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A Deuce Programme for the solution of Two-dimensional Heat-Flow Problems

by

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ROYAL AIRCRAFT ESTABLISHMENT

A DEUCE PROGRAMME FOR THE SOLUTION OF TWO-DIMENSIONAL HEAT-FLOW PROBLEMS

by

K. I. McKenzie

SUMMARY

A programme is devised to calculate by a finite difference method the transient temperature distributions in any cylindrical body, provided its cross-section can be mapped on a square network in such a way that no line of the net crosses its boundary more than twice. The programme requires the boundary conditions to be linear functions of temperature and heat flux, independent of time. The initial temperature distribution is arbitrary. The results of the programme are compared with analytical solutions in two simple cases, and two further problems are solved. The programme is stored in the R.A.E. Programme Library.

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1 LIST OF SYMBOLS

V	=	temperature
x, y	=	rectangular Cartesian co-ordinates
t	=	time
δx, δy	=	space intervals
δt	=	time interval
a	=	coefficient of $\frac{\partial V}{\partial x}$ in boundary condition
b	=	coefficient of $\frac{\partial V}{\partial y}$ in boundary condition
c	=	constant term in boundary condition
p	=	number of binary places of a and b
r	=	number of node points from the edge of the square to the boundary of the body under consideration
m	=	suffix referring to x
n	=	suffix referring to y
k	=	suffix referring to t

$$\rho = \frac{2(\delta x)^2}{\delta t} = \frac{2(\delta y)^2}{\delta t} = \frac{1}{450\ \delta t} .$$

2 INTRODUCTION

Analytical solutions to the equation of transient heat conduction in two dimensions are known only in the simplest cases. This equation applies to any long cylinder with no heat flow axially; which condition is approached in a number of practical structures, notably those of an angle piece or a multi-web wing with thick skin and web subjected to aerodynamic heating.

Owing to the paucity of standard solutions, it was decided to attempt a numerical solution using a digital computer. Of the numerical methods available, the most convenient was that put forward by Peaceman and Rachford[1]. This is a finite difference method which avoids[2,3,4] both the tendency to instability of some methods and the prohibitive amount of work entailed in others.

In order to achieve as much generality as possible, a programme was devised for the DEUCE digital computer which can be used to solve any two-dimensional heat conduction problem, provided that the boundary of the cross-section can be mapped on a square network in such a way that no line crosses the boundary more than twice. In addition, the boundary conditions must involve only linear functions of temperature and heat flux and be independent of time.

The programme was used to compute solutions for four particular problems: cylinders of square, circular and triangular cross-section, all with their boundaries held at constant temperatures; and a shape representing a section of ribbed sheet maintained at a constant temperature on the whole of the ribbed side and with an aerodynamic heating type of condition on the flat side. The theoretical solutions for the square and circle were compared with those obtained using the programme, and a good agreement was observed.

3 THE PROGRAMME

The differential equation for any practical problem must first of all be reduced, by suitable substitutions, to the non-dimensional form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial V}{\partial t} \tag{1}$$

where V is always less than 1; the boundary conditions being adjusted accordingly. These must be put into one of the forms

$$V = \frac{a}{30}\frac{\partial V}{\partial x} + \frac{b}{30}\frac{\partial V}{\partial y} + c \tag{2}$$

or

$$0 = \frac{a}{30}\frac{\partial V}{\partial x} + \frac{b}{30}\frac{\partial V}{\partial y} + c\ , \tag{3}$$

where a, b and c are the numbers actually used as data by the programme. The factor 1/30 arises because the programme uses a square network of 31 horizontal and 31 vertical lines, giving a mesh length

$$\delta x = \delta y = \frac{1}{30}\ . \tag{4}$$

Certain restrictions are placed on the magnitudes of the numbers a, b and c, namely

$$\frac{1}{2^p} \leqslant |a| < 2^{15-p}\ ,$$

$$\frac{1}{2^p} \leqslant |b| < 2^{15-p} \tag{5}$$

and

$$\frac{1}{2^{p+5}} \leqslant |c| < 2^{10-p}\ ,$$

where p can be any integer from 0 to 10 (determined by a parameter in the programme).

The boundary of the cross-section under consideration must be mapped on the network in such a way that no line of the net crosses the boundary more than twice. A boundary condition must be stated wherever a line of the net crosses the boundary (making a total of 116 points); at each of these points, in addition to a, b and c, the number r of nodal points of the net on the line, between the boundary and the edge of the square, must be specified. (When these two coincide, r is zero.)

Any number up to 15 different time intervals may be used, and each of these time intervals may be used any number of times. Each time interval must be greater than 1/900. The programme will compute the temperature distributions at ten successive times in about 1 hour.

Appendix 1 consists of a brief description of the numerical method used and an analysis of the order of magnitude of the truncation error. A detailed account of the preparation of data for the programme is given in Appendix 2, and an outline of its logical structure in Appendix 3.

4 RESULTS

In order to obtain some idea of the accuracy of the programme, two problems were solved to which theoretical solutions are known. These were the square and circle, with their boundaries held at a constant temperature, the initial temperature being zero everywhere. Comparisons of the results using the programme and those predicted by theory are given in Figs.2 and 3. It can be seen that, although agreement is good in both cases, that for the square is somewhat better than that for the circle. Again, for the square, agreement is slightly better for larger values of time, and with nearness to the centre of the square. Owing to the initial discontinuities at the boundary, this is to be expected for both square and circle, but is not observed in the circular case. Both of these facts are explained by the slight inaccuracies entailed in mapping the circle on to the square network, since if the cross-section actually mapped is not quite circular, the solution obtained cannot be expected to correspond to the theoretical solution for the circle, however accurately the numerical work is done.

The essential symmetry of these two problems was not made use of in the numerical solutions, so there is no reason why this should contribute to the accuracy of these particular cases. However if problems are solved which occupy a smaller area of the square network, somewhat less accuracy is clearly to be expected.

To illustrate the use of the programme a further two examples were solved. The first of these was an isosceles triangle whose height was equal to its base, under the same boundary and initial conditions as the first two problems. Results of this problem are shown in the form of contour maps of temperature in Figs.4 and 5. The fourth example chosen was a ribbed sheet as shown in Figs.1(d) and 1(e), with a constant temperature on the ribbed side and an aerodynamic heating type of condition on the flat side. The heat transfer resistance of the air was taken to be equal to that of the unribbed sheet. As an example, to fix the time scale, the material was taken to be stainless steel and the distance between ribs 6 inches. Figs.6, 7 and 8 show contour maps of the temperature distribution at 1, 2 and 3 minutes respectively.

