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by

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SUMMARY

The purpose of this note is to enable the crack propagation properties of sheet material to be determined in a much more economical way than has hitherto been possible. This objective is sought by establishing simple formulae for correlating the results for small flat sheet specimens under tension or cylindrical specimens under internal pressure, with those for larger but similar specimens, and for correlating results for a flat sheet with those for the corresponding (i.e. the same flat sheet rolled into a cylinder) cylindrical sheet under the same tension produced by internal pressure.

LIST OF CONTENTS

LIST OF ILLUSTRATIONS

1 INTRODUCTION

It will be agreed that in spite of intensive experimental work there is still much to be learnt about the mechanism of crack propagation in thin sheet material. There are a number of facts that are known with certainty. We know, for example, that the behaviour of a crack - the way it extends under repeated application of tensile forces across it and the unstable length it can reach before it finally self-propagates at speed - depends profoundly on the material of the sheet. Apart, however, from expecting ductile materials to give better results than those less ductile, there is no quantitative basis upon which to estimate the performance of a material by reference to its specification. It is also known that, under the same nominal applied tensile stress, a crack propagates faster and has a lower unstable length when the tension is induced by pressure in a cylindrical shell than when it is directly applied to a flat sheet. Moreover, for the same hoop stress, a crack becomes more ready to spread as the radius of the cylinder is reduced.

It follows from the above remarks that, in the absence of correlating factors, the behaviour of a crack in a pressure cabin of a particular diameter can only be determined by carrying out a test on a cylinder of equal diameter and identical sheet material. By the same token an estimate of the crack behaviour in the skin of a wing makes it necessary to carry out tests on a corresponding extent of flat sheet. Such multiplication of 'ad hoc' tests is highly uneconomic and wasteful of time and effort. What is obviously required, and what appears to be to some extent feasible, is the establishment of correlating factors that will allow the results of tests carried out on a flat sheet to be applied with some confidence in forecasting the behaviour of a sheet of the <u>same material</u> built into a pressure cabin of any diameter - and vice versa. It is with the object of suggesting methods - based on a physical interpretation of hitherto unexplained experimental results - for obtaining such factors that the present note is put forward. The phrase "same material" is underlined because it is not thought possible in the present state of knowledge to correlate the behaviour of a sheet of one material with that of a sheet of a different material on the basis either of their chemical composition or their material properties. Direct experiment appears here to be the only guide. It is, however, reasonable to expect that, if one material is shown by experiment to have better crack properties than another when both are tested as flat sheet, the same superiority will be shown when both are built into pressurised cylinders.

2 CRACKS IN FLAT SHEET

The behaviour of cracks in flat sheet will be considered first. An important question that needs to be considered is the degree to which, for the same applied stress, sheet thickness affects the results. If the sheet does not buckle in any way it is reasonable to suppose that a state of plane stress exists, i.e. the stress remains sensibly constant across the thickness. In that event one would not expect sheet thickness to enter into the problem so long as the nominal applied stress remains the same. Experiment seems to confirm this expectation not only for flat sheet[1], but also for pressurised cylindrical shells[2,3]. It can be assumed, therefore, that in making a scale model of a flat sheet or cylindrical specimen, there is no real need to scale down the sheet thickness correspondingly. One is confirmed in this view by the fact that, for sheets of comparable thickness under uniform tension in one direction, one would not expect the stress concentration factor at a circular hole to be other than the expected three-to-one value whatever their actual thicknesses, so long as in each case the boundaries of the sheet are far enough away from the hole.

2.1 Small hole or crack in large expanse of sheet

If we consider a small hole with a smooth boundary - round or oval for example - in a large expanse of sheet in which the stress is uniform, it will readily be accepted that any departure from that uniformity of stress is only local, and that at a good distance from the hole the stress retains its original uniformity. Furthermore, subject to the hole being small compared with the size of the sheet, the stress distribution around the hole should, to a different scale, be exactly the same whatever its size; and this should be true whether the local stress reaches beyond the elastic range or not. To fix ideas we may contemplate a circular hole in an infinite expanse of sheet in simple tension. As the hole is enlarged the disturbed area becomes progressively wider, but the stress distribution over that area remains the same. It follows that two such expanses of sheet with circular holes of different size would be expected to fail at the same nominal applied tensile stress.

If, however, the stress disturbance is caused by a narrow slit or crack lying across the direction of tension, it is no longer possible to ensure geometrical similarity between the short crack and the long crack. This is particularly true at the extremities of the crack where, if it has been formed naturally - by the extension of a shorter crack for example - its sharpness is the same whatever its length. As a result one would expect the sheet, under a progressively increasing applied stress, to fail at a lower stress for the longer than for the shorter crack, and one would also expect the difference to be greater the greater the ratio between the crack lengths.

Suppose, for example, that a short crack of length ℓ_o in a large sheet (which simulates a crack in an infinite sheet) leads to failure of the sheet under an applied stress σ_o. A crack of length $2\ \ell_o$ would be expected to cause failure at a lower stress $(\sigma_o - \delta\sigma_o)$ say, as a result of the end radii of the crack not being correspondingly doubled. Doubling the crack length thus reduces the failing stress in the ratio $(\sigma_o - \delta\sigma_o)/\sigma_o$ or r (say). If the crack is again doubled to a length of $4\ \ell_o$ a reduction ratio of r^2 would be expected.

On the basis of the above argument the relation between failing stress and crack length is known for all crack lengths once the reduction caused by a single increase is known. Suppose, under the effect of a short crack of length ℓ_o across the line of tension, the experimentally observed nominal applied failing stress (the uniform stress, i.e. remote from the crack) is σ_o and that an increase in crack length from ℓ_o to $n\ell_o$ is observed to reduce the failing stress to $r\sigma_o$. It follows that, if the crack length is increased to any length x, where

$$\frac{x}{\ell_o} = n^\theta \quad , \tag{1}$$

the failing stress correspondingly falls from σ_{ℓ_o} to σ_x, where

$$\frac{\sigma_x}{\sigma_{\ell_o}} = r^\theta \quad . \tag{2}$$

From (1)

$$\theta = \frac{\log (x/\ell_o)}{\log n} \qquad (3)$$

and therefore, from (2)

$$\frac{\sigma_x}{\sigma_{\ell_o}} = r^{\left(\frac{\log (x/\ell_o)}{\log n}\right)} , \qquad (4)$$

which gives the failing stress σ_x for any length of crack in terms of the reduction factor r initially obtained.

This applies only so long as the crack length is small compared with the width of the sheet. As soon as it becomes an appreciable fraction of that width the stress at points remote from the crack can no longer be assumed to remain constant as the crack grows, and a stage is soon reached where the reduction in failing stress calculated on the gross area is almost entirely due to the reduction in the net cross-sectional area of the sheet, and where equation (4) becomes irrelevant.

The only reliable way of checking the validity of the relation given by equation (4) is to compare the failing stresses of two sheets of different size, but of the same shape. As a matter of interest, however, use will also be made of the alternative but less reliable method of comparing, as indicated above, the failing stress of a wide sheet with short cracks of different lengths.

2.2 Similar cracks in similar sheets

We choose two sets of results from among the available data, one American and one British. The first set is taken from a paper[4] by McEvily et al., which quotes experimental results for two sheet specimens, one 12 in. wide and 36 in. long and the other 35 in. wide and 36 in. long. The widths are thus in the ratio 3:1, but the lengths are the same. However, so long as the cracks are well below half the length, the error in not having the length properly scaled has been shown experimentally by Harpur[5] not to be important.

The results given in Table 1 have been taken from Figs. 10 and 11 of McEvily's paper[4] and show the nominal failing stresses for the two sizes of sheet against corresponding crack lengths.

