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On the Numerical Calculation of Theodorsen's Transformation

By

B. Thwaites

LONDON: HER MAJESTY'S STATIONERY OFFICE

1964

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Theodorsen's Transformation

- By -
B. Thwaites

January, 1963

SUMMARY

In applied mathematics, many problems which are describable by the two-dimensional Laplace equation reduce to the determination of a conformal transformation between some prescribed region and one of standard shape. In such problems, it is a common practice to use simple subsidiary transformations in such a way that the final mapping to be determined is between a near-circle and a circle. The theory of Theodorsen (Ref.1) is then, as often as not, applicable.

In the belief that, simple as it is, Theodorsen's method should be available "off the shelf" as a standard computer programme, the author has prepared this note which might serve as the basis of such a standard routine. The particular programme reproduced in the Appendix is for a Ferranti Pegasus. Possibly the only point of serious mathematical interest in the paper concerns the rearrangement of the integral equation to make it amenable to the numerical techniques employed.

1. Notation and Conventions

ζ complex variable in the plane of the near-circle

$e^{\psi+i\theta}$ point on the near-circle

$\psi=\psi(\theta)$ the function which follows from the prescribed shape of the near-circle.

The origin in the ζ -plane is assumed to be chosen so that the deviation of $\psi(\theta)$ from some constant value is, in some sense, roughly a minimum.

It is convenient, though not necessary, for the scales in the ζ -plane to be chosen so that the average value of $\psi(\theta)$ is roughly zero.

Only shapes for which $\psi(\theta)$ is a single-valued function are considered, and for which $\psi(\theta)$ has continuous derivatives of all orders.

z complex variable in the plane of the circle

$e^{\bar{\psi}+i\phi}$ point on the circle whose radius is therefore $e^{\bar{\psi}}$

$\theta/$

θ and ϕ are both used, according to the context, as independent variables, though they are, of course, functionally related.

Any variable, when regarded as a function of θ , is denoted by a lower-case letter; and when one of ϕ , by an upper-case letter. Thus, for example, $\Psi(\phi) = \psi(\theta)$.

$E(\phi) \equiv \varepsilon(\theta) \equiv \phi - \theta$, in which θ and ϕ take corresponding values. Thus $\theta = \theta(\phi) = \phi - E(\phi)$ and $\phi = \phi(\theta) = \theta + \varepsilon(\theta)$.

$$\theta_m = \phi_m = m\pi/2N, \quad m = 0, 1, 2, \dots, 2N - 1.$$

Suffix m on a function denotes its value at θ_m or ϕ_m according to context.

$(f)_D$, $(f)_R$, $(f)_C$ denote, respectively, diagonal, row and column matrices whose elements are the $2N$ values $f_m \equiv f(\theta_m)$, $m = 0, 1, 2, \dots, 2N - 1$.

(C) a $2N \times 2N$ matrix, the conjugate matrix operator

(D) a $2N \times 2N$ matrix, the differential matrix operator

$(A)_R \equiv (1 \ 1 \ 1 \ \dots \ 1)/2N$, the mean-value matrix operator.

It is required to map the regions exterior to the two closed boundaries so that

$$\frac{d\zeta}{dz} \rightarrow 1 \quad \text{as } z \rightarrow \infty \quad \dots (1)$$

Fig.1 illustrates some of the notation.

2. Poisson's Integrals

Consider a function $f(z)$ expandible in the region $|z| \geq e^{\bar{\psi}}$ as a

Laurent series $\sum_{n=1}^{\infty} a_n z^{-n}$, and denote by $f(\phi)$ its value at $z = e^{\bar{\psi} + i\phi}$.

Cauchy's formula, when applied to the region between the circle $|z| = R$, $R > \bar{\psi}$ and the circle $|z| = e^{\bar{\psi}}$ indented outwards at $z = e^{\bar{\psi} + i\phi}$ as shown in Fig.2, gives in the limits $R \rightarrow \infty$ and $\delta \rightarrow 0$

$$f(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi + \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot \frac{t-\phi}{2} dt, \quad \dots (2)$$

where the Cauchy principal value of the last integral is taken.

On taking the mean value over $0 \leq \phi \leq 2\pi$ of Equation (2) we have

$$\int_0^{2\pi} f(\phi) d\phi = 0 \quad \dots (3)$$

and hence, from (2),

$$f(\phi) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \cot \frac{t-\phi}{2} dt \quad \dots (4)$$

If $f(\phi) = u(\phi) + iv(\phi) \quad \dots (5)$

then the imaginary part of Equation (4) gives

$$-v(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} u(t) \cot \frac{t-\phi}{2} dt, \quad \dots (6)$$

while the real part of (3) gives

$$\int_0^{2\pi} u(\phi) d\phi = 0. \quad \dots (7)$$

3. Theodoresen's Theory

The transformation between the two regions shown in Fig.1 is written in the form

$$\log(\zeta/z) = f(z). \quad \dots (8)$$

From Equation (1), $f(z) \rightarrow 0$ as $z \rightarrow \infty$ and it is assumed that $f(z)$ satisfies the same conditions as the function in Section 2.

With the values $\zeta = e^{\psi+i\theta}$ and $z = e^{\bar{\psi}+i\phi}$ on the boundaries, Equation (8) may be rewritten as

$$\psi - \bar{\psi} - i(\phi-\theta) = \text{value of } f(z) \text{ on the circle } |z| = e^{\bar{\psi}} \quad \dots (9)$$

and thus, from Equations (6) and (7),

$$\phi - \theta \equiv E(\phi) = -\frac{1}{2\pi} \int_0^{2\pi} \Psi(t) \cot \frac{t-\phi}{2} dt \quad \dots (10)$$

and

$$\bar{\psi} = \frac{1}{2\pi} \int_0^{2\pi} \Psi(\phi) d\phi. \quad \dots (11)$$

These are the two central equations in the mapping problem, put in terms of the unknown function $\Psi(\phi)$ rather than the known function $\psi(\theta)$. The form of Equation (10), rewritten as a non-linear integral equation for $\psi(\theta)$ and involving $\psi(\theta)$, is familiar enough, but is not particularly amenable to iterative solution.

