C.P. No. 705

C.P. No. 705

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL

CURRENT PAPERS

Limit Analysis of Circular Frames

By

E.O. Imegwu, B.Sc., Ph.D.

LONDON: HER MAJESTY'S STATIONERY OFFICE

1964

PRICE 4s. 6d. NET.

C.P. No.705

Limit Analysis of Circular Frames

- By -
E. O. Imegwu, B.Sc., Ph.D.*

January, 1961

SUMMARY

A method of limit analysis is given for aircraft fuselage frames subjected to known distributions of shear flow and vertical load. It is assumed that both axial force and bending moment are significant in causing collapse. An approximate method is adopted in most of the work but an accurate solution is illustrated for completeness.

---

1. Introduction

Although the method of limit analysis is used extensively in many types of structures its application to aircraft structures is almost unkown. This is partly due to the fact that aircraft structures tend to be complicated and in most cases present knowledge of limit analysis is inadequate to deal with the problem. In some problems, solutions can be obtained after considerable idealisation. The merit of limit analysis in aircraft structural design may lie in its use for estimating ratios of dimensions of a structure before an elastic method is applied.

The method of limit analysis for structures subjected to axial force and bending moment has been applied to two-hinged arches by Onat and Prager[6]. Any solution of the problem of fuselage frames by limit analysis is unknown to the Author.

The material of the frame is assumed to be rigid plastic and non-hardening. In consequence, no deformation results from any load which cannot cause collapse. At the critical load, plastic deformation starts and continues indefinitely without any further change in load, if changes in geometry were neglected. The stress-strain curve for a rigid plastic non-hardening material is shown in Fig.1 in comparison with that for mild steel. In what follows it will be assumed that the generalised stress-strain curve for a rigid plastic material has the same shape as OACDG ... in Fig.1.

For a rectangular cross section under a bending moment M and an axial force N the criterion for yield of the whole cross section can be written as (see Appendix and Ref.6);

$$\frac{M}{M_o} + \left(\frac{N}{N_o}\right)^2 \equiv m + n^2 \leq 1. \qquad \ldots (1)$$

In/

---

*Dr. Imegwu is now a Research Fellow at Oxford and this paper was written whilst he was working at the Fixed Wing Structures Department, Bristol Aircraft Limited, Filton, Bristol.

Replaces A.R.C.23 293.

In this expression M_o is the limiting value of the bending moment in absence of any other force and N_o is the limiting value of the axial force in absence of any other force; $m = M/M_o$, $n = N/N_o$. It can be shown in the manner of Ref.6 that a simpler approximation, on the safe side, is

$$m + n \leqslant 1, \qquad \ldots (2)$$

and that the deviation of expression (1) from the corresponding relation for sections other than the rectangular is small. In view of this insensitivity which is apparent also in combined bending and torsion[4], it will be assumed that Equation (1) applies generally.

Whenever Equation (1), or in an approximate solution, Equation (2), is satisfied at a point in a structure, a plastic hinge is formed at that point. If plastic hinges have formed in sufficient number and configuration to transform the structure into a mechanism, unrestricted plastic deformation sets in and the structure collapses. Between adjacent pairs of plastic hinges the segments of the structure are rigid and the inequality in (1) or (2) applies.

The fuselage frame shown in Fig.2(a) is made up of two circular arcs 3-1-3 of centre O and radius ρ and c-a-c of centre O' and radius ρ' rigidly joined to a cross beam c-c. The cross beam subtends an angle of 2ϕ at O and $2\phi'$ at O'. For simplicity the segments of the frame and cross beam are assumed to be each of uniform, but not necessarily identical, cross section. The cross-sectional dimensions are small compared with any radius. The frame is loaded symmetrically about the vertical axis of symmetry by a shear flow applied tangentially to the centroid of area of cross section. The intensity of shear flow is q per unit circumferential length of upper segment, q' for the lower segment; both are given by

$$q = q_o F(\theta), \quad q' = q_o' f(\theta'). \qquad \ldots (3)$$

In Equation (3) $F(\theta)$ and $f(\theta')$ are given as non-dimensional functions of the angles θ and θ' measured from the vertical axis of symmetry 1 - a, and q_o, q_o', of known ratio, are to be found. The distribution of shear flow is symmetrical about the axis 1 - a. In addition to the shear flow there are the loads P_1 and P_2, of known ratio, acting vertically downward on the cross beam and the lower segment of the frame respectively. The load P_2 is shown as a point load in Fig.2(a) but could be symmetrically distributed as P_1.

In determining the locations of plastic hinges points to be taken into consideration are rigid joints, points on the axis of symmetry 1 - a and points where the left-hand side of Equation (1) or (2) is a maximum. The fact that each plastic hinge is assumed bordered on either side by rigid material may affect the solution. This effect as well as the effect of shear force on the yield criterion is neglected.

2. Theorem of Limit Analysis

The first fundamental theorem of limit analysis for a structure under the load or group of loads Q will now be given. Of all loads or groups of loads Q' which satisfy both the yield criterion and the conditions of equilibrium while at the same time generating just enough plastic hinges to transform the structure into a mechanism, the largest is the actual collapse load or group of loads Q (see Ref.1-3).

This/

This theorem even in the absence of the second theorem of limit analysis is useful in solving many types of problems. Consider for example, the structure under the load or group of loads Q', of known ratios. Suitable cuts are made at appropriate points in the structure and any unknown reactions replaced by redundants. The expressions for bending moments and other effects of loading can now be written in terms of the redundants and the load. The positions of plastic hinges are assumed with due regard to the previous discussions. The number and configuration of plastic hinges must be such as to cause collapse. Equation (1) is used at the plastic hinges to obtain a number of equations in the load system and the redundants. In some problems the number of unknowns exceed the number of equations obtained in this manner, even when a sufficient number of hinges has been formed to reduce the structure to a mechanism. The complete set of equations is obtained by treating an appropriate number of unknowns as independent variables to be determined to make the load on the structure a maximum. In order to check conclusively that this maximum load is the actual collapse load it is essential to apply the second theorem of limit analysis (op. cit.). Without this check, however, the load calculated from the first theorem may in some cases, lie below the actual collapse load; it never exceeds the collapse load. A design based on the first theorem is therefore conservative. (For a complete discussion of this approach, see Ref.5.)

In each of the problems considered below the number of equations is the same as the number of unknowns. However, since the actual magnitudes of M and N are required for substitution into Equation (2), their signs must be known. These are determined by trial to make the load maximum; the actual steps are omitted for clarity.

The structure or part thereof is assumed to collapse by the formation of plastic hinges at all or some of the points 1,2,3; a, b, c; and e. The different modes of collapse arising from the distribution of plastic hinges will be illustrated with numerical examples. In these examples the shear flow distribution is specified as sinusoidal: $F(\theta) = \sin\theta$, $F \equiv f$ in Equation (3). The method is however, applicable to any manner of symmetrical shear flow distribution, and although the frame profile considered in the examples is made up of uniform circular arcs, the analysis can be adapted to arbitrary profiles with uniform or non-uniform cross sections so long as the structure has a vertical axis of symmetry.

