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Slender Not-So-Thin Wing Theory

by

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SLENDER NOT-SO-THIN WING THEORY

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J. C. Cooke, D.Sc.

SUMMARY

A method for making an approximate thickness correction to slender thin-wing theory is presented. The method is tested by applying it to cones with rhombic cross-sections and the agreement is found to be good if the cones are not too thick. It is then suggested that the thickness correction to slender thin-wing theory may be applied unchanged to linear thin-wing theory. This suggestion is compared with some experiments on delta wings and it is found that there is considerable improvement over thin-wing theory near the centre line, but that this improvement is not maintained as the wing tips are approached.

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1 INTRODUCTION

In order to calculate the flow of air past slender pointed wings the equations of motion are usually linearized, that is, the squares and products of the derivatives of the perturbation velocity potential are ignored. This method may be called "linear theory". A further simplification is usually introduced by applying the boundary conditions not on the wing surface but on a plane which is never far away from the surface of the wing. This we may call "linear thin wing theory".

Another approximation is "slender body theory" in which a term is dropped from the linearized equation of motion leaving the velocity potential φ to satisfy the equation

$$\frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{1}$$

at each station x = constant. With this we may also apply if we wish the same simplified boundary conditions as described above, and the result in this case is called "slender thin-wing theory". So far most calculations have used the second and fourth of these simplifications, namely "linear thin-wing theory" and "slender thin-wing theory". Both of these apply the boundary conditions in the same way, that is on some plane close to the surfaces of the wing, supposing that the wing is so shaped that it is possible to find such a plane. What one would like to do is to solve the linearized potential equation using the correct boundary conditions, since this would make the fullest use of the theory, which is still of course an approximation.

In this paper we do not do this directly, but we solve approximately the easier problem which we have called slender body theory. This provides a correction to slender thin wing theory, and it is suggested that by the principle of the "independence of small corrections" this correction may be applied to linear thin wing theory to give an improved solution of the linearized equation. That this is practicable at least in some cases may be shown by Fig.1, taken from Ref.1, which shows that, in the case of a thin slender ellipsoid in subsonic flow, the method gives better overall results than any of the other methods.

This paper gives an approximate solution of the slender body problem in supersonic flow. This is done by finding an approximate relation which transforms the wing section into a circle. Once this is done the problem is virtually solved, and the transformation may be improved by iteration if necessary, but we shall not do this here. The results are tested in the case of a cone with symmetric rhombic cross sections (for which the full slender body solution is known).

2 GENERAL

The wing is supposed to be close to the xy plane, the x axis being along wind or inclined to it at a small angle of incidence. We write

$$\zeta = y + iz \tag{2}$$

and suppose that the section of the wing by a plane x = constant is a curve symmetrical about the z axis. The velocity at infinity is V. Then to determine the perturbation velocity potential in slender body theory we must solve the equation (1) subject to the condition that

$$\frac{\partial\varphi}{\partial n} = V \frac{\partial z/\partial x}{\{1 + (\partial z/\partial y)^2\}^{\frac{1}{2}}} \tag{3}$$

on the boundary, where $z = z(x, y)$ is the equation of the wing surface[1]. In slender thin wing theory this is replaced by

$$\frac{\partial\varphi}{\partial n} = V \frac{\partial z}{\partial x} \tag{4}$$

at a point P' on some plane near to the surface of the wing. See Fig.2.

The ζ-plane is transformed into a T plane $(T = Y + iZ)$ such that the section becomes a circle of radius r. See Fig.3. If the section were a slit the transformation would be

$$\zeta = T + \frac{r^2}{T} , \tag{5}$$

and if the section differs only slightly from a slit the transformation will differ only slightly from this.

We shall consider two cases (both of which are symmetrical about the z-axis;) (1) the symmetrical case, when the section is symmetrical about the y-axis as well as the z-axis, and (2) the unsymmetrical case, corresponding to a cambered section. In both cases we consider only the case where the incidence is such that there is no flow around the edges.

If the transformation is known, then according to Weber[1,2] we shall have in the symmetrical case, if $s(x)$ is the semi-span at any section x = constant, and the flow is supersonic,

$$\varphi = \varphi_1 + \varphi_2 , \tag{6}$$

$$\varphi_1 = \frac{V}{\pi} \int_{-s}^{s} \frac{\partial z(x,y')}{\partial x} \log \frac{2|Y(y) - Y(y')|}{s} \, dy' , \tag{7}$$

$$\varphi_2 = \frac{V}{2\pi} \left\{ S'(x) \log \tfrac{1}{2} \beta s - \int_0^x S''(x') \log(x - x') \, dx' \right\} . \tag{8}$$

In equation (8) the function $S(x)$ is the area of a section of the wing, its derivatives are denoted by primes, and

$$\beta^2 = M^2 - 1 ,$$

where M is the Mach number at infinity. If the flow is subsonic φ_1 is not changed. φ_2 has a different value, however, and is given by Weber in Ref.1. The present method is concerned only with φ_1 and applies equally to supersonic or subsonic flow. In thin wing theory $2|Y(y) - Y(y')|$ is replaced by $|y - y'|$.

In the unsymmetrical case $Y(y) - Y(y')$ must be replaced by $Z(y) - Z(y')$, as may be shown by a method similar to that of Weber[1]. Since in both cases the section is symmetrical about the z axis, the latter form could be used in both cases. We shall not do this because it is more convenient not to, although it makes our treatment appear unsystematic.

3 FULLY SYMMETRICAL SECTIONS AT ZERO LIFT

3.1 General

We denote the ratio z_{max}/semi-span at any section by δ and assume that δ is small.

We write

$$\zeta = T + \frac{r^2}{T} - 2r\left(\frac{a_1 r}{T} + \frac{a_3 r^3}{T^3} + \ldots\right), \tag{9}$$

in order to transform the circle $T = re^{i\theta}$ into the required section in the ζ plane. This gives on the section

$$y = 2r\cos\theta - 2r(a_1\cos\theta + a_3\cos 3\theta + \ldots), \tag{10}$$

$$z = 2r(a_1\sin\theta + a_3\sin 3\theta + \ldots), \tag{11}$$

$$2Y = 2r\cos\theta .$$

This form of expansion gives the double symmetry required, since $z(-\theta) = -z(\theta)$, $y(\pi - \theta) = -y(\theta)$, $y(-\theta) = y(\theta)$. We write

$$-z_c(\theta) = 2r\{a_1\cos\theta + a_3\cos 3\theta + \ldots\},$$

the latter being known as the "conjugate of z". Thwaites[3] shows that

$$z_c(\theta) = -\frac{1}{\pi}\int_0^{\pi}\frac{z(\theta')\sin\theta'\,d\theta'}{\cos\theta - \cos\theta'} \tag{12}$$

if $z(\phi)$ is an odd function of ϕ. This may be written

$$z_c(\eta) = -\frac{1}{\pi}\int_{-1}^{1}\frac{z(\eta')\,d\eta'}{\eta - \eta'} \tag{13}$$

if $\eta = \cos\theta$, $\eta' = \cos\theta'$.

Equation (10) becomes

$$y = 2r\cos\theta + z_c(\theta) . \tag{14}$$

When $\theta = 0$ we have $y = s$ and so

$$s = 2r + z_c(0) . \tag{15}$$

We write $y = s\cos\phi$ and so we have

$$\frac{2r}{s}\cos\theta = \cos\phi - \frac{z_c(\theta)}{s} . \tag{16}$$

It is possible to find the relation between θ and ϕ by iteration, in much the same way as is done in the well-known Theodorsen two-dimensional wing theory[3]. This iteration is not necessary if we only require to go to one higher order in δ.

The a's in equation (11) are small quantities of order δ and hence θ and ϕ differ by a quantity of the first order in δ.

Now $z_c(\theta)/s$ and $z_c(0)/s$ are both $O(\delta)$ and so to this order we may write

$$\frac{2r}{s}\cos\theta = \cos\phi - \frac{\bar{z}_c(\phi)}{s} ,$$

where

$$-\bar{z}_c(\phi) = s\{a_1\cos\phi + a_3\cos 3\phi + \ldots\} .$$

This means that $\bar{z}$ is found from the equations

$$\left.\begin{aligned} y &= s\cos\phi , \\ \bar{z} &= s\{a_1\sin\phi + a_3\sin 3\phi + \ldots\} , \end{aligned}\right\} \tag{17}$$

and then the conjugate of $\bar{z}$ is determined. From now on we shall drop the bar over z and z_c.

Bearing in mind that $2Y = 2r\cos\theta$ we may write equation (7), to order δ

$$\varphi_1 = \frac{V}{\pi}\int_{-s}^{s}\frac{\partial z(x,y')}{\partial x}\log\left|\cos\phi - \cos\phi' - \frac{z_c(\phi)}{s} + \frac{z_c(\phi')}{s}\right| dy' , \qquad \ldots (18)$$

where $y' = s\cos\phi'$.

Since z_c and z_c' are $O(\delta)$ we may write this

$$\varphi_1 = \frac{V}{\pi}\int_{-s}^{s} \frac{\partial z(x,y')}{\partial x}\left\{\log\left|\cos\phi - \cos\phi'\right| - \frac{z_c(\phi) - z_c(\phi')}{s(\cos\phi - \cos\phi')}\right\} dy' , \qquad \ldots (19)$$

on expanding the logarithm.

The first term is that which would have been obtained by the usual slender thin wing theory. To this would be added φ_2 to give the full value of φ. The present analysis does not change φ_2, since φ_2 does not depend on the shape of the cross-section.

Thus our method produces a correction term of the next higher order in δ, and we shall denote this by $\Delta\varphi_1$, so that

$$\Delta\varphi_1 = -\frac{V}{\pi}\int_{-s}^{s} \frac{\partial z(x,y')}{\partial x} \cdot \frac{z_c(\phi) - z_c(\phi')}{s(\cos\phi - \cos\phi')}\, dy'$$

$$= -\frac{V}{\pi}\int_{-1}^{1} \frac{\partial z(x,\eta')}{\partial x} \cdot \frac{z_c(\eta) - z_c(\eta')}{\eta - \eta'}\, d\eta'$$

if $y' = s\eta' = s\cos\phi'$, and this may be written

$$\boxed{\Delta\varphi_1 = V\left\{z_c\left(\frac{\partial z}{\partial x}\right)_c - \left(\frac{\partial z}{\partial x}\, z_c\right)_c\right\}} , \qquad (20)$$

where f_c is defined as

$$f_c(\eta) = -\frac{1}{\pi}\int_{-1}^{1} \frac{f(\eta')\, d\eta'}{\eta - \eta'} \qquad (21)$$

which follows from equation (13).

Thus the thickness correction to φ may be found.