The modulus of the transformation on the boundaries is easily obtained from Equation (8) as

$$\left| \frac{d\zeta}{dz} \right| = e^{\psi(\theta) - \bar{\psi}} \frac{\{1 + [\psi'(\theta)]^2\}^{\frac{1}{2}}}{1 + \epsilon'(\theta)} \quad \dots (12)$$

The whole problem may be summarised thus: given $\psi(\theta)$, find $\epsilon(\theta)$ and $\bar{\psi}$ from a solution of Equations (10) and (11), and hence evaluate the required $|d\zeta/dz|$. But we must next discuss the practical side to the question.

4. The Practical Problem and its Solution Outlined

It is possible, of course, to go further than was implied in the preceding paragraph. For example, once $\Psi(\phi)$ and $E(\phi)$ are known, Cauchy's formula will give the value of $f(z)$, and hence the transformation, over the whole domain. In most practical applications, however, the greatest interest is attached to conditions on the boundaries and we shall in fact confine ourselves to these.

In practice also, $\psi(\theta)$ will almost invariably be specified as a set of numerical values rather than a closed function, and we suppose that ψ is given at M points, $\theta = \theta_r$, $r = 0, 1, 2, \dots, (M - 1)$. For the numerical solution of the problem, however, it is necessary to work in terms of equally-spaced values of θ (and of ϕ), and therefore to interpolate values for ψ at $2N$ equally-spaced values of θ . Each problem may suggest its own preferred method of interpolation, but the present author would normally use a Fourier-series interpolation. Thus, if M is even and equal to $2N$, we take

$$\psi(\theta) \simeq A_0 + \sum_{n=1}^{N-1} (A_n \cos n\theta + B_n \sin n\theta) + A_N \cos N\theta \quad \dots (13)$$

while if M is odd and equal to $(2N + 1)$, a further term, say $A_{N+1} \cos(N+1)\theta$, may be added to the series in (13). The coefficients A_n and B_n are evaluated so that the series takes the given values of ψ at the points θ_r , whereupon values at the $2N$ points

$$\theta = \theta_m = m\pi/N, \quad m = 0, 1, 2, \dots, (2N-1) \quad \dots (14)$$

are calculated at which

$$\psi = \psi_m = \psi(\theta_m). \quad \dots (15)$$

This interpolation is very easily programmed, and a Pegasus programme is available.

It is assumed henceforth that the $2N$ values ψ_m are known, and that it is required to determine the points on the circle which correspond to $\theta = \theta_m$ and the value of $|d\zeta/dz|$ at those points.

But/

But if equally-spaced values of θ specify the given function ψ , equally-spaced values of ϕ are needed for the computation of the singular integral (10). Now the standard first approximation to Theodorsen's theory takes $\psi(\theta) = \Psi(\theta)$ and $E(\phi) = \epsilon(\phi)$; but we aim at a solution of pre-specified error and so we need to relate more exactly values at equally-spaced θ to values at equally-spaced ϕ . The simple equations which now follow are the only novel feature in this paper and we may remark that they arise solely from the exigencies of computational methods.

The suffix m will be used on ϕ in the same way as it is on θ , namely

$$\phi_m = m\pi/N, \quad m = 0, 1, 2, \dots, (2N-1). \quad \dots (16)$$

Thus, $\theta_m = \phi_m$. It is, of course, important to appreciate that the suffix m , when attached to functions, does not denote correspondence: thus in general $\epsilon_m \neq E_m$ and $\psi_m \neq \Psi_m$. In fact, for ψ , we have the following equations:

$$\Psi_m = \Psi(\phi_m) = \psi(\phi_m - E(\phi_m)) = \psi(\phi_m - E_m), \quad \dots (17)$$

and a Taylor expansion of this gives

$$\Psi_m = \psi_m + \sum_{n=1}^{\infty} (-E_m)^n \psi_m^{(n)}/n! \quad \dots (18)$$

in which $\psi_m^{(n)} = [d^n \psi(\theta)/d\theta^n]_{\theta=\theta_m}$ and is regarded as known.

Equations (10) and (18) now become simultaneous equations for the $2N$ -dimensional vectors (Ψ_m) and (E_m) and here we assume that the obvious iterative procedure - starting with (ψ_m) as a first approximation to (Ψ_m) , calculating (E_m) from (10) and hence a better approximation for (Ψ_m) from the first so-many terms of (18), and so on - will converge. Nevertheless further study of alternative iterations might well be rewarding in terms of rapidity of convergence.

Now that (E_m) may be thought of as known, ϵ_m may be determined from the Equation

$$\epsilon_m = \epsilon(\theta_m) = E(\theta_m + \epsilon(\theta_m)) = E(\theta_m + \epsilon_m), \quad \dots (19)$$

which, in expanded form gives

$$\epsilon_m = E_m + \sum_{n=1}^{\infty} (\epsilon_m)^n E_m^{(n)}/n! \quad \dots (20)$$

in which $E_m^{(n)} = [d^n E(\phi)/d\phi^n]_{\phi=\phi_m}$ and is regarded now as known. An iteration between the two sides of the Equation, starting with E_m as the first approximation to ϵ_m uses almost the same process as that for the solution of (18); thus, part of the computer programme can be used equally for Equations (18) and (20), and it is probably better not to use the formal solution of (20) in series.

The solution to the mapping problem has been found in terms of the equally-spaced values θ_m and the derivative $\left| \frac{d\zeta}{dz} \right|$ follows at once from Equation (12).

5. Numerical Methods

Theodorsen's original treatment of the problem was based on Fourier representations of Ψ and E . Thus,

$$\left. \begin{aligned} \Psi(\theta) &= \bar{\Psi} + \sum_{n=1}^N (A_n \cos n\phi + B_n \sin n\phi), \\ E(\phi) &= \sum_{n=1}^N (A_n \sin n\phi - B_n \cos n\phi). \end{aligned} \right\} \dots (21)$$

His treatment, however, involved the determination of the Fourier coefficients and in this respect the comprehensive formulae discovered by Watson³ (1945) were a great advance. The full power of these formulae was exploited by Thwaites² (1961) who developed a matrix technique highly suitable for electronic computers. The reader is referred to the latter paper for details; here we only summarise the results needed for the present problem.