5 CONCLUSIONS

Using the programme, it is possible to solve a fairly wide variety of two-dimensional heat flow problems. The errors due to the finite difference approximation have been shown to be small, and do in fact prove to be so in practice. Thus a problem to which the analytical solution is unknown, and which would be very tedious for a human computer, can be worked out in about two hours, including the preparation of the data cards. The programme is stored in the R.A.E. Programme Library, together with a sheet of operating instructions.

LIST OF REFERENCES

Ref. No.	Author	Title, etc.
1	Peaceman, D. W. Rachford, H. H., Jr.	The numerical solution of parabolic and elliptic differential equations. J.Soc.Indust.Appl.Maths. Vol.3, No.1, March, 1955.
2	Douglas, J., Jr.	On the numerical solution of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$ by implicit methods. J.Soc.Indust.Appl.Maths. Vol.3, No.1, March, 1955.
3	Douglas, J., Jr.	A note on the alternating direction implicit method for the numerical solution of heat flow problems. For Humble Oil Co. Houston, Texas.
4	Douglas, J., Jr.	Numerical solution of two-dimensional heat flow problems. A.I.Ch.E. Journal, Vol.1, No.4, December, 1955.

APPENDIX 1

THE NUMERICAL METHOD

The method used was the alternating direction finite difference method of Peaceman and Rachford[1], which gives a considerable saving in computing time over the others available. Using this method the following finite difference approximations are taken for equation (1):

$$V_{m-1,n,2k+1} - (2+\rho)\, V_{m,n,2k+1} + V_{m+1,n,2k+1} = - V_{m,n-1,2k} + (2-\rho)\, V_{m,n,2k} - V_{m,n+1,2k} \tag{6}$$

and

$$V_{m,n-1,2k+2} - (2+\rho)\, V_{m,n,2k+2} + V_{m,n+1,2k+2} = - V_{m-1,n,2k+1} + (2-\rho) \times \times V_{m,n,2k+1} - V_{m+1,n,2k+1} \;. \tag{7}$$

A considerable amount has been written on the subject of the convergence and stability of this finite difference form[1,2,3,4], the general conclusion of which is that the solution of equations (6) and (7), taken together as a double set of finite difference equations, is stable for any time interval and converges to that of equation(1).

Equations (6) and (7), when written in terms of partial derivatives, become:

$$\left[\frac{\partial^2 V}{\partial x^2} - \frac{(\delta x)^2}{12}\frac{\partial^4 V}{\partial x^4} + O(\delta x)^4\right]_{m,n,2k+1} + \left[\frac{\partial^2 V}{\partial y^2} - \frac{(\delta y)^2}{12}\frac{\partial^4 V}{\partial y^4} + O(\delta y)^4\right]_{m,n,2k}$$

$$= \left[\frac{\partial V}{\partial t} - \frac{\delta t}{2!}\frac{\partial^2 V}{\partial t^2} + \frac{(\delta t)^2}{3!}\frac{\partial^2 V}{\partial t^3} + O(\delta t)^3\right]_{m,n,2k+1} . \tag{8}$$

and

$$\left[\frac{\partial^2 V}{\partial x^2} - \frac{(\delta x)^2}{12}\frac{\partial^4 V}{\partial x^4} + O(\delta x)^4\right]_{m,n,2k+1} + \left[\frac{\partial^2 V}{\partial y^2} - \frac{(\delta y)^2}{12}\frac{\partial^4 V}{\partial y^4} + O(\delta y)^4\right]_{m,n,2k+2}$$

$$= \left[\frac{\partial V}{\partial t} + \frac{\delta t}{2!}\frac{\partial^2 V}{\partial t^2} + \frac{(\delta t)^2}{3!}\frac{\partial^3 V}{\partial t^3} + O(\delta t)^3\right]_{m,n,2k+1} . \tag{9}$$

Adding(8) and (9):

$$2\left[\frac{\partial^2 V}{\partial x^2} - \frac{(\delta x)^2}{12}\frac{\partial^4 V}{\partial x^4} + O(\delta x)^4\right]_{m,n,2k+1} + \left[\frac{\partial^2 V}{\partial y^2} - \frac{(\delta y)^2}{12}\frac{\partial^4 V}{\partial y^4} + O(\delta y)^4\right]_{m,n,2k}$$

$$+ \left[\frac{\partial^2 V}{\partial y^2} - \frac{(\delta y)^2}{12}\frac{\partial^4 V}{\partial y^4} + O(\delta y)^4\right]_{m,n,2k+2} = \left[2\frac{\partial V}{\partial t} + 2\,\frac{(\delta t)^2}{3!}\frac{\partial^3 V}{\partial t^3} + O(\delta t)^2\right]_{m,n,2k+1}. \tag{10}$$

which reduces to:

$$\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - \frac{\partial V}{\partial t}\right]_{m,n,2k+1} = \left[\frac{(\delta x)^2}{12}\frac{\partial^4 V}{\partial x^4} + \frac{(\delta y)^2}{12}\frac{\partial^4 V}{\partial y^4} + (\delta t)^2\frac{\partial^2}{\partial t^2}\left(\frac{1}{6}\frac{\partial V}{\partial t} - \frac{\partial^2 V}{\partial y^2}\right) + O((\delta t)^3 + (\delta x)^4 + (\delta y)^4)\right]_{m,n,2k+1}. \tag{11}$$

Thus the magnitude of the truncation error is $O((\delta x)^2 + (\delta y)^2 + (\delta t)^2)$.

APPENDIX 2

PREPARATION OF DATA FOR THE PROGRAMME

The cards are fed into the machine in the following order:

(i) Programme cards.

(ii) Initial conditions.

The starting temperatures of all 961 node points of the network are punched on Hollerith cards. They are punched in 31 sets of 31, representing the temperatures along the nodal points on the horizontal lines of the net starting at the bottom left hand corner. The numbers are punched as four-digit unsigned decimal numbers, eight to a card, from column 21 to column 52. Each set of 31 temperatures is started on a new card, so that the last number of every fourth card is always punched zero. The 31 sets of 4 cards must be followed by four cards with all zeros punched, making 128 cards in all. A temperature must be punched for every nodal point, a zero being punched for points outside the boundary.