TABLE 1

(Sheet material 2024-T3)

Crack length (including central hole) as % of sheet width	Failing stress (lb/in.^{2} on gross area) ÷ 1000		Stress-reduction Ratio
	12 in. sheet	35 in. sheet	
10	52	42	0.81
20	46	32	0.7
30	36	26	0.72
40	30	21.5	0.72
45	27.5	20	0.73

It is not possible here to make comparisons of failing stress for a crack length (including central hole) less than 1/12 the sheet width because the central hole for the 12 in. sheet takes up 1/12 of the total width. It is unfortunate for our purpose here that in these experiments the same size of central hole - 1 in. diameter - was made in both the 12 in. and the 35 in. sheet. Thus, whereas the figure of 10 per cent in the first row of the above table is nearly all accounted for by the central hole in the case of the 12 in. sheet, less than half of it is so accounted for in the 35 in. sheet. Similarity between the two sheets for crack lengths between 10 and 15 per cent of the sheet width is not, therefore, achieved. That presumably is the reason why the stress ratio given in the last column is slightly higher for the shortest crack length. For the longer crack lengths the stress ratio is fairly constant so that an average value can be taken of about 0.73.

According to the argument already put forward, if the crack-end radius for the larger sheet had been made greater than that for the smaller sheet in the proper 3:1 ratio we should expect the same failing stress and, therefore, a stress ratio of unity. The drop in failing stress from unity to 0.73 is consequently to be attributed to lack of scaling up the crack-end radii in the proper ratio.

Following the analytical approach implied by equations (1) to (4) we are now in a position to estimate the failing stress corresponding to any size of crack in any size of (similar) sheet of the same material. Thus, let σ_o be the applied stress at failure for a crack extending a certain percentage width of a sheet of width ℓ_o and let this stress fall to $r\sigma_o$ when the sheet (and correspondingly the crack) is increased in size in the ratio n. It then follows that, for a similar sheet of width x and crack length x/ℓ_o times greater than that in the sheet of width ℓ_o, the failing stress is given by

$$\frac{\sigma_x}{\sigma_{\ell_o}} = r^{\left(\frac{\log x/\ell_o}{\log n}\right)} \quad . \tag{5}$$

In the present case we have

$$n = 3, \quad r = 0.73, \quad \ell_o = 12$$

and therefore

$$\frac{\sigma_x}{\sigma_o} = 0.73^{\left(\frac{\log x/12}{\log 3}\right)} \quad . \tag{6}$$

If therefore it is desired to estimate the applied failing stress (by which is always meant the applied stress at failure) σ_x for a sheet 100 in. wide (and length of the same order) for any percentage width of central crack, we write

$$\frac{\sigma_x}{\sigma_o} = 0.73^{\left(\frac{\log 100/12}{\log 3}\right)}$$

$$= 0.73^{1.93} = 0.55 \tag{7}$$

where σ_o is the applied failing stress for a 12 in. sheet with the same percentage crack length.

A crack of 30 in. for example in the 100 in. wide sheet is a 30 per cent crack, and from Table 1 we see that a 30 per cent crack corresponds to a failing stress of 36,000 lb/in.^{2} in a 12 in. sheet. The failing stress for the same percentage length of crack in the 100 in. sheet is therefore, by equation (7), 0.55 of 36,000 or 20,000 lb/in.^{2}.

Results are also given in Table 3 of Ref. 3 for the same sizes of sheet in material to specification 7075-T6. A smooth curve through these gives the values shown in Table 2 below.

TABLE 2

(Sheet material 7075-T6)

Crack length (including central hole) as % of sheet width	Failing stress (lb/in.^{2} on gross area) ÷ 1000		Stress-reduction Ratio
	12 in. sheet	35 in. sheet	
8.4	64	33	0.52
10	52	31	0.59
15	39	24	0.61
20	32	19	0.6
25	26.5	15.2	0.58
30	22.5	13	0.59
35	20	12	0.6

This shows that trebling of sheet size and crack length for sheet material 7075-T6 reduces the failing stress to an average of about 60 per cent of that for the smaller sheet. The corresponding figure for sheet material 2024-T3 as found above is 73 per cent.

Thus, as well as having much lower absolute values of applied stresses at failure than 2024-T3 sheet for the same percentage crack length, 7075-T6 sheet suffers a greater reduction in failing stress with increased size than 2024-T3. Following equation (6) and taking Table 2 as a basis*, we can write the failing stress σ_x for any width x of sheet to Specification 7075-T6 and for any percentage crack length in the form

$$\sigma_x = \sigma_o (0.6)^{\left(\frac{\log x/12}{\log 3}\right)} \quad . \qquad (8)$$

For example, again taking a 30 in. crack in a 100 in. wide sheet, we find the failing stress to be

$$\sigma_{100} = 10^3 \times 22.5 (0.6)^{\left(\frac{\log 100/12}{\log 3}\right)} = 8.4 \times 10^3 \text{ lb/in.}^2 . \qquad (9)$$

(Where the figure of 22.5×10^3 is taken from Table 2).

* This is a rough basis because the experimental work was never designed for the purpose of evaluating the scale effect for the particular material.

One notes that the applied failing stress for the same crack length in the same size sheet to Specification 2024-T3 as found above is more than three times greater.

These figures serve to illustrate the general trend of the scale effect for the two materials concerned, but, owing to the central hole being the same size in the 12 in. and 35 in. wide sheets, they cannot be taken as providing reliable data for use in quantitative calculations. A carefully carried out set of tests on two sheets with linear dimensions in the ratio 3:1 (say) should, however, provide a firm basis for calculating the scale effect that could then be used for estimating the failing stress of any size of (similar) sheet in the same material.

The second set of results we shall quote are taken from experiments[1] carried out for the Ministry of Supply by the Bristol Aircraft Company. The experimental values shown in Figs.1 and 2 for sheet material to Specification D.T.D.746 are typical in general character of others obtained for aluminium alloy sheets to other specifications. The points plotted in these figures are the actual experimental values and the smooth curves have merely been drawn in by eye to represent as nearly as possible the average failing stress for each crack length. Fig.1 refers to a sheet 20 in. × 10 in. × 0.04 in. and Fig.2 to a double-size sheet 40 in. × 20 in. × 0.04 in. The few experimental values that were obtained on a sheet 40 in. × 20 in. × 0.08 in. show that they differ to a negligible extent from those for the same size sheet of half the thickness. There is, therefore, no cockling effect so that, apart from the crack-end radii, the dimensions of the two sheets are effectively in the ratio 2:1.

Table 3, which is based on the smooth curves of Figs.1 and 2, shows the failing stresses for the two sheets at various crack lengths. It also shows in the last column the failing stress in the larger sheet as a fraction of that for the smaller. This fraction varies from 0.76 to 0.84 about an average value of approximately 0.8. According, therefore, to the theory put forward here, the effect of not scaling up the crack-end radii when other dimensions are doubled is to reduce the failing stress by some 20 per cent for sheet to Specification D.T.D.746.

TABLE 3

(Sheet material D.T.D.746)

Crack length / Sheet width	Failing stress (average on gross area): Sheet 20"×10"×0.04"	Failing stress (average on gross area): Sheet 40"×20"×0.04"	(Failing stress, larger sheet / Failing stress, smaller sheet) = Stress-reduction ratio
0.15	46	35	0.76
0.2	36.3	28	0.77
0.25	30.5	24	0.78
0.3	26.6	21.5	0.8
0.35	23.7	19.7	0.83
0.4	21.4	18	0.84
0.45	19.5	16	0.82
0.5	17.8	14.7	0.82
0.55	16	13	0.81
0.6	14.2	11.5	0.81
0.65	12.5	10	0.8
0.7	10.7	8.5	0.79
0.75	8.8	7.0	0.79
0.8	7.0	5.5	0.78
0.85	5.5	4.0	0.73
0.9	3.7	2.8	0.76

Using equation (5) and putting r equal to 0.8 and n equal to 2, we express the failing stress σ_x for a sheet of any width x in the form

$$\frac{\sigma_x}{\sigma_o} = 0.8^{\left(\frac{\log x/10}{\log 2}\right)} , \qquad (10)$$

where σ_o is the corresponding failing stress for the smaller sheet.

Thus, for a three-to-one increase in size of sheet, the failing stress falls to

$$\frac{(\sigma)_{30 \text{ in.}}}{(\sigma)_{10 \text{ in.}}} = 0.8^{(\log 3/\log 2)} = 0.695 \qquad (11)$$

of that for the smaller sheet.