Using the matrix notation described in Section 1, we may write

$$(E)_C = (C)(\Psi)_C \dots (22)$$

$$(E^{(1)})_C = (D)(E)_C \dots (23)$$

$$(\bar{\Psi})_C = (A)_R(\Psi)_C \dots (24)$$

where (C) , (D) and $(A)_R$ are matrices which may be calculated once and for all from Watson's formulae. These three Equations - and others derived from them, such as the obvious extension of Equation (23) to the calculation of higher derivatives - are exact when $\bar{\Psi}(\phi)$ and $E(\phi)$ are representable by the finite Fourier series (21) and the matrices are the appropriate size.

The two fundamental Equations, (10) and (18), may thus be written in matrix form as follows:

$$(E)_C = (C)(\Psi)_C, \dots (25)$$

$$(\Psi)_C = (\psi)_C + \sum_{n=1}^{\infty} [(E)_D]^{n-1} (-D)^n (\psi)_C / n!, \dots (26)$$

and the iteration already described in Section 4 is very simply carried out with a matrix interpretive scheme such as that for Pegasus. The coefficients $(-D)^n (\psi)_C / n!$ remain, of course, as constant column matrices throughout the iteration.

Once (Ψ) is determined, $\bar{\Psi}$ is calculated by the matrix equivalent of (11), namely

$$\bar{\Psi} = (A)_R(\Psi)_C \dots (27)$$

Also, $(\epsilon)_C$ follows from the iteration of Equation (20), and $(\epsilon')_C$ as $(D)(\epsilon)_C$. The modulus of the transformation can then be calculated from Equation (12).

A Pegasus programme for the complete procedure, starting from a table of values of ψ at equally-spaced values of θ , is described and given in full in the Appendix. It uses the Automat Scheme which is a combination of the standard Autocode and Matrix Interpretive Schemes. It was found that little advantage in machine time is gained by adjusting the number of terms in the expansions (18) and (20) according to the accuracy required in each case, and so the final programme is set to include first eleven terms of these series. This number of terms was determined partly by the intuitive conviction that no physical problem is likely to arise which would require more terms for the desired accuracy, and partly by the maximum number of points it is thought would normally be used around a contour. The storage capacity of Pegasus I is just adequate for 40 points and the programme input (on TAPE B) specifies this number of points.

The second number n_0 on the TAPE B input determines the final accuracy in the convergence of the iteration of Equations (18) and (20); more precisely the iterations cease when the norms of the vectors in successive iterations are equal to n_0 binary places.

6. An Example

To test the accuracy of the numerical analysis, one wants to choose a shape for which the transformation, while being exactly calculable, is not reducible to the finite Fourier series (21). An amusing and highly suitable example is a circle whose centre is not at the origin of co-ordinates. For this, as a final outcome, the derivative $|dz/dz|$ on the boundary is exactly unity.

In the notation of Fig.3,

$$\psi(\theta) = \log \{ \sqrt{1 - \delta^2 \sin^2 \theta} - \delta \cos \theta \}, \dots (28)$$

$$\epsilon(\theta) = - \sin^{-1} \{ \delta \sin \theta \}, \dots (29)$$

$$\Psi(\theta) = \frac{1}{2} \log \{ 1 + \delta^2 - 2\delta \cos \phi \}, \dots (30)$$

$$E(\phi) = - \tan^{-1} \left\{ \frac{\delta \sin \phi}{1 - \delta \cos \phi} \right\} \dots (31)$$

and $\bar{\Psi} = 1. \dots (32)$

It may be verified, of course, that $\Psi(\phi)$ and $E(\phi)$ are conjugate functions; also that $|dz/dz|$ as given by Equations (12), (28) and (29) is unity.

In any particular case, it is very difficult to disentangle, theoretically, the inaccuracies due respectively to the finite number of terms taken in the Taylor series (18) and (20), to the approximations of Watson's formulae, and to the degree of accuracy specified in the iteration. But for a given degree of accuracy, say of 4/5 significant decimal figures corresponding to a value of n_0 of 15, there is a value of $2N$, the total number of points, beyond which it is not profitable to go. This is shown up in the following Table in which it is clear that, to the degree of accuracy specified in the iteration, 20 points are quite sufficient.

$\frac{50}{\pi}$	$2N = \text{Number of Points}$		
	10	20	40
0	0.9953	0.9999	0.9999
1	1.0037	1.0000	1.0000
2	0.9984	0.9999	0.9999
3	0.9996	1.0001	1.0001
4	1.0021	1.0000	1.0000
5	0.9972	1.0000	1.0000

Table of rounded values of $|dZ/dz|$ for the case of a circle displaced by a third of the radius (for which the exact value is unity), the iterations ceasing at an accuracy of 4/5 decimal places.

For this special case of the displaced circle it has been found that, with 40 points (and with the first ten terms in the Taylor series), values correct to at least eight decimal places are obtainable. When therefore one considers that, in most aeronautical applications, errors of 0.1% are acceptable, it is thought that the programme developed here will be adequate for most cases.

7. Acknowledgment

I am grateful to Mr. W. S. Hall of Southampton University for his assistance with the programming.

References/

References

<u>No.</u>	<u>Author</u>	<u>Title, etc.</u>
1	T. Theodorsen	Theory of wing sections of arbitrary shape. N.A.C.A. Report 411. 1932.
2	B. Thwaites	The aerodynamic theory of sails I. Proc. Roy. Soc. A, Vol.261, p.402. 1961.
3	E. J. Watson	Formulae for the computation of the functions employed for calculating the velocity distribution about a given aerofoil. A.R.C. R & M 2176. May, 1945.

APPENDIX

APPENDIX

The programme which follows uses the AUTOMAT scheme which has been developed at Southampton University as a combination of the Autocode and the Matrix Interpretive schemes for the Ferranti Pegasus.

It is applicable to a function $\psi(\theta)$ which is given at an even number of equally-spaced values of θ through the period 2π ; thus it is assumed that an interpolation procedure has initially been carried through, as suggested in Section 4 of the paper.

The tapes required for the main tape reader are, in order:

- (i) the AUTOMAT scheme;
- (ii) a Matrix Interpretive scheme preset parameter tape, giving the value of $2N$, the number of points to be taken;
- (iii) the programme tape given below;
- (iv) an Autocode data tape given the value of $2N$;
- (v) an Autocode data tape, listing the values of ψ_m .

