\omega^2+\epsilon^2)^m},$$

the equation becomes

$$L_1 = H_0\Phi_1 - \Phi_2,$$

and on comparing this with (II-7) and its solution (4.2-6), we see that the solution is

$$H_0 = \frac{1}{\Phi_1{}^+}\left[K + \left\{\frac{\Phi_2}{\Phi_1{}^-}\right\}_+\right];$$

substituting for Φ_1 and Φ_2 and letting $\epsilon \to 0$, we have finally

$$H_0 = \frac{1}{\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}} + \Phi_N\right)^+} \left[K + \left\{ \frac{\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}}{\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}} + \Phi_N\right)^-} \right\}_+ \right]. \tag{4.3-12}$$

II.4. *The Solution of* (5.1-8).

The equation may be written as

$$\int_0^\infty g(y)[1 + \lambda F(-D_x)F(-D_y)]\,\varphi_{(m)}(x-y)dy + \lambda F(-D_x)F_1(D_x)\varphi_{(m)}(x) + \\ + (-1)^m \varphi_N{}^{(m)}(x) = 0, \qquad x \geqslant 0,$$

since for the second term of (5.1-8) $D_y = -D_x$. Denoting the Fourier Transform of the left-hand side by $L(i\omega)$, we have

$$L(i\omega) = \frac{1}{\pi}\int_{-\infty}^\infty e^{-i\omega x}dx \int_0^\infty g(y)[1 + \lambda F(-D_x)F(-D_y)]\varphi_{(m)}(x-y)dy + \\ + \frac{\lambda}{\pi}\int_{-\infty}^\infty F(-D_x)F_1(D_x)\varphi_{(m)}(x)e^{-i\omega x}dx + \frac{1}{\pi}(-1)^m \int_{-\infty}^\infty e^{-i\omega x}\varphi_N{}^{(m)}(x)dx.$$

On interchanging the order of integration,

$$L = (1+\lambda F\bar{F})G\Phi_{(m)} + \lambda \bar{F}F_1\Phi_{(m)} + (-i\omega)^m\Phi_N \tag{II-18}$$

Since $h_0(x) = g^{(m)}(x)$, we have $H_0 = 1 + (i\omega)^m G$, from (II-16). Also

$$\varphi_{N^{(m)}}(x) = (-1)^m \varphi_N{}^{(2m)}(x), \tag{4.3-9}$$

so that

$$\Phi_{N^{(m)}} = \omega^{2m}\Phi_N.$$

Using these relations in (II-18) yields

$$L = \frac{(H_0-1)}{(i\omega)^m}(1+\lambda F\bar{F})(\Phi_{T^{(m)}} + \omega^{2m}\Phi_N) + \lambda\bar{F}F_1(\Phi_{T^{(m)}} + \omega^{2m}\Phi_N) + (-i\omega)^m\Phi_N.$$

But

$$F(D) = D^m F_1(D), \tag{5.1-6}$$

or

$$F_1(i\omega) = \frac{1}{(i\omega)^m}F(i\omega),$$

giving

$$L = \frac{1}{(i\omega)^m}[(H_0-1)(1+\lambda F\bar{F})(\Phi_{T^{(m)}}+\omega^{2m}\Phi_N) + \lambda F\bar{F}(\Phi_{T^{(m)}}+\omega^{2m}\Phi_N) + \omega^{2m}\Phi_N],$$

or

$$L = \frac{1}{(i\omega)^m}[H_0(1+\lambda F\bar{F})(\Phi_{T^{(m)}}+\omega^{2m}\Phi_N) - \Phi_{T^{(m)}}] \tag{II-19}$$

This equation is similar to (II-17), with

$$(1+\lambda F\bar{F})(\Phi_{T^{(m)}}+\omega^{2m}\Phi_N)$$

replacing

$$(\Phi_{T^{(m)}}+\omega^{2m}\Phi_N),$$

so that from (4.3-12) the solution of (II-19) is

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}+\Phi_N\right)^+}\left[K+\left\{\frac{\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}}{(1+\lambda F\bar{F})^-\left(\dfrac{\Phi_{T^{(m)}}}{\omega^{2m}}+\Phi_N\right)^-}\right\}_+\right]. \tag{5.1-9}$$

II.5. *The Solution of* (5.3-9).

The equation is

$$(D_x^2-2c_TD_x)\varphi_N(x) + \lambda[F(-D_x)F_2(-D_y)\psi(x-y)]_{y=0} +$$
$$+\int_0^\infty g(y)[1+F(-D_x)F(-D_y)]\psi(x-y)dy +$$
$$+g(0)[(1+\lambda F(-D_x)F(-D_y))(D_x-2c_T)\varphi_{(1)}(x-y)]_{y=0} = 0, \qquad x\geqslant 0,$$

where we have written

$$\varphi_{(1)}(x-y) = \frac{1}{n}\sum_{r=1}^{n} D_x\theta_r(T-x)D_y\theta_r(T-y)$$
$$= \varphi_{T^{(1)}} + \varphi_{N^{(1)}},$$

the autocorrelation functions of the first derivatives of target and noise angles respectively. Now

$$g(0)[(1+\lambda F(-D_x)F(-D_y))(D_x-2c_T)\varphi_{(1)}(x-y)]_{y=0}$$
$$= g(0)[1+\lambda F(-D_x)F(D_x)](D_x-2c_T)\varphi_{(1)}(x),$$

and

$$\lambda[F(-D_x)F_2(-D_y)\psi(x-y)]_{y=0} = \lambda F(-D_x)F_2(D_x)\psi(x),$$

since $D_x = -D_y$. Thus the equation becomes

$$(D_x^2-2c_TD_x)\varphi_N(x) + \lambda F(-D_x)F_2(D_x)\psi(x) + \int_0^\infty g(y)[1+\lambda F(-D_x)F(-D_y)]\psi(x-y)dy +$$
$$+g(0)[1+\lambda F(-D_x)F(D_x)](D_x-2c_T)\varphi_{(1)}(x) = 0, \qquad x\geqslant 0.$$

Denoting the Fourier Transform of the left-hand side by $L(i\omega)$, we have

$$L = i\omega(i\omega-2c_T)\Phi_N + \lambda\bar{F}F_2\Psi + (1+\lambda F\bar{F})G\Psi + g(0)(1+\lambda F\bar{F})(i\omega-2c_T)\Phi_{(1)}, \tag{II-20}$$

where

$$\Psi = \frac{1}{\pi}\int_{-\infty}^{\infty}\psi(x)e^{-i\omega x}dx.$$

Now

$$h_0(x) = (D^2+2c_TD)g(x), \tag{5.3-4}$$

so that

$$H_0(i\omega) = \int_0^\infty (D^2+2c_TD)g(x)e^{-i\omega x}dx$$
$$= -Dg(0) - i\omega g(0) + (i\omega)^2G(i\omega) - 2c_Tg(0) + 2c_Ti\omega G(i\omega).$$

For no displacement lag $(D+2c_T)g(0) = -1$, giving

$$H_0(i\omega) = 1 - i\omega g(0) + i\omega(i\omega+2c_T)G(i\omega). \tag{II-21}$$

Also, from (5.3-8),

$$F_2(i\omega) = \frac{F(i\omega)}{i\omega(i\omega+2c_T)}. \tag{II-22}$$

From (II-20, 21 and 22) we have

$$L = i\omega(i\omega-2c_T)\Phi_N + \frac{\lambda F\bar{F}}{i\omega(i\omega+2c_T)}\Psi + \frac{(1+\lambda F\bar{F})\{H_0 - 1 + i\omega g(0)\}}{i\omega(i\omega+2c_T)}\Psi +$$

$$+ g(0)(1+\lambda F\bar{F})(i\omega-2c_T)\Phi_{(1)} \tag{II-23}$$

We can express $\Phi_{(1)}$ in terms of Ψ as follows:

$$\psi(x-y) = \frac{1}{n}\sum_{r=1}^{n}(D_x^2-2c_TD_x)(D_y^2-2c_TD_y)\theta_r(T-x)\theta_r(T-y)$$

$$= (D_x-2c_T)(D_y-2c_T)\frac{1}{n}\sum_{r=1}^{n}D_x\theta_r(T-x)D_y\theta_r(T-y)$$

$$= (D_x-2c_T)(D_y-2c_T)\varphi_{(1)}(x-y).$$

Thus, on transforming,

$$\Psi = -(i\omega-2c_T)(i\omega+2c_T)\Phi_{(1)},$$

or

$$(i\omega-2c_T)\Phi_{(1)} = -\frac{\Psi}{i\omega+2c_T}.$$

Substituting this last relation in (II-23) gives

$$L = i\omega(i\omega-2c_T)\Phi_N + \frac{(1+\lambda F\bar{F})H_0\Psi}{i\omega(i\omega+2c_T)} - \frac{\Psi}{i\omega(i\omega+2c_T)}. \tag{II-24}$$

But

$$\psi(x-y) = \psi_T(x-y) + (D_x^2-2c_TD_x)(D_y^2-2c_TD_y)\varphi_N(x-y),$$

from the definition of $\psi(x-y)$ in (5.3-7). Thus

$$\Psi = \Psi_T + (i\omega)^2(i\omega-2c_T)(i\omega+2c_T)\Phi_N \tag{II-25}$$

The combination of (II-24) and (II-25) yields

$$L = \frac{(1+\lambda F\bar{F})H_0[\Psi_T+\omega^2(\omega^2+4c_T^2)\Phi_N] - \Psi_T}{i\omega(i\omega+2c_T)}.$$

Now write $i\omega + \epsilon$ for $i\omega$, with ϵ positive. Then

$$L = \frac{(1+\lambda F\bar{F})H_0[\Psi_T+(\omega^2+\epsilon^2)(\omega^2+4c_T^2)\Phi_N] - \Psi_T}{(i\omega+\epsilon)(i\omega+2c_T)}. \tag{II-26}$$

There are two cases to consider, depending on the sign of c_T.