(iii) Boundary conditions

For the boundary conditions on horizontal lines, the constants a, b, c and r are punched in binary, two to each row of the card. a is punched in columns 21-36, the sign digit being in column 36, and b is punched in columns 37-52, the sign digit being in column 52. Similarly c is punched in columns 21-36 of the next row, but r is punched as an integer in columns 37-41. For a and b, the binary point may be placed anywhere, provided that if there are p digits in the fractional part a 1 is punched in column 37+ p of the 3-row of card 74 of the programme, and

$$p \leqslant 10 .$$

c always has 5 more binary places than a and b. The constants for every boundary condition must have the same number of binary places.

If 1 is punched in column 52 of the row in which r is punched, the condition takes the form of equation (3) rather than that of equation (2).

The two conditions for each row are punched consecutively, the left hand one first, and the rows are punched starting from the bottom and working up. If a row does not cross the boundary the r for the left hand condition is made 29 and that for the right hand condition zero. The conditions for the vertical lines are punched in a similar way, again starting at the bottom left hand corner. In this case the constants a and b interchange positions. The row conditions are punched on 10 cards, starting on the Y-row of the first card. The column conditions are punched starting on the 6-row of the next card and a blank card is inserted at the end making 22 cards in all.

(iv) The time intervals

Values of

$$\rho = \frac{2(\delta x)^2}{\delta t} = \frac{1}{450\ \delta t}$$

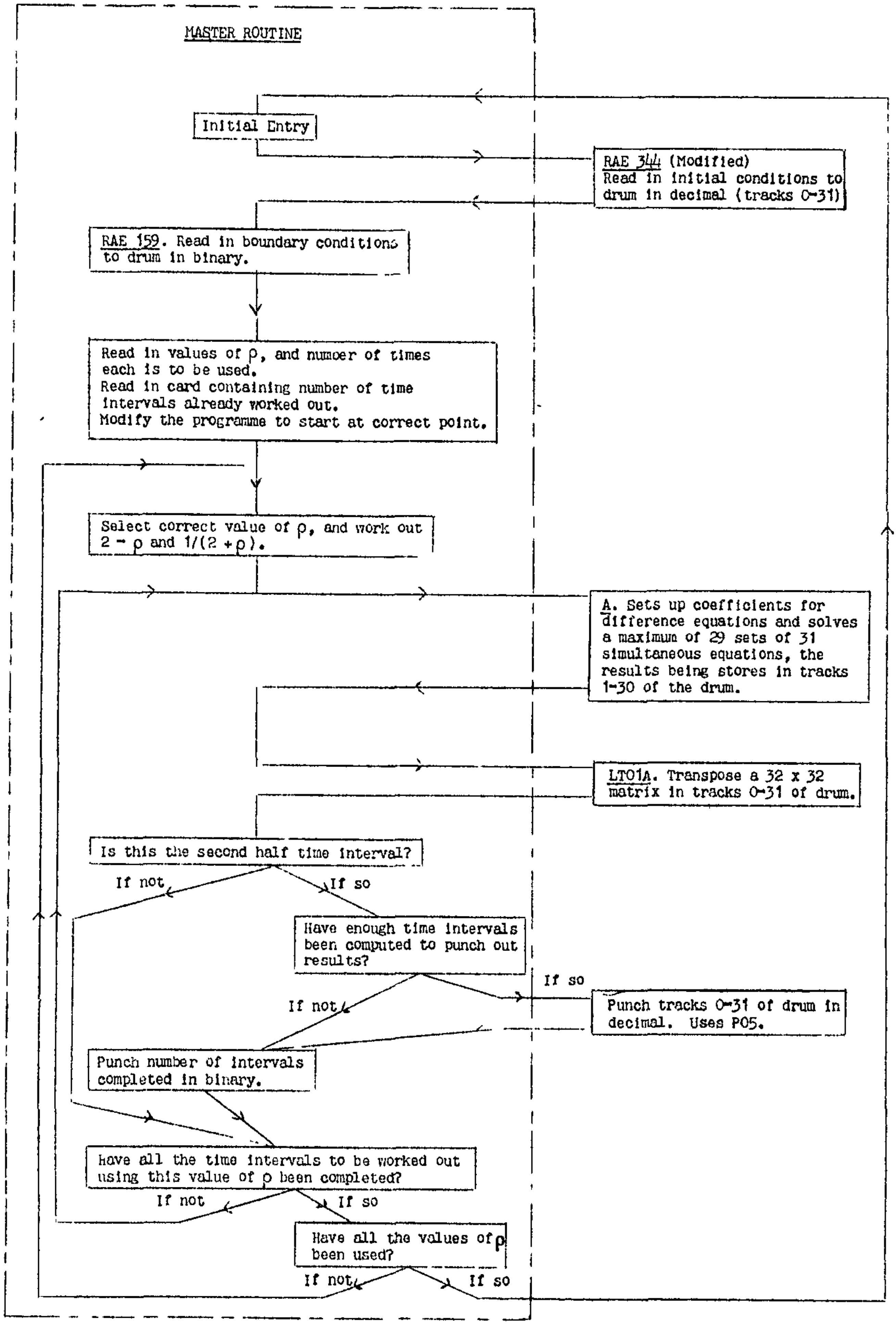
are punched to 30 binary places in columns 21-51. For each value of ρ the number of times it is to be used is punched as an integer in columns 38 onwards in the next row. A maximum of 15 different values of ρ may be taken, taking up 3 data cards. On the 4-row of the third card the number of values of ρ actually taken is punched, and in the 5-row is punched the number of time intervals to be used before the first set of results is punched out. Both these numbers are punched as integers in columns 37 onwards. This last provision is made because it is advisable to take a few very small time intervals at the beginning of the programme, in order to smooth out discontinuities in the initial conditions, but the results at those early times are usually of no interest. The values of ρ taken must all be less than 2, giving values of δt greater than 1/900.

(v) The last card.

The last data card contains the number of time intervals already worked out, punched as an integer in columns 37 onwards of the Y-row. This card is blank unless the programme is restarted in the middle of the problem. The results are punched out on 128 cards in the same form as the initial conditions, and each set is followed by a card containing the number of time intervals worked out, punched as an integer in columns 37 onwards of the Y-row. The programme can thus be restarted using this set of results as initial conditions and replacing the last data card by the last card of the set of results, the other data cards remaining unchanged.

