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Approximate Two-dimensional Aerofoil Theory

Part II. Velocity Distributions for Cambered Aerofoils

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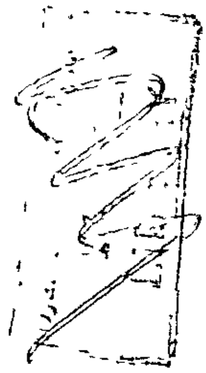

– By –

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29th September, 1942

1. Recapitulation.

As in Part I[1] we have three approximate formulae, of increasing accuracy and complexity, for the velocity at the surface of any aerofoil.

With the same nomenclature as in Part I x is the distance from the leading edge (measured along the chord) and y is the ordinate, both in fractions of the chord, q is the fluid velocity at the aerofoil surface, U the velocity of the undisturbed stream, and $-\beta$ the theoretical no-lift angle. Then with

$$x = \tfrac{1}{2}(1 - \cos\theta), \quad y = \tfrac{1}{2}\psi \sin\theta, \qquad \ldots\ldots (1)$$

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(t)dt, \qquad \ldots\ldots (2)$$

$$\varepsilon(\theta) \doteq -\frac{1}{2\pi} P \int_{-\pi}^{\pi} \psi(t) \cot \tfrac{1}{2}(t - \theta)dt$$

$$= -\frac{1}{2\pi}\int_{0}^{\pi} \{\psi(\theta + t) - \psi(\theta - t)\} \cot \tfrac{1}{2}t \, dt. \qquad \ldots\ldots (3)$$

(P denoting that the principal value of the integral is to be taken), the simplest approximation (Approximation I) is

$$q/U = 1 + g, \qquad \ldots\ldots (4)$$

where

$$g = C_0 + \varepsilon'(\theta) + (\varepsilon - \beta)\cot\theta + \frac{C_L}{a_0}\cot\theta + \frac{C_L}{2\pi}\operatorname{cosec}\theta \qquad \ldots\ldots (5)$$

when/

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when*

$$C_L = a_0 \sin(\alpha + \beta). \qquad \ldots\ldots (6)$$

θ is positive on the upper surface and negative on the lower surface; it is zero at the leading edge and $\pm\pi$ at the trailing edge. β is the value of ε at $\theta = \pi$. When the Kutta-Joukowski condition is satisfied at a sharp trailing edge, or the trailing edge, if rounded, is a stagnation point, the theoretical value of a_0 is approximately $2\pi e^{C_0}$.

Approximation I must fail when $\sin\theta$ is small; our next approximation (Approximation II) is

$$\frac{q}{U} = \frac{(1 + \frac{1}{2} C_0^2)\,|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g). \qquad \ldots\ldots (7)$$

More accurately still, especially at high lifts, we have Approximation III, namely†

$$\frac{q}{U} = \frac{e^{C_0}(1+\varepsilon')}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left|\left(1 - \frac{C_L^2}{a_0^2}\right)^{\frac{1}{2}}\sin(\theta+\varepsilon-\beta) + \frac{C_L}{a_0}\cos(\theta+\varepsilon-\beta) + \frac{C_L e^{-C_0}}{2\pi}\right|. \qquad \ldots\ldots (8)$$

2. The Contributions from the Thickness and the Camber.

For a cambered aerofoil we now use subscripts u and ℓ to denote quantities on the upper and lower surfaces respectively (remembering also that θ is positive on the upper and negative on the lower surface); we use the subscript c for the centre line and the subscript s for the half-thickness, i.e. for the corresponding symmetrical aerofoil. Thus we denote by y_c the ordinate of the centre line, and by y_s the half-thickness, which is the ordinate of the upper surface of the corresponding symmetrical aerofoil. Then×

$$y_u = y_c + y_s, \quad y_\ell = y_c - y_s. \qquad \ldots\ldots (9)$$

We/

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*If (6) does not hold, we must substitute $\sin(\alpha + \beta)$ for C_L/a_0 in (5). In obtaining (5) C_L^2/a_0^2 is neglected, so we may substitute $\alpha + \beta$ for C_L/a_0; in particular if $C_L = a_0(\alpha + \beta_1)$, where $\beta_1 \neq \beta$, then we simply add $(\beta - \beta_1)\cot\theta$ to the right-hand side of (5).

† When (6) does not hold we must substitute $\sin(\alpha + \beta)$ for C_L/a_0 in the first two terms of (8).

×The formulae (9) are approximate only if the fairing is put on (in the American manner) normal to the centre line; they are accurate if the fairing is put on (in the British manner) normal to the chord.

We write

$$\left.\begin{aligned} \psi_u(\theta) &= \psi_s(\theta) + \psi_c(\theta) \\ \psi_\ell(-\theta) &= \psi_s(\theta) - \psi_c(\theta) \end{aligned}\right\} \qquad \ldots\ldots (10)$$

where

$$\psi_s = 2y_s/\sin\theta, \quad \psi_c = 2y_c/\sin\theta. \qquad \ldots\ldots (11)$$

ψ_s is an even function and ψ_c an odd function of θ; for the derivatives of ψ we have

$$\left.\begin{aligned} \psi_u'(\theta) &= \psi_c'(\theta) + \psi_s'(\theta) \\ \psi_\ell'(-\theta) &= \psi_c'(\theta) - \psi_s'(\theta) \end{aligned}\right\}, \qquad \ldots\ldots (12)$$

ψ_s' being odd and ψ_c' even.

We next write

$$\left.\begin{aligned} \varepsilon_u(\theta) &= \varepsilon_c(\theta) + \varepsilon_s(\theta) \\ \varepsilon_\ell(-\theta) &= \varepsilon_c(\theta) - \varepsilon_s(\theta) \end{aligned}\right\} \qquad \ldots\ldots (13)$$

where

$$\varepsilon_s(\theta) = -\frac{1}{2\pi}\int_0^\pi \{\psi_s(\theta + t) - \psi_s(\theta - t)\}\cot\tfrac{1}{2}t\,dt.$$

$$= -\frac{\sin\theta}{\pi} P \int_0^\pi \frac{\psi_s(t)}{\cos\theta - \cos t}\,dt, \qquad \ldots\ldots (14)$$

and

$$\varepsilon_c(\theta) = -\frac{1}{2\pi}\int_0^\pi \{\psi_c(\theta + t) - \psi_c(\theta - t)\}\cot\tfrac{1}{2}t\,dt$$

$$= -\frac{1}{\pi} P \int_0^\pi \frac{\psi_c(t)\sin t}{\cos\theta - \cos t}\,dt, \qquad \ldots\ldots (15)$$

(See Part I, Lemma 8.) We also note that, as in Part I, if

$$\psi_s(\theta) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta \qquad \ldots\ldots (16)$$

then

$$\varepsilon_s(\theta) = \sum_{n=1}^{\infty} C_n \sin n\theta \qquad \ldots\ldots (17)$$

while/

while if

$$\psi_c(\theta) = \sum_{n=1}^{\infty} D_n \sin n\theta \qquad \ldots\ldots (18)$$

then

$$\varepsilon_c(\theta) = -\sum_{n=1}^{\infty} D_n \cos n\theta \qquad \ldots\ldots (19)$$

(Cf. Part I, Lemma 5.)

Thus ε_s is odd and ε_c is even; ε_s' is even and ε_c' is odd, and

$$\left.\begin{aligned} \varepsilon_u'(\theta) &= \varepsilon_s'(\theta) + \varepsilon_c'(\theta) \\ \varepsilon_\ell'(-\theta) &= \varepsilon_s'(\theta) - \varepsilon_c'(\theta) \end{aligned}\right\}, \qquad \ldots\ldots (20)$$

where

$$\varepsilon_s'(\theta) = -\frac{1}{2\pi} \int_0^\pi \{\psi_s'(\theta + t) - \psi_s'(\theta - t)\} \cot \tfrac{1}{2}t \, dt$$

$$= -\frac{1}{\pi} P \int_0^\pi \frac{\psi_s'(t) \sin t}{\cos\theta - \cos t} dt \qquad \ldots\ldots (21)$$

and

$$\varepsilon_c'(\theta) = -\frac{1}{2\pi} \int_0^\pi \{\psi_c'(\theta + t) - \psi_c'(\theta - t)\} \cot \tfrac{1}{2}t \, dt$$

$$= -\frac{\sin\theta}{\pi} P \int_0^\pi \frac{\psi_c'(t)}{\cos\theta - \cos t} dt. \qquad \ldots\ldots (22)$$

Finally we write

$$g(\theta) = g_s(\theta) + g_c(\theta) + g_L(\theta) \qquad \ldots\ldots (23)$$

where

$$g_s = C_o + \varepsilon_s'(\theta) + \varepsilon_s \cot\theta, \qquad \ldots\ldots (24)$$

$$g_c = \varepsilon_c'(\theta) + (\varepsilon_c - \beta)\cot\theta, \qquad \ldots\ldots (25)$$

g_L/

$$g_L = \frac{C_L}{a_0}\cot\theta + \frac{C_L}{2\pi}\operatorname{cosec}\theta$$

$$= \frac{C_L}{a_0}\cot\tfrac{1}{2}\theta - C_L\left(\frac{1}{a_0} - \frac{1}{2\pi}\right)\operatorname{cosec}\theta$$

$$= \tfrac{1}{2}C_L\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)\cot\tfrac{1}{2}\theta - \tfrac{1}{2}C_L\left(\frac{1}{a_0} - \frac{1}{2\pi}\right)\tan\tfrac{1}{2}\theta \quad \ldots(26)$$

In using Approximations I and II we must bear in mind that g_s is an even function and g_c, g_L are odd functions of θ. In using Approximations II and III we must use the values ψ_u, ε_u, ε_u' for ψ, ε, ε' on the upper surface, and the values ψ_ℓ, ε_ℓ, ε_ℓ' on the lower surface.