(a) $c_T > 0$. Divide each side of (II-26) by $(\epsilon - i\omega)(2c_T - i\omega)$. Then

$$\frac{L}{(\epsilon - i\omega)(2c_T - i\omega)} = (1+\lambda F\bar{F})H_0\left(\frac{\Psi_T}{(\omega^2+\epsilon^2)(\omega^2+4c_T{}^2)} + \Phi_N\right) - \frac{\Psi_T}{(\omega^2+\epsilon^2)(\omega^2+4c_T{}^2)},$$

and the left-hand side has no poles in the U.H.P. By analogy with (II-7) the solution is

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+\left(\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T{}^2)} + \Phi_N\right)^+}\left[K + \left\{\frac{\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T{}^2)}}{(1+\lambda F\bar{F})^-\left(\dfrac{\Psi_T}{\omega^2(\omega^2+4c_T{}^2)} + \Phi_N\right)^-}\right\}_+\right]. \quad (5.3\text{-}10)$$

(b) $c_T < 0$. Divide through (II-26) by $(\epsilon - i\omega)$. Then

$$\frac{L}{(\epsilon - i\omega)} = (1+\lambda F\bar{F})H_0\frac{\left(\dfrac{\Psi_T}{\omega^2+\epsilon^2} + (\omega^2+4c_T{}^2)\Phi_N\right)}{(i\omega+2c_T)} - \frac{\Psi_T}{(\omega^2+\epsilon^2)(i\omega+2c_T)},$$

and the solution is

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+\dfrac{\left(\dfrac{\Psi_T}{\omega^2+\epsilon^2} + (\omega^2+4c_T{}^2)\Phi_N\right)^+}{(i\omega+2c_T)^+}}\left[K + \left\{\frac{\dfrac{\Psi_T}{(\omega^2+\epsilon^2)(i\omega+2c_T)}}{(1+\lambda F\bar{F})^-\dfrac{\left(\dfrac{\Psi_T}{\omega^2+\epsilon^2} + (\omega^2+4c_T{}^2)\Phi_N\right)^-}{(i\omega+2c_T)^-}}\right\}_+\right],$$

which reduces to

$$H_0 = \frac{1}{(1+\lambda F\bar{F})^+\left(\dfrac{\Psi_T}{\omega^2} + (\omega^2+4c_T{}^2)\Phi_N\right)^+}\left[K + \left\{\frac{\dfrac{\Psi_T}{\omega^2}}{(1+\lambda F\bar{F})^-\left(\dfrac{\Psi_T}{\omega^2} + (\omega^2+4c_T{}^2)\Phi_N\right)^-}\right\}_+\right].$$

APPENDIX III

The Autocorrelation Function of the Target Acceleration Normal to the Sight Line

III.1. *The Autocorrelation Function in Terms of Probability Densities*

III.1.1. It was suggested in Section 5.3.1 that the acceleration of the targets normal to the line of sight might be regarded as forming a stationary ensemble, in the sense defined in the paper. We shall suppose that the mean path of the targets are straight lines, or nearly so, and that avoiding action is taken by weaving about the mean path in such a way that the actual paths are as unpredictable as possible.

Suppose that the acceleration is changed at random intervals, the duration y of the interval having the probability density $P_y(y)$, and that the acceleration during one interval is uncorrelated with the accelerations outside that interval. We wish to determine the function

$$R_T(x) = \frac{1}{n}\sum_{r=1}^{n} f_{Tr}(T)f_{Tr}(T-x),$$

where n is the number of engagements, assumed large, and $f_{Tr}(t)$ the acceleration of the rth target.

The chance that T, the time of interception, will be contained in an interval y is

$$\frac{yP_y(y)dy}{\int_0^\infty yP_y(y)dy} = \frac{yP_y(y)dy}{\bar{y}},$$

where $\bar{y}$ is the mean duration of the intervals.

The chance that both T and $T-x$ lie wholly within this interval is therefore

$$\left.\begin{aligned} &\left(1-\frac{|x|}{y}\right)\frac{yP_y(y)dy}{\bar{y}}, \qquad && y > |x|, \\ &\text{and zero,} && y < |x|. \end{aligned}\right\} \tag{III-1}$$

Suppose that the acceleration during each interval is sinusoidal, but that the amplitude, frequency and phase are changed from interval to interval. For the rth target, let the interval y which contains T start at time $T-z_r$. Then provided that T, $T-x$ are contained in the same interval y,

$$\frac{1}{n}\sum_{r=1}^{n} f_{Tr}(T)f_{Tr}(T-x) = \langle \hat{S}_r^2 \cos(\omega_r z_r + \varphi_r)\cos[\omega_r(z_r - x) + \varphi_r]\rangle^r,$$

where $\hat{S}_r$, ω_r and φ_r are the amplitude, frequency and phase during the interval in question, and $\langle\,\rangle^r$ denotes averaging with respect to r. Then

$$\begin{aligned}\frac{1}{n}\sum_{r=1}^{n} f_{Tr}(T)f_{Tr}(T-x) &= \langle \tfrac{1}{2}\hat{S}_r^2\{\cos[\omega_r(2z_r - x) + 2\varphi_r] + \cos\omega_r x\}\rangle^r \\ &= \tfrac{1}{2}\langle \hat{S}_r^2\rangle^r \langle \cos[\omega_r(2z_r - x) + 2\varphi_r] + \cos\omega_r x\rangle^r\end{aligned}$$

if the amplitude, frequency and phase are all independent of each other. If moreover the phases are random and equally likely,

$$\langle \cos[\omega_r(2z_r - x) + 2\varphi_r]\rangle^r = 0,$$

so that

$$\langle f_{Tr}(T)f_{Tr}(T-x)\rangle^r = \frac{\langle \hat{S}_r^2\rangle^r}{2}\langle \cos\omega_r x\rangle^r. \tag{III-2}$$

If $P_\omega(\omega)$ denotes the probability distribution of the frequencies, then

$$\langle f_{Tr}(T) f_{Tr}(T-x)\rangle^r = \alpha_T^2 \int_0^\infty P_\omega(\omega) \cos \omega x \, d\omega \,, \tag{III-3}$$

from (III-2), where $\alpha_T^2 = \langle \hat{S}_r^2 \rangle /2$, the mean square target acceleration.

Equation (III-3) applies when $T\,(T-x)$ are contained in the same interval. If they are not, then

$$\langle f_{Tr}(T) f_{Tr}(T-x)\rangle^r = 0 \,,$$

since we have assumed zero correlation between accelerations in different intervals.

The chance that $T,\ T-x$ lie within the same interval is given in (III-1), so that from (III-1) and (III-3),

$$R_T(x) = f_{Tr}(T) f_{Tr}(T-x) = \int_{|x|}^\infty \frac{y P_y(y) dy}{\bar{y}} \left(1 - \frac{|x|}{y}\right) \alpha_T^2 \int_0^\infty P_\omega(\omega) \cos \omega x \, d\omega \,, \tag{III-4}$$

which gives the autocorrelation function of the target accelerations in terms of the probability densities of the intervals and of the frequencies.

III.1.2. Consider now a particular case in which only one frequency is present—ω_0, say. There is still no correlation outside the same interval because of the random amplitude and phases, so that (III-4) holds. Then

$$P_\omega(\omega) = \delta(\omega - \omega_0)$$

and

$$\int_0^\infty \delta(\omega - \omega_0) \cos \omega x \, d\omega = \cos \omega_0 x \,.$$

From (III-4),

$$R_T(x) = \alpha_T^2 \frac{\cos \omega_0 x}{\bar{y}} \int_{|x|}^\infty y \left(1 - \frac{|x|}{y}\right) P_y(y) dy \,. \tag{III-5}$$

For a Poisson distribution of intervals we have

$$P_y(y) = \beta e^{-\beta y} \,,$$

and

$$\bar{y} = \int_0^\infty \beta y e^{-\beta y} dy = \frac{1}{\beta}$$

so that, from (III-5),

$$R_T(x) = \alpha_T^2 e^{-\beta |x|} \cos \omega_0 x \,. \tag{III-6}$$

Finally, if $\omega_0 = 0$,

$$R_T(x) = \alpha_T^2 e^{-\beta |x|} \,, \tag{III-7}$$

and this is the form used for the examples given in the paper. It will be seen that (III-7) applies to target paths in which the lateral acceleration is held constant for varying periods, the duration of the intervals being governed by a Poisson distribution with a mean duration of $1/\beta$. The amplitude of the acceleration is also a random variable with an r.m.s. value of α_T: the distribution of the amplitudes however does not affect the autocorrelation function. In Fig. 6, which shows a typical sequence of target accelerations under these conditions, the amplitude distribution is Gaussian, with a Poisson distribution for the intervals.

III.1.3. From the definition of $\psi_T(x)$ given in (5.3-2),

$$\psi_T(x) = \frac{R_T(x)}{r_T^2} = \sigma_T^2 e^{-\beta|x|} \tag{7.1-1}$$

where $\sigma_T = \alpha_T/r_T$, from (III-7).