This stress-reduction ratio for D.T.D.746 for a scale-up in sheet size of 3:1 may be compared with the corresponding reduction ratio of 0.73 for sheet material 2024-T3 and 0.6 for 7075-T6. This makes the scale effect for D.T.D.746 slightly more pronounced than for 2024-T3 sheet material, but much less pronounced than for 7075-T6.

2.3 Reduction of failing stress due to increasing crack length in wide sheet

As suggested in paragraph 2.1, an increase in length of a short crack in a large sheet is approximately equivalent to the same percentage increase in a crack of any length in an infinite sheet. In either case the stress at points remote from the crack is sensibly unchanged and so would be the stress concentration were the crack-end radii scaled up in proportion. It follows that doubling the size of a short crack in a large sheet should have much the same effect in reducing the failing stress as doubling the size of crack and sheet at the same time. This point can be checked against the experimental results by noting, for example, the effect of increasing a short crack length in the 35 in. wide sheet to Specification 7075-T6 whose failing stress against crack length is given in Table 3 of Ref. 4.

From this it is seen that doubling the crack-length) which always includes the central hole) from 0.05 of the sheet width to 0.1 drops the failing stress from 51 per cent to 38 per cent of $\sigma_{ult.}$ i.e. to 0.74 of its value before extension. The corresponding effect of scaling up both sheet and crack in the ratio 2:1 is seen from equation (8) to be

$$(0.6)^{\left(\frac{\log 24/12}{\log 3}\right)} = 0.72 \qquad (12)$$

The two figures are, therefore, in fair agreement.

A similar calculation can be made for D.T.D.746 on the basis of the results quoted in Table 3. Taking the 40 in. × 20 in. sheet and the two crack lengths 0.2 and 0.25 of sheet width, we find the failing-stress-reducing factor to be given by

$$\left(\frac{24}{28}\right)^{\frac{\log 2}{\log 0.25/0.2}} = 0.63 \qquad (13)$$

for a doubling of the crack length.

This compares with the value 0.77 in the last column of Table 3 obtained by scaling up sheet and crack together. The lower figure is to be expected in view of the reduced net cross-section.

2.4 General remarks on cracks in flat sheet

It seems fair to conclude from the above observations that

(a) In a sheet subjected to a uniform tensile stress along edges that are parallel to a central crack the value of the failing stress progressively drops as the size scale increases, i.e. as the length of the crack and the linear dimensions of the sheet are increased in the same ratio. This is most clearly shown when sheets of the same shape but different size are tested.

(b) Theoretically an increase in the length of a crack in a sheet of infinite size is equivalent to an increase in scale and should lead to the same drop in failing stress. This seems to be borne out by the few tests that have been made on short cracks in large sheets.

(c) The scale effect noted in (a) above, once determined for a particular scale ratio, can be found at once for any other scale ratio by a simple formula so long as the sheets are of the same material.

(d) The magnitude of the scale effect depends upon the specification of the sheet material. It is somewhat greater for D.T.D.746 than for 2024-T3 and much greater for 7075-T6.

(e) It seems reasonable to suppose that the cause of the scale effect here noted is the fact that, as the crack length and sheet size are scaled up proportionately, the sharpness of the crack extremities (i.e. the crack-end radius) remains unchanged and, therefore, in relation to the crack length, becomes more pronounced.

3 CRACKS IN PRESSURISED CYLINDERS

In passing from flat sheet under uniform tensile stress to circular cylinders under the hoop stress caused by internal pressure we need to consider two distinct aspects of the matter. In the first place we are interested in comparing the behaviour of cracks in cylinders of the same shape, but different size in order to see whether the same scale effect is present as that already noted for flat sheet. In the second place is the problem of correlating the behaviour of flat sheet under tensile stress with that of the same sheet formed into a cylinder under an equal hoop stress.

The experimental data necessary for discussing these matters is contained in a paper by Peters and Kuhn[2] who carried out tests on some fifty-eight unstiffened cylinders made up of sheet to Specification 2024-T3 and 7075-T6.

3.1 Failing stresses in cylinders of different sizes

A salient conclusion from the experiments carried out by the above authors is that sheet thickness is not a relevant parameter so long as the hoop stress is kept constant.

They varied the skin thickness of a particular cylinder over the range 0.006 to 0.025 in. without changing any other parameter than the internal pressure, which was adjusted so as to maintain a constant hoop stress, and found the nominal failing stress to be practically unaffected. It can be concluded from this that the cylindrical sheet in these experiments must have been approximately under 'plane-stress' conditions.

Two sizes of cylinder were used, one 3.6 in. radius and 20 in. long and a larger cylinder 14.4 in. radius and 74 in. long. Thus the ratio of the two radii - four to one - was slightly different from the ratio of the lengths, which was 3.72 to 1. If we can legitimately assume this small discrepancy in the length ratio to be unimportant, we can compare the behaviour of cracks of length ratio 4 to 1 and expect to obtain identical results except for the scale effect introduced by the constant crack-end radius.

On plotting the experimental results given in Ref. 2, it is found that for each cylinder size and each material a fairly smooth curve can be drawn through the plotted points. The following tables have been obtained by reading off from these smooth curves.

TABLE 4

(Sheet material 2024-T3)

Unstable crack length (in.)		Failing hoop-stress $lb/in.^2 \div 10^3$		Stress-reduction ratio
Cylinder diameter 3.6 in.	Cylinder diameter 14.4 in.	Cylinder diameter 3.6 in.	Cylinder diameter 14.4 in.	
0.3	1.2	40	30.5	0.8
0.5	2	30.6	24.0	0.78
1	4	19.5	14.0	0.72 (Average 0.77)
1.5	6	13.3	10.0	0.75
2	8	9.7	8.0	0.82

TABLE 5

(Sheet material 7075-T6)

Unstable crack length (in.)		Failing hoop-stress $lb/in.^2 \div 10^3$		Stress-reduction ratio
Cylinder diameter 3.6 in.	Cylinder diameter 14.4 in.	Cylinder diameter 3.6 in.	Cylinder diameter 14.4 in.	
0.3	1.2	32.5	24.5	0.75
0.5	2	25	18	0.72
0.75	3	19	12.5	0.66 (Average 0.7)
1	4	15	10	0.67
1.25	5	12.2	8.5	0.7

We note from these tables that scaling up cylinder size and crack length in the same ratio has the effect of reducing the nominal failing hoop-stress in both materials, the reduction being more pronounced for Specification 7075-T6. This is in accordance with what has already been observed and recorded in Tables 1 and 2 for flat sheets. There the average stress-reduction ratios for 2024-T3 and 7075-T6 materials and a size scale of 3:1 are 0.73 and 0.6 respectively. Using equation (6) we find the corresponding stress-reduction ratio for a 4:1 size scale for these two materials to be $0.73^{\log 4/\log 3}$ and $0.6^{\log 4/\log 3}$ i.e. 0.68 and 0.53 respectively. These are somewhat lower than the ratios (0.77 and 0.7 respectively) obtained above for the same sheet material in cylindrical form. It must be remembered, however, that apart from the fact that the flat sheet specimens had central holes, they were also without the buckling-preventing action of the longitudinal tension present in the cylindrical specimens.

4 RELATION BETWEEN THE FAILING TENSILE STRESS IN A FLAT SHEET AND THE FAILING HOOP STRESS IN A CYLINDER WHEN A CRACK CUTS ACROSS THE DIRECTION OF TENSION

It has been shown above that, once we have obtained by experiment the stress-reduction ratio for two similar flat sheets (of the same material) with similar cracks, we can estimate the failing stress in any similar sheet whatever its size. The failing hoop stress of a pressurised cylinder can be found in the same way based on the experimental values found by testing two similar cylinders of different sizes.

What has not so far been discussed is the second problem mentioned in para. 3 above, i.e. the relation between the failing tensile stress of a flat sheet with a given length of crack and the failing hoop stress of the same sheet (with the same crack) rolled up into a cylinder and subjected to internal pressure. At first glance these two stresses might be expected to be equal. In point of fact, however, as Peters and Kuhn have shown in Ref. 4, they are far from equal. These authors found that, as the length of the flat sheet - and hence the circumference of the corresponding cylinder - was reduced, the failing hoop stress in the cylinder fell far short of the failing tensile stress in the flat sheet. They were unable to explain this phenomenon and expressed the opinion that "the strong effect of curvature is not explained by known theory", and that "the physical nature of the curvature correction is obscure at present".