3. Expressions for Forces and Bending Moments

Consider the segment in Fig.2(b) near the point 1 on the axis of symmetry 1 - a subtending an angle θ at the centre O. Let the compressive axial force at 1 be H_1 and the clockwise bending moment M_1. Since the shear force at 1 vanishes, the bending moment M, the axial, vertical and horizontal forces N, V and H at θ are given by

$$M = M_1 - \rho H_1(1-\cos\theta) + \rho^2\int_0^\theta q\{1-\cos(\theta-\alpha)\}d\alpha \qquad \dots (4)$$

$$N = -H_1\cos\theta + \rho\int_0^\theta q\cos(\theta-\alpha)d\alpha \qquad \dots (5)$$

$$V = \rho\int_0^\theta q\sin\alpha d\alpha \qquad \dots (6)$$

H/

$$H = H_1 - \rho\int_0^{\theta} q \cos \alpha d\alpha. \qquad \ldots (7)$$

The following substitutions will be found useful:

$$\left.\begin{aligned} \frac{\rho q_o}{N_o} &= \mu, \quad \frac{\rho' q_o'}{N_o'} = \mu', \quad \frac{\rho N_o}{M_o} = \psi, \quad \frac{\rho' N_o'}{M_o'} = \psi', \\ \frac{M}{M_o} &= m, \quad \frac{N}{N_o} = n, \quad \frac{H}{N_o} = h, \quad \frac{P_2}{N_o'} = \eta\mu'. \end{aligned}\right\} \qquad \ldots (8)$$

For an arc near the point a subtending an angle θ' at 0 the corresponding expressions are

$$M' = M_a - \rho' H_a(1-\cos \theta') + \tfrac{1}{2}\rho' P_2 \sin \theta' + \rho'^2\int_0^{\theta'} \{1-\cos(\theta'-\alpha)\}d\alpha \qquad \ldots (9)$$

$$N' = H_a \cos \theta' + \tfrac{1}{2}P_2 \sin \theta' - \rho'\int_0^{\theta'} q' \cos(\theta'-\alpha)d\alpha \qquad \ldots (10)$$

$$V' = -\tfrac{1}{2}P_2 + \rho'\int_0^{\theta'} q' \sin \alpha d\alpha \qquad \ldots (11)$$

$$H' = -H_a + \rho'\int_0^{\theta'} q' \cos \alpha d\alpha \qquad \ldots (12)$$

where M_a is the clockwise moment and H_a the tensile horizontal force at a. The axial force N' is tensile, the vertical force V' is downward and the horizontal force H' acts from left to right as in Fig. 2(b).

If the effect of axial thrust on the cross beam is neglected, the bending moment M_e at the centre e is

$$M_e = M_3 + M_c + (V_3 + V_c)\rho \sin\phi - \int_0^{\rho \sin\phi} xp dx \qquad \ldots (13)$$

where M_3 and M_c are the clockwise moments and V_3 and V_c the vertical forces at 3 and c respectively, $p = p(x)$ is the intensity of load distribution where x is measured from the centre of the cross beam.

If the shear flow distribution is sinusoidal on either segment of the frame, then Equation (3) becomes

$$q = q_o \sin \theta, \quad q' = q_o' \sin \theta'. \qquad \ldots (14)$$

Using/

Using Equations (8) and carrying out the integrations in Equations (4) - (7), (9) - (12), we have

$$m = m_1 - \psi h_1(1-\cos\theta) + \psi\mu(1-\cos\theta - \tfrac{1}{2}\theta\sin\theta) \quad \ldots (15)$$

$$n = -h_1\cos\theta + \tfrac{1}{2}\mu\,\theta\sin\theta \quad \ldots (16)$$

$$v = \tfrac{1}{2}\mu\,(\theta - \tfrac{1}{2}\sin 2\theta) \quad \ldots (17)$$

$$h = h_1 - \tfrac{1}{2}\mu\sin^2\theta \quad \ldots (18)$$

$$m' = m_a - \psi' h_a(1-\cos\theta') + \psi'\mu'(1-\cos\theta' - \tfrac{1}{2}\theta'\sin\theta' + \tfrac{1}{2}\eta\sin\theta') \quad \ldots (19)$$

$$n' = h_a\cos\theta' + \tfrac{1}{2}\mu'(\eta-\theta')\sin\theta' \quad \ldots (20)$$

$$v' = \tfrac{1}{2}\mu'(\theta'-\eta-\tfrac{1}{2}\sin 2\theta') \quad \ldots (21)$$

$$h' = -h_a + \tfrac{1}{2}\mu'\sin^2\theta' \quad \ldots (22)$$

Let the cross beam carry only a concentrated load P_1 at e, then Equation (13) becomes

$$\frac{M_e}{M_o} = \frac{M_e}{M_{oB}} \times \frac{M_{oB}}{M_o} = \frac{\lambda M_e}{M_{oB}} = \lambda m_e$$

$$\lambda m_e = m_3 + m_c\frac{M_o'}{M_o} + \left(v_3 + v_c\frac{N_o'}{N_o}\right)\psi\sin\phi \quad \ldots (23)$$

Where Equations (8) have been used, M_{oB} is the limiting value of the bending moment on the cross beam in absence of any other force, $\lambda = M_{oB}/M_o$, $m_e = M_e/M_{oB}$. Resolving vertically for the entire structures, we have

$$P_1 + P_2 = 2(V_3 + V_c)$$

$$= \rho q_o(\theta_3 - \tfrac{1}{2}\sin 2\theta_3) + \rho' q_o'(\theta_c - \tfrac{1}{2}\sin 2\theta_c) \quad \ldots (24)$$

In the following analyses a frame with a single cell will first be considered; this is followed by a full consideration of the modes of collapse associated with the frame shown in Fig. 2(a).

4. Frame with a Single Cell

Consider the frame with a single cell, Fig. 3(a), of uniform cross section, Fig. 3(b), subjected to a sinusoidal shear flow, $q = q_o\sin\theta$, and a vertical concentrated load P_2 at the lowest point 3. Suppose plastic hinges form at the point 3 under the load and at 1 at the other end of the vertical diameter, then using the approximate Equation (2) at these points, we have

$$m_1 + h_1 = m_3 + n_3 = 1.$$

Substituting $\theta_3 = \pi$ into Equations (15) and (16) we further obtain

$m_1 + h_1$/

$$m_1 + h_1 = m_1 - 2\psi h_1 + 2\psi\mu + h_1 = 1$$

or

$$h_1/\mu = \xi = 1, \quad m_1 = 1 - \mu$$

Hence Equations (15) and (16) become

$$\left.\begin{aligned} m &= 1 - \mu(1 + \tfrac{1}{2}\psi\theta \sin\theta) \\ n &= \mu(\tfrac{1}{2}\theta \sin\theta - \cos\theta) \end{aligned}\right\} \qquad \ldots (25)$$

To transform the structure into a mechanism one more hinge on either side of the line 1 - 3 is necessary. At the point 2 where the plastic hinge forms the left-hand side of Equation (2) is maximum equal to unity; here the bending moment must, of course, be of opposite sign to those at 1 and 3. Therefore we require

$$-(m - n) = -1 + \mu\{1 - \cos\theta + \tfrac{1}{2}(\psi + 1)\theta \sin\theta\} \qquad \ldots (26)$$

to be maximum equal to unity. Hence

$$\frac{\partial(m - n)}{\partial\theta} = 0 < \frac{\partial^2(m - n)}{\partial\theta^2} \qquad \ldots (27)$$

Using the first of (27) in (26) we have, after simplifying

$$\theta \cot\theta + 1 + \frac{2}{\psi + 1} = R(\theta) = 0 \qquad \ldots (28)$$

To solve Equation (28) we first find a close approximation θ_r by trial, the corresponding values of R_r and $\partial R_r/\partial\theta_r = 1/\chi_r$ are also found.