Since ψ_c is odd

$$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\psi_s(t)dt = \frac{1}{\pi}\int_0^{\pi}\psi_s(t)dt, \quad \ldots(27)$$

and has the same value as for the corresponding symmetrical aerofoil.

Since ε_s is zero at $\theta = \pi$, β is the value of ε_c at $\theta = \pi$. Hence from (15),

$$\beta = \frac{1}{\pi}\int_0^{\pi}\frac{\psi_c(t)\sin t}{1+\cos t}dt = \frac{1}{\pi}\int_0^{\pi}\psi_c(t)\frac{1-\cos t}{\sin t}dt. \quad \ldots(28)$$

The calculation of g_s, C_0, ε_s, ε_s' was considered in Part I; we are here concerned with the calculation of g_c, $g_c + g_L$, β, ε_c, ε_c'.

3. <u>The Simplest Approximation for a Cambered Aerofoil.</u>

It follows from (15), (22), (25) and (28) that

$$g_c = -\frac{1}{\pi}P\int_0^{\pi}\left\{\frac{\psi_c'(t)\sin\theta + \psi_c(t)\sin t\cot\theta}{\cos\theta - \cos t} + \psi_c(t)\frac{\cos\theta\,(1-\cos t)}{\sin\theta\sin t}\right\}dt. \quad \ldots(29)$$

Since/

Since

$$\frac{\sin t \cot \theta - \sin \theta \cot t}{\cos \theta - \cos t} = \frac{\sin^2 t \cos \theta - \sin^2 \theta \cos t}{\sin \theta \sin t (\cos \theta - \cos t)}$$

$$= \frac{1 + \cos \theta \cos t}{\sin \theta \sin t} \qquad \ldots\ldots (30)$$

it follows that

$$\frac{\sin t \cot \theta}{\cos \theta - \cos t} + \frac{\cos \theta (1 - \cos t)}{\sin \theta \sin t} = \frac{\sin \theta \cot t}{\cos \theta - \cos t} + \frac{1 + \cos \theta}{\sin \theta \sin t}$$

$$= \frac{\sin \theta \cot t}{\cos \theta - \cos t} + \frac{\cot \frac{1}{2}\theta}{\sin t}; \qquad \ldots\ldots (31)$$

hence

$$g_c = -\frac{\sin \theta}{\pi} P \int_0^\pi \frac{\psi_c'(t) + \psi_c(t) \cot t}{\cos \theta - \cos t} dt - \frac{\cot \frac{1}{2}\theta}{\pi} \int_0^\pi \frac{\psi_c(t)}{\sin t} dt.$$

$$\ldots\ldots (32)$$

Now write

$$\frac{dy_c}{dx} = F(\theta). \qquad \ldots\ldots (33)$$

Then

$$F(\theta) = \frac{2}{\sin \theta} \frac{d}{d\theta} (\tfrac{1}{2}\psi_c \sin \theta) = \psi_c'(\theta) + \psi_c(\theta) \cot \theta. \quad \ldots (34)$$

In terms of Fourier series let

$$F(\theta) = \sum_{n=0}^{\infty} A_n \cos n\theta, \qquad \ldots\ldots (35)$$

where

$$A_0 = \frac{1}{\pi} \int_0^\pi F(t) dt \text{ and } A_n = \frac{2}{\pi} \int_0^\pi F(t) \cos nt\, dt \text{ for } n \geq 1. \quad \ldots (36)$$

We assume that $\psi_c(0) = \psi_c(\pi) = 0$, i.e. that $y_c/x^{\frac{1}{2}} \to 0$ as $x \to 0$ and $y_c/(1 - x)^{\frac{1}{2}} \to 0$ as $x \to 1$. These limits are valid with all usual centre lines (and their validity is necessary for the convergence of the second integral in (32)). Then (see Lemma 2, Appendix)

$$A_0 = \frac{1}{\pi} \int_0^\pi \psi_c(t) \cot t\, dt, \; A_1 = \frac{2}{\pi} \int_0^\pi \frac{\psi_c(t)}{\sin t} dt. \qquad \ldots\ldots (37)$$

Hence/

Hence

$$g_c = -\tfrac{1}{2} A_1 \cot \tfrac{1}{2}\theta - \frac{\sin\theta}{\pi} P \int_0^{\pi} \frac{F(t)dt}{\cos\theta - \cos t}; \quad \ldots\ldots (38)$$

also (from (28))

$$\beta = \tfrac{1}{2} A_1 - A_0. \quad \ldots\ldots (39)$$

4. Connection with the Vortex-Sheet Theory of Infinitely Thin Aerofoils.

The vortex-sheet theory of thin aerofoils as presented, for example, by Glauert[2], is a thoery of aerofoils of zero thickness and small camber at small lift coefficients, with all squares and products of the camber and lift coefficient neglected. When we make the same approximations, taking the thickness as zero, our formula for q/U is $q/U = 1 \pm [g_c(\theta) + g_L(\theta)]$; it follows that the strength of the vortex sheet by which the infinitely thin aerofoil is replaced in the vortex-sheet theory should be equal to $2U\,[g_c(\theta) + g_L(\theta)]$.

From (38) and Lemma 6 of Part I we see that, in terms of Fourier series

$$g_c = -\tfrac{1}{2} A_1 \cot \tfrac{1}{2}\theta + \sum_{n=1}^{\infty} A_n \sin n\theta, \quad \ldots\ldots (40)$$

so from (26)

$$g_c + g_L = -\tfrac{1}{2} A_1 \cot \tfrac{1}{2}\theta + \sum_{n=1}^{\infty} A_n \sin n\theta + \frac{C_L}{a_0} \cot \tfrac{1}{2}\theta$$

$$- C_L \left(\frac{1}{a_0} - \frac{1}{2\pi}\right) \operatorname{cosec}\theta. \quad \ldots\ldots (41)$$

Hence with $C_L = a_0(\alpha + \beta)$ and the value of β in (39)

$$g_c + g_L = (\alpha - A_0) \cot \tfrac{1}{2}\theta + \sum_{n=1}^{\infty} A_n \sin n\theta - C_L\left(\frac{1}{a_0} - \frac{1}{2\pi}\right) \operatorname{cosec}\theta. \quad \ldots\ldots (42)$$

Remembering that Glauert's A_0 is here denoted by $\alpha - A_0$ and that Glauert's $k/2V$ should be equal to $g_c + g_L$, we see that for $a_0 = 2\pi$ the above result agrees with Glauert's.

For the no-lift angle it follows from (39) that

$$\beta = -\frac{1}{\pi} \int_0^{\pi} F(\theta)(1 - \cos\theta)d\theta = -\frac{2}{\pi} \int_0^1 \frac{dy_c}{dx} \left(\frac{x}{1 - x}\right)^{\frac{1}{2}} dx, \quad \ldots\ldots (43)$$

and/

and if $y_c/(1-x)^{\frac{1}{2}} \to 0$ as $x \to 1$ a partial integration leads to the formula

$$\beta = \frac{1}{\pi} \int_0^1 \frac{y_c}{x^{\frac{1}{2}}(1-x)^{3/2}} \, dx, \qquad \ldots\ldots (44)$$

which may also be obtained directly from (28),[2] and is in agreement with the usual formula of thin-aerofoil theory. For our purposes, however, equation (39) is preferable as it stands.

We have deduced the results of the vortex-sheet theory of thin aerofoils from the exact theory of Theodorsen and Garrick (with an extension to values of a_0 other than 2π), and have found an integral sum* for the Fourier series; unless the Fourier series terminate we use the integral to calculate results for special cases. The Fourier series terminate if, and only if, y_c is expressible as a single polynomial in x over the whole range $0 \leqslant x \leqslant 1$.

5. The Moment Coefficient at Zero Lift.

The values of the moment coefficient C_M have not so far been considered; when taken, as is usual, about the quarter-chord point these values are sensitive to differences, due to the presence of the boundary layer and the wake, between the calculated and measured pressures at the surface towards the trailing edge; it also appears that for non-zero values of C_L some attention should be paid to the thickness of the aerofoil and the shape of the fairing. The matter is one of considerable complexity, but it appears that the thickness and shape of fairing do not, to the first order, affect the moment coefficient C_{M_0} at zero lift; moreover if the experimental no-lift angle $(-\beta_1)$ is equal to the theoretical no-lift angle $(-\beta)$, then C_{M_0} should apparently be given sufficiently accurately by the theoretical answer found by ignoring the effects of the boundary layer and the wake; and it is convenient to put on record here the theoretical values of C_{M_0} so found. For an aerofoil whose thickness and camber are not large the formula for C_{M_0}, taken as a "nose-up" moment coefficient, is, in the notation used here,

$$C_{M_0} = \frac{\pi}{4} D_2 - \frac{\pi}{2}\beta = \frac{\pi}{4}(A_2 - A_1), \qquad \ldots\ldots (45)$$

if $\beta = \beta_1$ ×. [The second equality in (45) follows from (39) and Lemma 2 (Appendix).]