The spectral density $\Psi_T(i\omega)$ for this autocorrelation function is

$$\Psi_T(i\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} \sigma_T^2 e^{-\beta|x|} e^{-i\omega x} dx = \sigma_T^2 \frac{2}{\pi}\frac{\beta}{\beta^2+\omega^2} \tag{7.1-2}$$

III.2. *An Alternative Derivation.*

The target acceleration autocorrelation function may be thought of as arising from a rather different kind of target motion, as follows. Suppose that the acceleration developed by the target f_T is related to the demand for acceleration f_{TD} by the operator $A_T(D)$:

$$f_T = A_T(D) f_{TD},$$

i.e. $A_T(p)$ is the transfer function of the target aircraft. Then

$$\frac{1}{n}\sum_{r=1}^{n} f_{Tr}(T-x) f_{Tr}(T-y) = A_T(-D_x)A_T(-D_y)\frac{1}{n}\sum_{r=1}^{n} f_{TDr}(T-x) f_{TDr}(T-y)$$

or

$$R_T(x-y) = A_T(-D_x)A_T(-D_y)R_{TD}(x-y),$$

so that

$$\Psi_T(i\omega) = A_T(i\omega)A_T(-i\omega)\Psi_{TD}(i\omega),$$

where Ψ_T, Ψ_{TD} are the spectral densities of the achieved and demanded target accelerations, divided by the range r_T. If we now assume that

$$\Psi_{TD}(i\omega) = k_{TD}^2,$$

i.e. that the demand for acceleration is 'white noise' over those frequencies to which the aircraft can respond, then

$$\Psi_T(i\omega) = |A_T(i\omega)|^2 k_{TD}^2.$$

In particular, if

$$A_T(D) = \frac{\beta}{\beta + D},$$

i.e. a simple time lag with time constant $1/\beta$, then

$$\Psi_T(i\omega) = k_{TD}^2 \frac{\beta^2}{\beta^2+\omega^2},$$

which is identical with (7.1-2) if

$$k_{TD}^2 = \frac{2\sigma_T^2}{\beta\pi}.$$

Thus the spectral density (7.1-2) may be considered as arising from a demand for target acceleration which is as unpredictable as possible, in that all frequencies to which the aircraft can respond are present with equal weight: the demanded and achieved acceleration being related by a simple time lag of time constant $1/\beta$, representative of the aircraft characteristics.

APPENDIX IV

On the Realisation of Networks defined by their Transfer Functions

IV.1. *Introduction.*

The process of optimisation leads to a transfer function of the overall system: and we have seen that in order to realise the optimum system within a given framework it is necessary that a given section of the system should have a particular transfer function. If, for example, the missile control loop of a beam rider is chosen arbitrarily, the tracker must have the transfer function as given in (6.1-2); alternatively, an arbitrary choice of the tracking servo system leads to an equation for the required missile control system (6.2-3)

In either case the required transfer function reduces to a ratio of polynomials

$$Z(p) = \frac{a_0 + a_1 p + a_2 p^2 + \ldots . a_m p^m}{b_0 + b_1 p + b_2 p^2 + \ldots . b_n p^n}, \tag{IV-1}$$

where the coefficients a_0, $a_1 \ldots .$ and b_0, $b_1 \ldots .$ are real and positive; and for the realisation of the optimum system it is necessary to produce a network having this transfer function.

Both the numerator and denominator of $Z(p)$ may be expressed as the product of quadratic factors, together with a linear factor if m or n is odd. Since the coefficients are real, the roots of the quadratics are either real or they occur in complex pairs. The linear factors (if any) must of course have real roots.

IV.2. *Real Poles and Zeros.*

The realisation of that part of $Z(p)$ which has real poles and zeros causes little difficulty, in that all such transfer functions may be realised by passive networks consisting of capacitances and resistances only. A number of well known networks, together with their transfer functions, are shown in Fig. 20. The restrictions on the coefficients result from the fact that the component values must be positive.

IV.3. *Complex Poles.*

The complex poles (and zeros) of $Z(p)$ require either inductances or active networks for their realisation. The frequency range in the present application precludes the use of inductance, so that active circuits with capacitances and resistances only are necessary.

Consider first the transfer function

$$\frac{1}{1 + \alpha p + \beta p^2},$$

where α and β are such that the roots of the denominator are complex. Such a transfer function may be obtained by applying feedback to an integrator and a simple time lag; the forward transfer function of this combination is

$$\frac{K}{p(1+pT)},$$

so that on applying feedback the overall transfer function is

$$\frac{\dfrac{K}{p(1+pT)}}{1 + \dfrac{K}{p(1+pT)}} = \frac{1}{1 + \dfrac{1}{K}p + \dfrac{T}{K}p^2},$$

and since the loop gain K and the time constant T are independent, the roots can be arranged to be real or complex.

A practical circuit for obtaining this type of transfer function is given in Fig. 21a. Its transfer function is given by

$$\frac{e_0(p)}{e_1(p)} = -\frac{1}{\dfrac{R_1}{R_0} + T_2\left(1 + \dfrac{R_1}{R_0} + \dfrac{R_1}{R_2}\right)p + T_1T_2p^2},$$

the negative sign arising from the reversing property of the amplifier.

IV.4. *Complex Zeros.*

The circuits so far described suffice to provide the real poles and zeros and the complex poles of $Z(p)$. There remain the complex zeros; it is useful to group each pair of these with a quadratic term of the denominator, whose roots may be either real or complex. Suppose for example that the expression

$$\frac{1 + \gamma p + \delta p^2}{1 + \alpha p + \beta p^2} \tag{IV-2}$$

is a factor of $Z(p)$ in which the quadratic in the numerator gives rise to a pair of complex zeros. We may write (IV-2) as

$$\frac{1}{1 + \alpha p + \beta p^2} + \frac{\gamma p(1+\delta/\gamma p)}{1 + \alpha p + \beta p^2}; \tag{IV-3}$$

and the term

$$\frac{\gamma p}{1 + \alpha p + \beta p^2}$$

may be obtained directly from Fig. 21a. For

$$v = -pT_2e_0,$$

so that

$$\frac{v}{e_1} = \frac{pT_2}{\dfrac{R_1}{R_0} + T_2\left(1 + \dfrac{R_1}{R_0} + \dfrac{R_1}{R_2}\right)p + T_1T_2p^2}.$$

The network of Fig. 21b has the transfer function

$$-\frac{R_4}{R_3}(1+pC_3R_3),$$

so that if the voltage v is applied to such a circuit (Fig. 21c) its output v_1 will be

$$v_1 = -\frac{R_4}{R_3}(1+pC_3R_3)v,$$

so that

$$\frac{v_1}{e_1} = \frac{-\dfrac{R_4}{R_3}(1+pT_3)pT_2}{\dfrac{R_1}{R_0} + T_2\left(1 + \dfrac{R_1}{R_0} + \dfrac{R_2}{R_0}\right)p + T_1T_2p^2},$$

and by a suitable choice of components this can be made equal to the second term of (IV-3). The first term of the latter is obtained at the point B in Fig. 21c, as before. If therefore the voltages at A and B are added in suitable proportions, we obtain for the overall transfer function of Fig. 21c:

$$\frac{e_0}{e_1} = -\frac{R_7}{R_5R_6 + R_6R_7 + R_7R_5}\left[\frac{R_5 + \frac{R_6R_1}{R_3}(1+pT_3)pT_2}{\frac{R_1}{R_0} + T_2\left(1 + \frac{R_1}{R_0} + \frac{R_1}{R_2}\right)p + T_1T_2p^2}\right],$$

which can be made equivalent to (IV-2).

Since the coefficients are independent there is no restriction on the poles or zeros—both may be real or complex. The circuit however is only useful when the zeros are complex, for if they are real they can be more readily obtained by using passive circuits.

The cathode follower of Fig. 21c is not essential—its absence modifies the coefficients as a result of the additional current taken by R_3 and C_3.

The complex zeros may be realised in a different way by using the circuit of Fig. 21a as the feedback element of a further amplifier. If (IV-2) is the required transfer function we first arrange the circuit of Fig. 21a to give

$$\frac{1}{1 + \gamma p + \delta p^2};$$

if this network is now used as a feedback element for an amplifier with gain G, the transfer function of the system will be

$$\frac{G}{1 + \dfrac{KG}{1 + \gamma p + \delta p^2}} = \frac{1 + \gamma p + \delta p^2}{KG + 1 + \gamma p + \delta p^2}\, G,$$

where K depends on the proportion of feedback. The coefficient of p in the denominator is now equal to that of p in the numerator. To avoid this an additional feedback from the point P of Fig. 21a is necessary: this leads to

$$\frac{G}{1 + \dfrac{K_1G + K_2Gp}{1 + \gamma p + \delta p^2}} = \frac{G(1+\gamma p+\delta p^2)}{1 + K_1G + (\gamma + K_2G)p + \delta p^2}$$

$$= \frac{1 + \gamma p + \delta p^2}{1 + \alpha p + \beta p^2}.$$

Fig. 22 shows the arrangement for $\beta < \delta$; other cases may be obtained by re-arranging the points at which the feedbacks are introduced to the amplifiers A_1 and A_2.

IV. 5. *An Example.*

A combination of the above circuits allows the realisation of any rational function such as $Z(p)$. The factors may be grouped in different ways, but there will usually be one arrangement which leads to the most economical circuit. As an example, consider the realisation of the transfer function given by (8.2-5):

$$S(p) = \frac{1}{1\cdot 04}\left(\frac{1 + 0\cdot 314p + 0\cdot 025p^2}{1 + 10\cdot 31p + 2\cdot 90p^2}\right)\left(\frac{1 + 4\cdot 23p + 7\cdot 87p^2}{1 + 0\cdot 60p + 0\cdot 233p^2}\right) \qquad (8.2\text{-}5)$$

This is the optimum missile network for the example of Section 8.2.1.