We show in Appendix 1 that an alternative method of writing equation (20) is

$$\boxed{\Delta\varphi_1 = V\left\{z\,\frac{\partial z}{\partial x} + \left(z\left(\frac{\partial z}{\partial x}\right)_c\right)_c\right\}} . \qquad (22)$$

This has the advantage that one less conjugate need be calculated.

3.2 Details

There seem to be at least three methods of proceeding. Firstly we may actually determine the a's by fitting a finite Fourier series to a finite number of points on the wing section using equations (17). One would hope that the a's would soon become small. All the necessary functions can then be calculated. Secondly we may note that Watson[4] has given a method of determining conjugates and their derivatives without actually finding the a's; the details are also given by Thwaites[3]. Finally consideration might also be given to finding conjugates by direct numerical integration of equation (21) in the form

$$f_c = \frac{1}{\pi}\int_{-1}^{1} \frac{f(\eta) - f(\eta')}{\eta - \eta'}\, d\eta' - \frac{1}{\pi} f(\eta) \log \frac{1+\eta}{1-\eta} . \qquad (23)$$

The derivative of f_c with respect to η is given by

$$\frac{\partial f_c}{\partial \eta} = -\frac{1}{\pi}\left\{\frac{f(1)}{1-\eta} + \frac{f(-1)}{1+\eta}\right\} + \frac{1}{\pi}\int_{-1}^{1} \frac{f'(\eta) - f'(\eta')}{\eta - \eta'}\, d\eta'$$

$$- \frac{1}{\pi} f'(\eta) \log \frac{1+\eta}{1-\eta} .$$

Primes attached to f denote derivatives.

If $f'(\eta')$ is infinite at $\eta' = \pm 1$ we may write this

$$\frac{\partial f_c}{\partial \eta} = \frac{1}{\pi}\int_{0}^{1} \left\{f'(\eta) - f'(\eta')\right\}\left\{\frac{1}{\eta - \eta'} - \frac{1}{\eta - 1}\right\} d\eta'$$

$$+ \frac{1}{\pi}\int_{-1}^{0} \left\{f'(\eta) - f'(\eta')\right\}\left\{\frac{1}{\eta - \eta'} - \frac{1}{\eta + 1}\right\} d\eta'$$

$$- \frac{1}{\pi} f'(\eta)\left\{\frac{1}{1-\eta} - \frac{1}{1+\eta}\right\} - \frac{1}{\pi} f'(\eta) \log \frac{1+\eta}{1-\eta}$$

$$- \frac{2}{\pi}\, \frac{f(0)}{1-\eta^2} .$$

If $f'(\eta)$ has a logarithmic singularity at $\eta = \pm 1$, the integrands in this equation will now vanish at $\eta' = \pm 1$. $\eta = 1$ itself must of course be excluded.

3.3 Singularities

Very often we deal with sections which have sharp edges at $\eta = \pm 1$. This leads in general to logarithmic singularities in φ_x and φ_y; φ itself is finite, but its derivatives are not so. The same applies to the correction term $\Delta\varphi$. The procedure used is in fact not uniformly valid at $\eta = \pm 1$, and we are using a singular perturbation procedure in going from equation (18) to equation (19) near $\eta = \pm 1$. As explained by Lighthill[5], the next term will usually contain a singularity of higher order. It was found, however, in examples that the results quite near to $\eta = \pm 1$ were as accurate as elsewhere. Attempts in these special examples to render the result uniformly valid led to finite velocity components at the edge, but except very near the edge the results were no more accurate than the direct procedure given above. As there seemed no simple extension of Lighthill's procedure to more general cases no further attempt to use it was made, since the results seemed to be sufficiently accurate without it.

4 UNSYMMETRICAL SECTIONS

A transformation to a circle in the T plane is made as before, but it is now more convenient to write

$$T = -ire^{i\theta} .$$

This makes $\theta = 0$ at the centre of the lower surface. See Fig.4. The transformation is

$$\zeta = T + \frac{r^2}{T} + 2\,ri\left(a_0 - \frac{ia_1 r}{T} + \frac{i^2 a_2 r_2}{T^2} - \frac{i^3 a_3 r^3}{T^3} + \ldots\right) , \tag{24}$$

which leads to

$$\left.\begin{aligned} z &= 2r\,a_0 + 2r(a_1 \cos\theta + a_2 \cos 2\theta + \ldots) \\ y &= 2r \sin\theta + 2r(a_1 \sin\theta + a_2 \sin 2\theta + \ldots) = 2r \sin\theta + z_c . \end{aligned}\right\} \quad \ldots (25)$$

If the section is thin we shall have $y = s$ when $\theta = \frac{1}{2}\pi - \beta$, where β is order δ. Hence, ignoring terms in β^2 and higher orders we have

$$s = 2r \cos\beta + z_c(\tfrac{1}{2}\pi) = 2r + z_c(\tfrac{1}{2}\pi) . \tag{26}$$

We let $y = s \sin\phi$, and suppose that $\theta = \phi + \Delta\phi$. Hence, from equations (25),

$$s \sin\phi = 2r\,\{\sin\phi + \Delta\phi \cos\phi\} + z_c(\theta) ,$$

keeping only first order terms in $\Delta\phi$. Using equation (26) we find to order δ

$$\Delta\phi = \frac{\sin\phi\ \bar{z}_c(\tfrac{1}{2}\pi) - \bar{z}_c(\phi)}{s\cos\phi} ,$$

where

$$\bar{z}_c(\phi) = s\,\{a_1 \sin\phi + a_2 \sin 2\phi + \dots\} .$$

As before we shall drop the bar over z and z_c.

We note that φ_1 is given by equation (7), with Y(y) replaced by Z(y), etc. Equation (7), modified in this way, reads

$$\varphi_1 = \frac{V}{\pi}\int \frac{\partial z(x,y')}{\partial x} \log \frac{2|Z(y) - Z(y')|}{s}\, dy' ,$$

where the limits are to be such that the path DAB in Fig.2 or Fig.3 is to be followed.

In the modified equation (7) the expression inside the logarithm is

$$E = \frac{2|Z(y) - Z(y')|}{s} = \frac{2r}{s}\,|\cos\theta - \cos\theta'| .$$

Now

$$\cos\theta = \cos\phi - \Delta\phi\sin\phi$$

$$= \cos\phi - \frac{\sin^2\phi}{\cos\phi}\frac{z_c(\tfrac{1}{2}\pi)}{s} + \tan\phi\,\frac{z_c(\phi)}{s} .$$

Hence we have

$$E = \left\{1 - \frac{z_c(\tfrac{1}{2}\pi)}{s}\right\} \Bigg|\cos\phi - \cos\phi' + \tan\phi\,\frac{z_c(\phi)}{s} - \tan\phi'\,\frac{z_c(\phi')}{s} - \frac{\sin^2\phi}{\cos\phi}\frac{z_c(\tfrac{1}{2}\pi)}{s} + \frac{\sin^2\phi'}{\cos\phi'}\frac{z_c(\tfrac{1}{2}\pi)}{s}\Bigg| .$$

On substituting in equation (7) and expanding, assuming that z_c/s is small, we find on putting $y = s\sin\phi$, that

$$\varphi_1 = \frac{Vs}{\pi} \int_0^\pi \frac{\partial z(x,\phi')}{\partial x} \log |\cos\phi - \cos\phi'| \cos\phi' \, d\phi'$$

$$+ \frac{V}{\pi} \int_0^\pi \frac{\partial z(x,\phi')}{\partial x} \left\{ \frac{\tan\phi \, z_c(\phi) - \tan\phi' \, z_c(\phi')}{\cos\phi - \cos\phi'} + \frac{z_c(\frac{1}{2}\pi)}{\cos\phi \cos\phi'} \right\} \cos\phi' \, d\phi' \, . \qquad \ldots (27)$$

Here z is even, z_c is odd and $\partial z(x,\phi')/\partial x$ is even. The first term is the slender thin wing value and the second term is evaluated in Appendix 2. Once more writing the correction to φ_1 as $\Delta\varphi_1$ we find that

$$\boxed{\Delta\varphi_1 = V \left\{ -z_c \left(\frac{\partial z}{\partial x}\right)_c + \left(z_c \frac{\partial z}{\partial x}\right)_c - h \frac{\sin\phi \, z_c(\phi) - z_c(\frac{1}{2}\pi)}{\cos\phi} \right.} \, , \qquad (28)$$

where

$$h = \frac{1}{\pi} \int_0^\pi \frac{\partial z(x,\phi')}{\partial x} \, d\phi' \, .$$

In this equation the conjugate of an odd function of ϕ is given by equation (12) and the conjugate of an even function $U(\phi)$ is given by

$$U_c(\phi) = - \frac{\sin\phi}{\pi} \int_0^\pi \frac{U(\phi') \, d\phi'}{\cos\phi - \cos\phi'} \qquad (29)$$

We show in Appendix 3 that equation (28) may be written

$$\boxed{\Delta\varphi_1 = V \left\{ -z \frac{\partial z}{\partial x} - \left(z \left(\frac{\partial z}{\partial x}\right)_c\right)_c - h \frac{\sin\phi \, z_c - z_c(\frac{1}{2}\pi)}{\cos\phi} + gh \right\}} \qquad (30)$$

where

$$g = \frac{1}{\pi} \int_0^\pi z \, d\phi, \qquad h = \frac{1}{\pi} \int_0^\pi \frac{\partial z}{\partial x} \, d\phi \, .$$

5 AN EXAMPLE. THE RHOMBIC CONE

In this case an exact conformal transformation is possible and Maskell (unpublished) has worked out the results for the symmetrical case for various edge angles. It is also possible to give an analytical solution of this problem by the present method.

We suppose the cone to be of unit length, that is, x is equal to unity at the base, which is situated downstream of the apex, the cone being at zero incidence. We write

$$s(x) = \underline{s}x , \quad \eta = y/\underline{s}x ,$$

so that $\underline{s}$ is the semi-span at the base. We further denote half the maximum thickness of any section x = constant by δs.

Hence we have

$$z = \delta s(1 - |\eta|), \quad \partial z/\partial x = \delta \underline{s} .$$

Using equation (22) we find, as shown in Appendix 4, that

$$\left.\begin{aligned} \Delta\varphi &= \frac{V \delta^2 \underline{s}^2 x}{\pi^2}\left\{\tfrac{1}{2}\pi^2(1 - 2|\eta|) + \tfrac{1}{2}\left(\log\frac{1+\eta}{1-\eta}\right)^2 + 4 \log 2 - \eta W\right\} \\ &= \frac{V \delta^2 \underline{s}^2 x}{\pi^2} B , \end{aligned}\right\} \quad (31)$$

where

$$\frac{\partial W}{\partial \eta} = -\frac{2}{1-\eta^2}\log\frac{1-\eta^2}{4\eta^2}, \quad W(o) = 0.$$

We have evaluated W in Appendix 4, in terms of Powell's[6] "Rℓ" functions or Mitchell's[7] "f" functions.