They failed to point out that, although they were unable to explain this strong effect of curvature, it was, nevertheless, an effect to be expected. For it is known from straightforward dimensional theory that, if two similar structures of different size are subjected to similarly distributed loads, the stresses induced are identical however complicated the structures may be. It follows that two similar cylinders with similar discontinuities - circular holes for example - will have identical stresses under equal internal pressures. Thus if each linear dimension - length, diameter, sheet thickness and diameter of circular hole - of the larger cylinder is n times that of the smaller, the stresses should everywhere be identical under the same pressure; but, on our assumption of 'plane stress' in the sheet, the stresses in the larger cylinder will remain unchanged if we reduce its sheet thickness in the ratio $1/n$ and reduce its internal pressure in the same ratio. We have now two cylinders, with the same sheet thickness and the same hoop tension, in which, according to dimensional theory, the stresses are identical in spite of the fact that the larger cylinder has a hole diameter n times that of the other. Thus, in comparing the results for the two sizes of cylinder, the authors should have expected the same failing stress not for cracks of the same length but for cracks proportional to cylinder size.

If the experimental results obtained by them for the two sizes of cylinders tested are compared on the latter basis, the strong curvature-effect they refer to disappears, leaving only the comparatively small discrepancy consequent upon the incorrect scaling of the crack-end radii. The result according to this, is that curvature has a small beneficial effect rather than a large deleterious effect on the nominal failing stress.

The above attempt at clarification is, it may be noted, of little assistance in solving the problem of correlating flat-sheet results with those for corresponding cylinders (i.e. cylinders made by rolling up the flat sheet). For, in order to deduce the nominal failing stress for a cylinder of infinite radius (representing the flat sheet case) from the results for cylinders of finite size, we have necessarily to contemplate cracks of infinite length in the infinite cylinder.

To overcome this difficulty use can be made of a simple empirical formula evolved by Peters and Kuhn to correlate experimental results for flat sheets with the results they obtained for their two cylinder sizes.

According to this formula the stress concentration at the end of a crack in a cylindrical sheet is given by

$$\sigma = \sigma_f \left(1 + \frac{k\ell}{r}\right) \qquad (14)$$

where σ_f = stress concentration in the corresponding flat sheet (i.e. the resultant stress at the crack-end)

ℓ = length of crack

r = radius of cylinder

k = empirical constant

For materials 2024-T3 and 7075-T6 they found the empirical constant k to have the same value, i.e. 4.6. At failure the stress given by equation (14) must equal the ultimate stress of the material $\sigma_{ult.}$ so that we write

$$\sigma_{ult.} = \sigma_o \left(\frac{\sigma_f}{\sigma_o}\right)\left(1 + \frac{k\ell}{r}\right) \qquad (14a)$$

where σ_o is the nominal applied stress (i.e. average stress on gross area of cross-section) at failure. It follows that

$$\sigma_o = \frac{\sigma_{ult.}}{\left(\frac{\sigma_f}{\sigma_o}\right)\left(1 + \frac{k\ell}{r}\right)} \qquad (14b)$$

Thus, if we know the stress-concentration factor $\left(\frac{\sigma_f}{\sigma_o}\right)$ for the flat sheet and also know the value of the constant k we can, for any length of crack and any radius of cylinder, derive the failing hoop stress from the failing tensile stress of corresponding flat sheet obtained by opening out the cylinder to form a plane sheet.

As already mentioned, formula (12) is purely an empirical relation that happens to fit the experimental results obtained for the two materials 2024-T3 and 7075-T6. Having no rational theoretical basis it cannot be applied with any confidence to other materials. Indeed Peters and Kuhn, on the basis of certain tests carried out by Griffith[6] on glass bulbs, issue a warning that formula (12) may be of limited scope and "should not be applied to other materials without check tests for verification".

Among the purposes of this note is to put formula (14) on a rational basis and so enable one to answer the question whether the constant k is likely or not to have much the same value for all structural materials.

The essence of relation (14) is that the added stress concentration around the crack extremities as a result of converting a flat sheet into the corresponding pressurised cylinder is equal to $\left(\frac{k\ell}{r}\right)$ times that in the flat sheet, i.e. it is directly proportional to the length ℓ of the crack and inversely proportional to the radius r of the cylinder. The following argument, in conjunction with the analysis given in the Appendix, in the first place demonstrates that this experimentally obtained relation is in accordance with theoretical considerations. In the second place it demonstrates that the value of the constant k should be much the same for all materials. This second conclusion is in conflict with Peters' and Kuhn's interpretation of Griffith's experiments on glass bulbs - a point that will be discussed later.

4.1 Basic argument of analysis in Appendix

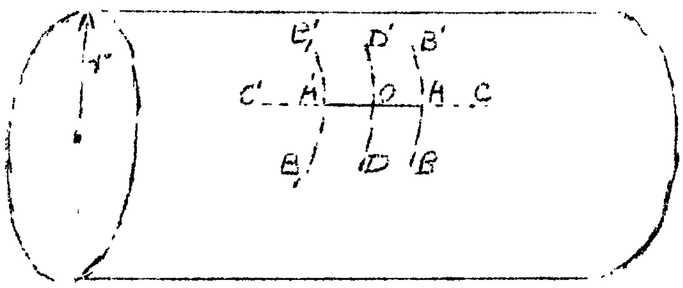

Fig. 3

Fig. 3 shows part of a cylinder of radius r under an internal pressure p_o. Before the crack A'OA appears the hoop tension is constant everywhere and has the value $p_o r$ per unit length of generator. Imagine now that a crack AA' is made in the skin, but that the hoop tension originally present across the two edges of the crack is maintained by an external agency so that (assuming no leakage) the hoop tension is nowhere changed.

We now assume a rigid sleeve to envelop the cylinder, which sleeve, while constraining the skin in the region of the crack against any radial expansion, allows free circumferential displacement. In the presence of this constraint the external agency maintaining the original hoop tension across the edges of the crack is next removed. In other words we apply a hoop compression $p_o r$ (per unit length) across the edges of the crack, which compression, by diffusion into the surrounding area, diminishes in value with distance from the edges in the same way as in a flat sheet. As a result of this and the prevailing curvature of the sheet a radial pressure amounting to p_o in the immediate vicinity of the free edges of the crack - and progressively smaller pressures with increasing distance from the crack - is applied to the constraining cylindrical sleeve, which may be considered kept in balance by a uniformly low pressure over the opposite side of the cylinder. The 'plane stress' distribution over the disturbed region BB'C'B₁'B₁C (including the stress concentrations at A and A') is now identical with that in the corresponding* flat sheet under an applied tension $p_o r$ per unit length and a cross tension of $p_o r/2$.

To obtain the additional stresses present in the cylindrical sheet, but absent from the corresponding flat sheet, we need now to find the effect of removing the constraining enveloping sleeve** while the original pressure p_o and the compressive forces at the crack edges are still maintained. Removing the sleeve, however, is equivalent to applying an additional internal pressure over the region BB₁B₁'B' of the same amount as the external pressure previously applied by the sleeve. This additional pressure is everywhere in direct proportion to the maximum value of such pressure, namely p_o close to the crack edges. In the absence of continuity of the sheet across the arcs AB, A'B' the area ABA'B' would fold back about the line BB' without any resistance. As it is, the pressure load on this area induces an additional hoop tension force that, in the absence of the crack, would be carried straight across the line AA'. Because of the crack, however, the hoop load across AA' has to be by-passed across AC and A'C' in the same way as the original hoop tension force $p_o r \ell$ already treated.

We know from general principles of stress diffusion that, in the flat sheet case (i.e. with the radial constraint still operative), the stress disturbance produced by the crack extends in the circumferential direction a distance proportional to the length ℓ (say) of the crack. It follows from what has been said above that the excess pressure (induced by removal of the constraint) extends the same distance. This means that the total load due to

---

* By 'corresponding flat sheet' we mean the sheet obtained by opening out the cylindrical sheet after cutting all along the generator diametrically opposite the crack.