A better value of θ is given by

$$\theta_{r+1} = \theta_r - \chi_r R_r$$

With this value $R(\theta)$ is again evaluated. The process is repeated until $R(\theta)$ is as small as desired. In some instances χ_{r+1} is found not to differ very much from χ_r in which case χ need not be evaluated more than twice.

Now if we substitute the value of θ obtained from Equation (28) into Equation (26) and make $-(m - n) = 1$, then

$$\mu = \frac{2}{1 - \cos\theta + \tfrac{1}{2}(\psi + 1)\theta \sin\theta} \qquad \ldots (29)$$

It is now verified that the inequality in (27) is fulfilled and that the yield criterion is not violated anywhere on the frame. It is also essential to establish that no other combination of signs of (m_1, n_1), (m_2, n_2) and (m_3, n_3) furnishes a higher value of μ than that given by Equations (25) - (29).

Example 1/

Example 1

Now consider a numerical example with the following data: $N_0 = M_0 = 0.1633\sigma_0$, Fig.3(b), $\rho = 70$. Then from (8) $\psi = 70$, equation (28) becomes

$$\theta \cot \theta + 1\cdot 0281690 = R(\theta) = 0.$$

With the approximate solution $\theta_r = 2\cdot 036\ 7992 = 116\cdot 7^{\circ}$ we have

$$R_r(\theta) = +\ 3765\cdot 7 \times 10^{-6}, \quad \chi_r = -\ 0\cdot 327\ 336$$

$$\theta_{r+1} = \theta_r + 0\cdot 327336 \times 3765\cdot 7 \times 10^{-6}$$

$$= 2\cdot 03803189 = 116^{\circ}\ 46'\ 14\cdot 25''$$

$$\chi_{r+1} = -\ 0\cdot 326666, \quad R_{r+1} = -\ 3\cdot 9 \times 10^{-6}$$

Finally $\theta = 2\cdot 03803176 = 116^{\circ}\ 46'\ 14''$, $R(\theta) = -\ 0\cdot 6 \times 10^{-6}$

$\mu = 0\cdot 0302820 = h_1$, $m_1 = 0\cdot 9697180$, $m_2 = -\ 0\cdot 958805$

$n_2 = +\ 0\cdot 041129$. Resolving vertically we have

$$P_2/N_0 = \mu\pi = 0\cdot 0951337.$$

It is easy to show that at 2, $\dfrac{\partial^2(m - n)}{\partial\theta^2} > 0$ as required in (27).

5. Modes of Collapse

In a frame with a single cell as in the last example, the computation of the collapse load is straightforward. Evidently the necessary equations are obtained by assuming only four plastic hinges. When the frame has two cells as in Fig.2(a), the mode of collapse depends on the proportions of the different parts of the frame. The modes of collapse considered may be classified as follows:

Mode A. Local collapse of either upper or lower segment alone.

Mode B. Simultaneous collapse of both upper and lower segments with different numbers of plastic hinges and an optional hinge on the cross beam.

Mode C. Simultaneous collapse of both upper and lower segments with the same number of plastic hinges and an optional hinge on the cross beam.

Mode D. Simultaneous collapse of both upper and lower segments with the same number of plastic hinges and a necessary hinge on the cross beam.

These modes of collapse illustrated in Fig.4 will now be investigated in detail.

6./

6. Mode A

In this mode collapse is local and is restricted to either the upper segment with hinges at 1, 2, 3 or the lower segment with hinges at a, b, c. If hinges form at 1 and 3, then using Equation (2) with the appropriate values of θ we have

$$m_1 + h_1 = m_3 + n_3 = 1$$

or

$$\frac{h_1}{\mu} = \xi = \frac{\psi(1 - \cos\theta_3 - \frac{1}{2}\theta_3 \sin\theta_3) + \frac{1}{2}\theta_3 \sin\theta_3}{\psi(1 - \cos\theta_3) + 1 + \cos\theta_3} \quad \ldots (30)$$

Hence Equations (15) and (16) become

$$\left.\begin{aligned} m &= 1 - \mu\{\xi + \psi\xi(1 - \cos\theta) - \psi(1 - \cos\theta - \tfrac{1}{2}\theta\sin\theta)\} \\ n &= \mu(-\xi\cos\theta + \tfrac{1}{2}\theta\sin\theta) \end{aligned}\right\} \quad \ldots (31)$$

For the additional hinges at 2, $(n - m)$ must be maximum at 2 and equal to unity. Now

$$n - m = -1 + \mu\{\tfrac{1}{2}(\psi + 1)\theta\sin\theta - (\psi - \psi\xi - \xi)(1 - \cos\theta)\} \quad \ldots (32)$$

Hence, if $\sin\theta \neq 0$,

$$\theta\cot\theta + 2\xi - \frac{\psi - 1}{\psi + 1} = R(\theta) = 0 \quad \ldots (33)$$

$$\mu = \frac{2}{\frac{1}{2}(\psi + 1)\theta\sin\theta - (\psi - \psi\xi - \xi)(1 - \cos\theta)} \quad \ldots (34)$$

For alternative hinges a, b, c we use Equations (19) and (20) to obtain the corresponding expressions:

$$m_a + h_a = m_c - n_c = 1$$

$$\frac{h_a}{\mu'} = \xi' = \frac{\psi'(1 - \cos\theta_c) - \frac{1}{2}(\psi' - 1)(\theta_c - \eta)\sin\theta_c}{\psi'(1 - \cos\theta_c) + 1 + \cos\theta_c} \quad \ldots (35)$$

$$-(m' + n') = -1 + \mu'\{\tfrac{1}{2}(\psi' + 1)(\theta' - \eta)\sin\theta' - (\psi' - \psi'\xi' - \xi')(1 - \cos\theta')\} \quad \ldots (36)$$

Since $-(m' + n')$ is maximum equal to unity at b

$$(\theta' - \eta)\cot\theta' + 2\xi' - \frac{\psi' - 1}{\psi' + 1} = R(\theta') = 0 \quad \ldots (37)$$

μ'/

$$\mu' = \frac{2}{\frac{1}{2}(\psi' + 1)(\theta' - \eta)\sin\theta' - (\psi' - \psi'\xi' - \xi')(1 - \cos\theta')} \quad \ldots (38)$$

The results of Equation (35) - (38) are valid for small values of $\eta = P_a/\mu' N_0'$. When η is large the following expressions apply:

$$h_a - m_a = -m_c + n_c = 1$$

$$\frac{h_a}{\mu'} = \xi' = \frac{\psi'}{\psi' - 1} + \frac{\frac{1}{2}(\eta - \theta_c)\sin\theta_c}{1 - \cos\theta_c} \quad \ldots (35a)$$

$$m' + n' = -1 + \mu'\{\xi'(1 + \cos\theta') + \tfrac{1}{2}(\psi' + 1)(\eta - \theta')\sin\theta' - \psi'(\xi' - 1)(1 + \cos\theta')\}$$