The first bracket contains a pair of real poles and real zeros, and inspection of the coefficients shows that it can be realised by the circuit of Fig. 20d. The second bracket has complex poles and zeros, and for this the network of Fig. 21c is required. The combination of these two circuits to give $S(p)$ is shown in Fig. 23, together with the component values.

The network is fairly elaborate,* but it can be condensed without materially departing from the correct transfer function. The degree to which the actual transfer function may deviate from that required by the theory is a matter which remains to be investigated: this is more expediently carried out by simulator experiments rather than theoretically.

* *See* footnote on p.65 to Section 12.2.

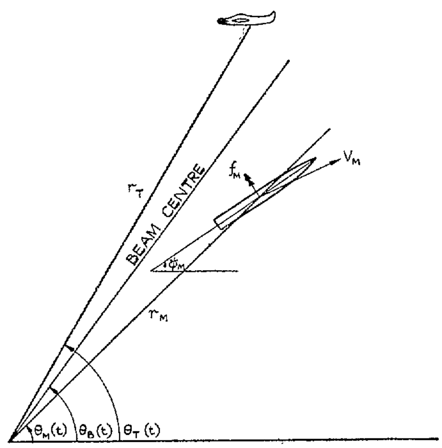


FIG. 1. The geometry of the beam-riding system in one plane.

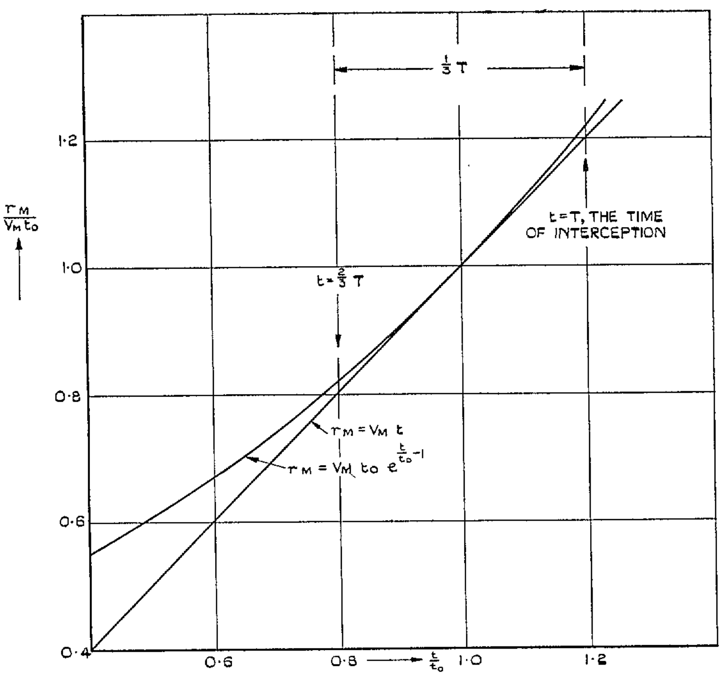


FIG. 2. An exponential approximation to the missile range.

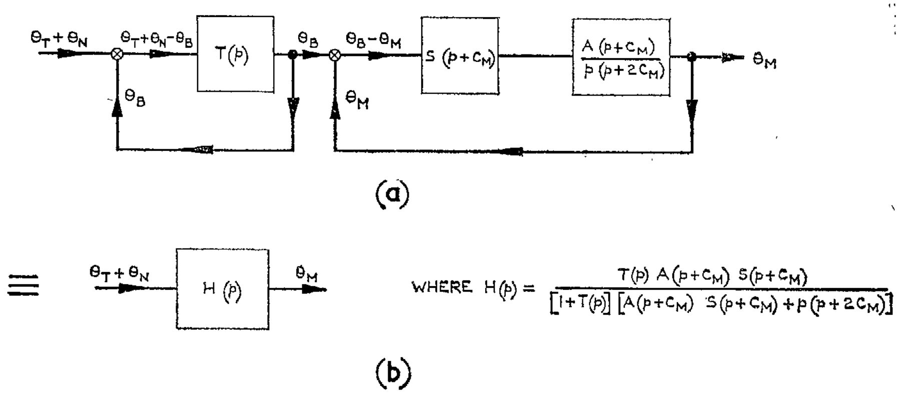


FIG. 3a and b. Transfer functions for the beam-riding system.

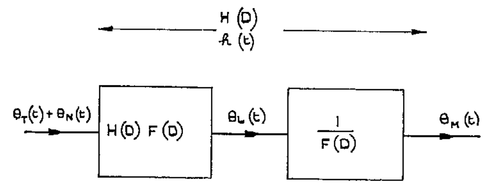


FIG. 4. Notation for the derivation of the optimum system subject to a constraint.

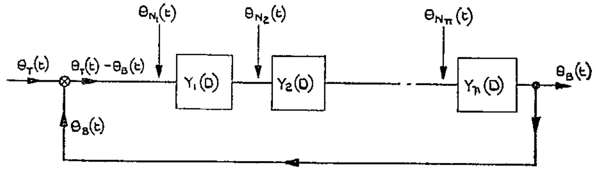


FIG. 5a. Noise sources in a target tracking system.

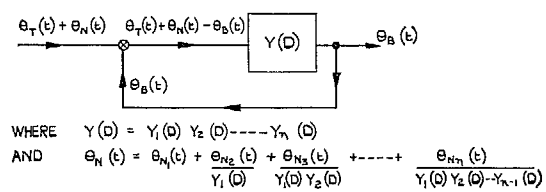


FIG. 5b. An equivalent representation.

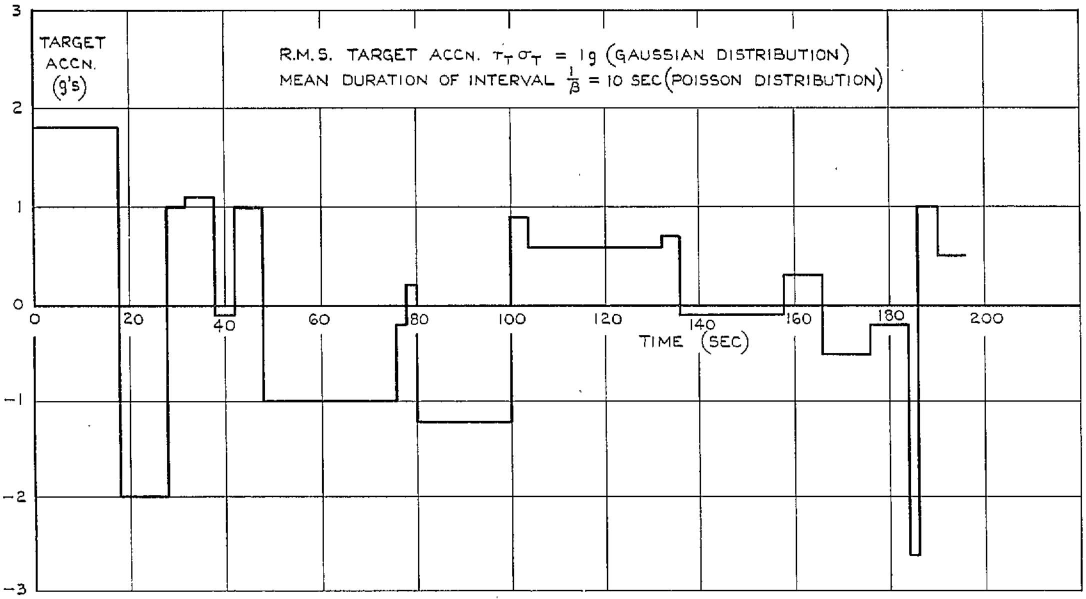
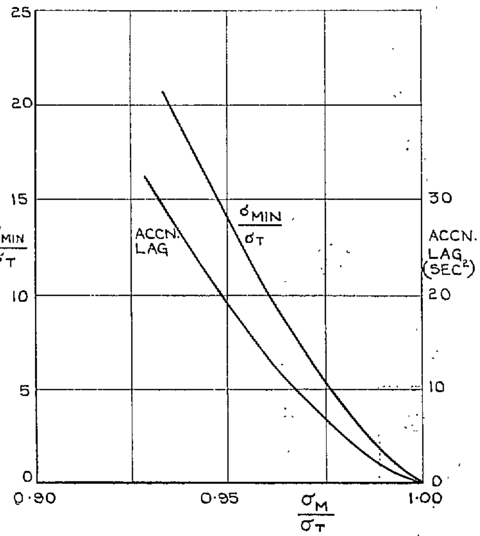


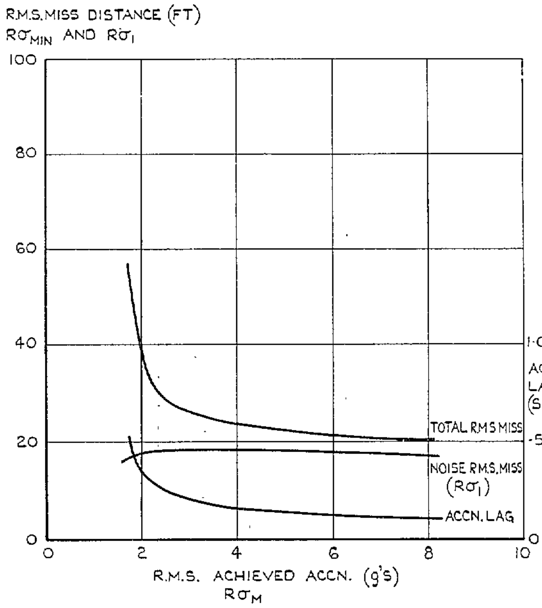
FIG. 6. An example of target acceleration for the autocorrelation function $\psi_T(x) = \sigma_T^2 e^{-\beta|x|}$.