We find

$$\left.\begin{aligned} \frac{\partial\Delta\varphi}{\partial x} &= \frac{V \delta^2 \underline{s}^2}{\pi^2}\left\{\tfrac{1}{2}\pi^2 + 4\log 2 + \tfrac{1}{2}\left(\log\frac{1-\eta}{1+\eta}\right)^2 + \frac{2\eta}{1+\eta}\log(1-\eta)\right. \\ &\qquad\left. - \frac{2\eta}{1-\eta}\log(1+\eta) + \frac{4\eta^2}{1-\eta^2}\log 2\eta\right\} \\ &= \frac{V \delta^2 \underline{s}^2}{\pi^2} E , \end{aligned}\right\} \quad (32)$$

$$\left.\begin{aligned} \frac{\partial\Delta\varphi}{\partial y} &= \frac{V \delta^2 \underline{s}}{\pi^2}\left\{-\pi^2 \operatorname{sgn}\eta + \frac{2}{1-\eta^2}\log\frac{1+\eta}{1-\eta} - W + \frac{2\eta}{1-\eta^2}\log\frac{1-\eta^2}{4\eta^2}\right. \\ &= \frac{V \delta^2 \underline{s}}{\pi^2} D . \end{aligned}\right\} \quad (33)$$

These are infinite at $\eta = \pm 1$, and we see that the singularity in $\Delta\varphi$ is of higher order than that in the slender wing value of φ, as was to be expected. It is easy to show that slender thin-wing theory leads to

$$\frac{\partial\varphi}{\partial x} = \frac{V\delta\underline{s}^2}{\pi}\left\{\log(1-\eta) + \log(1+\eta) + 2\log\tfrac{1}{2}\beta\underline{s}\right\},$$

$$\frac{\partial\varphi}{\partial y} = \frac{V\delta\underline{s}}{\pi}\log\frac{1+\eta}{1-\eta} .$$

Maskell has expressed the derivatives of φ on the cone in terms of two integrals G and F. We have

$$\frac{\partial\varphi}{\partial y} = \frac{V\delta\underline{s}}{\pi}\left\{\log\frac{1+\eta}{1-\eta} + \frac{\delta}{\pi}D\right\},$$

and Maskell equates this to

$$-\frac{2V\delta\underline{s}}{\pi}G . \tag{34}$$

Hence we have

$$G = -\tfrac{1}{2}\log\frac{1+\eta}{1-\eta} - \frac{\delta}{2\pi}D ,$$

and so our results can be compared directly with those of Maskell.

Maskell also writes

$$\frac{1}{V}\frac{\partial\varphi}{\partial x} = \frac{2}{\pi}\delta\underline{s}^2\left\{\log\frac{\beta\underline{s}}{\lambda} + 1\right\} + \frac{2\delta\underline{s}^2}{\pi}\eta G + \frac{2\delta\underline{s}^2}{\pi\lambda(1+\delta^2)^{\frac{1}{2}}}F , \tag{35}$$

where

$$\lambda = \frac{\pi n}{2^n}\cot\tfrac{1}{2}n\pi\,\frac{n!}{[(\frac{1}{2}n)!]^2} , \tag{36}$$

$$\delta = \tan\tfrac{1}{2}n\pi . \tag{37}$$

Hence we have

$$F = \lambda(1+\delta^2)^{\frac{1}{2}}\left\{\log\tfrac{1}{2}\lambda - 1 + \tfrac{1}{2}(1+\eta)\log(1+\eta) + \tfrac{1}{2}(1-\eta)\log(1-\eta) + (\delta/2\pi)B\right\},$$

where B is defined in equations (31).

We have plotted these results in two cases, corresponding to edge angles of 40° and 60°, and in Figs.4 and 5 they are compared with the thin-wing values. It will be seen that the latter are considerably in error. Since thin-wing theory gives logarithmically infinite values of the velocity components at the leading edge, Weber[8] has modified the formulae of slender thin-wing theory so as to produce uniformly valid results correct to the first order in δ. The effect is to replace the logarithmic term in G by

$$-\frac{\pi}{\omega}\left\{1 - (1-\eta)^{\frac{\omega}{2\pi}}\right\} - \tfrac{1}{2}(1-\eta)^{\frac{\omega}{2\pi}} \log(1+\eta) \quad ,$$

where

$$\omega = 2\tan^{-1}\delta \ ;$$

the F term is not changed. Values of G by her method are shown in Figs.4 and 5. The values near to the edge are improved, as is to be expected. The G curves show that we may go very close to the edge without the singularity causing any serious divergence from the correct solution, although actually the latter is finite at $\eta = 1$ whilst the approximation is infinite. Indeed, as the singularity is of higher order than that of the thin wing and as D is negative near $\eta = 1$ our approximate solution will tend to plus infinity at the edge, but it seems necessary to go very near to the edge before the curve turns, unless the section is very thick.

Thus we see that our approximation gives good results, even for cones with edge angles as large as 60°.

As shown above it was possible in this case to find F and G by direct integration and the use of tabulated functions. In a general case this would not be possible, and so Watson's method seems to be the best to use. This was done for comparison in our rhombic cone example.

It involves fitting a trigonometrical series of N terms to the section and then carrying out a simple numerical routine to find the conjugates. The values used were $N = 20$, 40 and 60, and near to the singularity the results differed from each other and from the correct integrated value by amounts which were larger than expected; however in the case $N = 40$ the error, in the worst case, was not enough to displace the curves for $G(\eta)$ and $F(\eta)$ in Figs.4 and 5 by more than one quarter of the amount they are already in error. That the discrepancy in Watson's method is as large as this is due to the singularity at the edge. However the approximation is improved by taking larger values of N; $N = 40$ seems to be adequate. Up to $\eta = 0.8$ the error for $N = 40$ is negligible. After that the curves for F and G oscillate, but it is possible to fair in curves which are sufficiently accurate for the purpose required.

6 PRESSURE DISTRIBUTION

The relation between the pressure coefficient c_p and the local velocity, on the assumption of zero shock or weak shock, is

$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho V^2} = \frac{2}{\gamma M^2}\left\{\left[1 + \tfrac{1}{2}(\gamma - 1)\, M^2\left(1 - \frac{q^2}{V^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\},$$

where ρ, V and M are the density, velocity and Mach number at infinity and q is the local velocity. This may be expanded in the form

$$c_p = \left(1 - \frac{q^2}{V^2}\right) + \frac{1}{4} M^2\left(1 - \frac{q^2}{V^2}\right)^2 + \frac{1}{40} M^4\left(1 - \frac{q^2}{V^2}\right)^3 + \dots \tag{38}$$

for $\gamma = 1.4$.

If φ is the perturbation velocity potential we have

$$q^2 = (V + \varphi_x)^2 + \varphi_y^2 + \varphi_z^2 \ , \tag{39}$$

where subscripts denote partial derivatives.

In linear or slender thin wing theory squares of φ_x, φ_y and φ_z are neglected and so there is no justification in going beyond the first term in equation (38); hence

$$c_p = -\frac{2}{V}\varphi_x \ .$$

In slender body theory φ_y^2 and φ_z^2 are of the same order as ϕ_x and cannot be ignored. Hence

$$c_p = -\frac{1}{V^2}\left(2 V \varphi_x + \varphi_y^2 + \varphi_z^2\right) \ . \tag{40}$$

Another point to be borne in mind is that our φ has been given as a function of x and y only, that is φ is known for <u>the point on the body</u> whose x and y coordinates are given. What has in effect been done is that $z = z(x, y)$ has been substituted in $\varphi(x,y,z)$ to produce a function

$$\varphi_1(x,y) + \varphi_2(x) = \varphi\{x,y,z(x,y)\} \ ,$$

φ being the perturbation velocity potential.

Now the velocity components on the body surface are $V + \varphi_x$, φ_y, φ_z. The velocity vector is perpendicular to the normal to the body surface and hence

$$(V + \varphi_x)\, z_x + \varphi_y\, z_y - \varphi_z = 0 \ .$$

In slender theory, where φ_x is small, this leads to

$$Vz_x + \varphi_y z_y = \varphi_z \ .$$

Since on the body

$$\varphi\{x,y,z(x,y)\} = \varphi_1(x,y) + \varphi_2(x)$$

we find

$$\varphi_{1x} + \varphi_{2x} = \varphi_x + \varphi_z z_x \ ,$$

$$\varphi_{1y} = \varphi_y + \varphi_z z_y \ .$$

These equations lead to the results

$$\varphi_y = \frac{\varphi_{1y} - Vz_x z_y}{1 + z_y^2} \ ,$$

$$\varphi_z = \frac{Vz_x + z_y \varphi_{1y}}{1 + z_y^2} \ ,$$

$$\varphi_x = \varphi_{1x} + \varphi_{2x} - z_x \frac{Vz_x + z_y \varphi_{1y}}{1 + z_y^2} \ .$$

To the first order in δ these values are

$$\left.\begin{aligned} \varphi_x &= \varphi_{1x} + \varphi_{2x} \ , \\ \varphi_y &= \varphi_{1y} \ , \\ \varphi_z &= Vz_x \ , \end{aligned}\right\} \qquad (41)$$

where φ_{1x} and φ_{1y} are the uncorrected values. To the second order in δ we may write

$$\left.\begin{aligned} \varphi_x &= \varphi_{1x} + \Delta\varphi_{1x} + \varphi_{2x} - Vz_x^2 \ , \\ \varphi_y &= \varphi_{1y} + \Delta\varphi_{1y} - Vz_x z_y \ , \\ \varphi_z &= Vz_x + z_y \varphi_{1y} \ . \end{aligned}\right\} \qquad (42)$$

Thus we find that in addition to the corrections $\Delta\varphi_1$ already made in the main body of this paper there must be included further corrections as set out in equations (42).

When we wish to calculate the pressure coefficient we note that in slender theory

$$q^2 = V^2 + 2V\varphi_{1x} + 2V\Delta\varphi_{1x} + 2V\varphi_{2x} - 2V^2 z_x^2$$

$$+ \varphi_{1y}^2 + V^2 z_x^2 ,$$

keeping only the second order in δ and ignoring second and higher powers of φ_{1x} and φ_{2x}.