From/

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*Kármán and Burgers[3] give (for zero incidence) an integral formula for the strength of the vortex sheet which after some manipulation may be shown to be identical with that obtained here; they appear to have made no use of the integral formula.

×If $\beta \neq \beta_1$ then theoretically, with the pressures calculated for the measured lift but with no other allowance for the boundary layer and the wake, we should add $\frac{1}{2}\pi(\beta - \beta_1)$ to the right-hand side of (45); there are indications that other allowances for the boundary layer and the wake would leave the addition proportional to $\beta - \beta_1$ but would change the factor $\frac{1}{2}\pi$.

From (45) we may deduce that

$$C_{M_0} = \int_0^1 \frac{y}{2x^{\frac{1}{2}}(1-x)^{3/2}} \left[4(1-x)(1-2x) - 1\right] dx, \quad \ldots\ldots (46)$$

which agrees with the usual formula of thin-aerofoil theory[3]. For our purposes, however, (45) is preferable as it stands.

6. The "Optimum" Lift and Angle of Attack.

Since

$$\cot \tfrac{1}{2}\theta = \left(\frac{1-x}{x}\right)^{\frac{1}{2}}, \quad \operatorname{cosec}\theta = \frac{1}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}, \quad \ldots\ldots (47)$$

it follows from (41) that in general $g_c + g_L$ becomes infinite at the leading and trailing edges like multiples of $x^{-\frac{1}{2}}$ or $(1-x)^{-\frac{1}{2}}$; these infinities occur therefore in g and in q/U according to Approximation I (equation (4)).* These infinities will not be present in the other approximate solutions or in the exact solution; for example they disappear in Approximation II (equation (7)) on multiplication by

$$\frac{(1 + \tfrac{1}{2}C_0^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}; \quad \ldots\ldots (48)$$

but in general there will still be a rather sharp maximum of q/U near the leading edge. There may also be a maximum of q/U near the trailing edge; the infinity in Approximation I at the tail will always be present unless $a_0 = 2\pi$ (more accurately, $2\pi e^{C_0}$), but any resulting maximum in q/U near the trailing edge is modified by the boundary layer and wake, and has no physical significance. Near the leading edge, however, there will be a danger of a suction peak unless the infinity is avoided. With C_L as in (6) the condition for the absence of an infinity at the leading edge is

$$C_L = C_{L\text{opt.}}, \quad \ldots\ldots (49)$$

where[x]

$$\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)C_{L\text{opt.}} = A_1, \quad \ldots\ldots (50)$$

and/

*We saw in Part I that there are logarithmic infinities in g_s at the nose and tail if the coefficients of x and of $1-x$, respectively, in the expansions of y_s in powers of x and of $1-x$, respectively, (or of $x^{\frac{1}{2}}$ and $(1-x)^{\frac{1}{2}}$ etc.) do not vanish. These infinities arise from discontinuities in ψ_s'; for example, if a_2 is the coefficient of x in the expansion of y_s in powers of x (or $x^{\frac{1}{2}}$), then ψ_s' changes abruptly at the leading edge from $-\frac{1}{2}a_2$ on the lower surface to $+\frac{1}{2}a_2$ on the upper surface. This discontinuity occurs for such N.A.C.A. aerofoils as N.A.C.A.0012, etc. The resulting infinities in g_s are of a lower order than those in $g_c + g_L$.

[x]If the experimental no-lift angle $-\beta_1$ differs from its theoretical value $-\beta$, we must add $\beta_1 - \beta$ to the right-hand side of (50).

and A_1 is given by (36). $C_{Lopt.}$ is the "optimum" lift coefficient, defined as the lift coefficient for which, on a theory neglecting squares of the camber, there is no singularity at the leading edge in the velocity distribution for an infinitely thin aerofoil of the shape of the given centre line. $C_{Lopt.}$ will be small for an aerofoil of small camber, and if we put

$$C_{Lopt.} = a_o (\alpha_{opt.} + \beta), \quad \ldots\ldots (51)$$

then the "optimum" incidence $\alpha_{opt.}$ is given by *

$$\alpha_{opt.} = \frac{2\pi}{2\pi + a_o} A_1 - \beta = A_o + \tfrac{1}{2}\left(\frac{2\pi - a_o}{2\pi + a_o}\right) A_1, \quad \ldots\ldots (52)$$

from (39). In particular, if $a_o = 2\pi$, then

$$C_{Lopt.} = \pi A_1, \ \alpha_{opt.} = A_o. \quad \ldots\ldots (53)$$

Now write

$$g_i = -\frac{\sin\theta}{\pi} P \int_0^{\pi} \frac{F(t)dt}{\cos\theta - \cos t} = \sum_{n=1}^{\infty} A_n \sin n\theta. \quad \ldots\ldots (54)$$

Then from (26), (38) and (50)

$$g_c + g_L = g_i + \tfrac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) (C_L - C_{Lopt.}) \cot \tfrac{1}{2}\theta$$
$$- \tfrac{1}{2}\left(\frac{1}{a_o} - \frac{1}{2\pi}\right) C_L \tan \tfrac{1}{2}\theta, \quad \ldots\ldots (55)$$

so that g_i is the value of $g_c + g_L$ when $C_L = C_{Lopt.}$ and $a_o = 2\pi$ (more accurately $2\pi e^{C_o}$).

A formula for $\alpha_{opt.}$ when $a_o = 2\pi$, equivalent to $\alpha_{opt.} = A_o$, was given by Theodorsen[4], who stressed its importance for avoiding a large suction peak near the nose. Theodorsen was considering the normal force distribution along the chord of an infinitely thin aerofoil with $a_o = 2\pi$, and obtained formulae equivalent to formulae for $g_c + g_L$ in our notation. Although expressed in terms

of/

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*If $C_L = a_o (\alpha_{opt.} + \beta_1)$, where $\beta_1 \neq \beta$, we must add $a_o (\beta - \beta_1)/(2\pi + a_o)$ to the right-hand side of (52).

of integrals, these formulae* are different from, and more complicated than, those given here.

Since A_1 is proportional to the amount of camber, the formula connecting $C_{L\,opt.}$ and A_1 may be used to find, for a given shape of centre line, the amount of camber which should be given to a wing for a specified C_L in order that a maximum of q/U near the nose may be avoided. Now $g_c + g_L$ has equal and opposite values on the upper and lower surfaces, and if the slope of the curve of $g_c + g_L$ for $C_L = C_{L\,opt.}$ exceeds the slope of the curve of g_s, then the curve of g for $C_L = C_{L\,opt.}$ on the lower surface will have a negative slope, and when the curve is rounded off by multiplication by the factor (48) we shall be left with a maximum of q/U somewhere near the nose on the lower surface. To avoid this maximum it will be necessary to take a somewhat higher value of C_L. Thus if the slope of the curve of $g_c + g_L$ for $C_L = C_{L\,opt.}$ exceeds the slope of the curve of g_s, the criterion in (50) will underestimate, for a given camber, the lift coefficient at which the maximum velocity may be kept back from the nose, and will therefore overestimate the amount of camber required for any specified C_L. It will appear also that if a wing is designed to have the maximum values of q/U on both surfaces at a considerable fraction of the chord back from the leading edge, then there is a certain range of values of C_L, which may be a moderate range or a very small one, for which the positions of the maxima of q/U may be considered satisfactory. This "favourable range of C_L" will form the object of a separate study, but we may note now that a diminution of this favourable range will be a consequence of the shifting of the "optimum" C_L value, as explained above, for a wing for which the slope of the curve of g_s is less than that of the curve of $g_c + g_L$ for $C_L = C_{L\,opt.}$, i.e. very nearly, of the curve of g_1.

7./

*Namely

$$g_c + g_L = (\alpha - \alpha_{opt.}) \left(\frac{1-x}{x}\right)^{\frac{1}{2}} + x^{\frac{1}{2}} (1-x)^{\frac{1}{2}} \frac{d\varepsilon}{dx}$$

$$+ \frac{1}{2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \{\varepsilon - \varepsilon_0 + x\,(\varepsilon_0 - 2\varepsilon + \varepsilon_1)\},$$

where

$$\alpha_{opt.} = \frac{1}{2\pi} \int_0^1 \frac{y\,(1-2x)}{[x(1-x)]^{3/2}}\,dx,$$

$$\varepsilon = \frac{1}{\pi} P \int_0^1 \frac{y\,(\xi)\,d\xi}{\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}(x-\xi)},$$

and ε_0, ε_1 are the values of ε at $x = 0$ and $x = 1$, respectively.

7. Summary of Formulae.

We have now obtained all the general formulae necessary to study the velocity distribution at the surface of a cambered aerofoil. Before considering examples it will be convenient to summarize the formulae.