APPENDIX 3

THE LOGICAL STRUCTURE OF THE PROGRAMME



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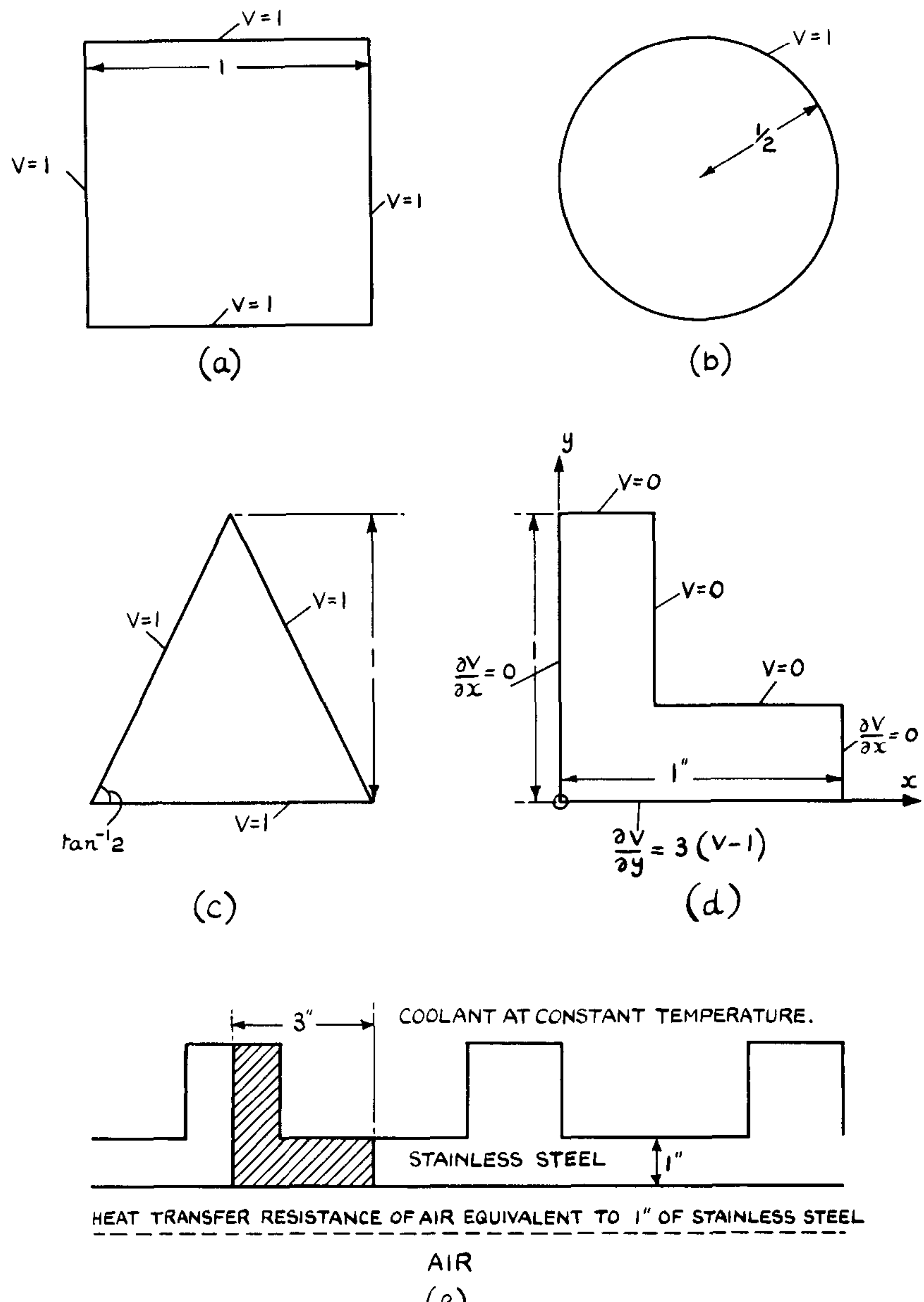