Hence the correction to c_p from slender thin wing theory to give slender not-so-thin wing theory is

$$\boxed{\Delta c_p = -\frac{2}{V}\Delta\varphi_{1x} - \frac{1}{V^2}\varphi_{1y}^2 + z_x^2 .} \qquad (43)$$

We may note that φ_1 is given by the first term in equations (19) or (27). By differentiating with respect to y we find that

$$\frac{1}{V}\frac{\partial\varphi_1}{\partial y} = -\left(\frac{\partial z}{\partial x}\right)_c , \qquad \text{(symmetrical case)}$$

$$\frac{1}{V}\frac{\partial\varphi_1}{\partial y} = h\tan\phi + \left(\frac{\partial z}{\partial x}\right)_c , \quad \text{(unsymmetrical case)}$$

where h is defined in equations (A.3.3).

Equation (43) gives the correction to be made to $1 - q^2/V^2$ to account for the thickness. We now say that the same correction can be applied to linear thin-wing theory to give what we might call linear not-so-thin wing theory. Finally we make the correction shown in equation (38). It is not easy to justify this last correction except that it seems to give better results; as other workers have found. (See for instance Ref.9.) We make no other attempt to justify its inclusion. The simplest way to do it is to take the c_p calculated by linear theory, incorporate the correction given in (43) and then add $\frac{1}{4}M^2c_p^2$ to this, where the c_p in this last formula may logically be any of those so far calculated; we shall take it as the $c_p + \Delta c_p$ obtained as just explained.

Hence we find that our estimate $\bar{c}_p$ for the pressure coefficient is given by

$$\bar{c}_p = \left[c_{p(thin)} + \Delta c_p\right] + \frac{1}{4}M^2\left[c_{p(thin)} + \Delta c_p\right]^2 \qquad (44)$$

where Δc_p is calculated from equation (43) and $c_{p(thin)}$ means the pressure coefficient obtained by linear thin wing theory.

7 EXAMPLES

The method of this paper has been applied to determine the pressure distribution on two delta wings with rhombic cross-sections, which were tested in the 8 ft tunnel at Bedford. The linear thin-wing values were worked out by Eminton[10]. The wings were such that

$$\text{Wing I:} \quad z(x,0) = 0.18\, x(1-x), \qquad \text{(Newby)}$$

$$\text{Wing V:} \quad z(x,0) = 0.0105\, x(1-x)(4-6x+4x^2-x^3). \qquad \text{(Lord V)}$$

The results for a Mach number of 2 are shown in Figs.6 and 7. It will be seen that near the centre line there is considerable improvement, but that as one moves outboard there is little or no improvement over linear thin-wing theory.

One is tempted to ascribe the discrepancy to the effect of the boundary layer. Some crude calculations for wing V have been done in some unpublished work and it was found that the boundary layer effects were of the correct sign, but not of sufficient magnitude to account entirely for the discrepancies. In the region of interest the correction to c_p which can be ascribed to the boundary layer is about +0.002 to +0.003, and this is too small. However the calculations were of a very crude nature and it may well be possible to ascribe the discrepancies to the effect of the boundary layer. However, they may be due to errors in small perturbation theory itself, and it may be necessary to apply second order corrections to that theory in order to obtain further improvement.

It should be pointed out that linear thin wing theory in the cases under consideration (in which the maximum thickness-chord ratios are as high as 9% and 11%) gives very good results, even before correction, much better than it did in the case of a two-dimensional aerofoil in subsonic flow. This is in spite of the fact that the basic equation (Laplace) was exact and not linearized as it is here. Moreover the effect of the boundary layer is greater in the subsonic case than it is here, mainly owing to the fact that in subsonic flow, inviscid theory demands strong adverse pressure gradients and a stagnation point at the trailing edge (unless the edge is a cusp) and the boundary layer has a strong effect there, making the velocity close to that of the main stream instead of its theoretical value zero. In supersonic flow the inviscid velocity at the trailing edge is already near to that of the main stream and the boundary layer only affects it slightly.

It should perhaps be pointed out that in some cases slender thin-wing theory gives quite good results compared with experiment, as indeed it does in the case of Wing I. When this happens it must be regarded as a fortunate cancellation of errors, such as that due to thickness and that due to non-slenderness.

8 CONCLUSIONS

The method given here gives the next higher order term in the thickness ratio δ in slender theory, and the examples show that, provided the wing is not too thick, it gives better values than the first approximation.

A further procedure is then suggested. This is to work out the not-so-thin correction to slender thin wing and apply the same correction to linear thin wing theory. The procedure is found to be quite successful in some cases. Logically it seems to be a legitimate operation provided that the linear and slender theories do not produce results which differ widely from one another. There are, however, cases in which such a large difference does occur, particularly when $S'(x)$ and $S''(x)$ have large values near to the trailing edge. For instance Firmin[11] has made some experiments on such a wing. In this case the two theories give widely different values of c_p near to the trailing edge and so it is probable that the argument about the independence of small corrections no longer applies; indeed, on attempting to use the methods of this paper to this case, it was found that the results did not give any significant improvement over linear thin-wing theory.

Finally the correction shown in equation (44) is introduced. It would seem that this correction cannot be justified, in view of the approximation made in deriving the linearized potential equation. One can only say that investigators have found that incorporating this correction does in fact lead to results agreeing more closely with experiment. It does so in general in the examples tested in this paper.

Although linear thin-wing theory is already quite good in predicting pressure distributions, the corrections given here are useful in that they do give improved values, and they also confirm that the linearized potential equation, fully exploited, is a useful and accurate approximation for thin wings.

It is necessary, however, for the wing to be "smooth". In Firmin's experiments[11] this was not so, and the results show that the theories are not so satisfactory in such cases.

LIST OF SYMBOLS

a_s	coefficient in expansions (9) and (24)
B	defined in equation (31)
c_p	pressure coefficient
D	defined by equation (33)
E	defined by equation (32)
f(x)	defined by equation (A.4.6)
F	defined by equation (35)
G	defined by equation (34)
g,h	defined by equation (A.3.3)
I	defined by equation (A.4.3)
K	defined by equation (A.4.1)
L	defined by equation (A.4.2)
M	Mach number
N	number of terms in Watson's formula, Section 3.2
n	defined by equation (37)

LIST OF SYMBOLS (Contd)

p	pressure
q	resultant velocity over the surface
r	radius of circle in T-plane
$R\ell$	defined by equation (A.4.5)
$s(x)$	semi-span at station x
$S(x)$	cross-sectional area at station x
$\underline{s}$	semi-span at the trailing edge
T	$Y + iZ$
V	velocity at infinity
v_n	normal component of velocity in cross-flow plane
W	defined by equation (A.4.5)
x,y,z	Cartesian coordinates
Y,Z	coordinates in T-plane
β	value of $\frac{1}{2}\pi - \theta$ in T-plane corresponding to the edge A of the wing. Fig.4
γ	ratio of specific heats
δ	t/s
ζ	$y + iz$
η	$y/s(x)$
λ	defined by equation (36)
ϕ	defined by equation (17)
φ	perturbation velocity potential

Subscript c applied to a function means its conjugate.

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APPENDIX 1

ALTERNATIVE FORM FOR $\Delta\varphi$, SYMMETRICAL CASE

Suppose that a and b are odd functions of ϕ. b_c will be even and ab_c will be odd. Writing $x = \cos\phi$, $x' = \cos\phi'$ and using equation (13) we have

$$(ab_c)_c = -\frac{1}{\pi}\int_{-1}^{1} \frac{a(x')\, b_c(x')}{x - x'}\, dx' = \frac{1}{\pi^2}\int_{-1}^{1} \frac{a(x')\, dx'}{x - x'} \int_{-1}^{1} \frac{b(x'')\, dx''}{x' - x''} .$$

This is a double principal value integral of Bertrand-Poincaré form. According to Muskhelishvili[12] the order of integration may be inverted to give

$$(ab_c)_c = \frac{1}{\pi^2}\left\{-\pi^2\, a(x)\, b(x) + \int_{-1}^{1} b(x'')\, dx'' \int_{-1}^{1} \frac{a(x')\, dx'}{(x - x')(x' - x'')}\right.$$

$$= -\, ab + \frac{1}{\pi^2}\int_{-1}^{1} \frac{b(x'')\, dx''}{x - x''} \int_{-1}^{1} a(x') \left\{\frac{1}{x - x'} - \frac{1}{x'' - x'}\right\} dx'$$

$$= -\, ab - \frac{1}{\pi}\int_{-1}^{1} \frac{b(x'')\, dx''}{x - x''} \left\{a_c(x) - a_c(x'')\right\}$$

$$= -\, ab + a_c b_c - (a_c b)_c \quad .$$

Hence we have

$$a_c b_c - (a_c b)_c = ab + (ab_c)_c ,$$

and so

$$z_c \left(\frac{\partial z}{\partial x}\right)_c - \left(z_c\, \frac{\partial z}{\partial x}\right)_c = z\,\frac{\partial z}{\partial x} + \left(z\left(\frac{\partial z}{\partial x}\right)_c\right)_c .$$

APPENDIX 2

SIMPLIFICATION OF EQUATION (27)

z and $\partial z/\partial x$ are even functions and z_c is an odd function of ϕ. Conjugates of odd and even functions are given by equations (12) and (29).

Now

$$\frac{1}{\pi}\int \frac{\partial z}{\partial x}\left\{\frac{\tan\phi\; z_c(\phi) - \tan\phi'\; z_c(\phi')}{\cos\phi - \cos\phi'}\right\}\cos\phi'\; d\phi'$$

$$= \frac{z_c(\phi)\tan\phi}{\pi}\int_0^\pi \frac{\partial z}{\partial x}\left[-1 + \frac{\cos\phi}{\cos\phi - \cos\phi'}\right] d\phi' - \frac{1}{\pi}\int_0^\pi \frac{\partial z}{\partial x}\;\frac{z_c(\phi')\sin\phi'}{\cos\phi - \cos\phi'}\, d\phi'$$

$$= -\, z_c\left(\frac{\partial z}{\partial x}\right)_c + \left(z_c\,\frac{\partial z}{\partial x}\right)_c - z_c\tan\phi \,.\, h \quad ,$$

where

$$h = \frac{1}{\pi}\int_0^\pi \frac{\partial z(x,\phi')}{\partial x}\, d\phi' \quad .$$

APPENDIX 3

ALTERNATIVE FORM FOR $\Delta\varphi$, UNSYMMETRICAL CASE

Suppose that a and b are even functions of ϕ. b_c and ab_c will be odd. Using equations (12) and (29) we have

$$(ab_c)_c = -\frac{1}{\pi}\int_0^\pi \frac{a(\phi')\ b_c(\phi')\ \sin\phi'\ d\phi'}{\cos\phi - \cos\phi'}$$

$$= \frac{1}{\pi}\int_0^\pi \frac{a(\phi')\ \sin\phi'\ d\phi'}{\cos\phi - \cos\phi'} \cdot \frac{\sin\phi'}{\pi}\int_0^\pi \frac{b(\phi'')\ d\phi''}{\cos\phi' - \cos\phi''}$$

$$= \frac{1}{\pi^2}\int_{-1}^{1} \frac{a(x')\ dx'(1-x'^2)^{\frac{1}{2}}}{x-x'}\int_{-1}^{1}\frac{b(x'')\ dx''}{(x'-x'')(1-x''^2)^{\frac{1}{2}}} \quad ,$$

on writing $x' = \cos\phi'$, $x'' = \cos\phi''$, $x = \cos\phi$.