If

$$\frac{dy_c}{dx} = F(\theta) = \sum_{n=0}^{\infty} A_n \cos n\theta \qquad \text{..... (56)}$$

and

$$\psi_c = \frac{2y_c}{\sin\theta} = \sum_{n=1}^{\infty} D_n \sin n\theta, \qquad \text{..... (57)}$$

then

$$\beta = \tfrac{1}{2}A_1 - A_0, \quad C_{M_0} = \frac{\pi}{4}(A_2 - A_1), \qquad \text{..... (58)}$$

$$C_{L\,opt.}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) = A_1, \quad \alpha_{opt.} = A_0 + \tfrac{1}{2}\left(\frac{2\pi - a_0}{2\pi + a_0}\right)A_1, \qquad \text{..... (59)}$$

$$g_i = -\frac{\sin\theta}{\pi} P \int_0^{\pi} \frac{F(t)\,dt}{\cos\theta - \cos t} = \sum_{n=1}^{\infty} A_n \sin n\theta, \qquad \text{..... (60)}$$

$$g_c = g_i - \tfrac{1}{2}A_1 \cot\tfrac{1}{2}\theta, \qquad \text{..... (61)}$$

$$\begin{aligned} g_c + g_L &= g_i + \tfrac{1}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)(C_L - C_{L\,opt.})\cot\tfrac{1}{2}\theta - \tfrac{1}{2}\left(\frac{1}{a_0} - \frac{1}{2\pi}\right)C_L \tan\tfrac{1}{2}\theta \\ &= g_i - \tfrac{1}{2}A_1\cot\tfrac{1}{2}\theta + \tfrac{1}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)C_L\cot\tfrac{1}{2}\theta - \tfrac{1}{2}\left(\frac{1}{a_0} - \frac{1}{2\pi}\right)C_L \tan\tfrac{1}{2}\theta, \end{aligned} \qquad \text{..... (62)}$$

and

$$\varepsilon_c = -\frac{2}{\pi} P \int_0^{\pi} \frac{y_c(\zeta)}{\cos\theta - \cos t}\,dt = -\sum_{n=1}^{\infty} D_n \cos n\theta, \qquad \text{..... (63)}$$

where/

where

$$\xi = \tfrac{1}{2}(1 - \cos t). \qquad \ldots\ldots (64)$$

The formulae for q/U are as follows.

Approximation I:

$$q/U = 1 + g_s + g_c + g_L \qquad \ldots\ldots (65)$$

Approximation II:

$$\frac{q}{U} = \frac{(1 + \frac{1}{2} C_o^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g_s + g_c + g_L)$$

$$= \frac{(1 + \frac{1}{2} C_o^2)|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g_s + g_i)$$

$$\pm \frac{1 + \frac{1}{2} C_o^2}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left[\frac{C_L}{2\pi} + \frac{C_L}{a_o}\cos\theta - \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)C_{Lopt.}(1 + \cos\theta)\right] \qquad \ldots\ldots (66)$$

Approximation III:

$$\frac{q}{U} = \frac{e^{C_o(1+\varepsilon')}}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left|\left(1 - \frac{C_L^2}{a_o^2}\right)^{\frac{1}{2}}\sin(\theta+\varepsilon-\beta) + \frac{C_L}{a_o}\cos(\theta+\varepsilon-\beta) + \frac{C_L e^{-C_o}}{2\pi}\right|. \qquad \ldots\ldots (67)$$

ψ is to be found from (10), ε from (13) and ε' from (20). g_s, ψ_s, ε_s, ε_s' refer to values for the corresponding symmetrical aerofoil, found as described in Part I.

The formulae show that in any example we have to find the values of A_0, A_1, A_2, g_i, ε_c, ε_c'. We use the Fourier series for g_i and ε_c if they terminate, which is the case if, and only if, y is a polynomial in x over the whole range $0 \leqslant x \leqslant 1$; otherwise we use the integrals. As in Part I, the integrals may be changed into proper integrals by the use of Lemma (7) of Part I; thus

$$g_i = \frac{\sin\theta}{\pi}\int_0^\pi \frac{F(\theta) - F(t)}{\cos\theta - \cos t}\,dt, \qquad \ldots\ldots (68)$$

$$\varepsilon_c = \frac{2}{\pi}\int_0^\pi \frac{y_c(x) - y_c(\xi)}{\cos\theta - \cos t}\,dt, \qquad \ldots\ldots (69)$$

and in these forms the integrals may be evaluated by numerical integration. This method of numerical evaluation is not recommended if suitable algebraic formulae can be fitted to the ordinate y_c of the centre line; in such a case use should be made of the analytical results of the following sections.

8./

8. Method of Obtaining Results when the Ordinates are given by Algebraic Formulae.

We shall now suppose that, with the chord divided into any number of segments, in each segment the ordinate y_c of the centre line is a polynomial in x.

Whereas in Part I it was most convenient to carry out the analysis in terms of x, it is here most conveniently carried out in terms of θ. The resulting formulae may be altered into formulae in x whenever such a change is thought desirable (just as the formulae of Part I may be altered into formulae in θ); the necessary formulae of transformation will be considered later. (See the end of §9 and Lemma 10 (Appendix)). For purposes of computation it appears that neither form has a clear-cut advantage over the other.

The functions y_c and $dy_c/dx = F(\theta)$ are given over the range $(0,\pi)$ for θ, and if the chord is divided into any number, say s, of segments, then the interval $(0,\pi)$ for θ is divided into s intervals, say $(0,\theta_1)$, (θ_1, θ_2), ... (θ_{r-1}, θ_r) ...... (θ_{s-1}, π). In each interval y_c and dy_c/dx are polynomials in x; it follows from Lemma 3 (Appendix) that in any interval, e.g. (θ_{r-1}, θ_r), y_c and $F(\theta)$ are represented by expressions of the form

$$y_c = \sum_{n=0}^{m_r} a_{nr} \cos n\theta, \quad F(\theta) = \sum_{n=0}^{m_r-1} b_{nr} \cos n\theta \quad (\theta_{r-1} \leqslant \theta \leqslant \theta_r) \quad .... \quad (70)$$

where the a_{nr} and b_{nr} are known, and m_r is the degree of the polynomial representation of y_c in terms of x in the interval considered. The values of m_r, a_{nr}, b_{nr} will in general be different in each of the different intervals (i.e. they will vary with r), so this representation is <u>not</u> a representation by Fourier cosine series.

The expressions for y_c and $F(\theta)$ in any interval are the sums of a number of contributions of which typical ones in the interval (θ_{r-1}, θ_r) are $\cos n\theta$ multiplied by a_{nr} and b_{nr} respectively. Thus A_0, A_1, A_2 and g_i are found by finding the contributions from a term $\cos n\theta$ in the expression for $F(\theta)$ in the interval (θ_{r-1}, θ_r), multiplying by b_{nr}, summing for n from $n = 0$ to $n = m_r - 1$, and finally summing the results so found for all the intervals; ε_c is similarly found from y_c. We therefore proceed to write down the contributions to A_0, A_1, A_2 and g_i from a term $\cos n\theta$ in the expression for $F(\theta)$ in the interval (θ_{r-1}, θ_r). A_0, A_1, A_2 are found from (36); thus the contribution

$$\text{to } A_0 = \frac{1}{\pi}\int_{\theta_{r-1}}^{\theta_r} \cos nt\, dt, \quad \text{to } A_1 = \frac{2}{\pi}\int_{\theta_{r-1}}^{\theta_r} \cos nt \cos t\, dt,$$

$$\text{to } A_2 = \frac{2}{\pi}\int_{\theta_{r-1}}^{\theta_r} \cos nt \cos 2t\, dt. \quad .... \quad (71)$$

Hence/

Hence the contribution to

$$\left.\begin{aligned} A_0 &= \frac{1}{\pi}(\theta_r - \theta_{r-1}) \text{ for } n = 0 \\ &= \frac{1}{n\pi}(\sin n\theta_r - \sin n\theta_{r-1}) \text{ for } n \geqslant 1, \end{aligned}\right\} \quad \ldots\ldots (72)$$

to

$$\left.\begin{aligned} A_1 &= \frac{2}{\pi}(\sin \theta_r - \sin \theta_{r-1}) \text{ for } n = 0 \\ &= \frac{1}{\pi}\left(\theta_r - \theta_{r-1} + \frac{\sin 2\theta_r - \sin 2\theta_{r-1}}{2}\right) \text{ for } n = 1 \\ &= \frac{1}{\pi}\left\{\frac{\sin (n-1)\theta_r - \sin (n-1)\theta_{r-1}}{n-1} + \frac{\sin (n+1)\theta_r - \sin (n+1)\theta_{r-1}}{n+1}\right\} \\ &\qquad \text{for } n \geqslant 2, \end{aligned}\right\} (73)$$

and to

$$\left.\begin{aligned} A_2 &= \frac{1}{\pi}(\sin 2\theta_r - \sin 2\theta_{r-1}) \text{ for } n = 0 \\ &= \frac{1}{\pi}\left(\sin \theta_r - \sin \theta_{r-1} + \frac{\sin 3\theta_r - \sin 3\theta_{r-1}}{3}\right) \text{ for } n = 1 \\ &= \frac{1}{\pi}\left(\theta_r - \theta_{r-1} + \frac{\sin 4\theta_r - \sin 4\theta_{r-1}}{4}\right) \text{ for } n = 2 \\ &= \frac{1}{\pi}\left\{\frac{\sin (n-2)\theta_r - \sin (n-2)\theta_{r-1}}{n-2} + \frac{\sin (n+2)\theta_r - \sin (n+2)\theta_{r-1}}{n+2}\right\} \\ &\qquad \text{for } n > 2. \end{aligned}\right\} (74)$$

Now write

$$I_n(\theta_r) = P \int_0^{\theta_r} \frac{\cos nt}{\cos \theta - \cos t}\, dt. \quad \ldots\ldots (75)$$