FIG. 1. DIAGRAMS DESCRIBING THE FOUR PROBLEMS SOLVED USING THE PROGRAMME

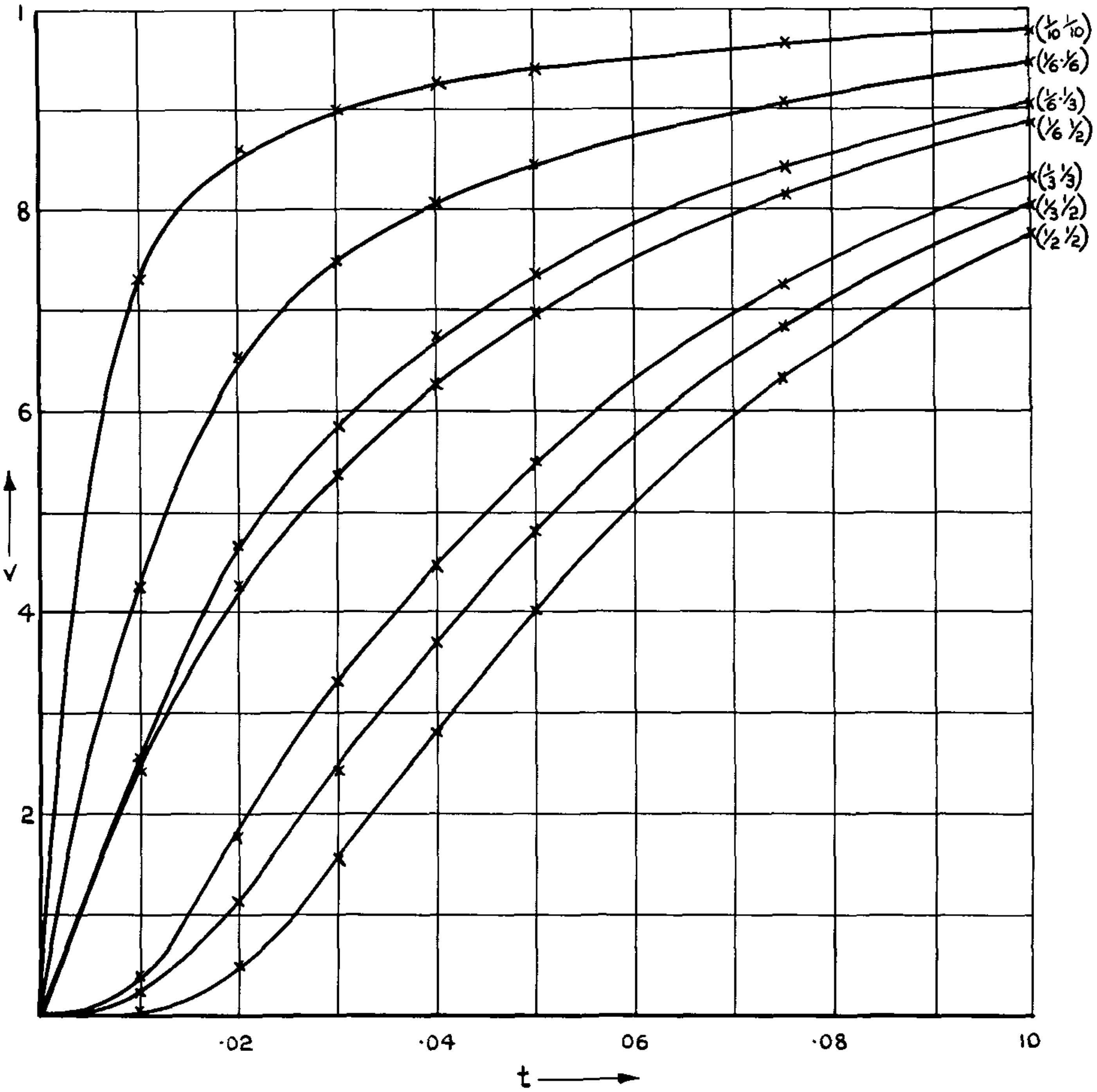

FIG. 2. COMPARISON OF THE ANALYTICAL SOLUTION AND THE NUMERICAL RESULTS FOR THE UNIT SQUARE SHOWN IN FIG 1. (a). TEMPERATURE IS PLOTTED AGAINST TIME FOR VARIOUS POINTS ON THE SQUARE

CONTINUOUS LINE = ANALYTICAL SOLUTION.

CROSSES = NUMERICAL RESULTS.

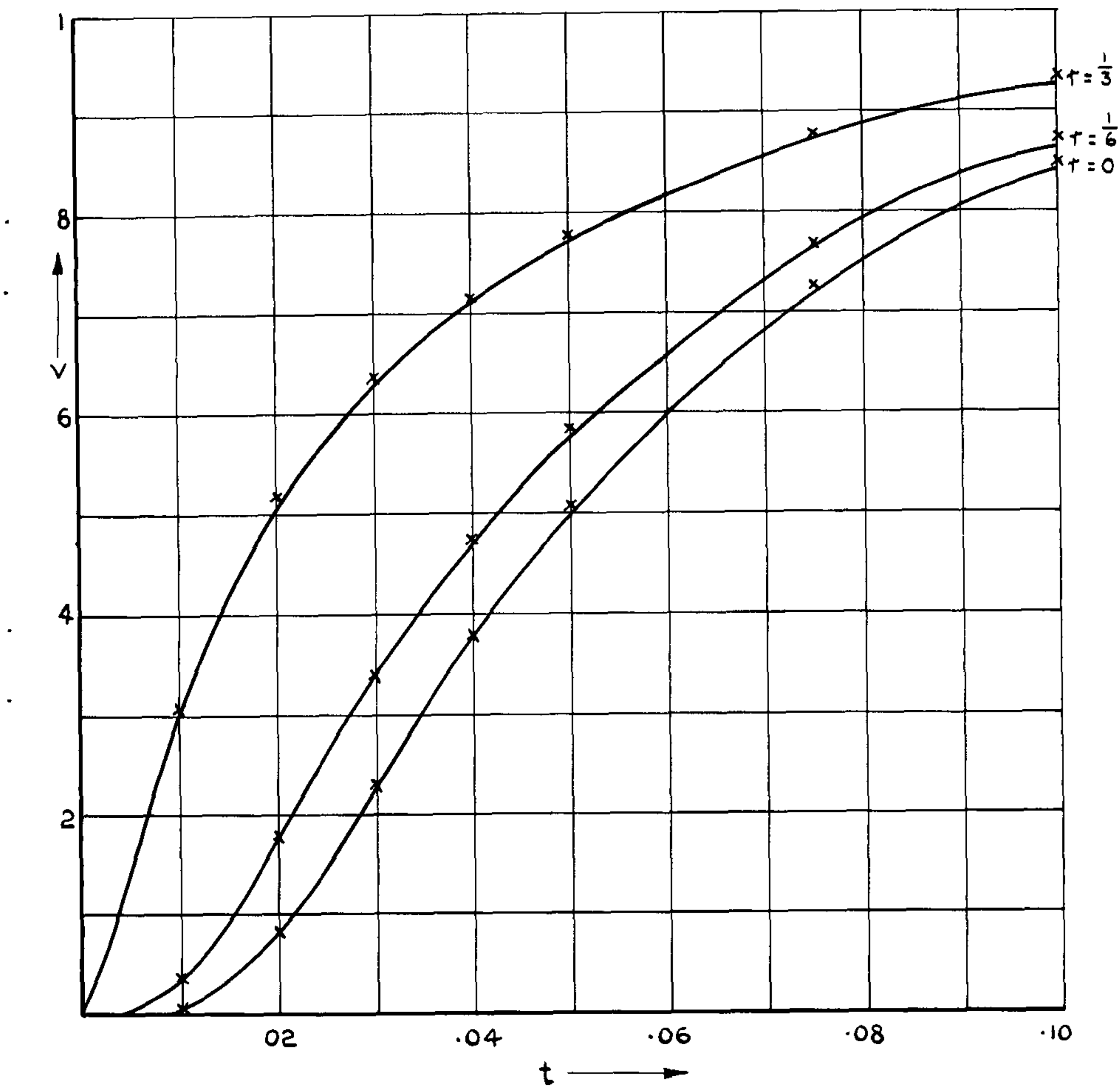

FIG. 3. COMPARISON OF ANALYTICAL SOLUTION AND NUMERICAL RESULTS FOR THE CIRCLE SHOWN IN FIG. I. (b). TEMPERATURE IS PLOTTED AGAINST TIME FOR VARIOUS POINTS ALONG A RADIUS.