At the point b $(m' + n')$ is maximum equal to unity:

$$(\eta - \theta')\cot\theta' + \frac{\psi' - 1}{\psi' + 1} - 2\xi' = R(\theta') = 0 \quad \ldots (37a)$$

$$\mu' = \frac{2}{\xi'(1 + \cos\theta') + \frac{1}{2}(\psi' + 1)(\eta - \theta')\sin\theta' - \psi'(\xi' - 1)(1 - \cos\theta')} \quad \ldots (38a)$$

<u>Example 2</u>

The methods described for Mode A are now illustrated by examples. Let

$\theta = 2\pi/3, \quad \phi' = \phi = \theta_c = \pi/3, \quad \rho = 70, \quad N_0/M_0 = N_0'/M_0' = 1.$

Then $\psi = \psi' = \rho = \rho' = 70, \quad \xi = 0{\cdot}4021225.$

(a) For hinges at 1, 2, 3 Equation (33) gives

$$\theta\cot\theta - 0{\cdot}1675860 = R(\theta) = 0.$$

The solution of this Equation is $\theta = \theta_2 = 1{\cdot}4562154 = 83^\circ\ 26{\cdot}0998'$. From Equations (30), (31), (34), (17), (18) we have

$$\mu = 0{\cdot}1365534, \quad h_1 = 0{\cdot}0549112, \quad m_1 = 0{\cdot}9450888,$$

$$m_2 = -0{\cdot}9075043, \quad n_1 = 0{\cdot}0924956, \quad m_3 = 0{\cdot}8487042,$$

$$n_3 = 0{\cdot}1512945, \quad v_3 = 0{\cdot}1725631, \quad h_3 = 0{\cdot}0037037.$$

(b) For hinges at a, b, c when η is small, say $\eta = 0{\cdot}125$, Equations (35) and (37) give

$$\xi' = 0{\cdot}20401906$$

$$(\theta' - 0{\cdot}125)\cot\theta' - 0{\cdot}5637929 = R(\theta') = 0.$$

The solution of this Equation is $\theta' = \theta_b = 0{\cdot}9966834 = 57^\circ\ 6{\cdot}3451'$. Hence from Equations (38), (35), (19) - (22) we have

μ'/

$$\mu' = 3{\cdot}2345167, \quad h_a = 0{\cdot}6599031, \quad m_a = 0{\cdot}3400969,$$

$$m_b = -0{\cdot}1746666, \quad n_b = -0{\cdot}8253335, \quad m_c = 0{\cdot}0383336,$$

$$n_c = -0{\cdot}9616662, \quad v_c = 0{\cdot}7911383, \quad h_c = 0{\cdot}5530407.$$

(c) For hinges at a, b, c when η is large, say $\eta = \pi/2$ Equations (35a) and (37a) give

$$\xi' = 1{\cdot}46794258$$

$$\left(\frac{\pi}{2} - \theta'\right)\cot\theta' - 1{\cdot}9501709 = R(\theta') = 0.$$

The solution is $\theta' = \theta_b = 0{\cdot}5015312 = 28^\circ\ 44{\cdot}1373'$. And from Equations (38a), (35a), (19) - (22),

$$\mu' = 0{\cdot}1178510, \quad h_a = 0{\cdot}1729986, \quad m_a = -0{\cdot}8270015,$$

$$m_b = 0{\cdot}8180150, \quad n_b = 0{\cdot}1819853, \quad m_c = -0{\cdot}8867809,$$

$$n_c = 0{\cdot}1132190, \quad v_c = -0{\cdot}0563688, \quad h_c = -0{\cdot}1288044.$$

The more accurate relation of Equation (1) can now be applied to these examples; a systematic approach is shown in a later section. Using the values obtained above from the approximate relation of Equation (2) we can find which segment of the ring collapses first if N_o, N_o', q_o'/q_o are given. Let ε denote the ratio of the shear flow intensity at c to that at 3,

$$\varepsilon = q_c/q_3 = \frac{q_o' \sin\phi'}{q_o \sin\phi} \qquad \ldots (39)$$

then since $\rho \sin\phi = \rho' \sin\phi'$ we have

$$q_o'/q_o = \varepsilon\rho'/\rho. \qquad \ldots (40)$$

Since ρ'/ρ is known so is ε. In all our analyses it will be assumed that the shear flow distribution is nowhere discontinuous: $q_c = q$, $\varepsilon = 1$. Suppose $M_o = N_o = 0{\cdot}1633\sigma_c = M_o' = N_o'$, $\varepsilon = 1$. For the failure of the upper segment we use the first of Equations (8), the results of Example 2(a) and Equation (40):

$$q_o/\sigma_o = \frac{\mu N_o}{\rho\sigma_o} = \frac{0{\cdot}1365534 \times 0{\cdot}1633}{70}$$

$$= 3{\cdot}185595 \times 10^{-4} = q_o'/\sigma_o.$$

For the failure of the lower segment when $\eta = 0{\cdot}125$, we use Example 2(b):

$$q_o'/\sigma_o = \frac{\mu' N_o'}{\rho'\sigma_o} = \frac{3{\cdot}2345167 \times 0{\cdot}1633}{70}$$

$$= 75{\cdot}45665 \times 10^{-4} = q_o/\sigma_o.$$

Since/

Since the value of q_o obtained from the upper is lower than that from the lower segment, the former is weaker under this loading and fails first. If instead of the value in Example 2(b) we use the value $\eta = \pi/2$ in Example 2(c) then

$$q_o'/\sigma_o = \frac{\mu' N_o'}{\rho' \sigma_o} = \frac{0{\cdot}117851 \times 0{\cdot}1633}{70}$$

$$= 2{\cdot}749295 \times 10^{-4} = q_o/\sigma_o.$$

In this case the lower segment fails first. In both cases the cross beam is assumed not to collapse.

7. Mode B

This mode of collapse is characterised by the formation of five plastic hinges on the upper segment and three on the lower or vice versa. This happens if at the point of collapse of one segment by forming five plastic hinges the other has developed three plastic hinges. For the upper segment to form three hinges at 1 and 3 Equation (30) is satisfied and the yield criterion must not be violated at θ given by Equation (33). If the lower segment forms three hinges at a and c Equation (35) or (35a) must be satisfied and the yield criterion must not be violated at θ' given by Equation (37) or (37a).

Example 4

Consider the problem of Example 2(a) and (c).

If $M_o = N_o = 0{\cdot}15\sigma_o$ and $M_o' = N_o' = 0{\cdot}1633\sigma_o$ then from our investigation of Mode A for $\eta = \pi/2$ we find that the lower segment collapses with five hinges:

$$q_o'/\sigma_o = q_o/\sigma_o = 2{\cdot}749295 \times 10^{-4}.$$

Investigating the upper segment for collapse with five hinges we have,

$$q_o/\sigma_o = q_o'/\sigma_o = 2{\cdot}9261443 \times 10^{-4}.$$

Thus, the upper segment cannot collapse with five hinges. If three hinges form on the upper segment then

$$\mu = \rho q_o/N_o = \frac{70 \times 2{\cdot}749295 \times 10^{-4}}{0{\cdot}15} = 0{\cdot}1283004$$

$$h_1 = 0{\cdot}0515925, \quad m_1 = 0{\cdot}9484075, \quad m_3 = +0{\cdot}8578477,$$

$$n_3 = 0{\cdot}1421518, \quad m_2 = -0{\cdot}7922188, \quad n_2 = 0{\cdot}0869054.$$

It is seen that a hinge does not form at 2.