μ	H_0	$\frac{\sigma_{MIN}}{\sigma_T}$	$\frac{\sigma_M}{\sigma_T}$	ACCN. LAG
β	$\frac{1+10\sqrt{2}p+50\sqrt{2}p^2}{1+10\sqrt{2}p+100p^2}$	20·70	·933	29·3
$\sqrt{2}\beta$	$\frac{1+10p+40p^2}{1+10p+50p^2}$	7·75	·9675	10·0
2β	$\frac{1+5\sqrt{2}p+21{\cdot}81p^2}{1+5\sqrt{2}p+25p^2}$	2·66	·986	3·2
3β	$\frac{1+\frac{10\sqrt{2}}{3}p+10{\cdot}33p^2}{1+\frac{10\sqrt{2}}{3}p+11{\cdot}11p^2}$	·69	·997	·78
10β	$\frac{1+\sqrt{2}p+0{\cdot}99p^2}{1+\sqrt{2}p+p^2}$	·01	1	·01
α	1	0	1	0

$\beta = 0{\cdot}1$

FIG. 7. Limited achieved missile acceleration: $T = k^2 = 0$.

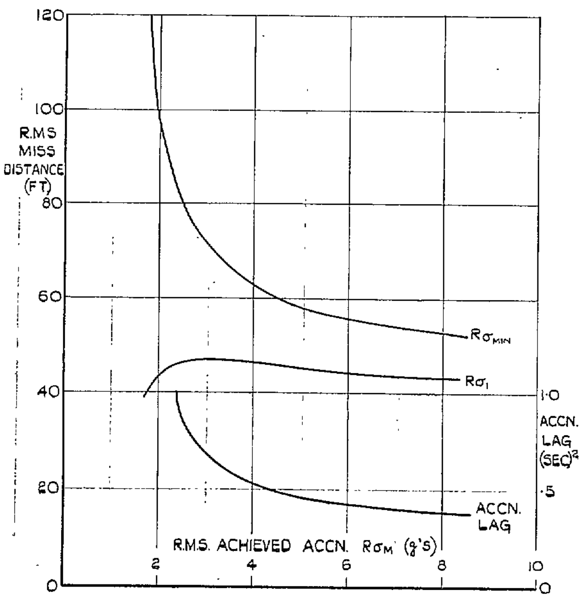


R.M.S. MISS DISTANCE AND ACCELERATION LAG v R.M.S. ACHIEVED ACCN.

μ^4	$H_0(p)$	ACCN LAG (SEC^2)	$R\sigma_1$ (FT)	$R\sigma_2$ (FT)	$R\sigma_{MIN}$ (FT)	$R\sigma_M$ (g's)
1	$\frac{1{\cdot}262+4{\cdot}121p+6{\cdot}128p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(1+1{\cdot}414p+p^2)}$	·477	16·7	49·4	52·1	1·79
4	$\frac{2{\cdot}523+7{\cdot}198p+9{\cdot}455p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(2+2p+p^2)}$	·320	18·0	31·2	36·0	2·12
10	$\frac{3{\cdot}989+10{\cdot}564p+12{\cdot}966p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(3{\cdot}162+2{\cdot}515p+p^2)}$	·254	18·1	22·8	29·2	2·50
25	$\frac{6{\cdot}308+15{\cdot}676p+18{\cdot}164p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(5+3{\cdot}162p+p^2)}$	·207	18·3	19·0	26·3	3·02
49	$\frac{8{\cdot}831+21{\cdot}082p+23{\cdot}551p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(7+3{\cdot}742p+p^2)}$	·181	18·3	16·7	24·8	3·52
100	$\frac{12{\cdot}616+29{\cdot}015p+31{\cdot}324p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(10+4{\cdot}472p+p^2)}$	·161	18·0	14·9	23·3	4·32
484	$\frac{27{\cdot}744+59{\cdot}768p+60{\cdot}836p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(22+6{\cdot}633p+p^2)}$	·127	17·6	12·1	21·3	6·26
1000	$\frac{38{\cdot}894+83{\cdot}945p+83{\cdot}636p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(31{\cdot}623+7{\cdot}953p+p^2)}$	·116	17·2	11·1	20·5	8·15

THE OPTIMUM TRANSFER FUNCTIONS

FIG. 8a and b. Limited achieved missile acceleration: $T = 0$; $k^2 = 0{\cdot}5 \times 10^{-8}$ rad^2/rad/sec.



R.M.S. MISS DISTANCE AND ACCN. LAG vs. R.M.S. ACHIEVED ACCN.

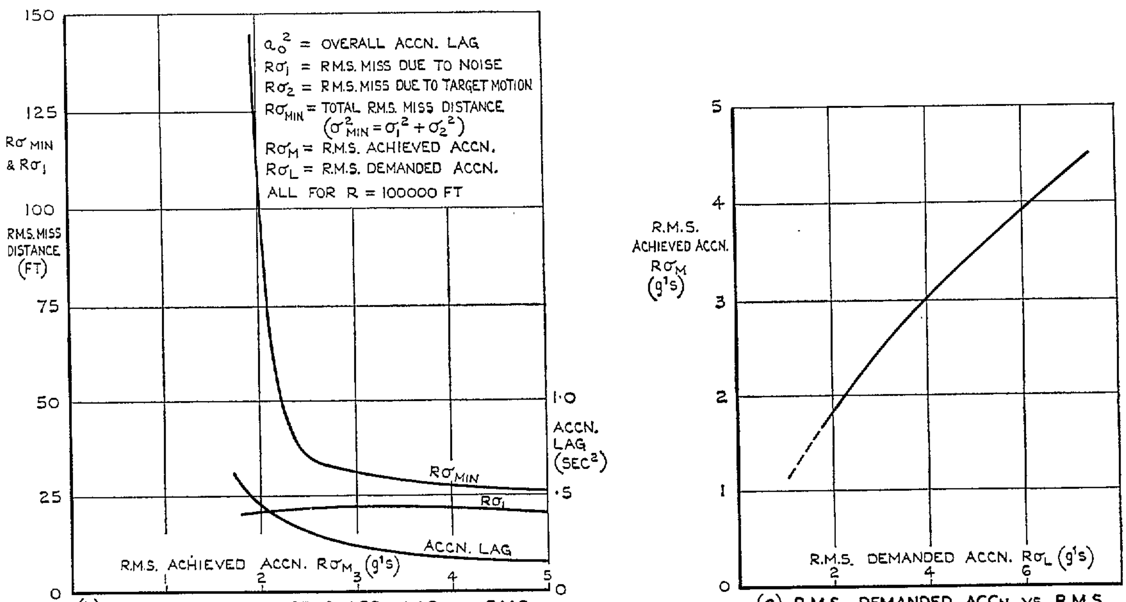
μ^4	$H_0(p)$	ACC^N LAG (SEC^2)	$R\sigma_1$ (FT.)	$R\sigma_2$ (FT.)	$R\sigma_{MIN}$ (FT)	$R\sigma_M$ (g's)
·1	$\frac{{\cdot}141+{\cdot}725p+1{\cdot}604p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)({\cdot}316+{\cdot}795p+p^2)}$		40·8	117	124	1·76
·25	$\frac{{\cdot}223+1{\cdot}031p+2{\cdot}080p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)({\cdot}500+1{\cdot}000p+p^2)}$		43·4	94·1	104	1·94
·50	$\frac{{\cdot}315+1{\cdot}357p+2{\cdot}572p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)({\cdot}707+1{\cdot}189p+p^2)}$		44·9	80·0	92	2·07
1·0	$\frac{{\cdot}446+1{\cdot}800p+3{\cdot}224p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(1+\sqrt{2}p+p^2)}$	·912	46·0	68·9	82·8	2·39
2·0	$\frac{{\cdot}631+2{\cdot}404p+4{\cdot}093p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(\sqrt{2}+1{\cdot}682p+p^2)}$	·761	46·7	59·4	75·6	2·70
10	$\frac{1{\cdot}411+4{\cdot}819p+7{\cdot}453p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(3{\cdot}162+2{\cdot}515p+p^2)}$	·552	46·5	43·8	63·9	3·84
20	$\frac{1{\cdot}995+6{\cdot}563p+9{\cdot}818p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(4{\cdot}472+2{\cdot}991p+p^2)}$	·489	45·7	38·9	60·0	4·57
100	$\frac{4{\cdot}460+13{\cdot}69p+9{\cdot}257p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(10+4{\cdot}472p+p^2)}$	·402	44·0	31·2	53·9	7·16

THE OPTIMUM TRANSFER FUNCTIONS

FIG. 9a and b. Limited achieved missile acceleration: $T = 0$; $k^2 = 4 \times 10^{-8}$ rad^2/rad/sec

μ^4	$H_0(p)$	a_0^2 (SEC^2)	$R\sigma_1$ (FT)	$R\sigma_2$ (FT)	$R\sigma_{MIN}$ (FT)	$R\sigma_L$ (g's)	$R\sigma_M$ (g's)
1	$\dfrac{126{\cdot}15+436{\cdot}4p+681{\cdot}0p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(100{\cdot}01+20{\cdot}001p+p^2)(1+1{\cdot}407p+p^2)}$	·586	20·4	262	263	1·92	1·79
2	$\dfrac{178{\cdot}4+576{\cdot}8p+846{\cdot}3p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(100{\cdot}03+20{\cdot}003p+p^2)(1{\cdot}414+1{\cdot}670p+p^2)}$	·483	21·2	132	134	2·11	1·93
4	$\dfrac{252{\cdot}3+767{\cdot}6p+1066p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(100{\cdot}06+20{\cdot}005p+p^2)({\cdot}999+1{\cdot}979p+p^2)}$	·405	21·8	58·8	62·7	2·36	2·11
10	$\dfrac{398{\cdot}7+1131p+1472p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(100{\cdot}15+20{\cdot}012p+p^2)(3{\cdot}156+2{\cdot}472p+p^2)}$	·329	22·1	30·3	37·5	2·80	2·41
100	$\dfrac{1262+3127p+3602p^2}{(1{\cdot}082+p)(1{\cdot}166+1{\cdot}082p+p^2)(101{\cdot}43+20{\cdot}119p+p^2)(9{\cdot}860+4{\cdot}220p+p^2)}$	·217	22·2	20·4	30·1	4·63	3·31

FIG. 10a. The optimum transfer functions.