CONTINUOUS LINE = ANALYTICAL SOLUTION

CROSSES = NUMERICAL RESULTS

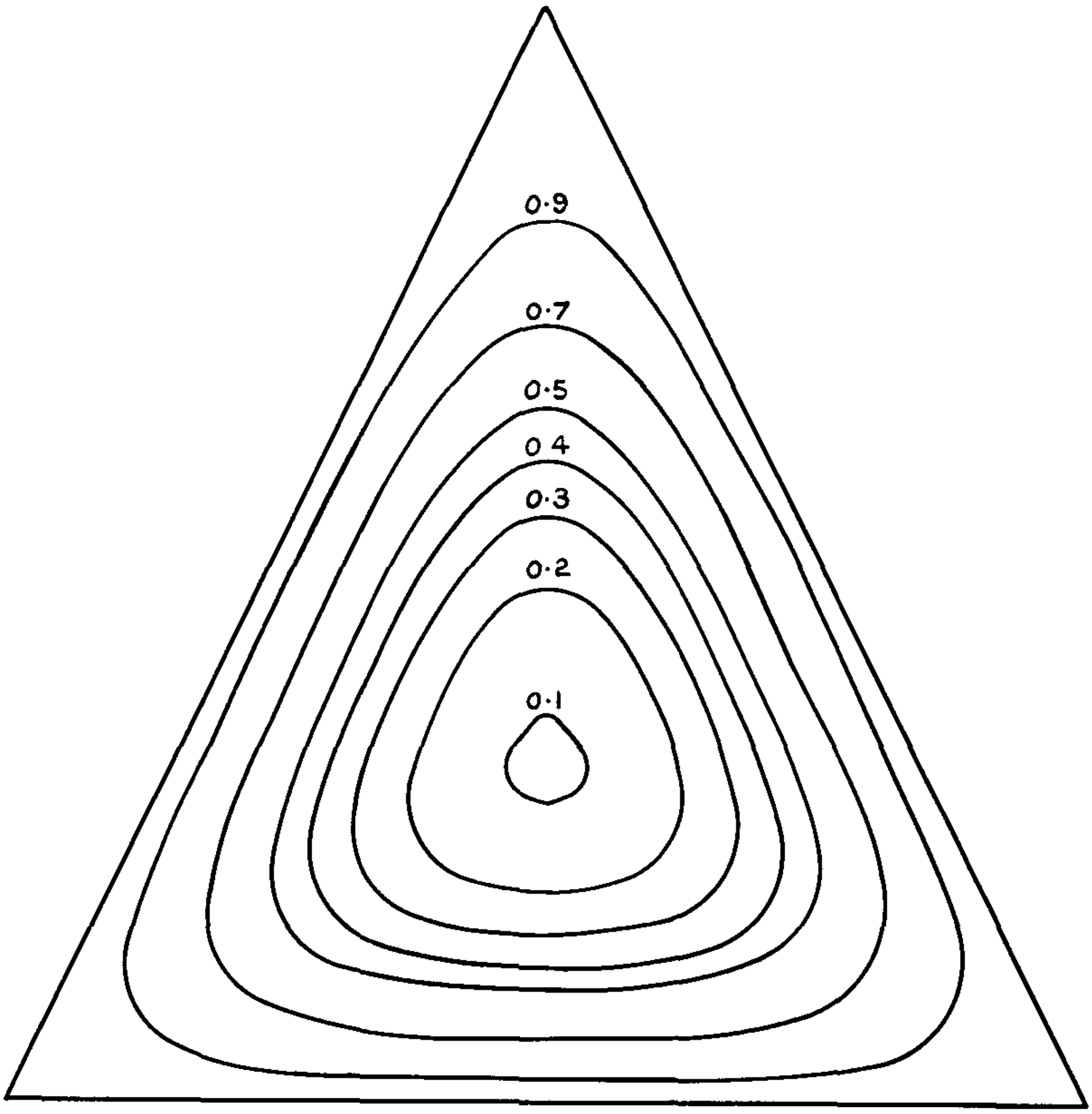

FIG. 4. CONTOUR MAP OF ISOTHERMALS FOR THE TRIANGLE SHOWN IN FIG. 1.(C) AT TIME $t = 0 \cdot 01$.