Inverting the order of integration as in Appendix 1 we have

$$(ab_c)_c = \frac{1}{\pi^2}\left\{-\pi^2\ a(x)\ b(x) + \int_{-1}^{1}\frac{b(x'')\ dx''}{(1-x''^2)^{\frac{1}{2}}}\int_{-1}^{1}\frac{a(x')(1-x'^2)^{\frac{1}{2}}\ dx'}{(x-x')(x'-x'')}\right.$$

$$= -ab + \frac{1}{\pi^2}\int_{-1}^{1}\frac{b(x'')\ dx''}{(x-x'')(1-x''^2)^{\frac{1}{2}}}\int_{-1}^{1} a(x')\left\{\frac{1}{x-x'} - \frac{1}{x''-x'}\right\}(1-x'^2)^{\frac{1}{2}}\ dx'$$

$$= -ab + \frac{1}{\pi^2}\int_0^\pi \frac{b(\phi'')\ d\phi''}{\cos\phi - \cos\phi''}\int_0^\pi\left\{\frac{a(\phi')}{\cos\phi - \cos\phi'} - \frac{a(\phi')}{\cos\phi'' - \cos\phi'}\right\}\sin^2\phi'\ d\phi' \quad . \qquad \ldots \quad (A.3.1)$$

Now

$$\frac{1}{\pi}\int_0^\pi \frac{a(\phi')\ \sin^2\phi'\ d\phi'}{\cos\phi - \cos\phi'} = \frac{1}{\pi}\int_0^\pi a(\phi')\left\{\cos\phi' + \cos\phi + \frac{\sin^2\phi}{\cos\phi - \cos\phi'}\right\}d\phi'$$

$$= \frac{1}{\pi}\int_0^\pi a(\phi')\ \cos\phi'\ d\phi' + \frac{\cos\phi}{\pi}\int_0^\pi a(\phi')\ d\phi' - \sin\phi a_c(\phi) \ . \qquad \ldots \quad (A.3.2)$$

The first term cancels on substituting in equation (A.3.1). We therefore find

$$(ab_c)_c = -ab + \frac{1}{\pi}\int_0^\pi \frac{b(\phi'')}{\cos\phi - \cos\phi''}\left\{-\sin\phi\ a_c(\phi) + \sin\phi''\ a_c(\phi'')\right\} d\phi''$$

$$+ \frac{1}{\pi^2}\int_0^\pi b(\phi'')\ d\phi'' \int_0^\pi a(\phi')\ d\phi'$$

$$= -ab + a_c b_c - (a_c b)_c + \frac{1}{\pi^2}\int_0^\pi a(\phi)\ d\phi \int_0^\pi b(\phi)\ d\phi \quad .$$

Hence putting

$$a = z\ , \qquad b = \frac{\partial z}{\partial x}\ ,$$

$$g = \frac{1}{\pi}\int_0^\pi z\ d\phi\ , \qquad h = \frac{1}{\pi}\int_0^\pi \frac{\partial z}{\partial x}\ d\phi\ , \qquad \text{(A.3.3)}$$

we find

$$-z_c\left(\frac{\partial z}{\partial x}\right)_c + \left(z_c\ \frac{\partial z}{\partial x}\right)_c = -z\frac{\partial z}{\partial x} - \left(z\left(\frac{\partial z}{\partial x}\right)_c\right)_c + gh \quad .$$

APPENDIX 4

THE RHOMBIC CONE

We have

$$z = \delta \underline{s}\, x(1 - |\eta|), \qquad \partial z/\partial x = \delta \underline{s} .$$

Hence

$$z_c = -\frac{\delta \underline{s}\, x}{\pi} \int_{-1}^{1} \frac{1 - |\eta'|}{\eta - \eta'}\, d\eta'$$

$$= -\frac{\delta \underline{s}\, x}{\pi} K(\eta) \quad ,$$

where

$$K(\eta) = (1+\eta)\log(1+\eta) - (1-\eta)\log(1-\eta) - 2\eta\log\eta \qquad \text{(A.4.1)}$$

by straightforward integration.

We also have

$$\left(\frac{\partial z}{\partial x}\right)_c = -\frac{\delta \underline{s}}{\pi} \log\frac{1+\eta}{1-\eta} .$$

Hence

$$\left(z\left(\frac{\partial z}{\partial x}\right)_c\right)_c = \frac{\delta^2 \underline{s}^2 x}{\pi^2} L \quad ,$$

where

$$L = \int_{-1}^{1} \frac{1 - |\eta'|}{\eta - \eta'} \log\frac{1+\eta'}{1-\eta'}\, d\eta' . \qquad \text{(A.4.2)}$$

To calculate L we note first that if

$$I = \int_{-1}^{1} \frac{d\eta'}{\eta - \eta'} \int_{-1}^{1} \frac{d\eta''}{\eta' - \eta''} = \int_{-1}^{1} \frac{1}{\eta - \eta'} \log\frac{1+\eta'}{1-\eta'}\, d\eta' , \qquad \text{(A.4.3)}$$

then, on changing the order of integration[11],

$$I = -\pi^2 + \int_{-1}^{1} d\eta'' \int_{-1}^{1} \frac{d\eta}{(\eta-\eta')(\eta'-\eta'')}$$

$$= -\pi^2 + \int_{-1}^{1} \frac{d\eta''}{\eta-\eta''} \int_{-1}^{1} \left\{\frac{1}{\eta-\eta'} - \frac{1}{\eta''-\eta'}\right\} d\eta'$$

$$= -\pi^2 + \left(\log\frac{1+\eta}{1-\eta}\right)^2 - I \quad ,$$

and so

$$I = -\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\left(\log\frac{1+\eta}{1-\eta}\right)^2 \quad .$$

We have

$$L = \int_{-1}^{1} \frac{1}{\eta-\eta'} \log\frac{1+\eta'}{1-\eta'}\, d\eta' - \int_{0}^{1} \frac{\eta'}{\eta-\eta'} \log\frac{1+\eta'}{1-\eta'}\, d\eta' + \int_{0}^{1} \frac{\eta'}{\eta+\eta'} \log\frac{1+\eta'}{1-\eta'}\, d\eta'$$

$$= I + \int_{0}^{1} \left\{1 - \frac{\eta}{\eta-\eta'} + 1 - \frac{\eta}{\eta+\eta'}\right\} \log\frac{1+\eta'}{1-\eta'}\, d\eta$$

$$= -\tfrac{1}{2}\pi^2 + \tfrac{1}{2}\left(\log\frac{1+\eta}{1-\eta}\right)^2 + 4\log 2 - \eta\, W \quad , \qquad (A.4.4)$$

where

$$W = \int_{0}^{1} \left\{\frac{\log(1+\eta')}{\eta-\eta'} + \frac{\log(1+\eta')}{\eta+\eta'} - \frac{\log(1-\eta')}{\eta-\eta'} - \frac{\log(1-\eta')}{\eta+\eta'}\right\} d\eta' \quad . \qquad \ldots \quad (A.4.5)$$

To evaluate these four integrals write in them

$1 + \eta' = (1+\eta)t, \quad 1 + \eta' = (1-\eta)t, \quad 1 - \eta' = (1-\eta)t, \quad 1 - \eta' = (1+\eta)t$

respectively.

The first integral is

$$- \int_{\frac{1}{1+\eta}}^{\frac{2}{1+\eta}} \frac{\log(1+\eta) + \log t}{t-1}\, dt = - \log(1+\eta) \log \frac{1-\eta}{\eta} - R\ell\left(\frac{2}{1+\eta}\right) + R\ell\left(\frac{1}{1+\eta}\right) ,$$

where $R\ell(x)$ is defined by Powell[6] as

$$R\ell(x) = \int_1^x \frac{\log|t|}{t-1}\, dt \quad . \tag{A.4.6}$$

Powell tabulates $R\ell(x)$ and Mitchell[7] tabulates a function $f(x)$, where

$$f(x) = - R\ell(1-x) \quad . \tag{A.4.7}$$

The other integrals are evaluated in the same way and then use is made of the relations

$$R\ell(x) + R\ell(1-x) = \log|x| \log|1-x| - \pi^2/6 \quad ,$$

$$\left.\begin{aligned} R\ell(1/x) &= \tfrac{1}{2}(\log x)^2 - R\ell(x) \quad , \\ R\ell(-1/x) &= \tfrac{1}{2}(\log x)^2 - R\ell(-x) - \tfrac{1}{2}\pi^2 \; . \end{aligned}\right\} (x > 0)$$

We finally obtain

$$W = \tfrac{1}{2}\left(\log \frac{1-\eta}{2}\right)^2 - \tfrac{1}{2}\left(\log \frac{1+\eta}{2}\right)^2 + R\ell\left(\frac{1+\eta}{2}\right) - R\ell\left(\frac{1-\eta}{2}\right) - 2\, R\ell(\eta) + 2\, R\ell(-\eta) \; .$$

This may be also written in terms of Mitchell's "f" functions by equation (A.4.7).

Differentiating, we have

$$\frac{dW}{d\eta} = - \frac{2}{1-\eta^2} \log \frac{1-\eta^2}{4\eta^2} \quad . \tag{A.4.8}$$

By equations (A.4.5) and (22) we find

$$\Delta\varphi \;=\; \frac{V\,\delta^2 \underline{s}^2 x}{\pi^2}\left\{\tfrac{1}{2}\,\pi^2(1-2|\eta|) + \tfrac{1}{2}\left(\log\frac{1+\eta}{1-\eta}\right)^2 + 4\log 2 - \eta\,W\right\} ,$$

which is equation (31).

Differentiating with respect to x and y and using equation (A.4.8) we obtain equations (32) and (33).