** It is interesting to note that there is an allusion in Ref. 3 to some experiments carried out by the Douglas Aircraft Co. in which the effect of such a sleeve was measured in terms of resistance to fatigue. The following is quoted from Ref. 3 (para. 27). "In one unpublished test by Douglas Aircraft Co., a crack in a cylindrical specimen was covered by a plexiglass sheet to prevent excessive bulging of the crack lips. Without the plexiglass radial support to the edges of the crack a specimen failed at 2247 cycles at a hoop stress of 9130 p.s.i., but with the support a similar specimen at the same stress took 39,875 cycles before failure. This gives further indication that the problem in a cylindrical structure under pressure is very different from a flat tension specimen containing a crack".

the excess pressure over the region $BB'B_1B_1'$ is proportional to ℓ^2 and to the maximum value p_o of the excess pressure. We can therefore write

$$\text{Excess pressure load} \qquad Q = k_1 p_o \ell^2 \tag{15}$$

where k_1 is a factor that, depending as it does on the pressure distribution over the region $AA_1B_1'B_1$ (Fig.3), is directly dependent on the pressures on the constraining sleeve and hence on the 'flat-sheet' stress distribution over that region. It is dependent therefore on the ratios ℓ/r and r/L of crack-length to radius and of radius to cylinder length L.

It has been shown in the Appendix - equation (38) - that, if a sheet ring of width b and radius r_o under internal pressure p_o is further subjected over an arc $r_o\theta$ equal to ℓ in length, where θ is small (< 40° say) compared with 180°, to a radial load Q no greater than $b\,\ell p_o$ (equilibrated by a uniform pressure over the opposite half of the ring) the extra pull or hoop load in the ring is equal to $Q/2$.

If now, in Fig.3, we imagine the arcs BB_1 and $B'B_1'$ extended into complete circles, they may be considered as representing the edges of a sheet ring of width ℓ subjected to a uniform internal pressure p_o and locally to an excess pressure load Q, that, according to equation (15), is directly proportional to $p_o\ell^2$. Since the ring is broken along the crack AA' the pressure load Q, as already explained, is transferred to the adjoining areas beyond the boundaries of the "ring", and there gives rise to a total hoop force across AC and A'C' of amount, according to the above argument, equal to $Q/2$. This is in addition to, but distributed in a way very different from, the hoop force $p_o r \ell$ that is distributed over AC and A'C' as in a flat sheet and that also must be by-passed.

The maximum hoop tension due to each occurs at A (Fig.3). That due to the force $p_o r \ell$ may be written in the form

$$(T_A)_1 = F\left(\frac{\ell}{r}, \frac{r}{L}\right) \cdot p_o r \ell \div \ell \tag{16}$$

where $F\left(\frac{\ell}{r}, \frac{r}{L}\right)$ is a function of the length ℓ of the crack as a fraction of the radius and of the ratio of cylinder radius to cylinder length. Division by ℓ is necessary because the length AC (and A'C') over which the hoop force is by-passed is proportional to ℓ.

The corresponding hoop tension caused by the hoop force Q, which has a very different distribution* over AC, may similarly be written in the form

$$(T_A)_2 = f\left(\frac{\ell}{r}, \frac{r}{L}\right) \cdot p_o \ell^2 \div \ell \; . \qquad (16a)$$

The total hoop tension at A is thus

$$T_A = (T_A)_1 + (T_A)_2 = F\left(\frac{\ell}{r}, \frac{r}{L}\right) p_o r \left\{ 1 + \left(\frac{\ell}{r}\right) \frac{f\left(\frac{\ell}{r}, \frac{r}{L}\right)}{F\left(\frac{\ell}{r}, \frac{r}{L}\right)} \right\} . \qquad (17)$$

As already noted the resultant pressure-load Q is the integral of pressures whose distribution over the region BB'B$_1$'B$_1$ (Fig.3) (by removal of the constraining sleeve) is identical with the distribution of 'flat sheet' compression forces (with constraining sleeve present and compression forces applied to the edges of the crack). On this basis the ratio $f\left(\frac{\ell}{r}, \frac{r}{L}\right) / F\left(\frac{\ell}{r}, \frac{r}{L}\right)$ can be regarded as a constant k independent of both ratios ℓ/r and r/L. Equation (17) may thus be written in the form

$$T_A = (T_A)_{\text{flat sheet}} \left(1 + \frac{k\ell}{r}\right) \; . \qquad (18)$$

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* This 'very different' distribution is caused (in the writer's opinion) by the quite different mechanisms by which the 'flat-sheet' load $p_o r \ell$ and the local pressure load Q are transmitted round the crack. The former takes place by familiar stress diffusion in the plane of the sheet, but the latter, since bending of the sheet is a negligible factor, must be transmitted by membrane forces that perforce must be directed at an angle to the crack direction and therefore in a direction along which the sheet curvature is small (depending as it does on the square of the sine of that angle). Only thus can the very large value of the experimentally derived constant k (i.e. 4.6) be explained.

Since the stress concentration at the crack-end A is directly proportional to the hoop tension at A

$$\sigma_A = (\sigma_A)_{\text{flat sheet}} \left(1 + \frac{k\ell}{r}\right) . \qquad (19)$$

When the pressure p_o in the cylinder reaches bursting value σ_A becomes equal to the failing stress σ_{ult} for the sheet material, so that, under these conditions,

$$\sigma_{ult} = \sigma_o \left(\frac{(\sigma_A)_{\text{flat sheet}}}{\sigma_o}\right) \left(1 + \frac{k\ell}{r}\right) \qquad (20)$$

where $\sigma_o = \frac{pr}{t}$ = nominal hoop stress in cylinder,

and t = sheet thickness.

Thus

$$(\sigma_o)_{\text{at failure}} = \frac{\sigma_{ult}}{\left\{\frac{(\sigma_A)_{\text{flat sheet}}}{\sigma_o}\right\}\left(1 + \frac{k\ell}{r}\right)} . \qquad (20a)$$

This is identical with the empirical relation (12b) deduced by Peters and Kuhn directly from experiment, and in which they found k to have the value 4.6.

An important point to note in equation (19) is that the expression for σ_A is independent of the size of the cylinder and remains the same so long as the ratio ℓ/r is the same. This is because the scale effect is completely taken account of in the leading factor $(\sigma_A)_{\text{flat sheet}}$. A further point that is fundamental in regard to this equation is that any effect of sheet material on the value of σ_A is also taken account of in that factor. That is why the constant k is independent of sheet material.

4.2 Effect of axial tension in pressurised cylinders

The biaxial character of the stress in a pressurised cylinder does not affect the value of the constant k. This is because it is derived as the result of comparing a cylinder under biaxial stress with a flat sheet under the same biaxial stress. What we are after however is the relation between a flat sheet in simple tension and the corresponding pressurised cylinder under the same (hoop) tension and an axial stress of half that amount.

For a constant stress ratio (here 2:1) we can express the stress concentration at the crack end A in a flat sheet in the form

$$\sigma_{A(\text{biaxial stress})} = k' \, \sigma_{A(\text{simple tensile stress})} \qquad (21)$$

with the result that equation (19) becomes

$$(\sigma_A)_{cyl.} = (\sigma_A)_{\text{(flat sheet biaxially stressed)}} \left(1 + \frac{k\ell}{r}\right)$$

$$= k' \, (\sigma_A)_{\substack{\text{flat sheet} \\ \text{simple tensile}}} \left(1 + \frac{k\ell}{r}\right) \quad . \qquad (22)$$

Because, however, the longitudinally applied stress gives rise to practically no stress concentration at the crack end, and what stress it does produce normal to the crack direction relieves rather than accentuates that due to the main tensile stress, the value of k' is very nearly, but slightly less than, unity. The result is that, on introducing the factor k' in the denominator of equation (20a) we find the applied failing stress σ_o to be slightly increased in the ratio $1/k'$. By assuming k' to have unit value, i.e. by deriving the constant k by directly comparing the cylinder failing stress with that of the corresponding flat sheet in plain tension, we ensure that formula (20a) gives a slightly conservative estimate of the failing stress for a cylinder.