Then from (60) and Lemma 4 (Appendix), the contribution to

g_1/

$$\begin{aligned} g_i &= -\frac{\sin\theta}{\pi} P\int_{\theta_{r-1}}^{\theta_r} \frac{\cos nt}{\cos\theta - \cos t}\,dt \\ &= -\frac{1}{\pi}\{I_n(\theta_r)\sin\theta - I_n(\theta_{r-1})\sin\theta\} \\ &= \frac{1}{\pi}\left\{\log_e \frac{\sin\frac{1}{2}|\theta-\theta_{r-1}|}{\sin\frac{1}{2}(\theta+\theta_{r-1})} - \log_e \frac{\sin\frac{1}{2}|\theta-\theta_r|}{\sin\frac{1}{2}(\theta+\theta_r)}\right\} \text{ for } n = 0 \\ &= \frac{\cos\theta}{\pi}\left\{\log_e \frac{\sin\frac{1}{2}|\theta-\theta_{r-1}|}{\sin\frac{1}{2}(\theta+\theta_{r-1})} - \log_e \frac{\sin\frac{1}{2}|\theta-\theta_r|}{\sin\frac{1}{2}(\theta+\theta_r)}\right\} \\ &\quad + \frac{\sin\theta}{\pi}(\theta_r - \theta_{r-1}) \text{ for } n = 1 \\ &= \frac{\cos n\theta}{\pi}\left\{\log_e \frac{\sin\frac{1}{2}|\theta-\theta_{r-1}|}{\sin\frac{1}{2}(\theta+\theta_{r-1})} - \log_e \frac{\sin\frac{1}{2}|\theta-\theta_r|}{\sin\frac{1}{2}(\theta+\theta_r)}\right\} \\ &\quad + \frac{\sin n\theta}{\pi}(\theta_r - \theta_{r-1}) \\ &\quad + \frac{2}{\pi}\sum_{s=0}^{n-2} \frac{\sin(n-s-1)\theta_r - \sin(n-s-1)\theta_{r-1}}{n-s-1}\sin(s+1)\theta \text{ for } n \geq 2. \end{aligned} \quad (76)$$

The contributions to β, C_{M_0}, $C_{L\,opt}$, α_{opt}, ε_c, $g_c + g_L$ follow from (58), (59), (61) and (62), and it appears unnecessary to write them out.

Similarly the contribution from a term $\cos n\theta$ in the expression for y_c in the interval (θ_{r-1}, θ_r) to

$$\varepsilon_c(\theta) = -\frac{2}{\pi} P\int_{\theta_{r-1}}^{\theta_r} \frac{\cos nt}{\cos\theta - \cos t}\,dt = -\frac{2}{\pi}\{I_n(\theta_r) - I_n(\theta_{r-1})\}, \quad \ldots\ldots (77)$$

so the contribution to $\frac{1}{2}\varepsilon_c(\theta)\sin\theta$ is equal to the contribution to g_i in (76) above for the same value of n.

We may find β from (39) and the values of A_0 and A_1; if we calculate ε_c then β is the value of $\varepsilon_c(\pi)$, which is the value given by (28); the values calculated by the two methods must agree, so we may check our calculations in any example. In transforming (28) to (39) it was assumed, of course, that $\psi_c(1 - \cos\theta)$

is/

is continuous, i.e. that $y_c \tan\frac{1}{2}\theta$ is continuous, from segment to segment, and in order to show that the two values of β agree, when found as algebraic formulae and not numerically, we may have to use the conditions of continuity of $y_c \tan\frac{1}{2}\theta$.

We shall find $\varepsilon_c'(\theta)$ by differentiation of the formula for $\varepsilon_c(\theta)$, and if we also find g_c we shall have another check from equation (25).

9. Centre-Line Ordinate Represented by Two Quartics.

We now consider the calculations in detail when y_c is represented by two quartics in x, one in each of two segments of the chord. It appears that such algebraic formulae as have so far been proposed for centre-line ordinates are all* included as special cases.

Let

$$\left.\begin{aligned} y_c &= y_1(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 \text{ for } 0 \leqslant x \leqslant X_1 \\ &= y_2(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 \text{ for } X_1 \leqslant x \leqslant 1, \end{aligned}\right\} \quad (78)$$

so that

$$\left.\begin{aligned} \frac{dy_c}{dx} &= y_1'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 \text{ for } 0 \leqslant x \leqslant X_1 \\ &= y_2'(x) = b_1 + 2b_2x + 3b_3x^2 + 4b_4x^3 \text{ for } X_1 \leqslant x \leqslant 1. \end{aligned}\right\} \quad (79)$$

Since y_c is measured from the chord line, which is the join of the leading and trailing edges, it must vanish at the trailing edge, so

$$\sum_{n=0}^{4} b_n = 0. \quad \ldots\ldots (80)$$

Let

$$X_1 = \tfrac{1}{2}(1 - \cos\theta_1) \quad \ldots\ldots (81)$$

so that

$$\theta_1 = 2\sin^{-1} X_1^{\frac{1}{2}}. \quad \ldots\ldots (82)$$

Then it follows from Lemma 3 (Appendix) that, in terms of θ, ...

y_c/

*Except centre lines designed to give, according to thin-aerofoil theory, specified load distributions along the chord at $C_L = C_{L\text{opt}}$. See Part IV.

$$y_c = y_1 = \frac{1}{128}(64a_1 + 48a_2 + 40a_3 + 35a_4)$$

$$- \frac{1}{32}(16a_1 + 16a_2 + 15a_3 + 14a_4)\cos\theta$$

$$+ \frac{1}{32}(4a_2 + 6a_3 + 7a_4)\cos 2\theta - \frac{1}{32}(a_3 + 2a_4)\cos 3\theta$$

$$+ \frac{a_4}{128}\cos 4\theta \text{ for } 0 \leqslant \theta \leqslant \theta_1, \qquad \ldots\ldots (83)$$

$$y_c = y_2 = \frac{1}{128}(128b_0 + 64b_1 + 48b_2 + 40b_3 + 35b_4)$$

$$- \frac{1}{32}(16b_1 + 16b_2 + 15b_3 + 14b_4)\cos\theta$$

$$+ \frac{1}{32}(4b_2 + 6b_3 + 7b_4)\cos 2\theta - \frac{1}{32}(b_3 + 2b_4)\cos 3\theta$$

$$+ \frac{b_4}{128}\cos 4\theta \text{ for } \theta_1 \leqslant \theta \leqslant \pi, \qquad \ldots\ldots (84)$$

$$F(\theta) = \frac{dy_1}{dx} = \frac{1}{8}(8a_1 + 8a_2 + 9a_3 + 10a_4) - \frac{1}{8}(8a_2 + 12a_3 + 15a_4)\cos\theta$$

$$+ \frac{3}{8}(a_3 + 2a_4)\cos 2\theta - \frac{a_4}{8}\cos 3\theta \text{ for } 0 \leqslant \theta \leqslant \theta_1, \quad \ldots\ldots (85)$$

$$F(\theta) = \frac{dy_2}{dx} = \frac{1}{8}(8b_1 + 8b_2 + 9b_3 + 10b_4) - \frac{1}{8}(8b_2 + 12b_3 + 15b_4)\cos\theta$$

$$+ \frac{3}{8}(b_3 + 2b_4)\cos 2\theta - \frac{b_4}{8}\cos 3\theta \text{ for } \theta_1 \leqslant \theta \leqslant \pi. \qquad (86)$$

In the results of §8 we take $\theta_r = \theta_1$, $\theta_{r-1} = 0$ for the first interval and $\theta_r = \pi$, $\theta_{r-1} = \theta_1$ for the second interval, and we multiply the various contributions by the appropriate coefficients in the expressions above. In this way, if we write

$$c_r = a_r - b_r \; (r = 1, 2, 3, 4), \qquad \ldots\ldots (87)$$

we find, using the coefficients in (85) and (86), that

A/

$$A_o = \frac{1}{8}(8b_1 + 8b_2 + 9b_3 + 10b_4) + \frac{\theta_1}{8\pi}(8c_1 + 8c_2 + 9c_3 + 10c_4)$$
$$- \frac{\sin\theta_1}{8\pi}(8c_2 + 12c_3 + 15c_4) + \frac{3\sin 2\theta_1}{16\pi}(c_3 + 2c_4) - \frac{c_4 \sin 3\theta_1}{24\pi}, \quad \ldots\ldots (88)$$

$$A_1 = -\frac{1}{8}(8b_2 + 12b_3 + 15b_4) - \frac{\theta_1}{8\pi}(8c_2 + 12c_3 + 15c_4)$$
$$+ \frac{\sin\theta_1}{8\pi}(16c_1 + 16c_2 + 21c_3 + 26c_4) - \frac{\sin 2\theta_1}{4\pi}(2c_2 + 3c_3 + 4c_4)$$
$$+ \frac{\sin 3\theta_1}{8\pi}(c_3 + 2c_4) - \frac{c_4 \sin 4\theta_1}{32\pi}, \quad \ldots\ldots (89)$$

$$A_2 = \frac{3}{8}(b_3 + 2b_4) + \frac{3\theta_1}{8\pi}(c_3 + 2c_4) - \frac{\sin\theta_1}{2\pi}(2c_2 + 3c_3 + 4c_4)$$
$$+ \frac{\sin 2\theta_1}{8\pi}(8c_1 + 8c_2 + 9c_3 + 10c_4) - \frac{\sin 3\theta_1}{24\pi}(8c_2 + 12c_3 + 15c_4)$$
$$+ \frac{3\sin 4\theta_1}{32\pi}(c_3 + 2c_4) - \frac{c_4 \sin 5\theta_1}{40\pi}. \quad \ldots\ldots (90)$$