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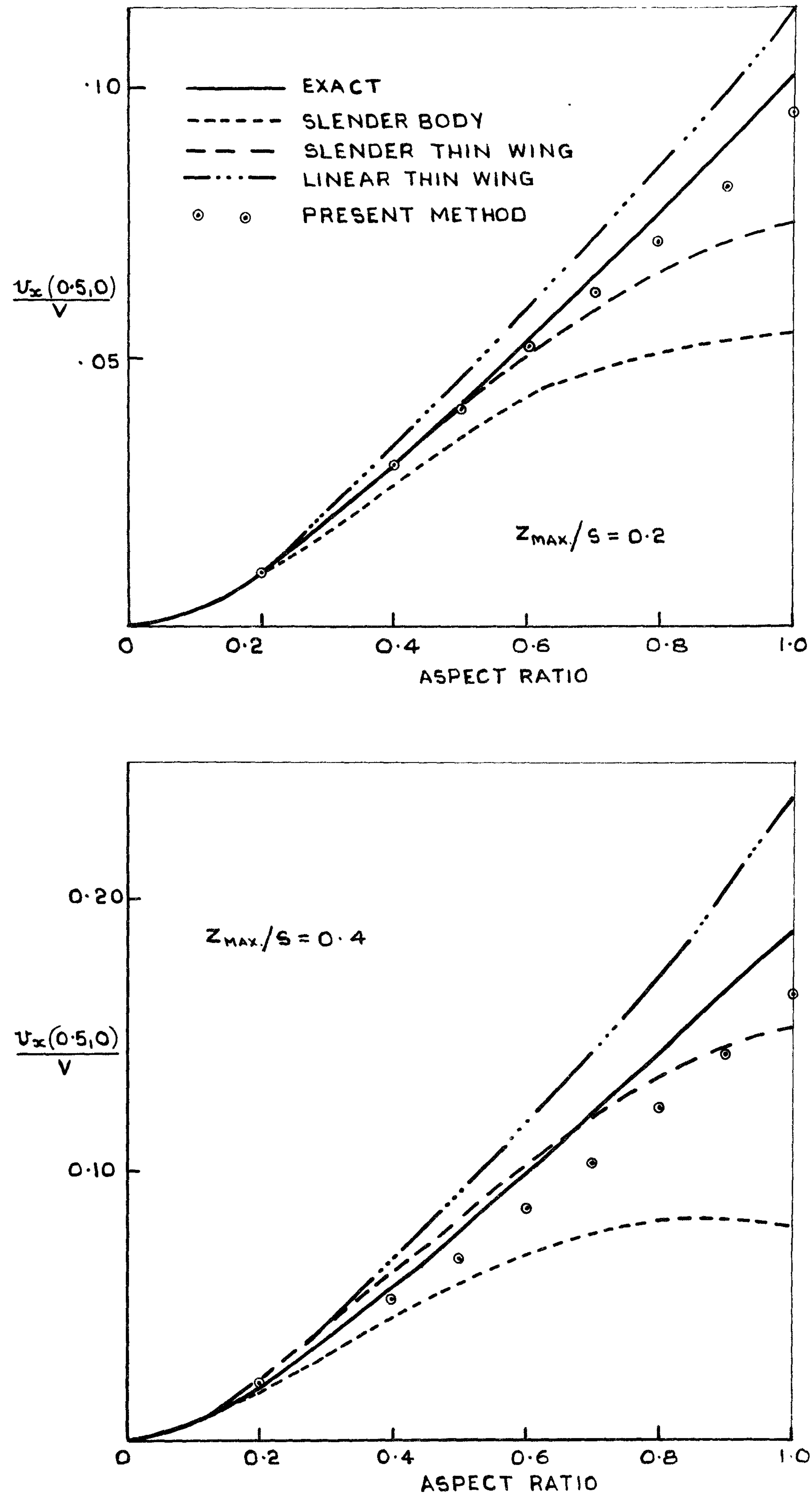

FIG. 1. VELOCITY INCREMENTS AT THE POINT OF MAXIMUM THICKNESS IN ELLIPSOIDS (WEBER[1])

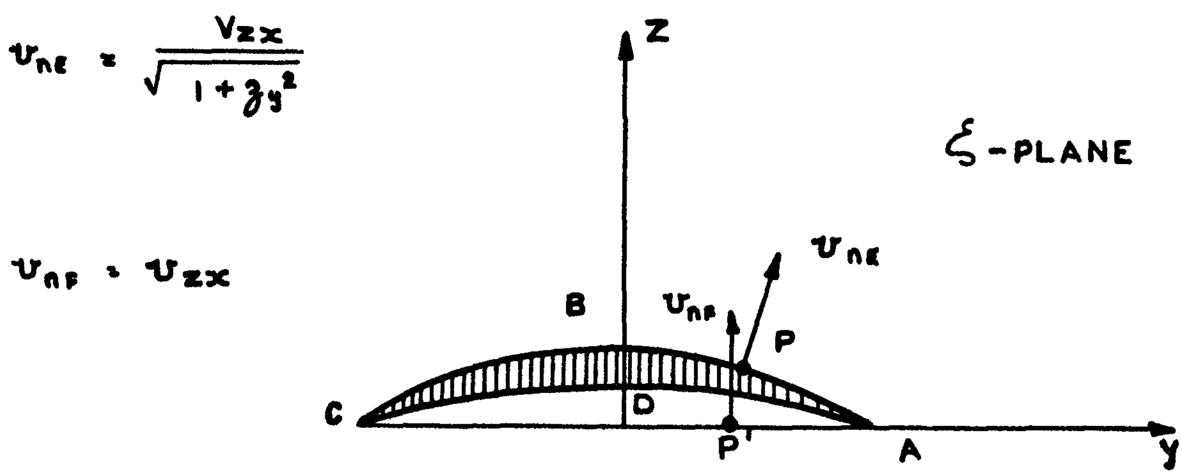

FIG. 2. SLENDER BODY (v_{nE}) AND SLENDER THIN WING (v_{nF}) BOUNDARY CONDITIONS.

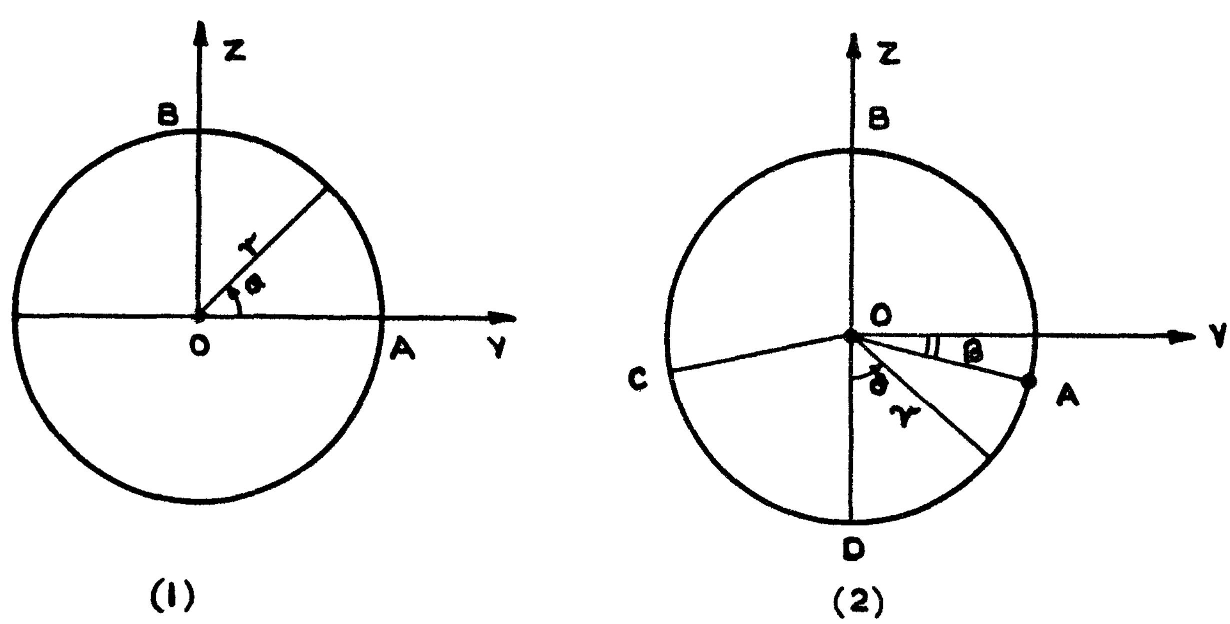

FIG. 3. T- PLANE (1) SYMMETRICAL CASE. (2) UNSYMMETRICAL CASE.

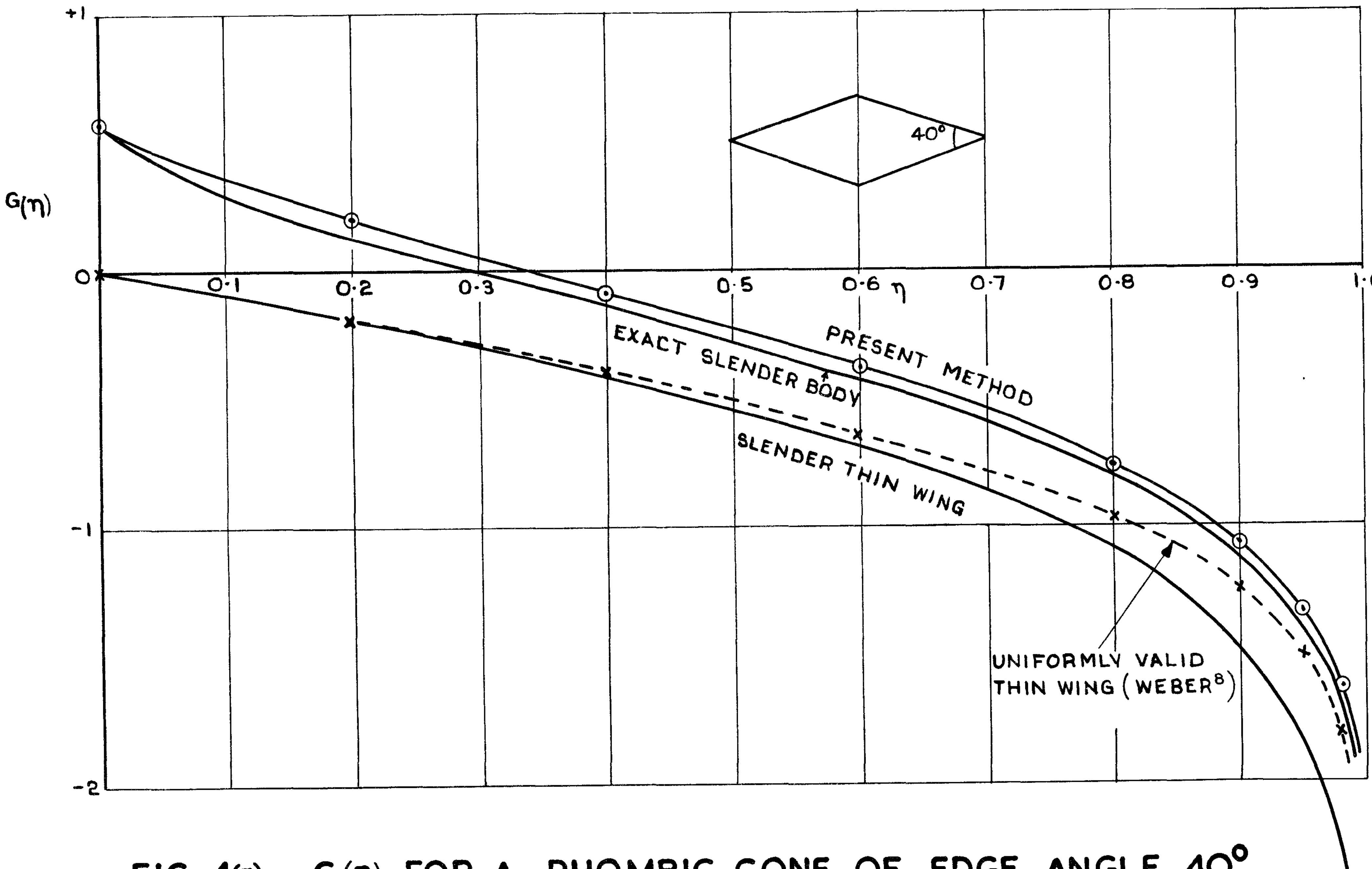

FIG 4(a) G(η) FOR A RHOMBIC CONE OF EDGE ANGLE 40°.