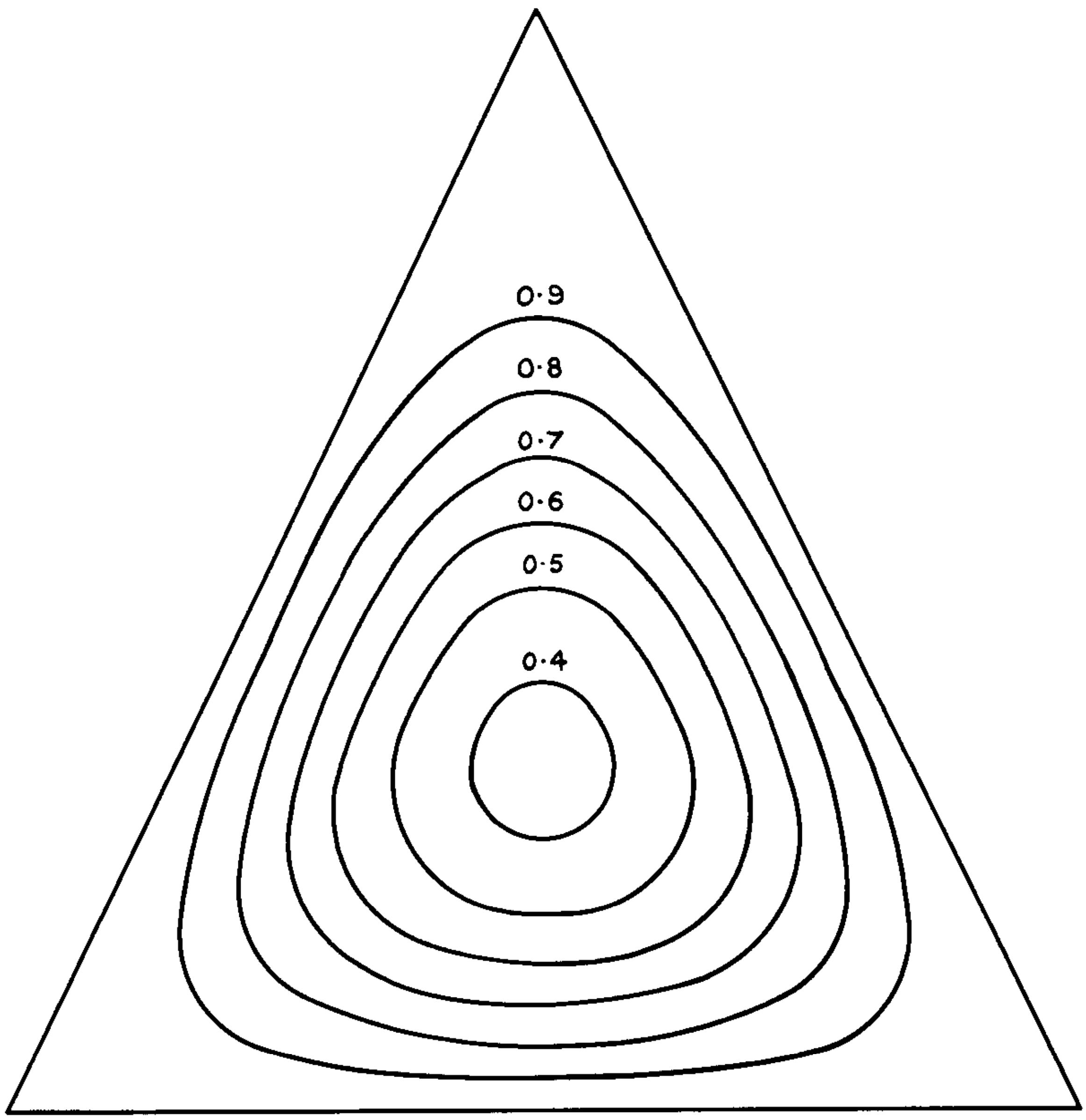

FIG. 5. CONTOUR MAP OF ISOTHERMALS FOR TRIANGLE SHOWN IN FIG. 1.(C) AT TIME $t = 0 \cdot 02$

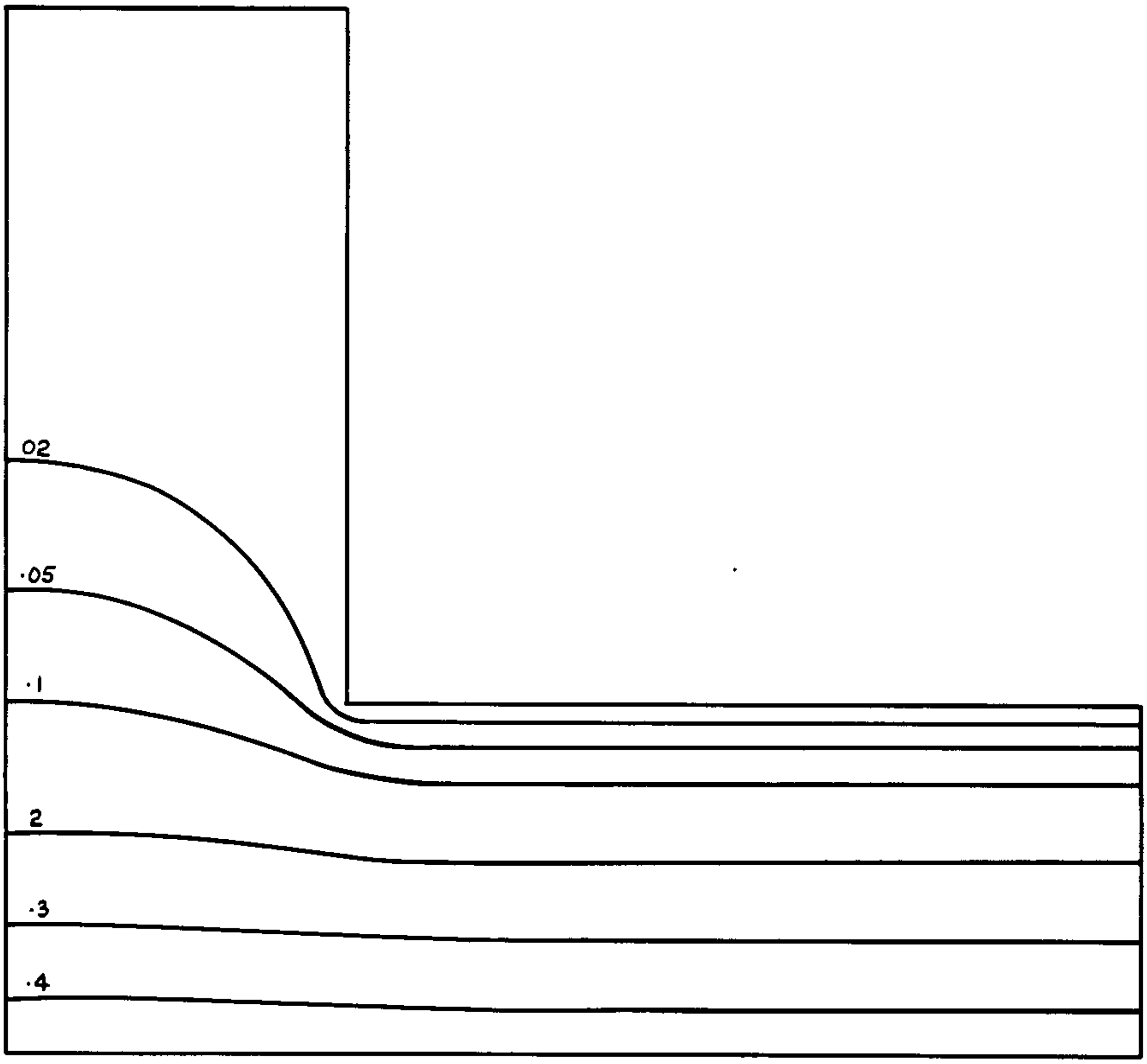

FIG. 6. CONTOUR MAP OF ISOTHERMALS FOR PROBLEM DESCRIBED BY FIGS. 1 (d) & (e) AT TIME 1 MINUTE.