8. Mode C

In this mode the proportions of the structure are such that the upper and the lower segments collapse simultaneously with five plastic hinges each. The yield criterion must not be violated on the cross beam. The accurate solution using Equation (1) instead of the approximate relation of Equation (2) is illustrated here.

If/

If hinges form at 1 and 3 in Fig. 2(a) then

$$m_1 + h_1^2 = m_3 + n_3^2 = 1.$$

Hence substituting for θ_3 from Equations (15) and (16) we have

$$(\tfrac{1}{2}\mu\theta_3 \sin\theta_3)^2 - \mu\{h_1\theta_3\sin\theta_3\cos\theta_3 - \psi(1 - \cos\theta_3 - \tfrac{1}{2}\theta_3\sin\theta_3)\}$$
$$= (h_1\sin\theta_3)^2 + \psi h_1(1 - \cos\theta_3)$$

or

$$\left.\begin{aligned}
\mu &= A + \sqrt{A^2 + B} = \mu(h\ ); \\
A &= \frac{\tfrac{1}{2}h_1\theta_3\sin\theta_3\ \cos\theta_3 - \tfrac{1}{2}\psi(1 - \cos\theta_3 - \tfrac{1}{2}\theta_3\sin\theta_3)}{(\tfrac{1}{2}\theta_3\sin\theta_3)^2} \\
B &= \frac{(h_1\sin\theta_3)^2 + \psi h_1(1 - \cos\theta_3)}{(\tfrac{1}{2}\theta_3\sin\theta_3)^2}
\end{aligned}\right\} \quad \ldots (41)$$

At any angle θ we have

$$n^2 - m = (\tfrac{1}{2}\mu\theta\ \sin\theta - h\cos\theta)^2 - 1 + h_1^2 + \psi h_1(1 - \cos\theta)$$
$$- \psi\mu(1 - \cos\theta - \tfrac{1}{2}\theta\ \sin\theta)$$

A hinge forms at 2 where

$$\frac{\partial}{\partial\theta}(n^2 - m) = n^2 - m - 1 = 0$$

Hence

$$(\tfrac{1}{2}\mu\theta\ \sin\theta - h_1\ \cos\theta)\{2h_1 + \mu(1 + \theta\cot\theta)\}$$
$$+ \psi h_1 - \tfrac{1}{2}\psi\mu(1 - \theta\cot\theta) = R_1(\theta) = 0, \quad \ldots (42)$$

$$2 + \psi\mu(1 - \cos\theta - \tfrac{1}{2}\theta\ \sin\theta) - h_1^2 - \psi h_1(1 - \cos\theta)$$
$$- (\tfrac{1}{2}\mu\theta\ \sin\theta - h_1\ \cos\theta)^2 = R_2(h_1) = 0. \quad \ldots (43)$$

To solve these equations we first obtain an approximate solution as in Equations (30) - (34). The corresponding value of h_1 is then used in Equation (41) to obtain μ. We are now in a position to find

$$\frac{1}{\chi_1} = \frac{\partial R_1}{\partial\theta}, \quad \frac{1}{\chi_2} = \frac{\partial R_2}{\partial h_1} \quad \ldots (44)$$

Equation (42) is solved in the manner of Equation (33). We now substitute into Equation (43) to obtain R_2. A new value of h_1 to make R_2 closer to zero is

$$h_1 - \chi_2 R_2.$$

The process of calculating μ, solving Equation (42) and evaluating R_2 is repeated until R_2 is as small as desired.

The/

The corresponding expressions for the lower segment are as follows ($u = +1$ for small values of η and for large values of η, $u = -1$):

$$h_a^2 + um_a = um_c + n_c^2 = 1$$

$$\left.\begin{aligned} \mu' &= A' + \sqrt{(A')^2 + B'} = \mu'(h_a); \\ A' &= \frac{\frac{1}{2}h_a(\theta_c - \eta)\sin\theta_c\cos\theta_c - \frac{1}{2}u\psi\{1 - \cos\theta_c - \frac{1}{2}(\theta_c - \eta)\sin\theta_c\}}{\{\frac{1}{2}(\theta_c - \eta)\sin\theta_c\}^2} \\ B' &= \frac{(h_a\sin\theta_c)^2 + u\psi' h_a(1 - \cos\theta_c)}{\{\frac{1}{2}(\theta_c - \eta)\sin\theta_c\}^2} \end{aligned}\right\} \quad \ldots (45)$$

At any angle θ' we have

$$(n')^2 - um' = \{h_a\cos\theta' - \tfrac{1}{2}\mu'(\theta' - \eta)\sin\theta'\}^2 - 1 + h_a^2$$
$$+ u\psi' h_a(1 - \cos\theta') - u\psi'\mu'(1 - \cos\theta' - \tfrac{1}{2}(\theta' - \eta)\sin\theta')$$

Hence the equations to be satisfied by θ' and h_a are

$$- \{h_a\cos\theta' - \tfrac{1}{2}\mu'(\theta' - \eta)\sin\theta'\}\{\mu'(\theta' - \eta)\cot\theta' + \mu' + 2h_a\}$$
$$+ u\psi' h_a - \tfrac{1}{2}u\psi'\mu'\{1 - (\theta' - \eta)\cot\theta'\} = R_1(\theta') = 0 \quad \ldots (46)$$

$$2 - h_a^2 + u\psi'\mu'\{1 - \cos\theta' - \tfrac{1}{2}(\theta' - \eta)\sin\theta'\} - u\psi' h_a(1 - \cos\theta')$$
$$- \{h_a\cos\theta' - \tfrac{1}{2}\mu'(\theta' - \eta)\sin\theta'\}^2 = R_2(h_a) = 0 \quad \ldots (47)$$

<u>Example 5</u>

Suppose $\rho = 70$, $M_o = N_o = 0\cdot1633\sigma_o$, $\theta_s = 2\pi/3$ for the upper segment and $\rho' = 65$, $N_o' = 4M_o' = 0\cdot1167838\sigma_o$, $\eta = 0$ for the lower segment. Then $\theta_c =$

$$\arcsin\frac{70\sin\theta_s}{65}$$

$= 1\cdot2016602$, $\psi = 70$, $\psi' = 520$. The approximate values for the upper segment are given in Example 2(a). Using Equations (35) - (38) we obtain the following approximate values for the lower segment:

$$\theta' = 0\cdot8492990, \quad \mu' = 0\cdot1725632, \quad h_a = 0\cdot0214855,$$

$$m_b = -0\cdot9591719, \quad n_b = -0\cdot0408282, \quad m_c = 0\cdot9110472,$$

$$n_c = -0\cdot0889450, \quad h_c = -0\cdot0535637, \quad v_c = 0\cdot0746561.$$

From these the accurate solution is carried out. The following results are obtained by means of an electronic computer:

θ/

$$\theta = 1{\cdot}4315916, \quad \mu = 0{\cdot}1498058, \quad h_1 = 0{\cdot}0594607,$$

$$m_1 = 0{\cdot}9964644, \quad m_2 = -0{\cdot}9904072, \quad n_2 = 0{\cdot}0979426,$$

$$m_3 = 0{\cdot}9725796, \quad n_3 = 0{\cdot}1655891, \quad h_3 = 0{\cdot}00328354,$$

$$v_3 = 0{\cdot}1893101;$$

and

$$\theta' = 0{\cdot}8403107, \quad \mu' = 0{\cdot}1806188, \quad h_a = 0{\cdot}0223006,$$

$$m_a = 0{\cdot}9995027, \quad m_b = 0{\cdot}9982657, \quad m_b = 0{\cdot}0416455,$$

$$m_c = 0{\cdot}9913196, \quad n_c = 0{\cdot}0931649, \quad h_c = 0{\cdot}0562520,$$

$$v_c = 0{\cdot}0781316.$$

For the failure of the upper segment

$$q_o/\sigma_o = \frac{\mu N_o}{\rho\sigma_o} = 3{\cdot}494755 \times 10^{-4},$$

$$\therefore \quad q_o'/\sigma_o = \frac{\varepsilon\rho' q_o}{\rho\sigma_o} = 3{\cdot}245130 \times 10^{-4};$$

and for the failure of the lower segment

$$q_o'/\sigma_o = \frac{\mu' N_o'}{\rho'\sigma_o} = 3{\cdot}245131 \times 10^{-4}$$

$$\therefore \quad q_o/\sigma_o = \frac{\rho q_o'}{\varepsilon\rho'\sigma_o} = 3{\cdot}494756 \times 10^{-4}.$$

Since the values of q_o obtained from consideration of failure for both segments are the same, it follows that both fail simultaneously.

The bending moment on the cross beam is given by Equation (23):

$$\lambda m_e = 16{\cdot}0134135.$$

If the yield criterion is not to be violated on the cross beam and if the effect of axial force is ignored, we must have

$$\frac{M_{oB}}{M_o} = \lambda \geqslant 16{\cdot}0134135.$$

For lower values of λ another mode of collapse must be investigated.

9. Mode D

Collapse in this mode takes place by the formation at the same time of one hinge on the cross beam and three hinges on each segment of the ring. The hinges are assumed to form at 1, 3, a, c and e in Fig.2(a); the yield criterion must not be violated at 2 and b. For hinges at 1 and 3 we need Equation (30) and for hinges at a and c we use Equation (35) or (35a). Since λ is given we can find q_o by putting $m_e = 1$ in Equation (23) which is now used with Equation (40).

Example 6

Consider the data of Example 5: $\rho = 70$, $M_o = N_o = 0{\cdot}1633\sigma_o$, $\theta_3 = 2\pi/3$ for the upper segment and $\rho' = 65$, $N_o' = 4M_o' = 0{\cdot}1167838\sigma_o$, $\eta = 0$ and in addition $\lambda = 12$. Using Equations (8) and (40) together with Equations (30), (15) and (17) at 3 and Equations (35), (19) and (21) at c we have

$$\mu'/\mu = \frac{N_o}{N_o'}\left(\frac{\rho'}{\rho}\right)^2 = 1{\cdot}2056860,$$

$$m_3 = 1 - 1{\cdot}1079648\mu, \quad v_3 = 1{\cdot}2637039\mu,$$

$$m_c = 1 - 0{\cdot}5157438\mu, = 1 - 0{\cdot}6218251\mu,$$

$$v_c = 0{\cdot}4325770\mu, = 0{\cdot}521552\mu,$$

Hence

$$\lambda m_e = 1{\cdot}1787872 + 97{\cdot}9999826\mu = 12,$$

$$\mu = 0{\cdot}1104206, \quad \mu' = 0{\cdot}1331325.$$

It is easy to show that the yield criterion is not violated at 2 or b.

10. Concluding Remarks

The assumption that the material is rigid plastic is not strictly true of any structural material. A material which closely approaches this in behaviour is mild steel as shown in Fig.1. For materials without a sharp yield point the limiting stress σ_o to be used in the analysis is obtained by consideration of strain among others.

By comparing Equations (30) - (34) with (41) - (44), it is observed that considerable algebraic and numerical work is saved by using the approximate Equation (2) instead of (1). The error in the collapse load arising from this approximation is small if the values of n are not close to $\pm\frac{1}{2}$ when the error may be large. Whenever a high degree of accuracy is desired Equation (1) must be adopted.

In the analyses it has been assumed that the material of the frame remains rigid until the collapse load is reached when it becomes plastic at hinge points. Thus the assumption of a sinusoidal distribution of shear flow is exact for the frame of Example 1. For the rest of the examples in which the frame has a cross beam the assumption of a sinusoidal shear flow is only an approximation. However, the general method of analysis can always be applied when the distribution of shear flow is given.

The/

The method of limit analysis may be used to estimate the proportions of a structure using a suitable load factor. It is then essential to use elastic methods to investigate the effects of heating, creep, fatigue, elastic stability and the magnitude of deflections under working load. Nevertheless the simplicity of limit analysis compared with formal elastic methods cannot be overestimated.

References/

References

No.	Author(s)	Title, etc.
1	D. C. Drucker, H. J. Greenberg and W. Prager	The safety factor of an elastic-plastic body in plane strain. J. Appl. Mech., Vol.18, p.371, 1951.
2	D. C. Drucker, W. Prager and H. J. Greenberg	Extended limit design theorems for continuous media. Quart. Appl. Math., Vol.9, p.381, 1951.
3	H. J. Greenberg and W. Prager	Limit design of beams and frames. Proc. Am. Soc. Civ. Eng., Vol.77, Separate No.59, 1951.
4	E. O. Imegwu	Plastic flexure and torsion. J. Mech. Phys. Solids, Vol.8, No.2, p.141, 1960.
5	E. O. Imegwu	Ultimate strength of structural sections under combined torsion and flexure with special reference to arbitrarily curved girders. Ph.D. Thesis, University of St. Andrews, Chap.III, IV, 1960.
6	E. T. Onat and W. Prager	Limit analysis of arches. J. Mech. Phys. Solids, Vol.I, No.2, p.77, 1953.

APPENDIX

From Fig.5 we obtain the following expressions for the bending moment M and the axial force N acting simultaneously on a given rectangular cross section at yield:

$$M = \sigma_0 by(d - y), \quad N = \sigma_0 b(d - 2y).$$

In simple tension $y = 0$ and $M = 0$,

$$N = \sigma_0 bd = N_0,$$

where N_0 denotes the limiting value of the axial force in absence of any other force. In simple compression $y = d$ and