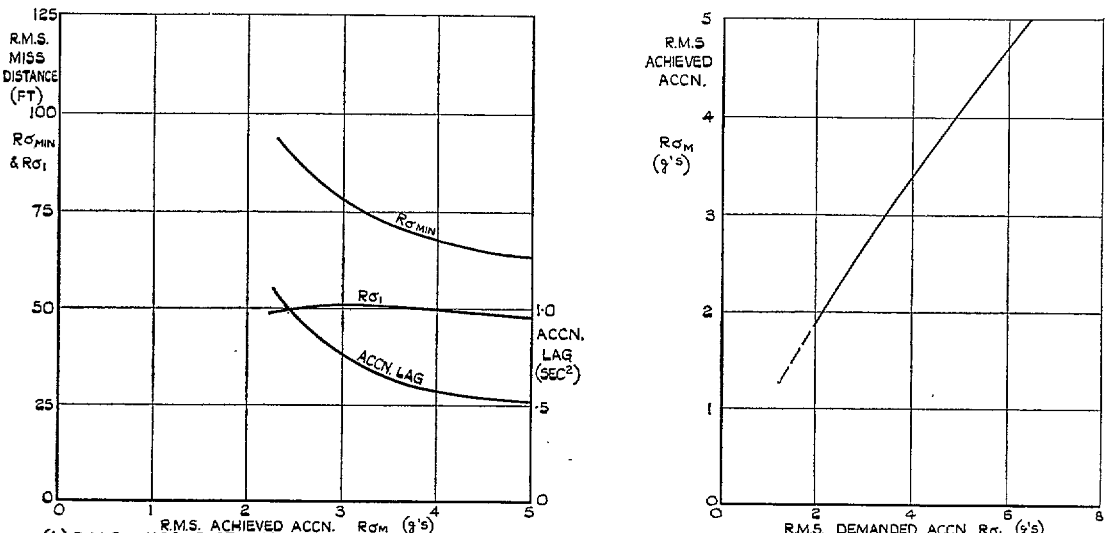
(b) R.M.S. MISS DISTANCE & ACCN LAG VS R.M.S ACHIEVED ACCN.

(c) R.M.S. DEMANDED ACCN VS R.M.S. ACHIEVED ACCN.

FIG. 10b and c. Limited demanded missile acceleration: $T = 0{\cdot}1$ sec; $k^2 = 0{\cdot}5\times 10^{-8}$ rad^2/rad/sec.

μ^4	$H_0(p)$	a_0^2 (SEC²)	$R\sigma_1$ (FT)	$R\sigma_2$ (FT)	$R\sigma_{MIN}$ (FT)	$R\sigma_L$ (g's)	$R\sigma_M$ (g's)
1	$\dfrac{44{\cdot}60+188{\cdot}6p+351{\cdot}1p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(100{\cdot}01+20{\cdot}001p+p^2)(1+1{\cdot}407p+p^2)}$	1·07	49·4	79·2	93·3	2·56	2·33
2	$\dfrac{63{\cdot}09+252{\cdot}4p+448{\cdot}3p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(100{\cdot}03+20{\cdot}003p+p^2)(1{\cdot}414+1{\cdot}670p+p^2)}$	·90	50·1	68·9	85·2	2·92	2·62
4	$\dfrac{89{\cdot}21+339{\cdot}9p+577{\cdot}4p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(100{\cdot}06+20{\cdot}005p+p^2)(1{\cdot}999+1{\cdot}979p+p^2)}$	·79	50·9	60·6	79·1	3·44	2·96
10	$\dfrac{141{\cdot}0+508{\cdot}9p+826{\cdot}7p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(100{\cdot}15+20{\cdot}012p+p^2)(3{\cdot}156+2{\cdot}472p+p^2)}$	·63	49·8	51·2	71·4	4·24	3·55
50	$\dfrac{315{\cdot}4+1052p+1592p^2}{({\cdot}766+p)({\cdot}582+{\cdot}766p+p^2)(100{\cdot}73+20{\cdot}061p+p^2)(7{\cdot}020+3{\cdot}614p+p^2)}$	·55	48·1	40·8	63·1	6·57	5·04

FIG. 11a. The optimum transfer functions.



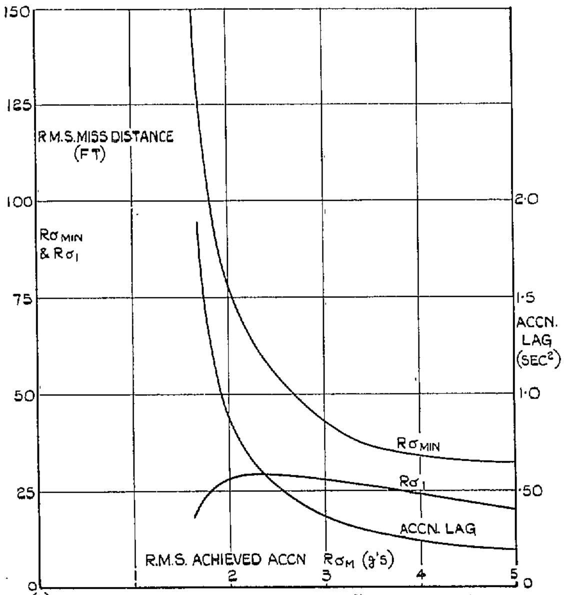
(b) R.M.S. MISS DISTANCE & ACCN LAG *vs.* R.M.S ACHIEVED ACCN

(c) R.M.S. DEMANDED ACCN *vs* R.M.S. ACHIEVED ACCN

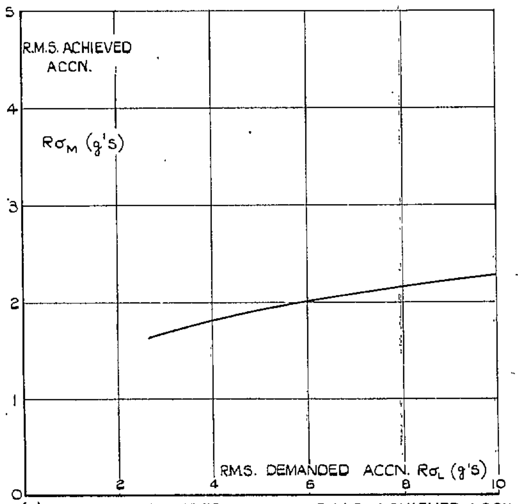
FIG. 11b and c. Limited demanded missile acceleration: $T = 0{\cdot}1$ sec; $k^2 = 4 \times 10^{-8}$ rad²/rad/sec.

μ^4	$H_0(p)$	a_0^2 (SEC²)	$R\sigma_1$ (FT)	$R\sigma_2$ (FT)	$R\sigma_{MIN}$ (FT)	$R\sigma_L$ (g's)	$R\sigma_M$ (g's)
·5	$\dfrac{\cdot 892+4\cdot 530p+9\cdot 857p^2}{(1\cdot 082+p)(1\cdot 166+1\cdot 082p+p^2)(1\cdot 307+2\cdot 242p+p^2)(\cdot 541+\cdot 817p+p^2)}$	1·85	21·5	131	133	2·96	1·68
1	$\dfrac{1\cdot 262+5\cdot 999p+12\cdot 33p^2}{(1\cdot 082+p)(1\cdot 166+1\cdot 082p+p^2)(\cdot 443+2\cdot 342p+p^2)(\cdot 693+\cdot 887p+p^2)}$	1·53	23·3	114	117	3·39	1·74
2	$\dfrac{1\cdot 784+7\cdot 977p+15\cdot 53p^2}{(1\cdot 082+p)(1\cdot 166+1\cdot 082p+p^2)(1\cdot 614+2\cdot 463p+p^2)(\cdot 876+\cdot 957p+p^2)}$	1·29	25·1	100	103	3·94	1·80
4	$\dfrac{2\cdot 523+10\cdot 65p+19\cdot 71p^2}{(1\cdot 082+p)(1\cdot 166+1\cdot 082p+p^2)(1\cdot 825+2\cdot 604p+p^2)(\cdot 096+1\cdot 031p+p^2)}$	1·09	26·0	88	91	4·66	1·88
10	$\dfrac{3\cdot 990+15\cdot 69p+27\cdot 31p^2}{(1\cdot 082+p)(1\cdot 166+1\cdot 082p+p^2)(2\cdot 175+2\cdot 822p+p^2)(1\cdot 454+1\cdot 137p+p^2)}$	·87	27·8	78	82	5·90	1·97

FIG. 12a. The optimum transfer functions.