For the second tape reader, an autocode data tape is required carrying a single number, namely the number of significant binary digits required in the iterations.

The outputs from the programme are, in turn, $\Psi(\phi)$, $E(\phi)$, $\varepsilon(0)$ and $|dz/dz|$.

An outline flow diagram is also given.

PEGASUS AUTOMAT

D
N
THEODORSENS TRANSFORMATION

Notes

Number of
matrix
instruc-
tion

J7090-1
J512
O 7168

<p>07060 (100,24IXI)→100 07060 (1700,24IXI)→1700 07101</p>	<p>Completion of (C) and (D) matrices. (C) from location 100; (D) from location 1700.</p>
<p>(0,24IXI)→3300 (3300,24IXI)→3940 (3300,IX24I)X(3300,24IXI)→48 07101</p>	<p>Read in $\phi(\theta)$, from location 3300. ϕ in location 48.</p>
<p>10 07060 (1700,24IX24I)X(3300,24IXI)→3300 07060 (12)X(3340,24IXI)→3300 07101</p>	<p>Calculation of $(-1)^n \phi^{(n)} / n!$ if $n28 = 0$ or $E^{(n)} / n!$ if $n28 = 1$ from location (3260 + 40n).</p>
<p>15 (100,24IX24I)X(3940,24IXI)→3980 (3980,24IXI)→4020 (4020,24I/)X(3340,24IXI)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3380,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3420,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3460,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3500,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3540,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3580,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3620,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3660,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3980,24I/)X(4020,24IXI)→4020 (4020,24I/)X(3700,24IXI)→4060 (3940,24I/)+(4060,24I/)→3940 (3940,24I/)+(3300,24I/)→3940 (3940,IX24I)X(3940,24IXI)→49</p>	<p>Calculation of 10 terms of series for $\psi(\phi)$ if $n28 = 0$ or $\epsilon(\theta)$ if $n28 = 1$ from location 3940. ψ or ϵ in location 49.</p>

48	07101 (3940,24IXI)→0	Print $\phi(\phi)$.																																				
	(50,IX24I)X(3940,24IXI)→99 (6)X(99)→99 (99)→0	Print $\bar{\psi}$.																																				
	(3300,24IXI)→3380 (100,24IX24I)X(3940,24IXI)→3300 (3300,24IXI)→n	Print $E(\phi)$.																																				
	(3380,24IXI)→140 (3300,24IXI)→3940 (3940,24IXI)→3980 (3340,24I/)X(3340,24IXI)→100 (50,24I/)+(100,24I/)→100	$1 + (\phi')^2$ in location 100.																																				
	(3300,IX24I)X(3300,24IXI)→48 07101	E in location 48.																																				
62	(3940,24IXI)→0 (1700,24IX24I)X(3940,24IXI)→180 (180,24I/)+(50,24I/)→180 07101	Print $\epsilon(\theta)$.																																				
66	(220,24IXI)→0 07101	Print $ d\zeta/dz $.																																				
68	(3940,24I/)→3980 J16																																					
	I J 320 J7090 J8	Initial Orders.																																				
	STOP	First order to be obeyed.																																				
	<p>$v_0 = -1$ $v_1 = \text{ARCCOS } v_0$ $v_2 = \text{TAPE}$ Read in 2N $v_3 = v_2/2$ $n_2 = v_3$ $n_1 = v_2$ $v_4 = v_1/v_2$ $v_5 = v_4 \times 2$ $v_6 = 1/v_2$</p> <p>$n_3 = \text{TAPEB}$ Read in iteration accuracy $n_0 = n_3$</p>	<table border="1"> <thead> <tr> <th colspan="2">Constants</th> <th>v</th> <th>n</th> </tr> </thead> <tbody> <tr> <td>π</td> <td>in location</td> <td>1</td> <td></td> </tr> <tr> <td>2N</td> <td></td> <td>2</td> <td></td> </tr> <tr> <td>N</td> <td></td> <td>3</td> <td></td> </tr> <tr> <td>N</td> <td></td> <td></td> <td>2</td> </tr> <tr> <td>2N</td> <td></td> <td></td> <td>1</td> </tr> <tr> <td>$\pi/2N$</td> <td></td> <td>4</td> <td></td> </tr> <tr> <td>π/N</td> <td></td> <td>5</td> <td></td> </tr> <tr> <td>$1/2N$</td> <td></td> <td>6</td> <td></td> </tr> </tbody> </table>	Constants		v	n	π	in location	1		2N		2		N		3		N			2	2N			1	$\pi/2N$		4		π/N		5		$1/2N$		6	
Constants		v	n																																			
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$1/2N$		6																																				
	<p>$n_3 = 0$ $4) v(50+n_3) = 1$ $n_3 = n_3 + 1$ $\rightarrow 4, n_1 > n_3$</p>	Unit matrix from location 50.																																				
	<p>$v_{100} = 0$ $v(100+n_2) = 0$ $v_{1700} = 0$ $v(1700+n_2) = 0$ $v_{10} = v_4$ $n_3 = 1$ $v_{11} = -1$ $1) v_{12} = \text{COT } v_{10}$ $n_4 = n_1 - n_3$ $\rightarrow 2, v_{11} = -1$ $v(100+n_3) = 0$ $\rightarrow 6$ $2) v(100+n_3) = v_{12}/v_3$ $6) v(100+n_4) = -v(100+n_3)$ $v_{12} = v_{12} \times v_{11}$ $v(1700+n_3) = v_{12}/2$ $v(1700+n_4) = -v(1700+n_3)$ $v_{11} = -v_{11}$ $v_{10} = v_{10} + v_4$ $n_3 = n_3 + 1$ $\rightarrow 1, n_1 > n_3$</p>	Calculation of first columns of (C) and (D) matrices.																																				


```

n3=n1
n8=n1
3)V(100+n3)=V(99+n3)
V(1700+n3)=V(1699+n3)
n29=n3-n1
n31=n3+1
>M7103,1
n8=n8+1
n3=n3+n1
>3,n1>n8
V1700=0