Hence

$$\beta = \tfrac{1}{2}A_1 - A_o = -\frac{1}{16}(16b_1 + 24b_2 + 30b_3 + 35b_4)$$
$$- \frac{\theta_1}{16\pi}(16c_1 + 24c_2 + 30c_3 + 35c_4)$$
$$+ \frac{\sin\theta_1}{16\pi}(16c_1 + 32c_2 + 45c_3 + 56c_4)$$
$$- \frac{\sin 2\theta_1}{16\pi}(4c_2 + 9c_3 + 14c_4)$$
$$+ \frac{\sin 3\theta_1}{48\pi}(3c_3 + 8c_4) - \frac{c_4 \sin 4\theta_1}{64\pi}, \quad \ldots\ldots (91)$$

C_{M_0}/

$$C_{M_0} = \frac{\pi}{4}(A_2 - A_1) = \frac{\pi}{32}(8b_2 + 15b_3 + 21b_4) + \frac{\theta_1}{32}(8c_2 + 15c_3 + 21c_4)$$

$$- \frac{\sin\theta_1}{32}(16c_1 + 24c_2 + 33c_3 + 42c_4)$$

$$+ \frac{\sin 2\theta_1}{32}(8c_1 + 12c_2 + 15c_3 + 18c_4)$$

$$- \frac{\sin 3\theta_1}{96}(8c_2 + 15c_3 + 21c_4)$$

$$+ \frac{\sin 4\theta_1}{128}(3c_3 + 7c_4) - \frac{c_4 \sin 5\theta_1}{160} . \qquad \ldots\ldots (92)$$

Also the values of $C_{L\,opt.}$ and $\alpha_{opt.}$ are found at once from (59), and it appears unnecessary to write them out. For g_i we find that

$$g_i = -\frac{1}{\pi}\left(\frac{dy_1}{dx} - \frac{dy_2}{dx}\right)\log_e \frac{\sin\frac{1}{2}|\theta - \theta_1|}{\sin\frac{1}{2}(\theta + \theta_1)}$$

$$- \sin\theta\left[\frac{1}{8}(8b_2 + 12b_3 + 15b_4) + \frac{\theta_1}{8\pi}(8c_2 + 12c_3 + 15c_4)\right.$$

$$\left.- \frac{3\sin\theta_1}{4\pi}(c_3 + 2c_4) + \frac{c_4 \sin 2\theta_1}{8\pi}\right]$$

$$+ \sin 2\theta\left[\frac{3}{8}(b_3 + 2b_4) + \frac{3\theta_1}{8\pi}(c_3 + 2c_4) - \frac{c_4\sin\theta_1}{4\pi}\right]$$

$$- \sin 3\theta\left[\frac{b_4}{8} + \frac{c_4\theta_1}{8\pi}\right] , \qquad \ldots\ldots (93)$$

and g_c, $g_c + g_L$ follow from (61) and (62).

In the same way, using the coefficients in (83) and (84), we find that

½/

$$\tfrac{1}{2}\varepsilon_c \sin\theta = -\frac{1}{\pi}\left[y_1(x) - y_2(x)\right] \log_e \frac{\sin\tfrac{1}{2}|\theta - \theta_1|}{\sin\tfrac{1}{2}(\theta + \theta_1)}$$

$$- \sin\theta\left[\frac{1}{32}(16b_1 + 16b_2 + 15b_3 + 14b_4)\right.$$

$$+ \frac{\theta_1}{32\pi}(16c_1 + 16c_2 + 15c_3 + 14c_4)$$

$$- \frac{\sin\theta_1}{16\pi}(4c_2 + 6c_3 + 7c_4)$$

$$\left. + \frac{\sin 2\theta_1}{32\pi}(c_3 + 2c_4) - \frac{c_4 \sin 3\theta_1}{192\pi}\right]$$

$$+ \sin 2\theta\left[\frac{1}{32}(4b_2 + 6b_3 + 7b_4) + \frac{\theta_1}{32\pi}(4c_2 + 6c_3 + 7c_4)\right.$$

$$\left. - \frac{\sin\theta_1}{16\pi}(c_3 + 2c_4) + \frac{c_4 \sin 2\theta_1}{128\pi}\right]$$

$$- \sin 3\theta\left[\frac{1}{32}(b_3 + 2b_4) + \frac{\theta_1}{32\pi}(c_3 + 2c_4) - \frac{c_4 \sin\theta_1}{64\pi}\right]$$

$$+ \sin 4\theta\left[\frac{b_4}{128} + \frac{c_4\theta_1}{128\pi}\right]. \qquad \ldots\ldots (94)$$

We divide by $\tfrac{1}{2}\sin\theta$ and use the expansion of $\sin n\theta/\sin\theta$ given at the end of Lemma 4 (Appendix). We then collect together the various terms, and substitute b_0 for $-(b_1 + b_2 + b_3 + b_4)$ in the constant term (see equation (80)). We also write

$$\psi_1(\theta) = \frac{2y_1}{\sin\theta}, \quad \psi_2(\theta) = \frac{2y_2}{\sin\theta}. \qquad \ldots\ldots (95)$$

The resulting formula for ε_c is

$\varepsilon_c(\theta)$/

$$\varepsilon_c(\theta) = -\frac{1}{\pi}[\psi_1(\theta)-\psi_2(\theta)] \log_e \frac{\sin \frac{1}{2}|\theta-\theta_1|}{\sin \frac{1}{2}(\theta+\theta_1)} + b_0 - \frac{\theta_1}{\pi}(c_1+c_2+c_3+c_4)$$

$$+ \frac{\sin \theta_1}{32\pi}(16c_2+24c_3+29c_4) - \frac{\sin 2\theta_1}{16\pi}(c_3+2c_4) + \frac{c_4 \sin 3\theta_1}{96\pi}$$

$$+ \cos \theta \left[\frac{1}{32}(16b_2+24b_3+29b_4) + \frac{\theta_1}{32\pi}(16c_2+24c_3+29c_4)\right.$$

$$\left. - \frac{\sin \theta_1}{4\pi}(c_3 + 2c_4) + \frac{c_4 \sin 2\theta_1}{32\pi}\right]$$

$$- \cos 2\theta \left[\frac{1}{8}(b_3+2b_4) + \frac{\theta_1}{8\pi}(c_3+2c_4) - \frac{c_4 \sin \theta_1}{16\pi}\right]$$

$$+ \cos 3\theta \left[\frac{b_4}{32} + \frac{c_4\theta_1}{32\pi}\right] . \qquad \ldots\ldots (96)$$

It is not difficult to show that

$$\lim_{\theta \to 0} [\psi_1(\theta) - \psi_2(\theta)]\log_e \frac{\sin \frac{1}{2}|\theta-\theta_1|}{\sin \frac{1}{2}(\theta+\theta_1)} = 2b_0 \cot\tfrac{1}{2}\theta_1, \qquad \ldots\ldots (97)$$

and that, with equation (80) satisfied,

$$\lim_{\theta \to \pi} [\psi_1(\theta)-\psi_2(\theta)] \log_e \frac{\sin \frac{1}{2}|\theta-\theta_1|}{\sin \frac{1}{2}(\theta+\theta_1)} = -2(a_1+a_2+a_3+a_4) \tan\tfrac{1}{2}\theta_1 . \qquad \ldots\ldots (98)$$

Hence

$$\varepsilon_c(0) = -\frac{2b_0}{\pi} \cot\tfrac{1}{2}\theta_1 - \frac{1}{16}(16b_1+8b_2+6b_3+5b_4) - \frac{\theta_1}{16\pi}(16c_1+8c_2+6c_3+5c_4)$$

$$+ \frac{\sin \theta_1}{32\pi}(16c_2+16c_3+15c_4) - \frac{\sin 2\theta_1}{32\pi}(2c_3+3c_4) + \frac{c_4 \sin 3\theta_1}{96\pi}, \qquad \ldots\ldots (99)$$

and

$$\beta = \varepsilon_c(\pi) = \frac{2}{\pi}(a_1+a_2+a_3+a_4) \tan\tfrac{1}{2}\theta_1 - \frac{1}{16}(16b_1+24b_2+30b_3+35b_4)$$

$$- \frac{\theta_1}{16\pi}(16c_1+24c_2+30c_3+35c_4) + \frac{\sin \theta_1}{32\pi}(16c_2+32c_3+47c_4)$$

$$- \frac{\sin 2\theta_1}{32\pi}(2c_3 + 5c_4) + \frac{c_4 \sin 3\theta_1}{96\pi} . \qquad \ldots\ldots (100)$$

The/

The two values of β in (91) and (100) agree if

$$\frac{2}{\pi}(a_1+a_2+a_3+a_4)\tan\tfrac{1}{2}\theta_1 = \frac{\sin\theta_1}{32\pi}(32c_1+48c_2+58c_3+65c_4)$$

$$-\frac{\sin 2\theta_1}{32\pi}(8c_2+16c_3+23c_4)$$

$$+\frac{\sin 3\theta_1}{32\pi}(2c_3+5c_4) - \frac{c_4\sin 4\theta_1}{64\pi}, \qquad \ldots\ldots (101)$$

and from the formulae for y_c in (83) and (84) and for $\tan\frac{1}{2}\theta\cos n\theta$ in Lemma 5 (Appendix), it follows that this relation expresses the continuity of $y_c\tan\frac{1}{2}\theta$ at $\theta = \theta_1$.