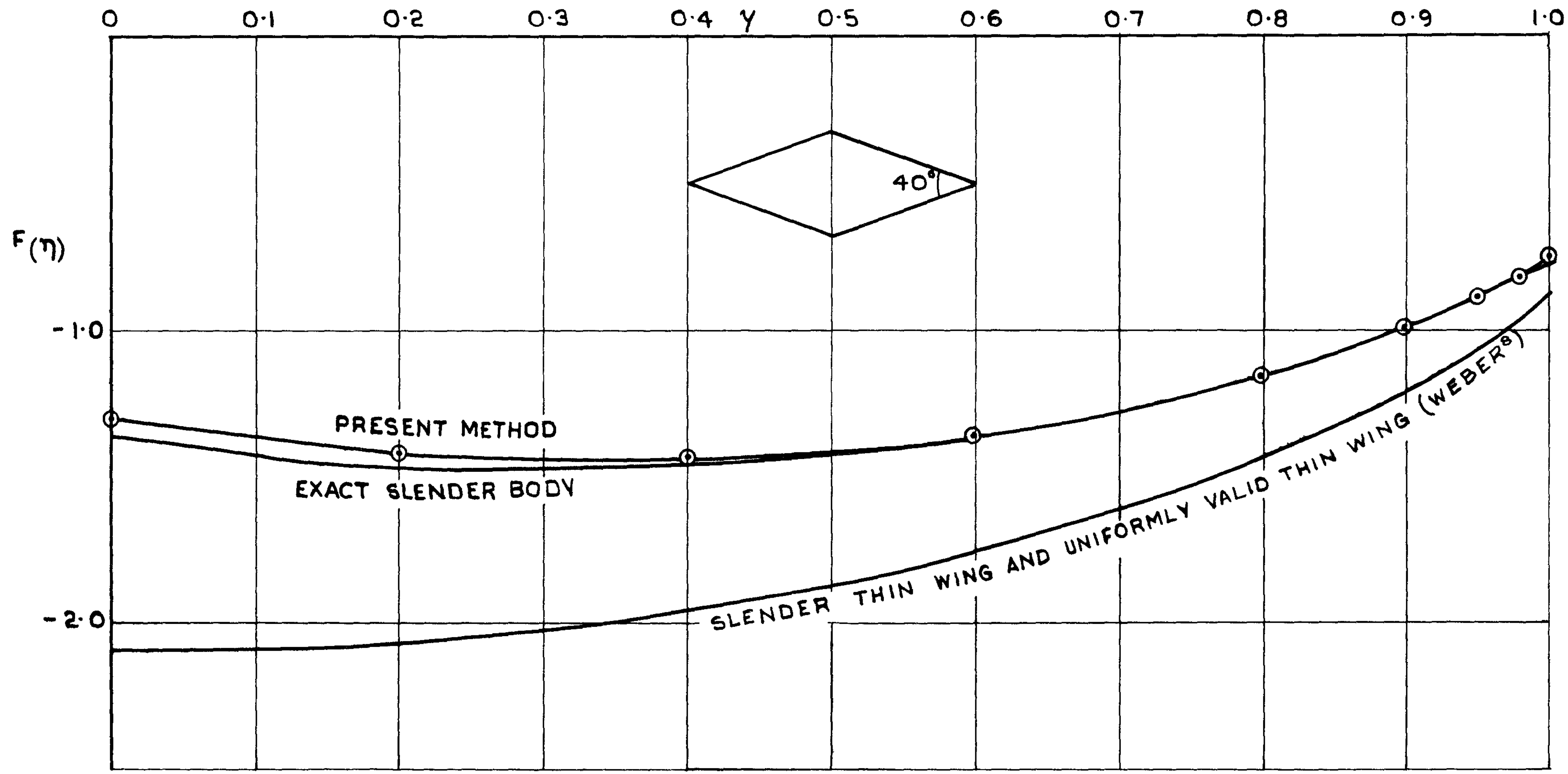

FIG 4(b). F(η) FOR A RHOMBIC CONE OF EDGE ANGLE 40°.

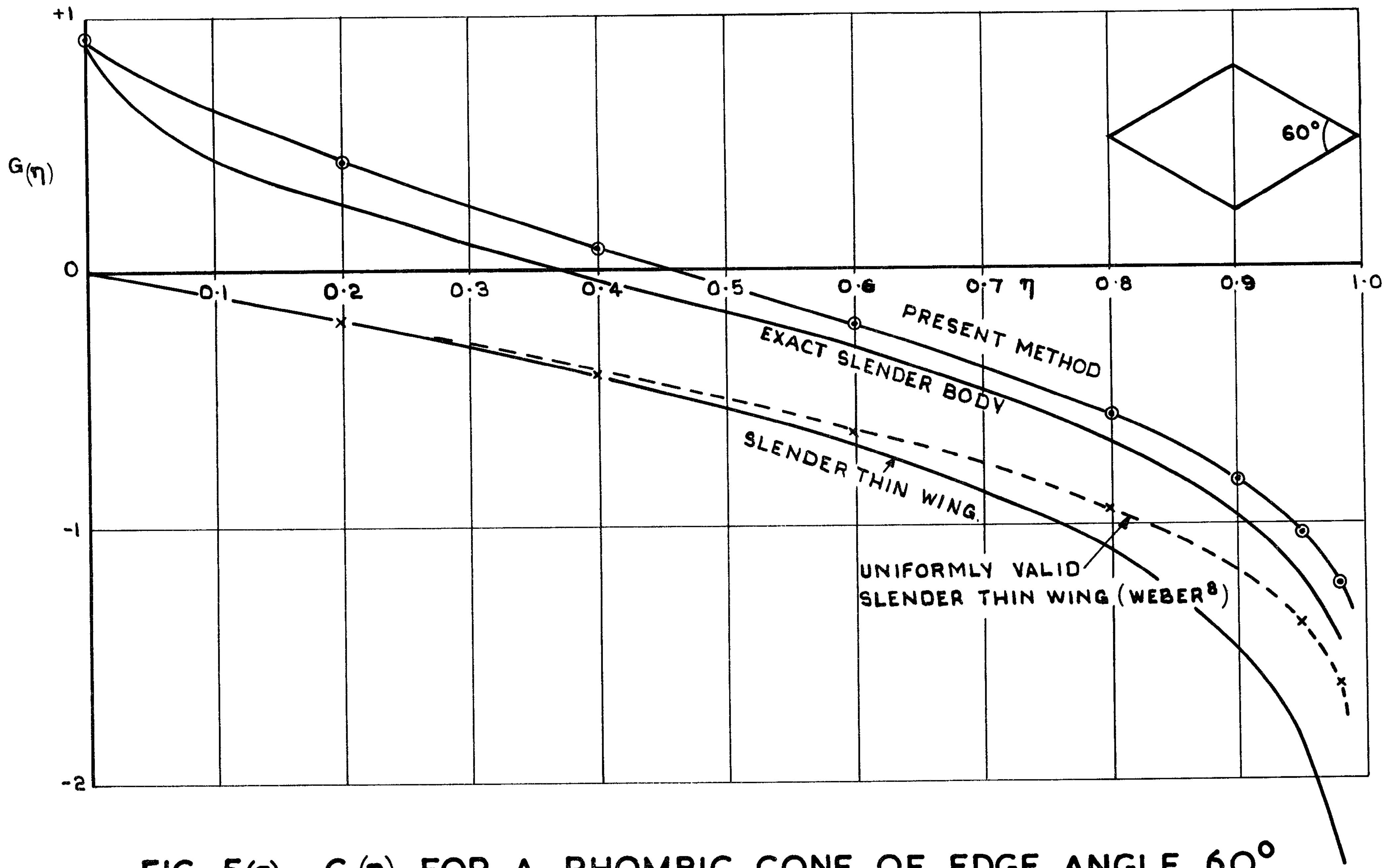

FIG. 5(a). G (η) FOR A RHOMBIC CONE OF EDGE ANGLE 60°.