5 POINTS CLARIFIED BY THE THEORETICAL TREATMENT

With the empirical formula of Peters and Kuhn now established on a theoretical basis, it becomes possible to derive certain conclusions that were previously inadmissible.

The main conclusion is that the constant k, already found by Peters and Kuhn to be identical for two widely different aluminium alloys 2024-T3 and 7075-T6, is likely to be the same for all structural materials and for all sizes of cylinder. This view is put forward in spite of the results quoted by the above authors from the work of Griffith[6] on glass bulbs and tubes. They found the results for glass tubes to be somewhat inconclusive because of the small range of tube diameters covered. In Griffith's experiments on glass-bulbs, however, in which $\frac{\ell}{r}$ varied from 0.2 to 0.9, they found the quantity $\left(1 + \frac{k\ell}{r}\right)$ to differ from unity by only 4 per cent at most, so giving, by equation (20a), a nominal failing stress σ_o only 4 per cent less than that for the corresponding flat sheet. If k had the same value, namely 4.6, for glass as for the two aluminium alloys, the factor $\left(1 + \frac{k\ell}{r}\right)$ would amount to $(1 + 4.6 \times 0.9)$ or 5.14, which, if used in equation (20a) gives a nominal failing stress of about $1/5$ of that for the corresponding flat sheet. In other words, to make the formula agree with experiment, the value of k for glass should be $(0.04 \div 5.14)$ or less than one-hundredth of its value for the two alloys.

There is a simple explanation to account for this apparent discrepancy and hence to confirm the view that the factor k for glass is unlikely to be different from that for any other structural material.

It will be recalled that the hoop tensions produced in sheet cylinders, extra to those in the corresponding flat sheet, came about as a result of removing the enveloping sleeve that resisted the outward radial forces caused by releasing the tensile forces holding the edges of the crack together.

Consider now what happens when the pressurised vessel takes the form of a bulb - a spherical bulb let us say. As before, if the edges of the crack are held together by an outside agency the hoop tension - originally uniform over the whole surface remains unchanged, but at a value half that of the cylinder of the same radius. This is because the pressure in the bulb is resisted half by the hoop tension across the direction of the crack and half by the hoop tension in the same direction as the crack.

Before removing the external constraint holding the crack edges together we introduce, as before, an enveloping (now spherical) surface to prevent any radial displacement consequent upon the local internal pressure caused by applying the compressive edge stresses required to cancel the external constraint. The important point, however, is that <u>this local pressure</u> - unlike the corresponding pressure in the cylinder - <u>never comes into action so far as the enveloping surface is concerned.</u> What happens is that, as the hoop tension across the crack goes out of action the hoop tension in the direction of the crack is doubled. Instead of the hoop tensions in the two directions taking equal shares of the pressure, the hoop tension in the direction of the crack now takes it all - accompanied moreover by radial displacements at the crack edges that are negligible compared with those at the crack edges of the corresponding cylinder. In other words there are no extra stresses round the crack ends caused by the local pressure induced by removal of the constraining surface. The stress concentrations around the crack ends are, therefore, practically the same as those for the parallel case of the flat sheet.

6 CONCLUSIONS

The conclusions to be drawn from the above work (and the references mentioned) may be summarised as follows.

6.1 The failing stress of thin sheet under tensile forces across a crack is largely independent of its thickness whether the sheet is flat or constitutes the skin of a pressurised cylinder. It depends heavily, however, on the sheet material.

6.2 Comparison of results for similar flat sheet specimens (same planform, and crack lengths proportional to the linear dimensions) shows that there is a scale effect - different for different materials - that makes the larger specimen fail at a lower applied stress than the smaller.

6.3 That it is a true scale effect is indicated by the shape of the curve that has applied stress at failure for ordinate and crack length as abscissa. When the crack is still short compared with the width of the sheet, so that the stresses at distances from the crack large compared with its length are relatively unaffected as the crack extends, the curve follows closely that given by equation (4).

6.4 The amount of this scale effect is readily obtained by comparing the applied stresses at failure of two similar sheet specimens (with similar crack lengths) that are substantially different in size. Once this is obtained for a particular sheet material the applied stress at failure of any size of sheet (of the same shape and material) can be estimated from a formula such as (4).

6.5 A similar scale effect occurs in pressurised sheet cylinders with cracks. Once this is evaluated by comparing two similar cylinders (with similar cracks) the stress at failure for any size of (similar) cylinder and any length of crack can be estimated from the curve of stress at failure against crack-length for one of the original cylinders.

6.6 For corresponding flat sheet and cylindrical specimens the scale effects referred to in (6.4) and (6.5) above should theoretically be the same for the same material.

6.7 The relation (12b) between the nominal stress at failure for a flat sheet in plain tension and for the same sheet rolled up to form a pressurised cylinder with the same hoop tension and same crack length was obtained empirically by Peters and Kuhn[2]. The theoretical basis for this relation (based on the analysis in the Appendix) has here been established and makes it possible to judge whether the empirical constant k included in formula (12b) is likely to vary appreciably from one material to another.

6.8 On the basis of the theoretical considerations discussed above one would not expect the empirical constant k in equation (12b) to vary much from one structural material to another. The results quoted by Peters and Kuhn[2] from Griffith's work on glass tubes and bulbs, which seem at first sight to indicate that the value of k for glass is less than one-hundredth of that for aluminium alloys, have here been explained and the apparent small value of k shown to be caused by the double curvature of the glass bulbs. For glass tubes there is no reason to suppose that k has a different value from that of any other material. The constancy of the factor k widens the scope of the formula and therefore greatly enhances its usefulness.

6.9 As a result of the above points the curve of failing stress against crack length for a large unstiffened cylinder can be derived from that of a similar small cylinder by using the scale effect for the particular material, and the curve for the small cylinder can in turn be derived from that of the 'corresponding' flat sheet by using formula (20a). The result is that the behaviour of a large cylinder can be estimated from that of a small flat sheet. An alternative procedure would be to use formula (20a) to derive first the failing stress of the large cylinder from that of the 'corresponding' flat sheet and secondly to derive the failing stress of the latter from that of a similar smaller flat sheet by using the appropriate scale effect.

6.10 As stated in the introduction the main purpose of this note is to enable tests on crack-propagation in sheet metal structures to be carried out with greater economy - to enable the nominal stress at failure for large flat sheets and cylinders to be deduced from results, obtained quickly and cheaply, for small sheets and cylinders, and to enable results for cylinders to be obtained from tests on flat sheets. The formulae derived above would seem to go some way towards achieving this end.

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APPENDIX

EFFECT OF LOCAL INCREASE OF RADIAL PRESSURE ON LOCAL HOOP TENSION

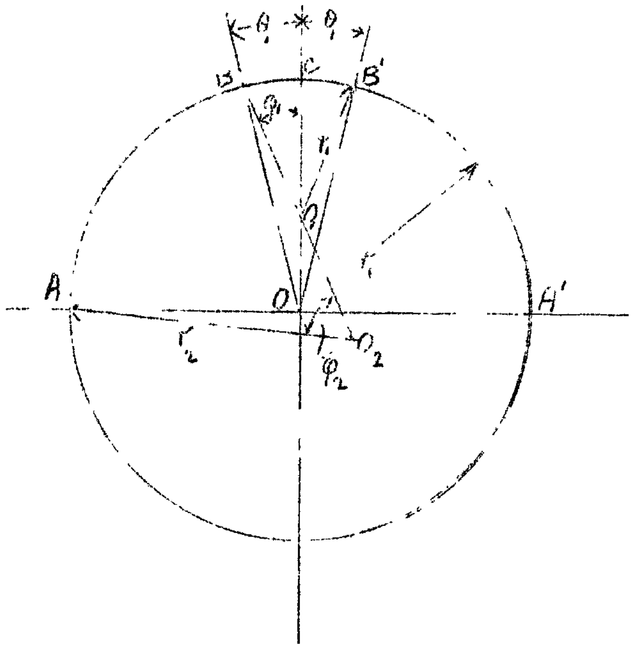

Fig. A.1

We consider first the hoop tensions in a unit length of a pressurised cylinder made of thin sheet. We treat, in other words, the two-dimensional problem of the pressurised ring of unit width. The upper half of this is represented in Fig.A.1 by the semi-circle ACA'. The ring is initially under an internal pressure, and hence a force per unit arc of p_o, and the problem is to determine the new hoop tensions introduced as a result of applying a further pressure $(p_1 - p_o)$ of about the same magnitude as p_o over the arc BB' that makes the resultant local pressure equal to p_1.