```

Completion of (C) and (D) matrices.

```
>M7103,6
```

```
n28=0
```

Set for iteration of $\psi(\phi)$.

```

19)n3=1
n29=0
n30=0
n31=40
V11=1
17)>11,n28=0
V12=1/V11
>18
11)V12=-1/V11
18)>M7103,10
n30=n30+40
n31=n31+40
n3=n3+1
V11=V11+1
>17,10>n3

```

Calculation of $(-1)^n \phi^{(n)}/n!$ if $n28 = 0$
or $E^{(n)}/n!$ if $n28 = 1$.

```

5)>15,n28=1
>M7103,15
7)>14,V48=V49
V48=V49
>5

```

Iteration for $\psi(\phi)$, ceasing when inner products are unchanged to $n0$ binary significant figures.

```

14)>16,n28=1
>M7103,48
n28=1
>19

```

```

15)>M7103,68
>7

```

Iteration for $\epsilon(\theta)$.

```

16)V99=EXPV99
>M7103,62
n3=0
24)V10=EXPV(140+n3)
V10=V10/V99
V11=SQRTV(100+n3)
V10=V10*V11
V(220+n3)=V10/V(180+n3)
n3=n3+1
>24,n1>n3
>M7103,66

```

Calculation of $|d\xi/dz|$.

```

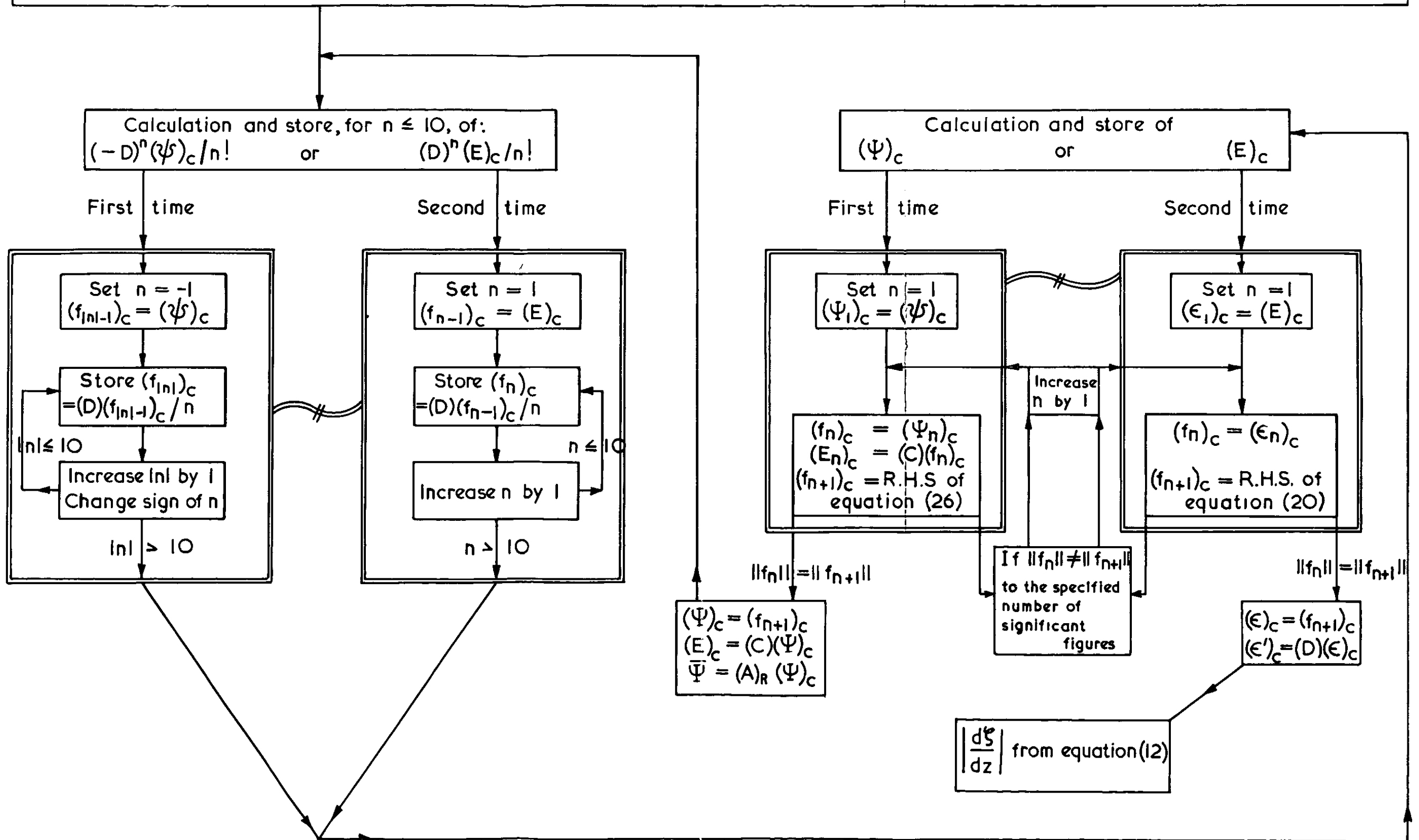
(→0)
XXXXXXXXXXXXXXXXXXXX

```


Outline flow diagram

Initially: set the number, $2N$, of values of ψ which are given, N being even. Set associated constants such as $\pi, N, 1/N, \pi/N, \pi/2N$. Set also the number of significant figures required for convergence of iterations. Read in the given values of ψ and store as the column matrix $(\psi)_c$.