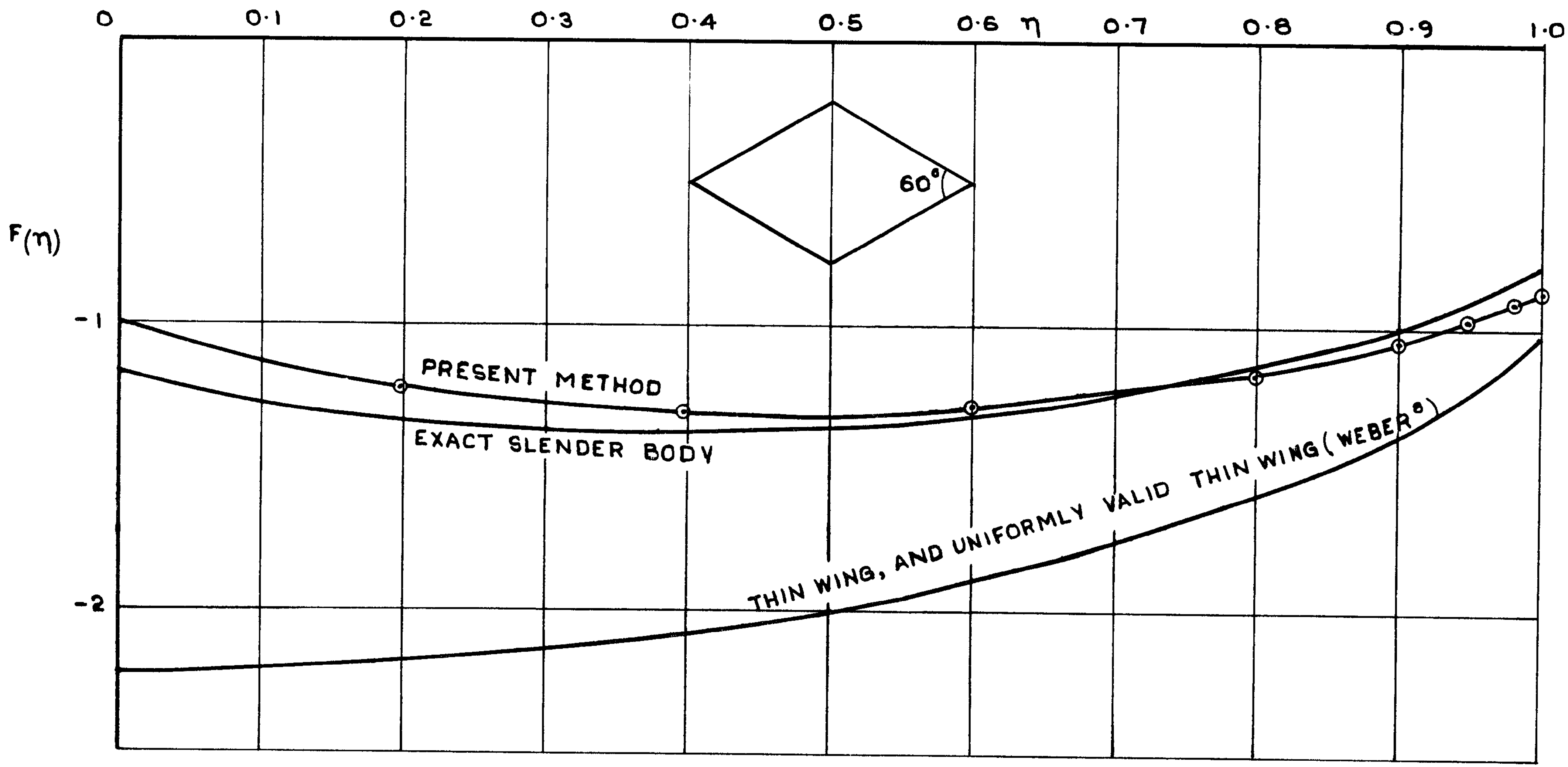

FIG. 5(b) $F(\eta)$ FOR A RHOMBIC CONE OF EDGE ANGLE 60°.

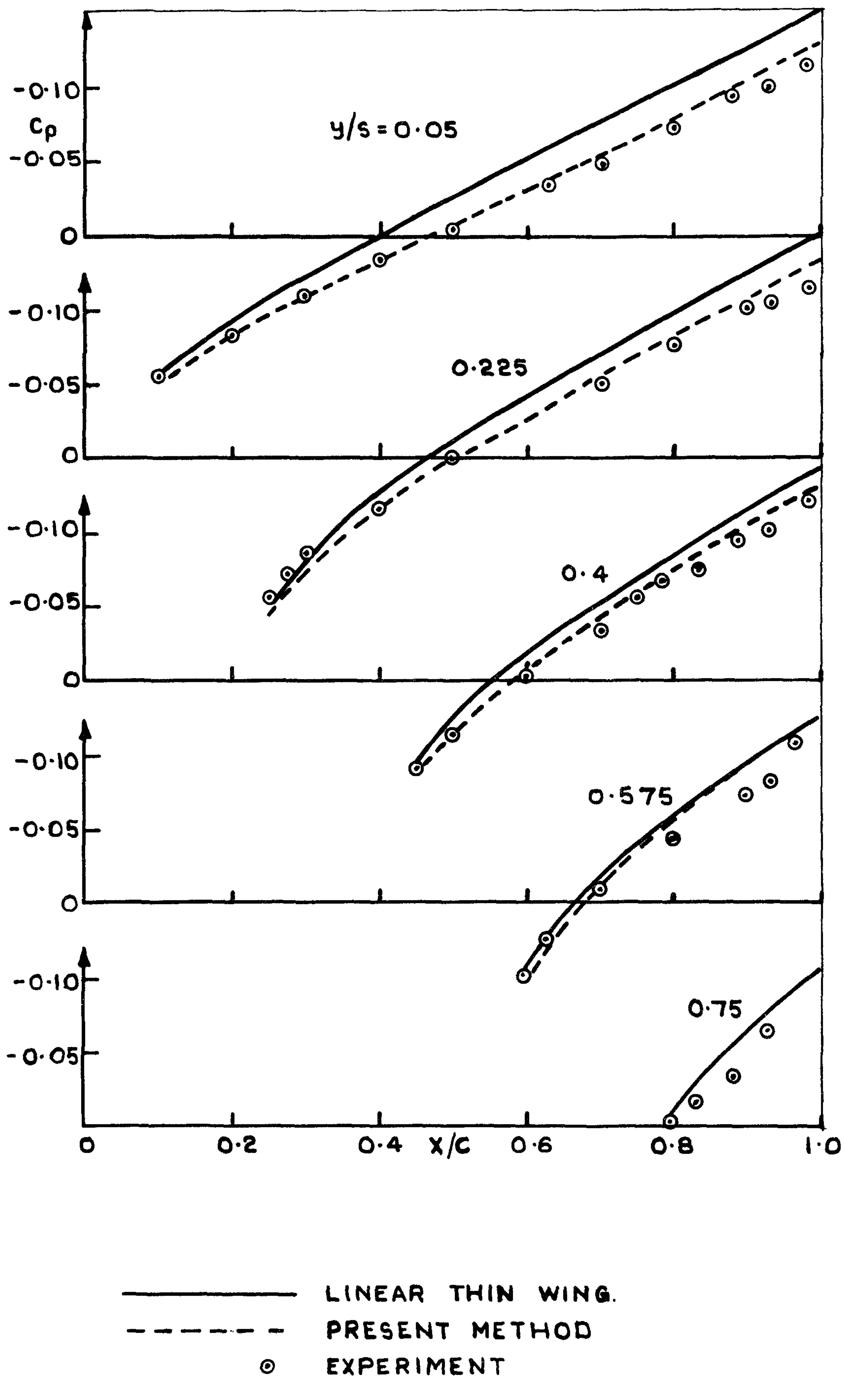

FIG. 6. PRESSURE DISTRIBUTION ON WING I. M=2.

(TWO METHODS INDISTINGUISHABLE FOR y/s = 0·75.)

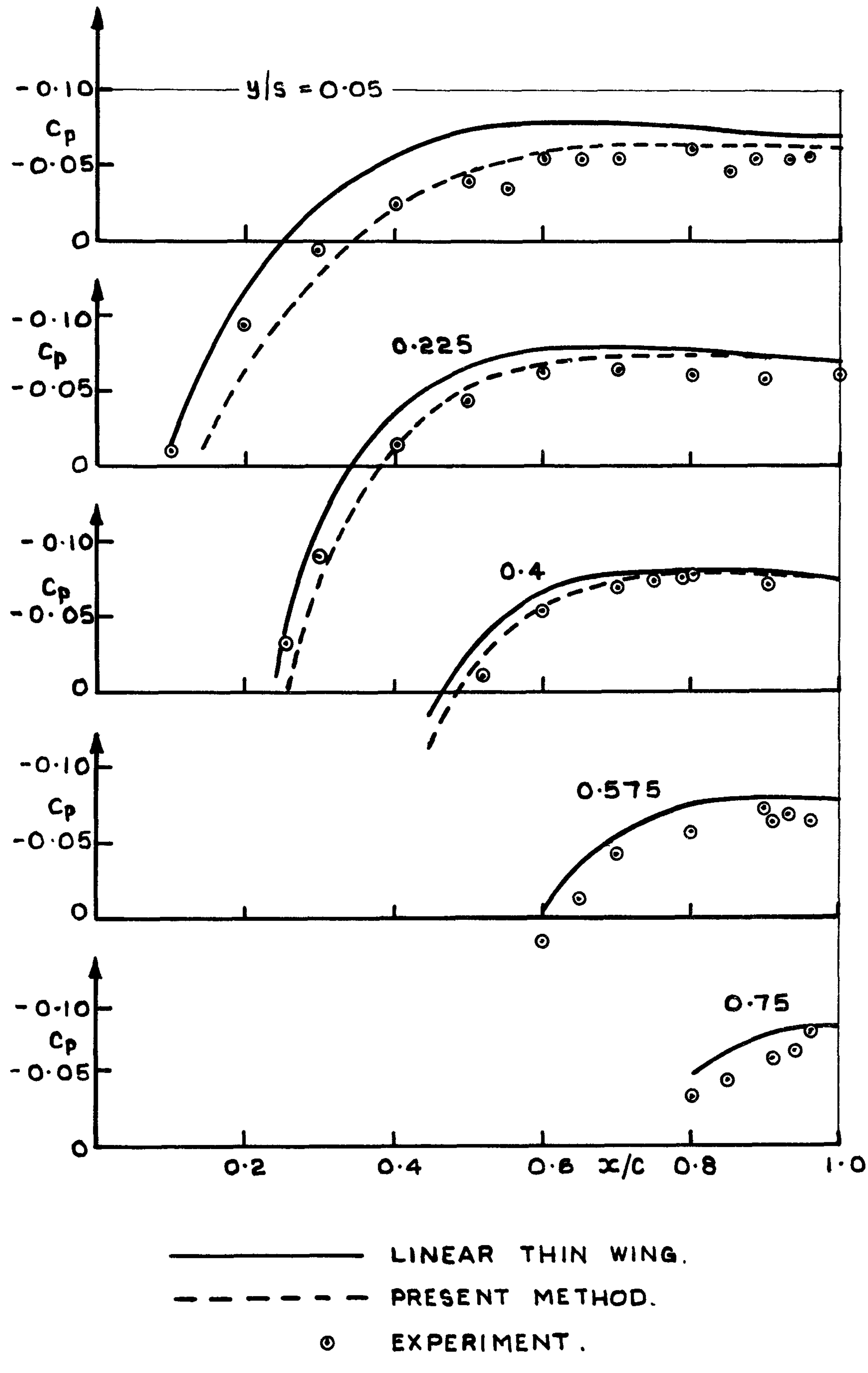

FIG. 7. PRESSURE DISTRIBUTION ON WING V. M=2.
(TWO METHODS INDISTINGUISHABLE FOR y/s = 0·575 AND 0·75.)

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