$$M = 0, \quad N = -\sigma_0 bd = -N_0.$$

When $y = d/2$ the section is stressed in pure bending

$$M = bd^2\sigma_0/4 = M_0, \quad N = 0$$

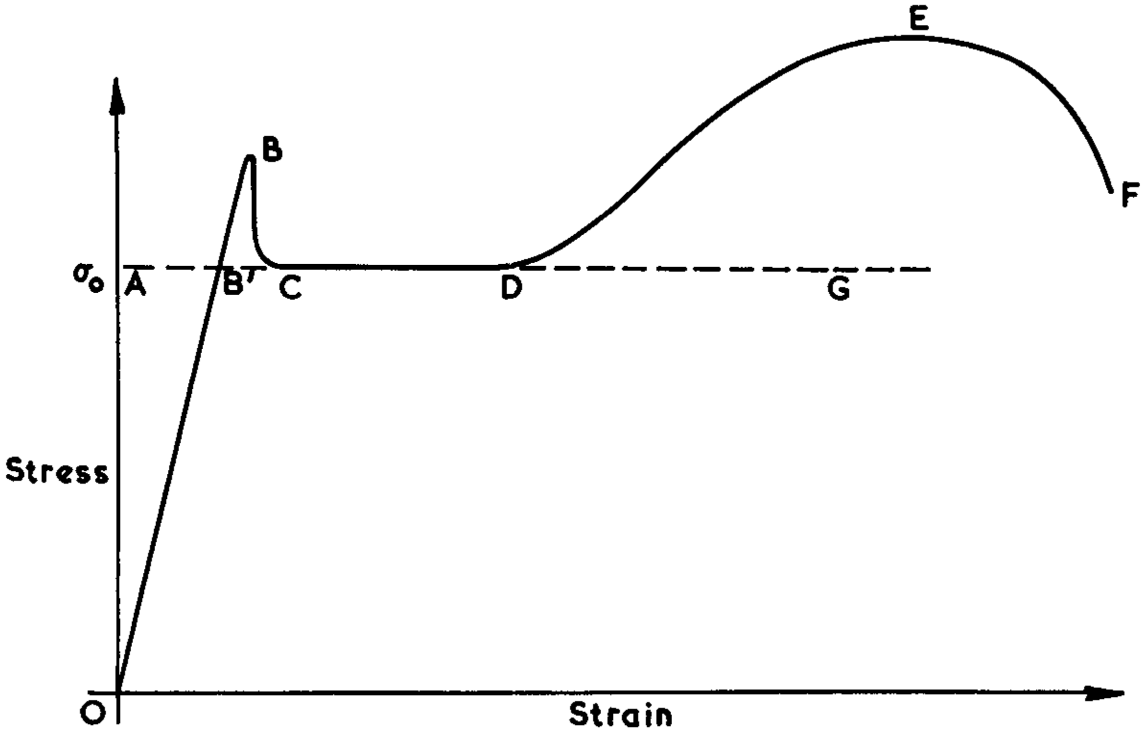
where M_0 denotes the limiting value of the bending moment in the absence of any other force. Eliminating between M and N we have

$$\frac{M}{M_0} + \left(\frac{N}{N_0}\right)^2 = m + n^2 = 1.$$

For any given cross section M_0 and N_0 can be found using the stress distributions shown in Fig.5(b) and (c).

AT

FIG. 1



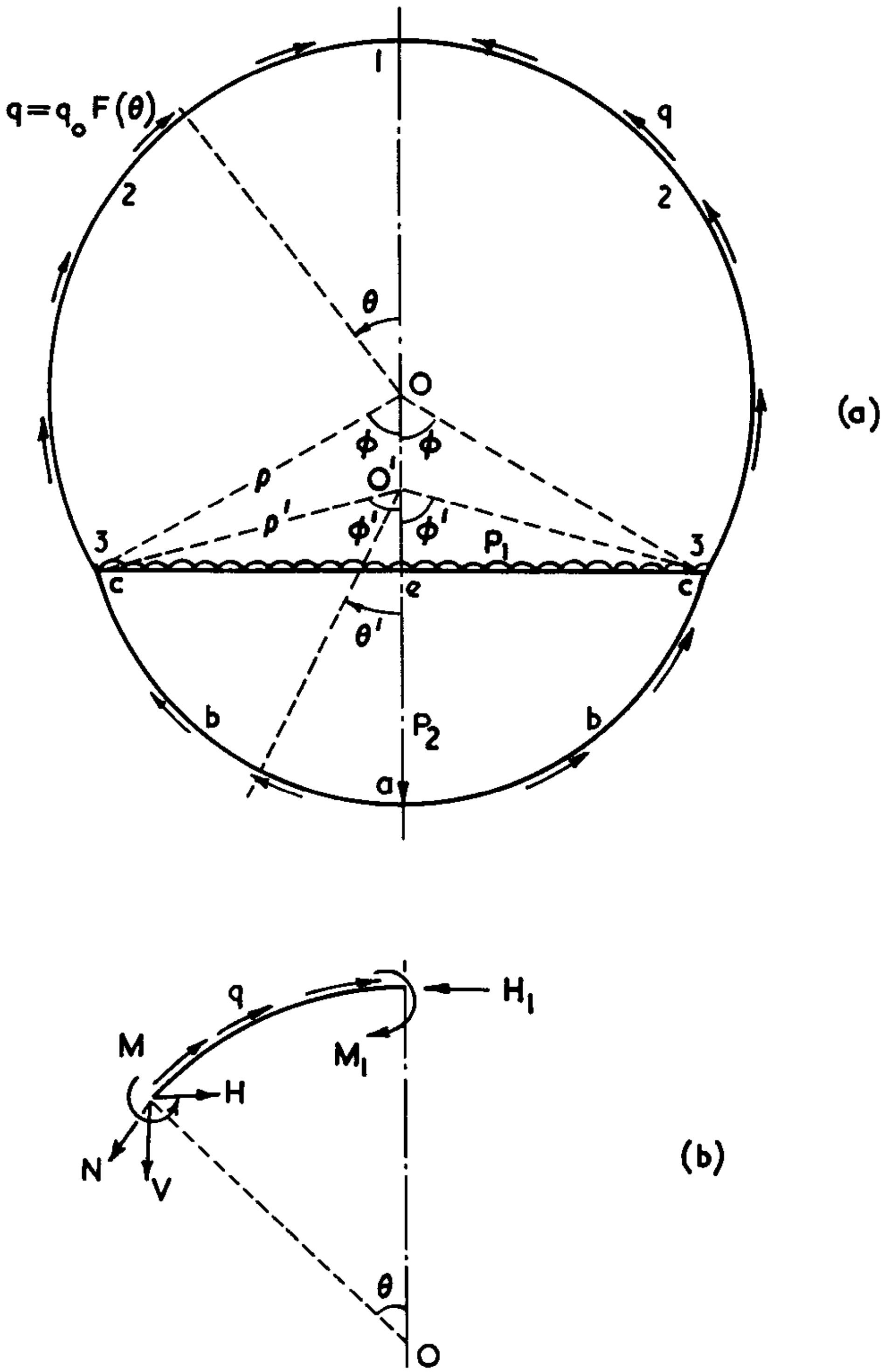
Stress – strain curve for mild steel and ideal material (not to scale)

(a) Mild steel: OB′BCDEF, B is upper yield point, C lower yield point and E ultimate load.