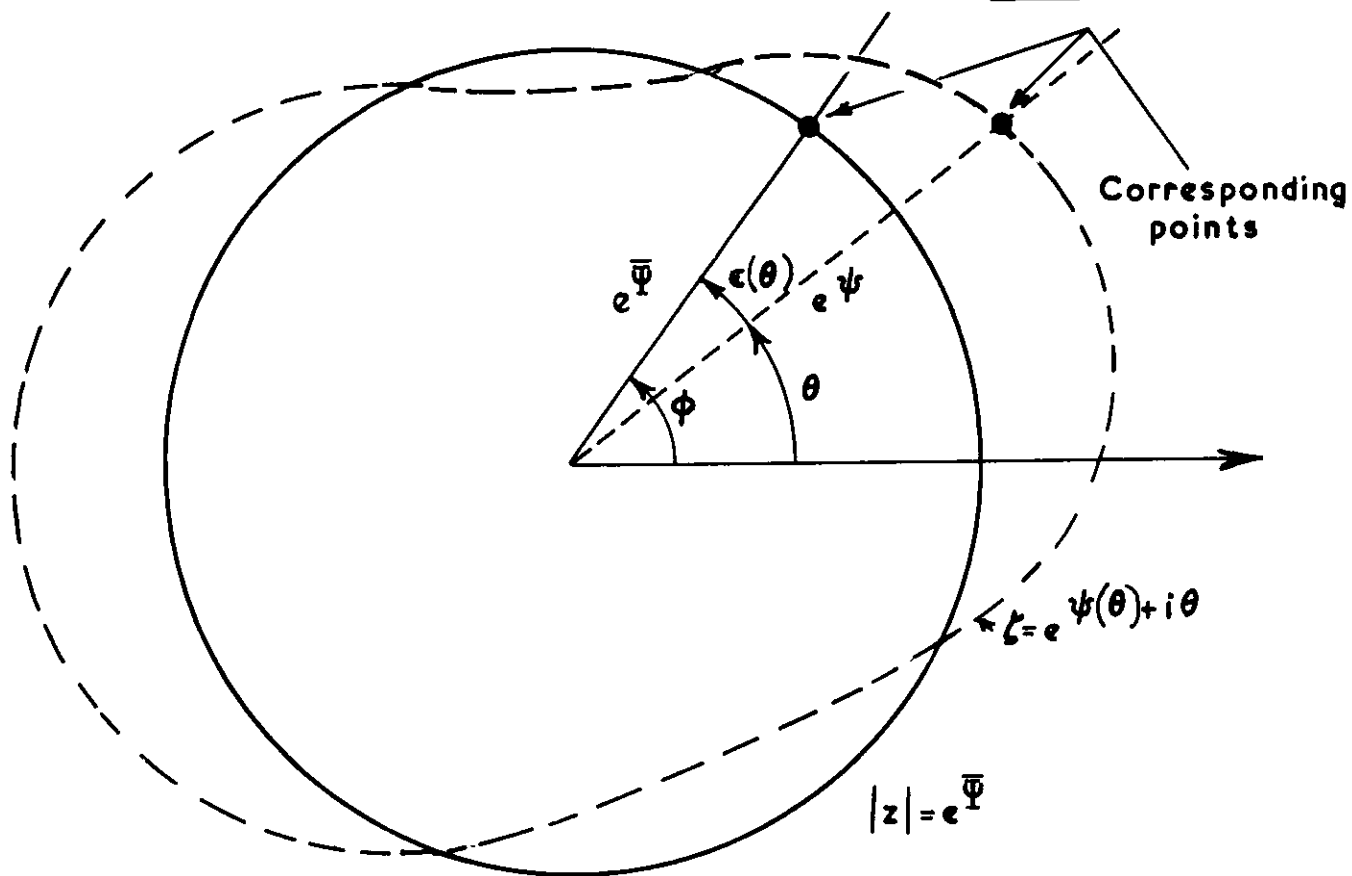
Calculate and store the elements of mean-value, differential and conjugation matrices, respectively $(A_R) = (A_i), (D) = (D_{i,j}), (C) = (C_{i,j}); i, j = 1, 2, \dots, 2N$
 $A_i = 1/2N; D_{1,1} = D_{2N,1} = 0; D_{i,1} = (-1)^i \frac{1}{2} \cot[(i-1)\pi/2N], i \neq 1, 2N; C_{i,1} = 0, i \text{ odd}; C_{i,1} = -(1/N) \cot[(i-1)\pi/2N], i \text{ even}$
 The remaining rows, $j = 2, \dots, 2N$, of (D) and (C) are given by the fact that the two matrices are circulants.



Note \rightleftharpoons joins the two subroutines containing almost analogous calculations, and a full programme would take advantage of this