It will be seen later that the conclusions reached from this analysis are not affected by the simplifying boundary condition that the points A and A' are fixed in position. It will also be shown later that it is legitimate to neglect the bending stiffness of the sheet forming the ring, which may, therefore, be regarded as an inextensible membrane.

Let r_o = original radius of curvature

= radius of curvature under uniform pressure p_o

θ_1 = angle BOC

r_1 = new radius of curvature for arc BB' with centre of curvature at O_1

ϕ_1 = angle subtended by arc BB' at its centre O_1 (making $r_1\phi_1$ equal to $r_o\theta_1$)

r_2 = new radius of curvature over arc AB (and A'B')

ϕ_2 = angle subtended by arc AB at its centre O_2

p_o = original pressure in cylinder

$(p_1 - p_o)$ = additional pressure applied over the central arc BB'

From equilibrium considerations the centre O_2 must lie on the same straight line as the radius BO_1.

By symmetry it is enough to consider the quadrant AC alone. The four variables r_1, r_2, ϕ_1, ϕ_2 may be evaluated from four equations, which are derived as follows

To keep the peripheral length constant we have

$$r_1\phi_1 + r_2\phi_2 = r_o \frac{\pi}{2} \quad . \tag{23}$$

For equilibrium at the junction B of the two arcs

$$p_o r_2 = p_1 r_1 \quad , \tag{24}$$

and from geometrical considerations

$$r_1\phi_1 = r_o\theta_1 \quad . \tag{25}$$

Because A and A' are fixed in position, we have the further geometrical condition

$$r_2(\sin \overline{\phi_1 + \phi_2} - \sin \phi_1) + r_1 \sin \phi_1 = r_o \sin \frac{\pi}{2} = r_o \, . \tag{26}$$

Making use of equations (23), (24) and (25) in (26) we obtain the following equation in the single variable ϕ_1 :-

$$\frac{p_1}{p_o} \{\sin \overline{(1 + c)\ \phi_1} - \sin \phi_1\} + \sin \phi_1 = \frac{\phi_1}{\theta_1}$$

or

$$\frac{p_1}{p_o} \cdot \frac{\sin \overline{(1 + c)\ \phi_1}}{\phi_1} - \left(\frac{p_1}{p_o} - 1\right) \frac{\sin \phi_1}{\phi_1} = \frac{1}{\theta_1} \quad , \tag{27}$$

where

$$c = \left(\frac{\frac{\pi}{2} - \theta_1}{\theta_1} \cdot \frac{p_o}{p_1}\right) \quad . \tag{28}$$

For any given value of θ_1 this equation can readily be solved by trial and error. We take the case in which we are particularly interested* where p_1 is twice p_o and θ_1 has various values. For $\theta_1 = 10^o$.

We find that

$$\left.\begin{array}{llll} \phi_1 = 17^o, & r_1 = 0.589\ r_o \\ \phi_2 = 68^o, & r_2 = 1.178\ r_o \end{array}\right\} \quad . \tag{29}$$

---

* See para. 4.1 of main text.

The tension in the ring is therefore

$$T = p_o r_2 = p_1 r_1 = 1.178\, p_o r_o \quad . \qquad (30)$$

The effect of doubling the pressure over the central 20° of arc is thus to increase the original tension $p_o r_o$ by the fraction 0.178.

It is important to note that the angle which the tangent at A makes with the horizontal is given by

$$(\phi_1 + \phi_2) = 85^\circ \qquad (31)$$

i.e. only 5° off the vertical. This suggests at once an easy approximate way for finding the new tension in the ring. We merely assume that the tangent at A makes 90° instead of the actual 85° with the horizontal and that the change in the projection of the central arc BB' on the horizontal is negligible (i.e. that $r_1 \sin\phi_1 \sim r_o \sin\theta_1$).

In the present case the one assumption introduces an error that is the difference between sin 90° and sin 85° i.e. an error of 0.4 per cent. The other assumption involves the very slight differences before and after applying the extra pressure $(p_1 - p_o)$ in the projections on the horizontal of the arcs AB and BC. With these assumptions, the condition for equilibrium of the vertical forces may be written in the form

$$T = p_1 r_o \sin\theta_1 + p_o r_o (1 - \sin\theta_1) \qquad (32)$$

$$= p_o r_o (1 + \sin\theta_1), \text{ if } p_1 = 2p_o \quad , \qquad (32a)$$

or

$$T = p_o r_o (1.174) \quad . \qquad (32b)$$

Thus the approximate fractional increase in the hoop tension is 0.174 as against the "correct" value of 0.178, an underestimate of only 2¼ per cent. For $\theta_1 = 20^\circ$ and with p_1 still twice p_o we obtain from equation (27)

$$\left.\begin{aligned} \phi_1 &= 29.75^\circ, \quad r_1 = 0.673\, r_o \\ \phi_2 &= 52.1^\circ, \quad r_2 = 1.346\, r_o \\ (\phi_1 + \phi_2) &= \text{angle of tangent at A} = 82^\circ \end{aligned}\right\} \qquad (33)$$

$$T = p_o r_2 = p_1 r_1 = 1.346\, p_o r_o \qquad (34)$$

a fraction increase of 0.346.

The approximate method gives by equation (32)

$$T = p_o r_o (1 + \sin 20^o) = 1.342\ p_o r \ , \qquad (35)$$

which makes this fractional increase equal to 0.342 as against the "correct" value of 0.346 - an underestimate of $1\frac{1}{3}$ per cent. It is clear in fact that (assuming points A and A' fixed) the greater the extent of the loaded central arc the more near to the trial-and-error answer the value given by equation (32) becomes; when this arc covers the whole semi-circle the two values are, of course, identical.

Neglect of sheet bending stiffness justified

Since the change of curvature produced in the ring is now known the energy absorbed in bending the sheet is easily calculated and so also is the work done by the applied pressure. The relative magnitudes of these have been worked out for the above numerical examples and it is found that the bending energy constitutes a negligible proportion of the total. This fact justifies the original assumption that bending energy of the sheet may be neglected in the present problem.

What has been established so far is that, in the two-dimensional case of a ring the extra hoop-tension (per unit length of generator) caused by increasing the pressure from p_o to p_1 over an arc θ_1 of the ring is given, according to equation (32), by the approximate formula

$$(T - T_o) = (p_1 - p_o)\ r_o \sin \theta_1$$

$$= (p_1 - p_o)\ r_o\ \theta_1 \left(\frac{\sin \theta_1}{\theta_1}\right) \ , \qquad (36)$$

where T_o = original hoop-tension $p_o r_o$ before application of the excess pressure.

If, over the arc BB' of Fig.A1, the applied excess pressure $(p_1 - p_o)$ is not constant we can by the above adjustment still express the total pull P over the ring cross-section in the form

$$P \propto b(T_C - T_o)$$

$$\propto b \left\{(p_1)_C - p_o\right\} r_o \theta_1 \left(\frac{\sin \theta_1}{\theta_1}\right) \qquad (37)$$

where $(p_1)_C$ is the pressure (constant along the generator) at the mid-point C of the arc BB'. This equation differs from (36) in bringing in the width b of the ring and substituting the sign of direct proportionality for that of equality.

If, further, the angle θ_1 is small compared with 90° (less than 20°, say) equation (37) takes the form

$$P \propto b\left\{(p_1)_C - p_o\right\}\ell \quad , \qquad (38)$$

where

$$\ell = r_o\theta_1 \quad .$$

This relation, according to the argument developed here, holds equally well even though the pressure at C is not constant along the generator, so long as the excess pressure $\{(p_1)_C - p_c\}$, where $(p_1)_C$ now stands for the maximum value of p_1 in the generator through C, is no greater than p_o, which it is not in the practical case one is here interested in. This is because, under the stipulated conditions, the amount of the extra pull in the ring under a local radial load superposed on uniform internal pressure and equilibrated by a uniform reaction over the opposite half of the ring, is equal to half the amount of that load.