(b) Elastic plastic material: OB′CDG ––––––

(c) Rigid plastic material: OAB′CDG –––––

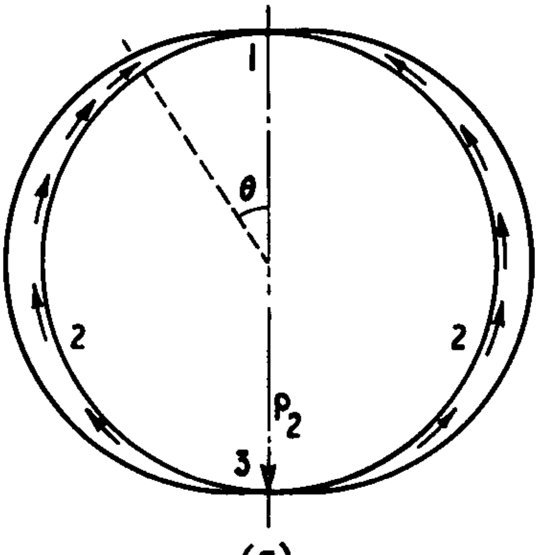
FIG. 2

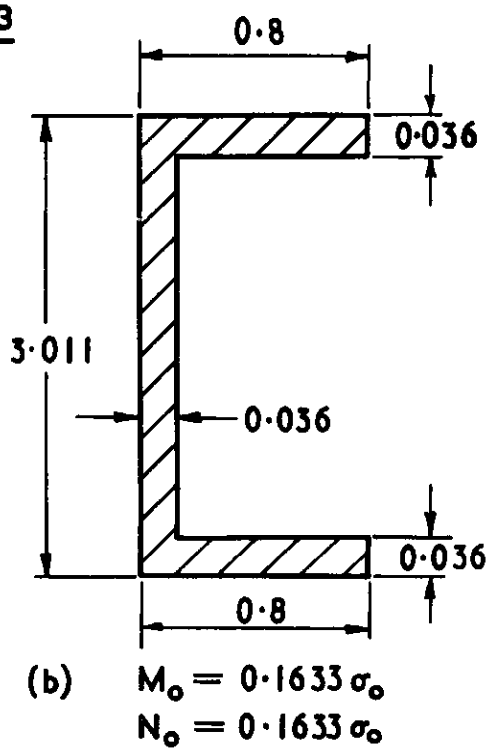


Fuselage frame

FIGS. 3 & 4

FIG. 3

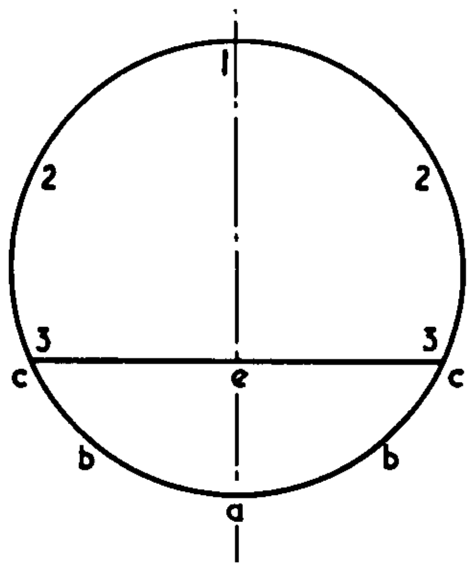




(a) Frame with a single cell

(b) Cross section of frame (not to scale)

FIG. 4.



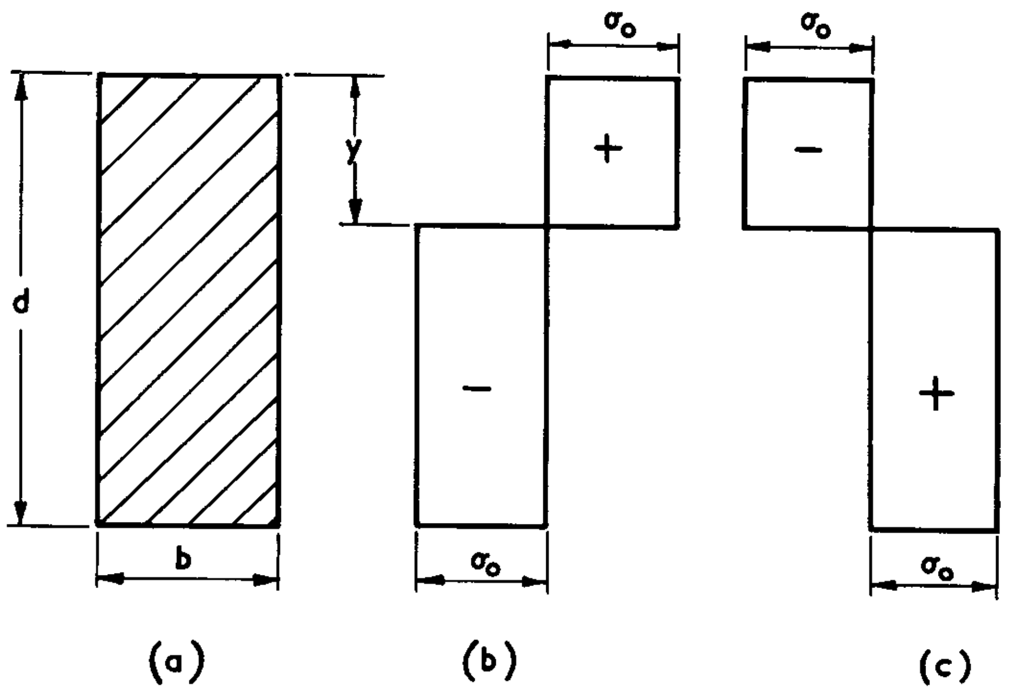
Mode A: hinges at 1, 2, 3, or a, b, c.

Mode B: hinges at 1, 2, 3, a, c. or 1, 3, a, b, c.

Mode C: hinges at 1, 2, 3, a, b, c.

Mode D: hinges at 1, 3, a, c, e.

FIG. 5



(a) Rectangular cross section
(b),(c) Stress distribution for rigid plastic material
(in bending and thrust)

A.R.C. C.P. No.705
January, 1961.
Imegwu, E. O., Bristol Aircraft Ltd.

LIMIT ANALYSIS OF FUSELAGE FRAMES

A method of limit analysis is given for aircraft fuselage frames subjected to known distributions of shear flow and vertical load. It is assumed that both axial force and bending moment are significant in causing collapse. An approximate method is adopted in most of the work but an accurate solution is illustrated for completeness.

A.R.C. C.P. No.705
January, 1961.
Imegwu, E. O., Bristol Aircraft Ltd.

LIMIT ANALYSIS OF FUSELAGE FRAMES

A method of limit analysis is given for aircraft fuselage frames subjected to known distributions of shear flow and vertical load. It is assumed that both axial force and bending moment are significant in causing collapse. An approximate method is adopted in most of the work but an accurate solution is illustrated for completeness.

A.R.C. C.P. No.705
January, 1961.
Imegwu, E. O., Bristol Aircraft Ltd.

LIMIT ANALYSIS OF FUSELAGE FRAMES

A method of limit analysis is given for aircraft fuselage frames subjected to known distributions of shear flow and vertical load. It is assumed that both axial force and bending moment are significant in causing collapse. An approximate method is adopted in most of the work but an accurate solution is illustrated for completeness.

C.P. No. 705

Printed and published by
HER MAJESTY'S STATIONERY OFFICE
To be purchased from
York House, Kingsway, London w.c.2
423 Oxford Street, London w.1
13A Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff
39 King Street, Manchester 2
50 Fairfax Street, Bristol 1
35 Smallbrook, Ringway, Birmingham 5
80 Chichester Street, Belfast 1
or through any bookseller

Printed in England

C.P. No. 705

S O Code No 23-9015-5