Finally we must calculate $\varepsilon_c'(\theta)$. By straightforward differentiation of (96) we find that

$$\varepsilon_c'(\theta) = -\frac{[\psi_1(\theta)-\psi_2(\theta)]\sin\theta_1}{\pi(\cos\theta_1-\cos\theta)} - \frac{1}{\pi}[\psi_1'(\theta)-\psi_2'(\theta)]\log_e\frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)}$$

$$-\sin\theta\left[\frac{1}{32}(16b_2+24b_3+29b_4)+\frac{\theta_1}{32\pi}(16c_2+24c_3+29c_4)\right.$$

$$\left.-\frac{\sin\theta_1}{4\pi}(c_3+2c_4)+\frac{c_4\sin 2\theta_1}{32\pi}\right]$$

$$+\sin 2\theta\left[\frac{1}{4}(b_3+2b_4)+\frac{\theta_1}{4\pi}(c_3+2c_4)-\frac{c_4\sin\theta_1}{8\pi}\right]$$

$$-\sin 3\theta\left[\frac{3b_4}{32}+\frac{3c_4\theta_1}{32\pi}\right], \qquad \ldots\ldots (102)$$

where

$$\left.\begin{aligned}\psi_1'(\theta) &= y_1'(x) - \frac{2\cos\theta}{\sin^2\theta}y_1(x) = y_1'(x) - \psi_1(\theta)\cot\theta \\ \psi_2'(\theta) &= y_2'(x) - \frac{2\cos\theta}{\sin^2\theta}y_2(x) = y_2'(x) - \psi_2(\theta)\cot\theta\end{aligned}\right\}, \qquad (103)$$

but since $\psi_1(\theta_1) = \psi_2(\theta_1)$ the numerator and denominator of the first term in (102) both vanish when $\theta = \theta_1$, and a transformation of this term is desirable. If $y_1(\theta)$, $y_2(\theta)$ are used to denote the values of y_1 and y_2 in (83) and (84), then $y_1(\theta_1) = y_2(\theta_1)$

and/

and

$$\frac{[\psi_1(\theta)-\psi_2(\theta)]\ \sin\theta_1}{\cos\theta_1-\cos\theta}$$

$$= \frac{2\sin\theta_1}{\sin\theta(\cos\theta_1-\cos\theta)}\ \{y_1(\theta)-y_2(\theta)-[y_1(\theta_1)-y_2(\theta_1)]\}$$

$$= \frac{\sin\theta_1}{\sin\theta}\left[\frac{1}{16}(16c_1+16c_2+15c_3+14c_4)\right.$$

$$-\frac{1}{16}(4c_2+6c_3+7c_4)\ \frac{\cos 2\theta_1-\cos 2\theta}{\cos\theta_1-\cos\theta}$$

$$\left.+\frac{1}{16}(c_3+2c_4)\ \frac{\cos 3\theta_1-\cos 3\theta}{\cos\theta_1-\cos\theta}-\frac{c_4}{64}\ \frac{\cos 4\theta_1-\cos 4\theta}{\cos\theta_1-\cos\theta}\right]$$

$$= \cot\tfrac{1}{2}\theta\left[\frac{\sin\theta_1}{64}(32c_1+16c_2+10c_3+7c_4)-\frac{\sin 2\theta_1}{64}(8c_2+8c_3+7c_4)\right.$$

$$\left.+\frac{\sin 3\theta_1}{64}(2c_3+3c_4)-\frac{c_4}{128}\sin 4\theta_1\right]$$

$$+\tan\tfrac{1}{2}\theta\left[\frac{\sin\theta_1}{64}(32c_1+48c_2+58c_3+65c_4)-\frac{\sin 2\theta_1}{64}(8c_2+16c_3+23c_4)\right.$$

$$\left.+\frac{\sin 3\theta_1}{64}(2c_3+5c_4)-\frac{c_4}{128}\sin 4\theta_1\right]$$

$$-\sin\theta\left[\frac{\sin\theta_1}{4}(c_3+2c_4)-\frac{c_4}{16}\sin 2\theta_1\right]+\frac{c_4\sin\theta_1}{16}\sin 2\theta \qquad \ldots\ldots (104)$$

from Lemma 8 (Appendix).

Hence/

Hence

$$\varepsilon_c'(\theta) = -\frac{1}{\pi}\left[\psi_1'(\theta)-\psi_2'(\theta)\right]\log_e \frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)}$$

$$-\cot\tfrac{1}{2}\theta\left[\frac{\sin\theta_1}{64\pi}(32c_1+16c_2+10c_3+7c_4)-\frac{\sin 2\theta_1}{64\pi}(8c_2+8c_3+7c_4)\right.$$

$$\left.+\frac{\sin 3\theta_1}{64\pi}(2c_3+3c_4)-\frac{c_4\sin 4\theta_1}{128\pi}\right]$$

$$-\tan\tfrac{1}{2}\theta\left[\frac{\sin\theta_1}{64\pi}(32c_1+48c_2+58c_3+65c_4)\right.$$

$$\left.-\frac{\sin 2\theta_1}{64\pi}(8c_2+16c_3+23c_4)+\frac{\sin 3\theta_1}{64\pi}(2c_3+5c_4)-\frac{c_4\sin 4\theta_1}{128\pi}\right]$$

$$-\sin\theta\left[\frac{1}{32}(16b_2+24b_3+29b_4)+\frac{\theta_1}{32\pi}(16c_2+24c_3+29c_4)\right.$$

$$\left.-\frac{\sin\theta_1}{2\pi}(c_3+2c_4)-\frac{3c_4\sin 2\theta_1}{32\pi}\right]$$

$$+\sin 2\theta\left[\frac{1}{4}(b_3+2b_4)+\frac{\theta_1}{4\pi}(c_3+2c_4)-\frac{3c_4\sin\theta_1}{16\pi}\right]$$

$$-\sin 3\theta\left[\frac{3b_4}{32}+\frac{3c_4\theta_1}{32\pi}\right]. \qquad \ldots\ldots (105)$$

By considering limits as $\theta \to 0$ and as $\theta \to \pi$ we may verify that

$$\varepsilon_c'(0) = \varepsilon_c'(\pi) = 0. \qquad \ldots\ldots (106)$$

Finally by using Lemma 9 (Appendix) to transform $(\varepsilon_c - \beta)\cot\theta$ we may verify that the equation

$$\varepsilon_c' + (\varepsilon_c - \beta)\cot\theta = g_i - \tfrac{1}{2}A_1\cot\tfrac{1}{2}\theta \qquad \ldots\ldots (107)$$

is satisfied, as it should be since each side is equal to g_c.

If we wish to express our formulae in terms of x we have

$$\sin\theta = 2x^{\frac{1}{2}}(1-x)^{\frac{1}{2}},\ \cot\tfrac{1}{2}\theta = \left(\frac{1-x}{x}\right)^{\frac{1}{2}},\ \tan\tfrac{1}{2}\theta = \left(\frac{x}{1-x}\right)^{\frac{1}{2}}, \qquad \ldots\ldots (108)$$

$$\frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} = \frac{X_1+x(1-2X_1)-2\left[X_1(1-X_1)x(1-x)\right]^{\frac{1}{2}}}{|x-X_1|}, \qquad \ldots\ldots (109)$$

$\psi_1(\theta)$/

$$\psi_1(\theta) = \frac{y_1(x)}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}}, \; \psi_1'(\theta) = y_1'(x) - \frac{1-2x}{2x(1-x)} y_1(x), \quad \ldots (110)$$

and similarly for ψ_2, ψ_2'. Formulae for $\cos n\theta$ and $\sin n\theta/\sin\theta$ in terms of x are given in Lemma 10 (Appendix). The relations between θ_1 and X_1 are the same as those between θ and x.

10. Special Examples.

Example 1. Clark Y.

R. C. Pankhurst has suggested formulae+ for a Clark Y aerofoil. The under surface of this aerofoil is flat aft of a certain percentage of the chord (33.17 percent according to Pankhurst's formulae), so the camber is proportional to the thickness. Results for a 12 percent thick fairing were set out in Part I; we shall therefore set out here results for the centre line of a 12 percent thick aerofoil. All the results (A_0, A_1, A_2, β, C_{M_0}, C_{Lopt}, α_{opt}, g_i, g_c, ε_c, ε_c'), except those depending on the lift ($g_c + g_L$), are proportional to the camber*, and therefore to the thickness ratio for a Clark Y aerofoil.

For a 12 percent thick aerofoil, Pankhurst's equations for the centre line are

$$y_c = y_1 = 0.2431368x - 0.6994284x^2 + 0.9882636x^3 - 0.5411604x^4 \quad \text{for } 0 \leqslant x \leqslant 0.3317$$

$$= y_2 = 0.0023916 + 0.1690320x - 0.2583216x^2 + 0.0868980x^3 \quad \text{for } 0.3317 \leqslant x \leqslant 1.$$

Hence

$$\frac{dy_1}{dx} = 0.2431368 - 1.3988568x + 2.9647908x^2 - 2.1646416x^3,$$

$$\frac{dy_2}{dx} = 0.1690320 - 0.5166432x + 0.2606940x^2.$$

The maximum camber for any thickness ratio occurs at $x = 0.4134$ according to Pankhurst's formulae; for a 12 percent thick aerofoil $(y_c)_{max}$ is 0.03426, and for a thickness/chord ratio of t/c is $0.2855\,t/c$.

In/

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*$C_{Lopt.}$ and $\alpha_{opt.}$ depend on a_0, and are strictly proportional to the camber only if a_0 is not altered.