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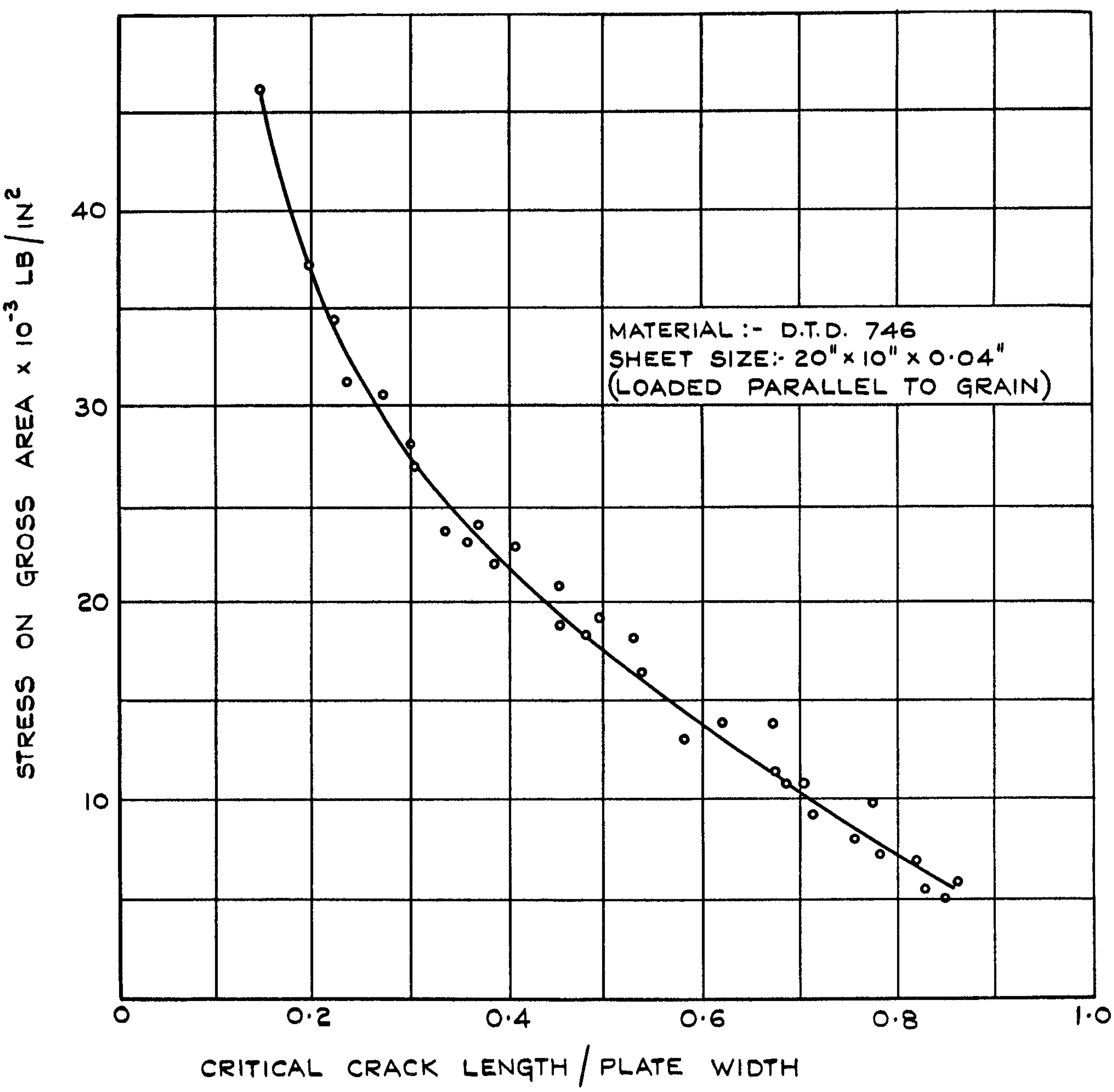

FIG. 1. BRISTOL TEST ON D.T.D. 746 SHEET. (1st SPECIMEN)

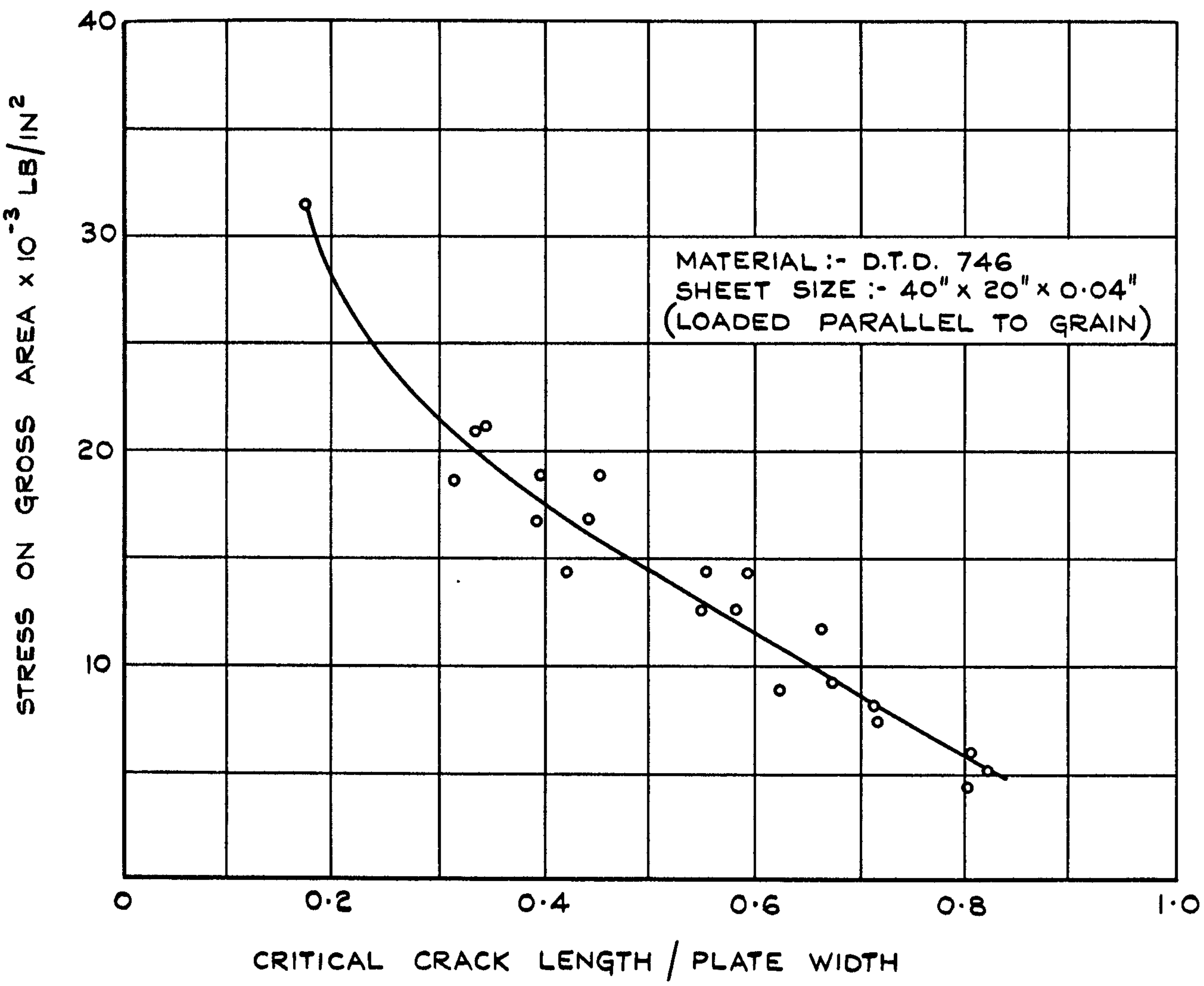

FIG. 2. BRISTOL TEST ON D.T.D. 746 SHEET (2nd SPECIMEN)

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