(b) R.M.S. MISS DISTANCE & ACCN LAG vs. R.M.S ACHIEVED ACCN

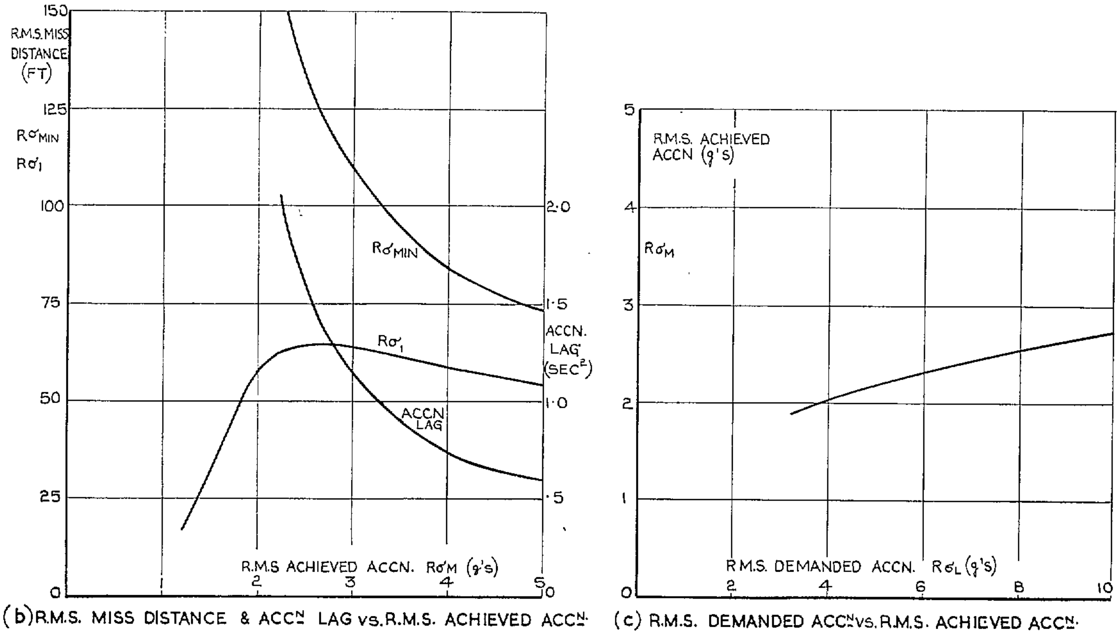


(c) R.M.S. DEMANDED ACCN vs. R.M.S. ACHIEVED ACCN

FIG. 12b and c. Limited demanded missile acceleration: $T = 1$ sec; $k^2 = 0\cdot 5 \times 10^{-8}$ rad²/rad/sec.

μ^4	$H_0(p)$	a_0^2 (SEC²)	$R\sigma_1$ (FT)	$R\sigma_2$ (FT)	$R\sigma_{MIN}$ (FT)	$R\sigma_L$ (g's)	$R\sigma_M$ (g's)
·5	$\dfrac{\cdot315+1\cdot844p+4\cdot514p^2}{(\cdot766+p)(\cdot582+\cdot766p+p^2)(1\cdot307+2\cdot242p+p^2)(\cdot541+\cdot817p+p^2)}$	2·79	57·8	188	197	3·73	2·00
1	$\dfrac{\cdot446+2\cdot464p+5\cdot746p^2}{(\cdot766+p)(\cdot582+\cdot766p+p^2)(1\cdot443+2\cdot342p+p^2)(\cdot693+\cdot887p+p^2)}$	2·37	60·7	167	176	4·40	2·10
2	$\dfrac{\cdot631+3\cdot305p+7\cdot373p^2}{(\cdot766+p)(\cdot582+\cdot766p+p^2)(1\cdot614+2\cdot463p+p^2)(\cdot876+\cdot957p+p^2)}$	2·04	62·7	146	159	5·23	2·22
4	$\dfrac{\cdot892+4\cdot450p+9\cdot517p^2}{(\cdot766+p)(\cdot582+\cdot766p+p^2)(1\cdot825+2\cdot604p+p^2)(1\cdot096+1\cdot031p+p^2)}$	1·78	64·0	130	145	6·29	2·35
10	$\dfrac{1\cdot410+6\cdot630p+13\cdot49p^2}{(\cdot766+p)(\cdot582+\cdot766p+p^2)(2\cdot175+2\cdot822p+p^2)(1\cdot454+1\cdot137p+p^2)}$	1·48	62·4	111	128	8·10	2·56

FIG. 13a. The optimum transfer functions.



(b) R.M.S. MISS DISTANCE & ACCN. LAG vs. R.M.S. ACHIEVED ACCN. (c) R.M.S. DEMANDED ACCN. vs. R.M.S. ACHIEVED ACCN.

FIG. 13b and c. Limited demanded missile acceleration: $T = 1$ sec; $k^2 = 4 \times 10^{-8}$ rad²/rad/sec.

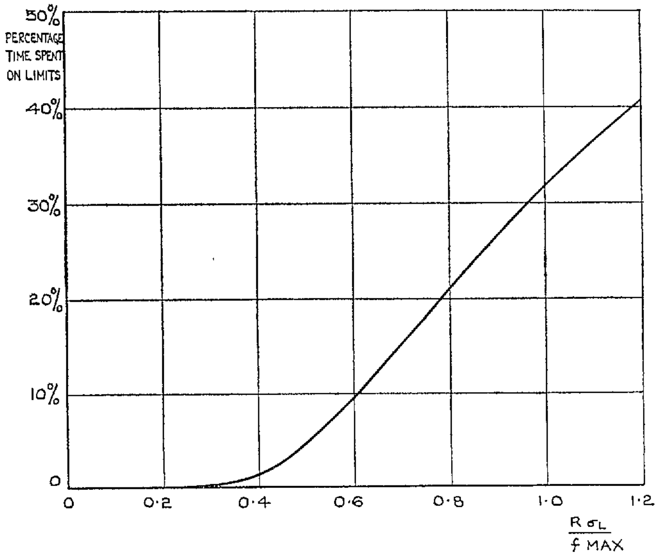


Fig. 14. The percentage time spent on the limits when the r.m.s. demanded acceleration $R\sigma_L$ is a given fraction of the limiting acceleration $f_{\max}$.

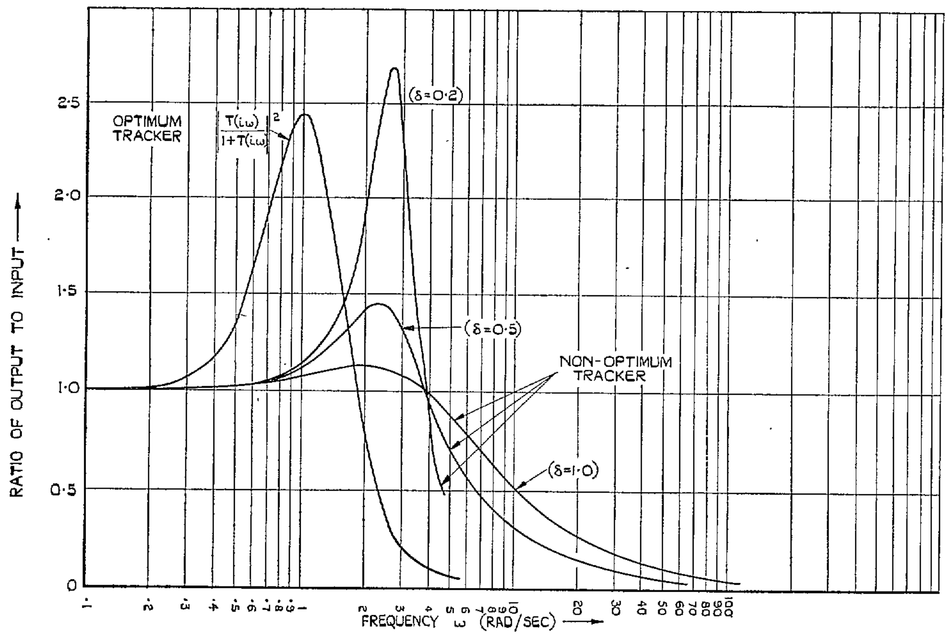


FIG. 15. Frequency responses of tracking systems (Section 8.1.2).

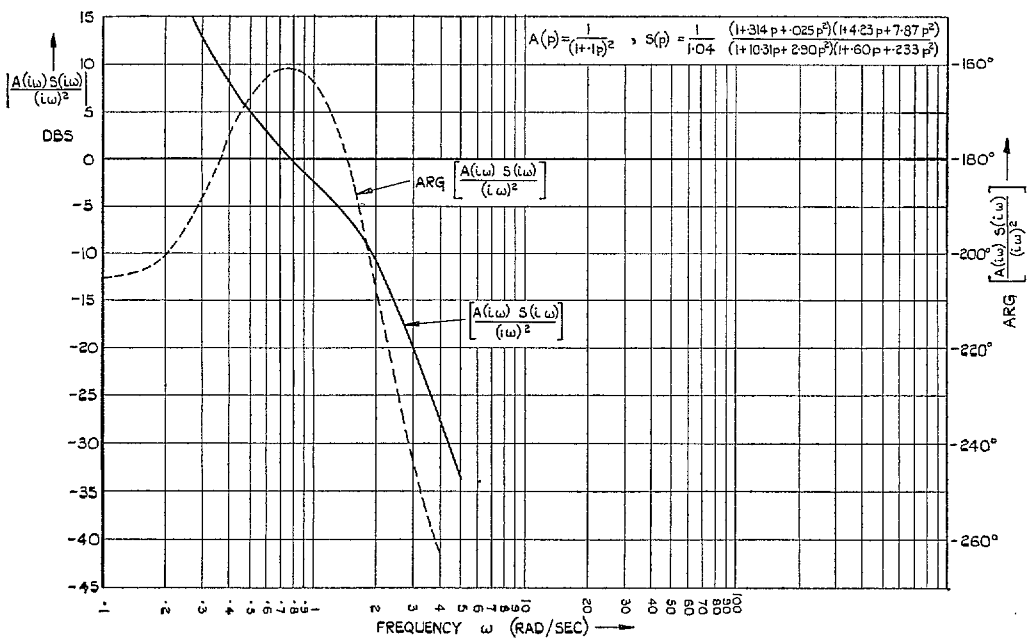


FIG. 16. Amplitude—phase diagram for the transfer function relating ($\theta_B - \theta_M$) and θ_M (Section 8.2.3).