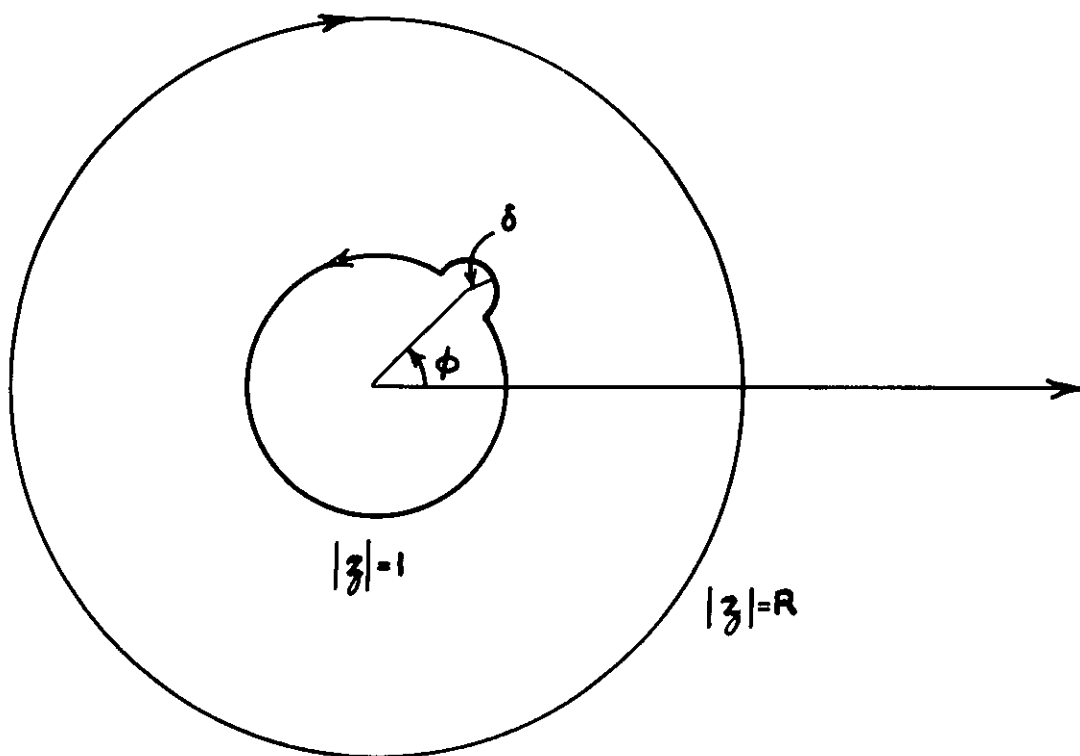
FIGS. 1 & 2

FIG. 1



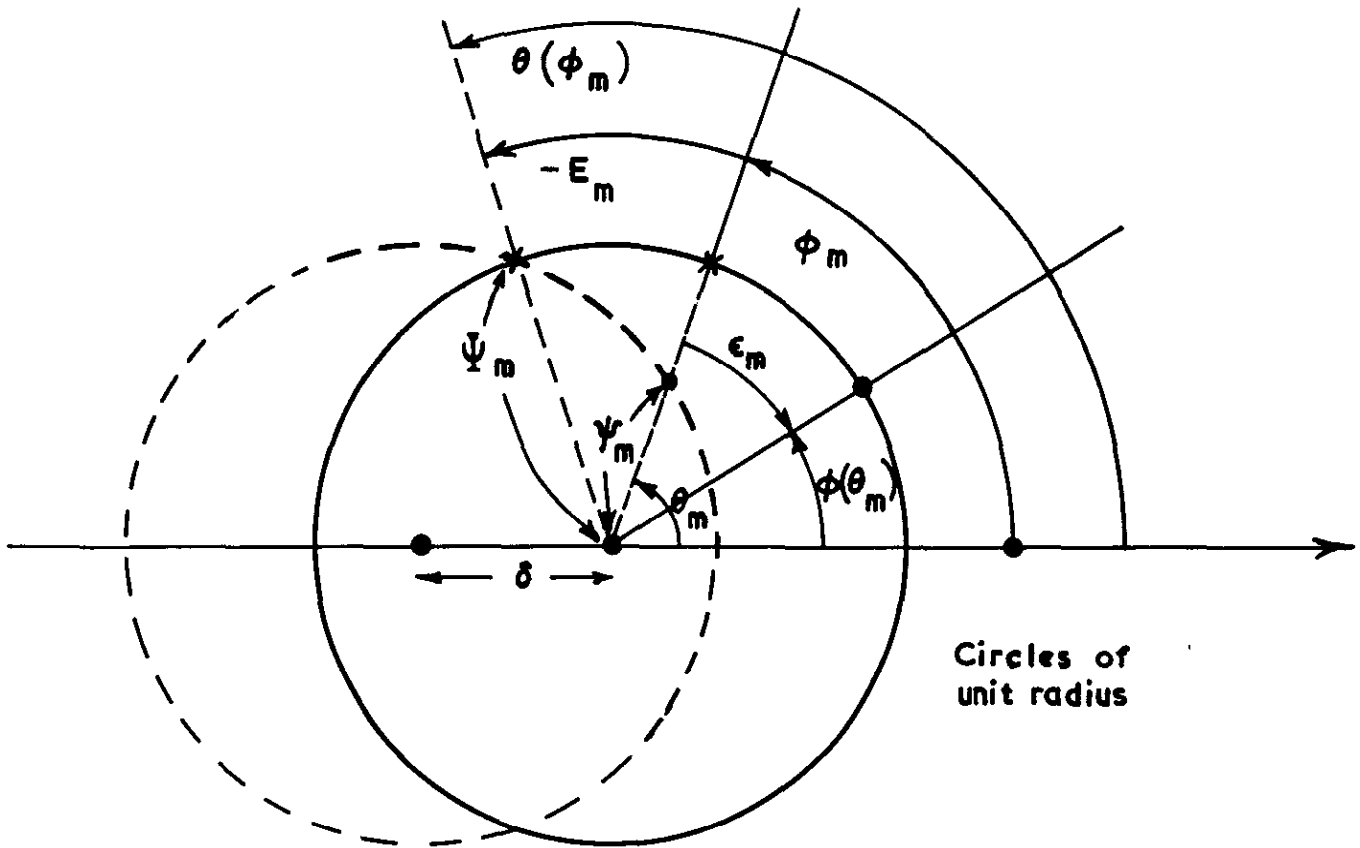
Notation

FIG. 2



Contour for Poisson's integrals

FIG. 3



Notation in the test case of a displaced circle

A.R.C. C.P.No.691
January, 1963.
Thwaites, B.

ON THE NUMERICAL CALCULATION OF
THEODORSEN'S TRANSFORMATION

In applied mathematics, many problems which are describable by the two-dimensional Laplace equation reduce to the determination of a conformal transformation between some prescribed region and one of standard shape. In such problems, it is a common practice to use simple subsidiary transformations in such a way that the final mapping to be determined is between a near-circle and a circle. The theory of Theodorsen (Ref.1) is then, as often as not, applicable.