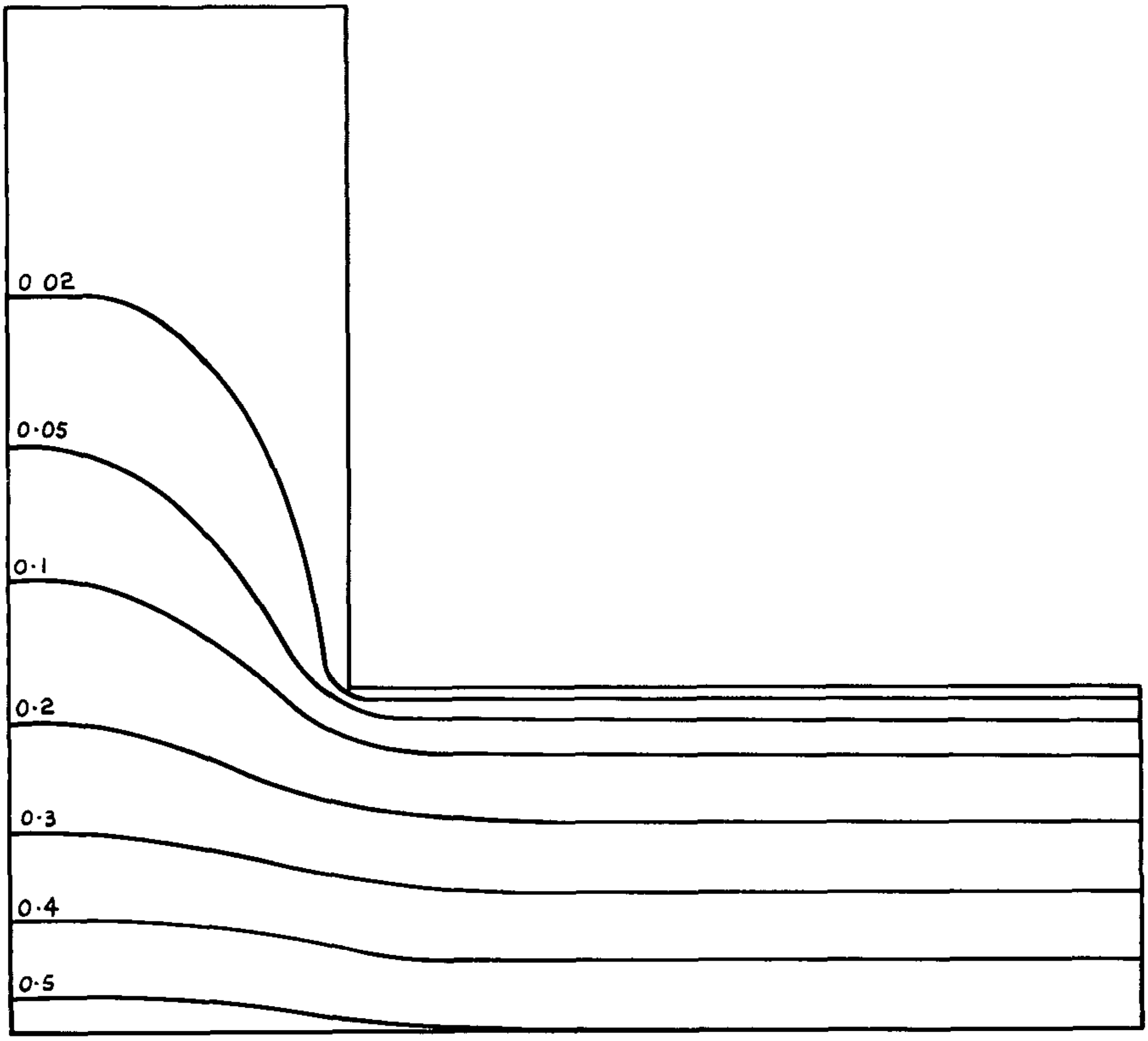

FIG. 7. CONTOUR MAP OF ISOTHERMALS FOR PROBLEM DESCRIBED BY FIGS. 1.(d) & 1.(e) AT TIME 2 MINUTES.

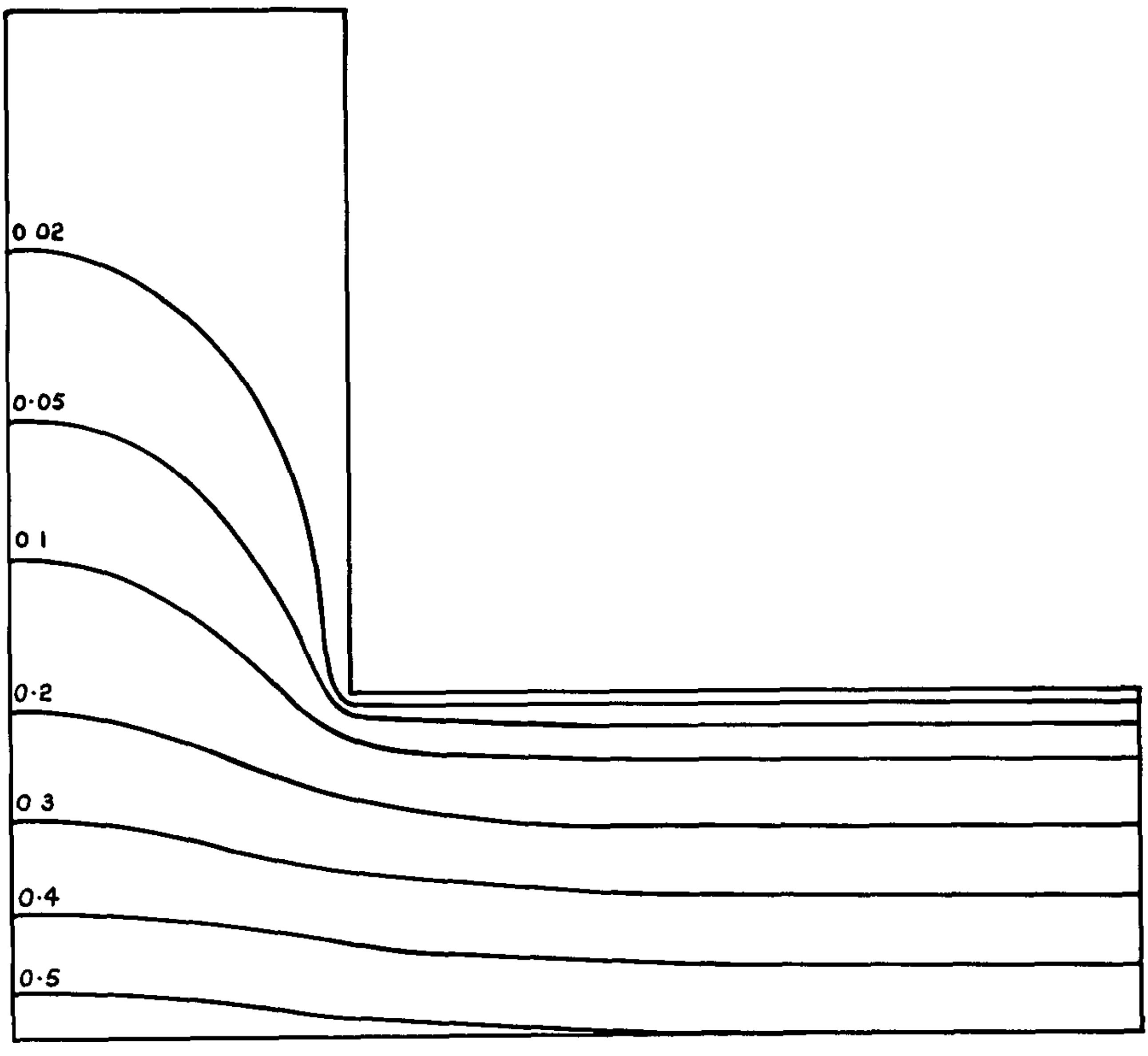

FIG. 8. CONTOUR MAP OF ISOTHERMALS FOR PROBLEM DESCRIBED BY FIG. 1.(d) & 1.(e) AT TIME 3 MINUTES.

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