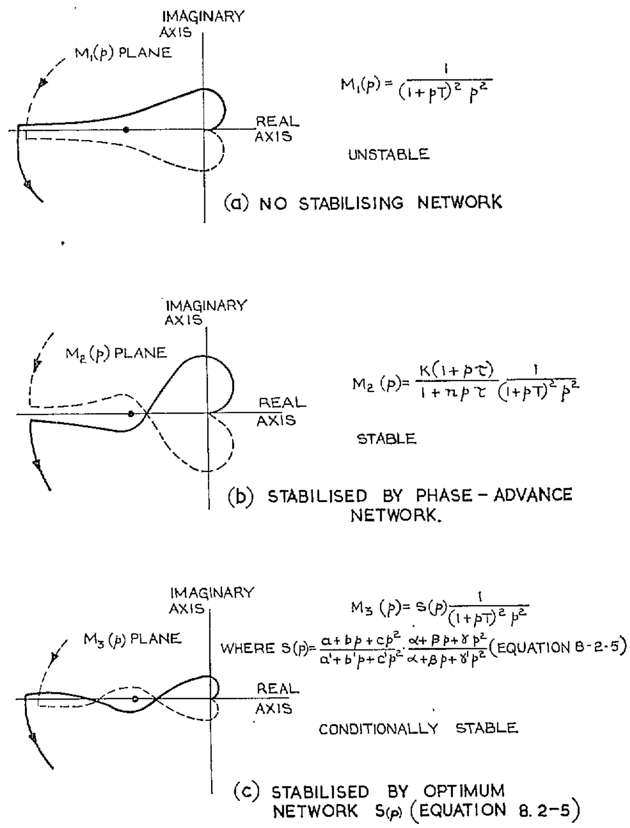


FIG. 17a to c. Nyquist diagrams for the missile system.

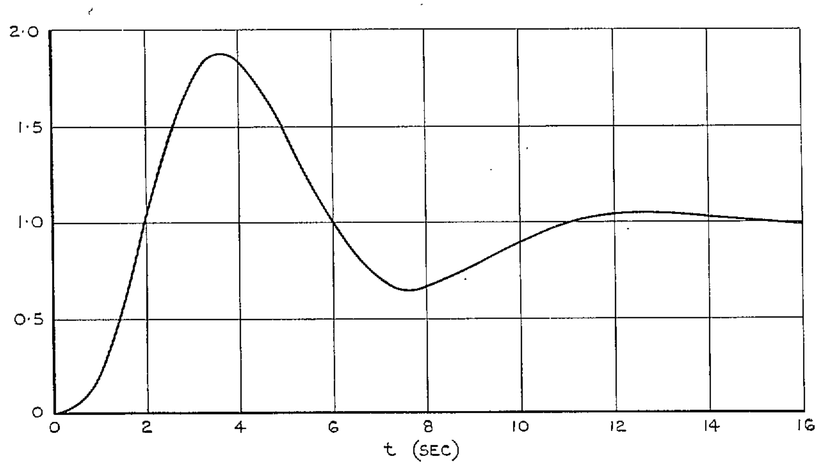
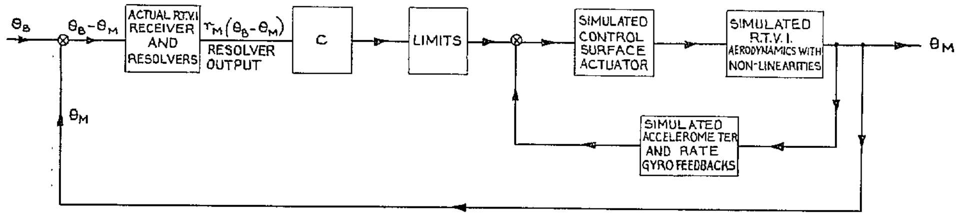


FIG. 18. Missile response to a step displacement of the beam, with the optimum network of (8.2–5).



FOR THE NORMAL R.T.V.I. SYSTEM, C IS A PHASE-ADVANCE NETWORK PLUS FILTERS
FOR THE OPTIMUM SYSTEM, IT IS THE NETWORK OF FIG 23, GIVING THE TRANSFER FUNCTION OF EQ.8.2–5.

FIG. 19. The simulator arrangement for comparing the phase-advance and optimum systems.

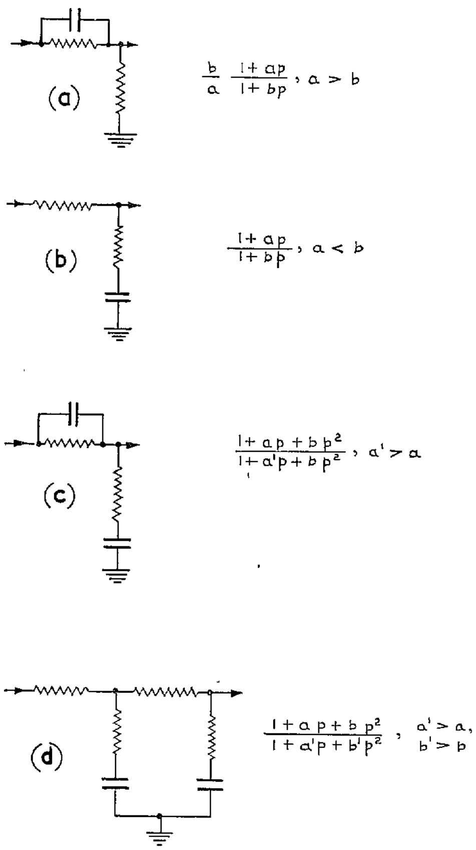


FIG. 20a to d. Networks for the realisation of real poles and zeros.

(92372) Wt. 67/7689 K.5 7/66 Hw.

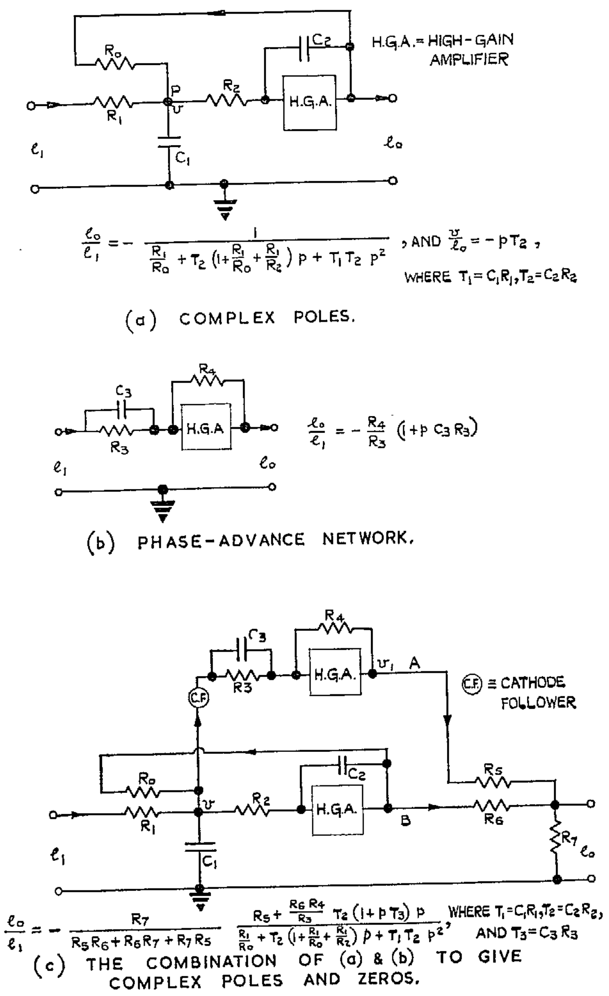


FIG. 21a to c. Networks for the realisation of complex poles and zeros.

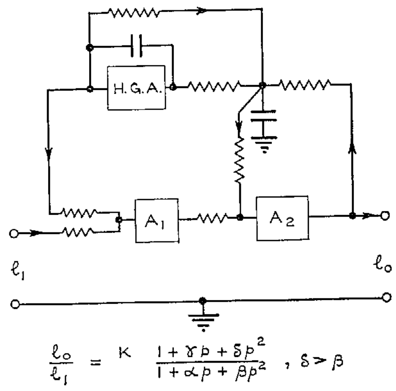


FIG. 22. An alternative circuit for the realisation of complex poles and zeros.

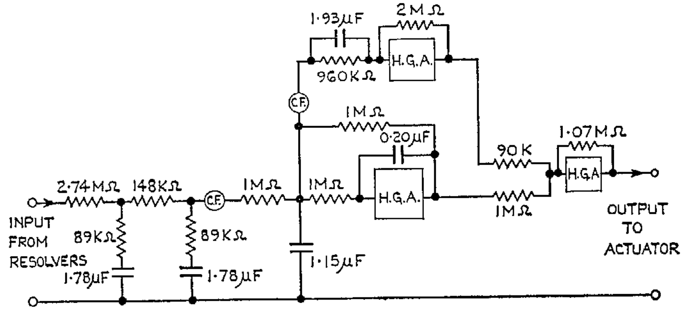


FIG. 23. The circuit used in the simulator for the realisation of $S(p)$, equation (8.2–5).

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