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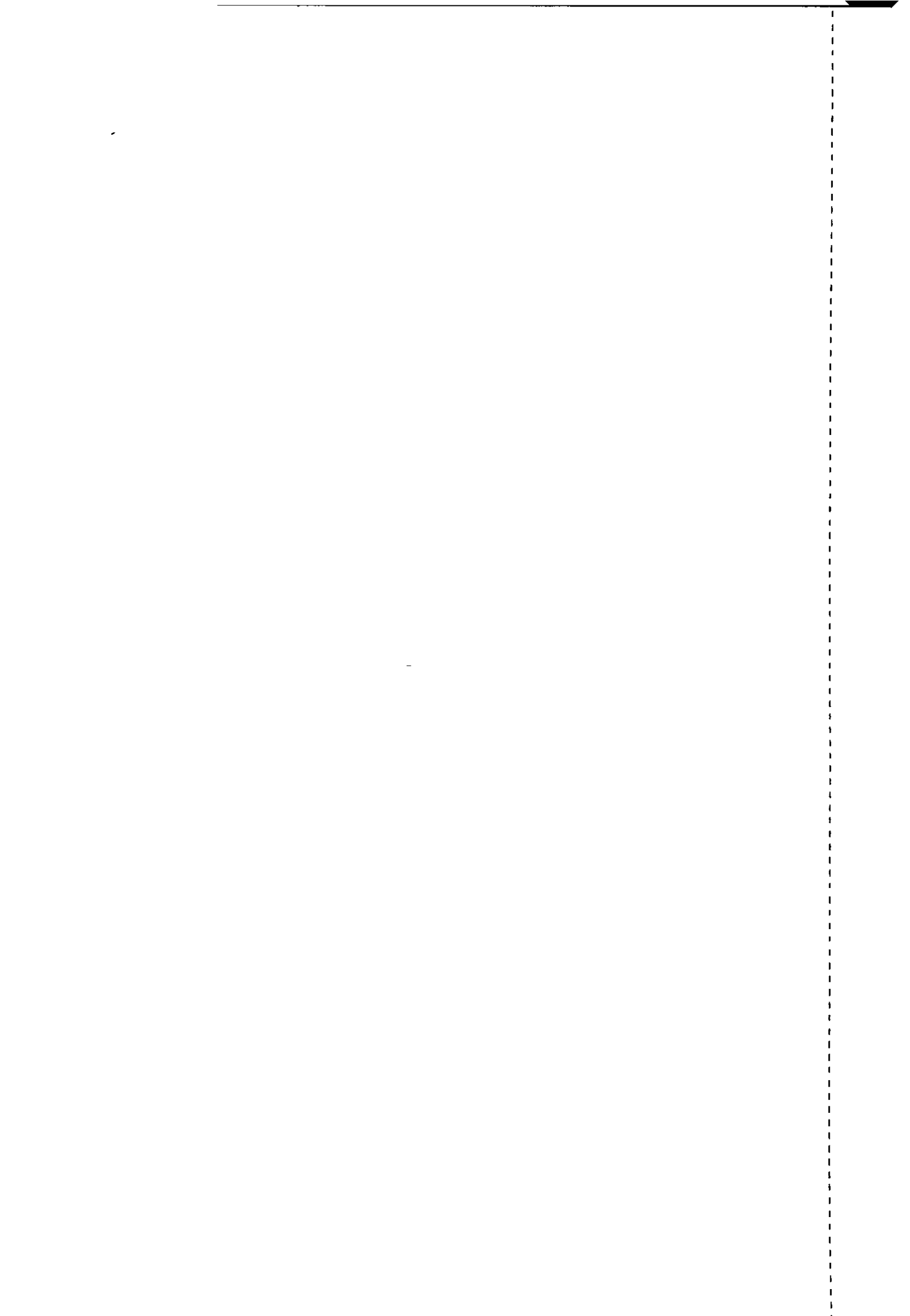
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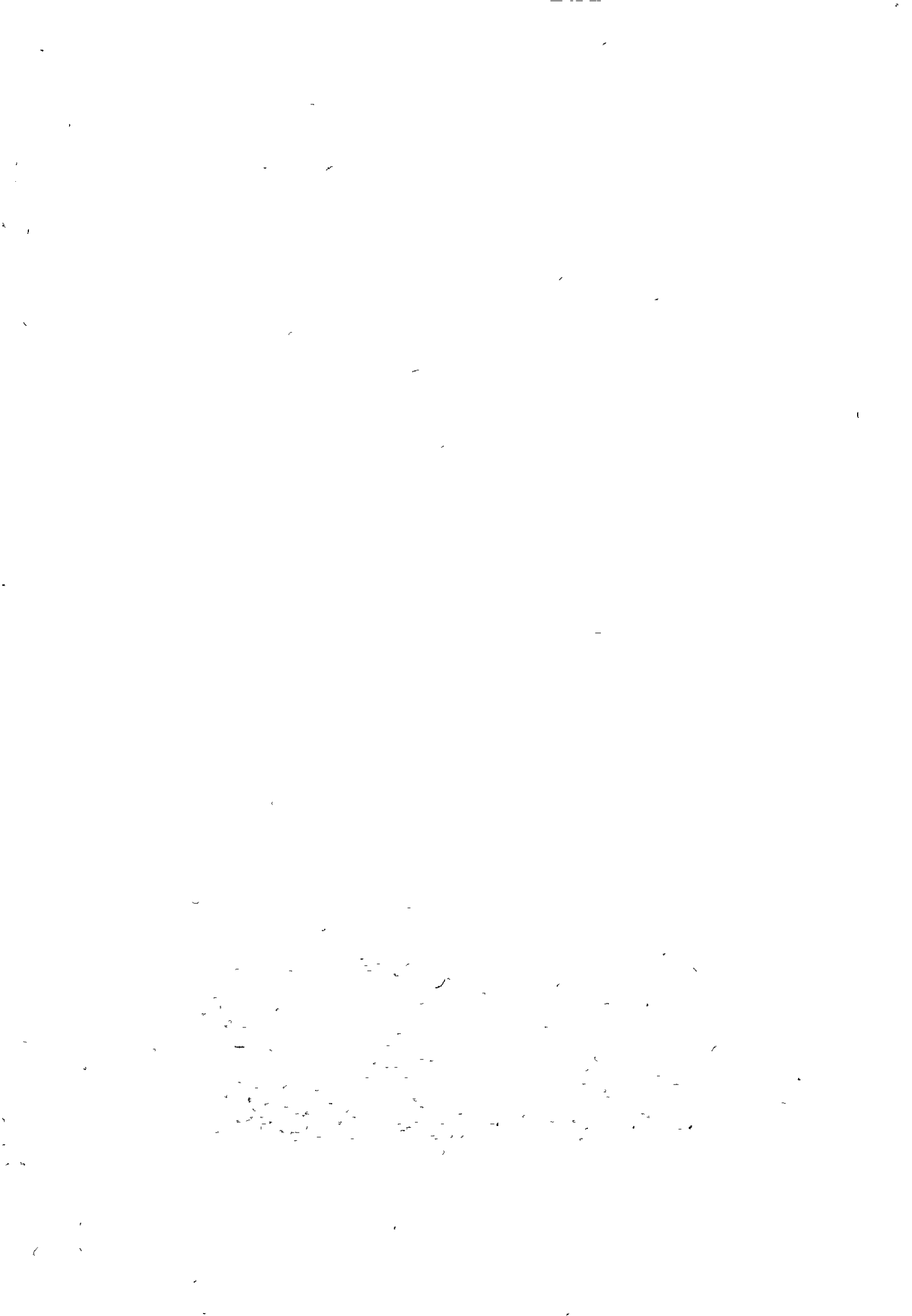
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