+R. and M. 2130.

In the notation of §9

$a_1 = 0.2431368$, $a_2 = -0.6994284$, $a_3 = 0.9882636$, $a_4 = -0.5411604$,

$b_0 = 0.0023916$, $b_1 = 0.1690320$, $b_2 = -0.2583216$, $b_3 = 0.0868980$, $b_4 = 0$

$c_1 = 0.0741048$, $c_2 = -0.4411068$, $c_3 = 0.9013656$, $c_4 = -0.5411604$,

$X_1 = 0.3317$, $\cos\theta_1 = 0.3366$, $\sin\theta_1 = 0.9416477$, $\theta_1/\pi = 0.3907230$,

$\frac{1}{\pi}\cot\frac{1}{2}\theta_1 = 0.4518176$, $\frac{1}{\pi}\tan\frac{1}{2}\theta_1 = 0.2242524$, $\frac{1}{\pi}\sin\theta_1 = 0.2997358$,

$\frac{1}{\pi}\sin 2\theta_1 = 0.2017821$, $\frac{1}{\pi}\sin 3\theta_1 = -0.1638960_5$, $\frac{1}{\pi}\sin 4\theta_1 = -0.3121169_5$

$\frac{1}{\pi}\sin 5\theta_1 = -0.0462211$.

When we insert these values in the formulae of §9 we find the following numerical results.

$$A_0 = 0.017528, \quad A_1 = 0.146252, \quad A_2 = 0.050508,$$

$$\beta = 0.055598 \text{ radians}, \quad C_{M_0} = -0.075197,$$

$$\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{L\,opt.} = 0.146252,$$

$$\alpha_{opt.} = 0.017528 + 0.073126\left(\frac{2\pi - a_0}{2\pi + a_0}\right) \text{ radians},$$

$$g_i = -\frac{1}{\pi}\left(\frac{dy_1}{dx} - \frac{dy_2}{dx}\right)\log_e \frac{1 - 0.3366\cos\theta - 0.9416477\sin\theta}{|\cos\theta - 0.3366|}$$

$$+ 0.141476\sin\theta + 0.046624\sin 2\theta$$

$$+ 0.026430\sin 3\theta,$$

$$g_c + g_L = g_i + \left[\tfrac{1}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_L - 0.073126\right]\cot\tfrac{1}{2}\theta$$

$$- \tfrac{1}{2}\left(\frac{1}{a_0} - \frac{1}{2\pi}\right) C_L \tan\tfrac{1}{2}\theta,$$

$$\varepsilon_c = -\frac{1}{\pi}\left[\psi_1(\theta) - \psi_2(\theta)\right]\log_e \frac{1 - 0.3366\cos\theta - 0.9416477\sin\theta}{|\cos\theta - 0.3366|}$$

$$- 0.002224 - 0.067498\cos\theta - 0.012162\cos 2\theta - 0.006608\cos 3\theta$$

$\varepsilon_c(0)$/

$\varepsilon_c(0) \;=\; -0.090652$

$$\varepsilon_c{}'(\theta) = -\frac{1}{\pi}\left[\psi_1{}'(\theta)-\psi_2{}'(\theta)\right]\log_e \frac{1 - 0.3366\cos\theta - 0.9416477\sin\theta}{|\cos\theta - 0.3366|}$$

$$- 0.00108093\cot\tfrac{1}{2}\theta + 0.00206070\tan\tfrac{1}{2}\theta + 0.060763\sin\theta$$

$$+ 0.034462\sin 2\theta + 0.019823\sin 3\theta,$$

where

$\psi_1(\theta) = 2y_1/\sin\theta,\ \psi_1{}'(\theta) = y_1{}'(x) - \psi_1(\theta)\cot\theta,$

and similarly for ψ_2, $\psi_2{}'$.

Example 2. First N.A.C.A. Centre Line.

The equations to the first series of centre lines suggested by the N.A.C.A.[5] are

$$y_c = y_1 = \frac{m}{p^2}(2px - x^2) \text{ for } 0 \leqslant x \leqslant p$$

$$= y_2 = \frac{m}{(1-p)^2}(1 - 2p + 2px - x^2) \text{ for } p \leqslant x \leqslant 1.$$

Hence

$$\frac{dy_1}{dx} = \frac{2m}{p^2}(p - x),\quad \frac{dy_2}{dx} = \frac{2m}{(1-p)^2}(p - x).$$

The maximum camber occurs at $x = p$, and $(y_c)_{max.} = m$. The centre line of such an aerofoil as N.A.C.A.2412, for example, belongs to this series with $m = 0.02$, $p = 0.4$ (and the fairing is 12 percent thick.)

In the notation of §9

$$a_1 = \frac{2m}{p},\quad a_2 = -\frac{m}{p^2},\quad a_3 = a_4 = 0,$$

$$b_0 = \frac{m(1-2p)}{(1-p)^2},\quad b_1 = \frac{2mp}{(1-p)^2},\quad b_2 = -\frac{m}{(1-p)^2},\quad b_3 = b_4 = 0.$$

Write

$$\pi M = \frac{m}{p^2} - \frac{m}{(1-p)^2} = \frac{m(1-2p)}{p^2(1-p)^2}.$$

Then

$$c_1 = 2\pi pM,\quad c_2 = -\pi M,\quad c_3 = c_4 = 0.$$

Also/

Also

$$\frac{dy_1}{dx} - \frac{dy_2}{dx} = 2\pi M (p - x),$$

$$\psi_1(\theta) - \psi_2(\theta) = \frac{2}{\sin\theta}\{y_1(x) - y_2(x)\} = -\frac{2\pi M}{\sin\theta}(p - x)^2,$$

$$\psi_1'(\theta) - \psi_2'(\theta) = y_1'(x) - y_2'(x) - \frac{1-2x}{2x(1-x)}[y_1(x) - y_2(x)]$$

$$= \pi M(p-x)\left[2 + \frac{(1-2x)(p-x)}{2x(1-x)}\right]$$

Further

$$X_1 = p;$$

hence

$$\theta_1 = 2\sin^{-1}p^{\frac{1}{2}}, \ \sin\theta_1 = 2p^{\frac{1}{2}}(1-p)^{\frac{1}{2}}, \ \cot\tfrac{1}{2}\theta_1 = \frac{\sin\theta_1}{1-\cos\theta_1} = \frac{\sin\theta_1}{2p},$$

$$\sin 2\theta_1 = 2(1-2p)\sin\theta_1; \quad \sin 3\theta_1 = (3-16p+16p^2)\sin\theta_1,$$

$$\frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} = \frac{p + x(1-2p) - \frac{1}{2}\sin\theta_1\sin\theta}{|p - x|}.$$

When we make these substitutions in §9 and write

$$2N = \frac{m}{(1-p)^2} + M\theta_1,$$

we find the following results.

$$A_0 = M[\sin\theta_1 - (1-2p)\theta_1 - \pi p^2]$$

$$A_1 = 2N - (1-2p)M\sin\theta_1$$

$$A_2 = \frac{8}{3}p(1-p)\,M\sin\theta_1$$

$$\beta = N + M[\pi p^2 + (1-2p)\theta_1 - \tfrac{1}{2}(3-2p)\sin\theta_1]$$

$$C_{M_0} = -\frac{\pi}{4}\left\{2N - \frac{M\sin\theta_1}{3}(3 + 2p - 8p^2)\right\}$$

$$\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)C_{L\,opt.} = A_1, \ \alpha_{opt.} = A_0 + \tfrac{1}{2}\left(\frac{2\pi - a_0}{2\pi + a_0}\right)A_1,$$

gi/

$$g_i = -2M(p-x)\log_e \frac{p+x(1-2p)-\frac{1}{2}\sin\theta_1\sin\theta}{|p-x|} + 2N\sin\theta,$$

$$g_c + g_L = g_i + \left[\tfrac{1}{2}\left(\frac{1}{a_o}+\frac{1}{2\pi}\right)C_L - \tfrac{1}{2}A_1\right]\cot\tfrac{1}{2}\theta - \tfrac{1}{2}\left(\frac{1}{a_o}-\frac{1}{2\pi}\right)C_L\tan\tfrac{1}{2}\theta,$$

$$\varepsilon_c(\theta) = \frac{2M(p-x)^2}{\sin\theta}\log_e \frac{p+x(1-2p)-\frac{1}{2}\sin\theta_1\sin\theta}{|p-x|}$$

$$+ M[\pi p^2 + (1-2p)\theta_1 - \tfrac{1}{2}\sin\theta_1] - N\cos\theta,$$

$$\varepsilon_c(0) = M[\pi p^2 + (1-2p)\theta_1 - \tfrac{1}{2}(1+2p)\sin\theta_1] - N,$$

$$\varepsilon_c'(\theta) = -M(p-x)\left[2 + \frac{(1-2x)(p-x)}{2x(1-x)}\right]\log_e \frac{p+x(1-2p)-\frac{1}{2}\sin\theta_1\sin\theta}{|p-x|}$$

$$-\tfrac{1}{2}pM\sin\theta_1\cot\tfrac{1}{2}\theta + \tfrac{1}{2}(1-p)M\sin\theta_1\tan\tfrac{1}{2}\theta + N\sin\theta\ .$$

Example 3. Second N.A.C.A. Centre Line.

The equations to the second series of centre lines suggested by the N.A.C.A.[6,7] are*

$$y_c = y_1 = K[m^2(3-m)x - 3mx^2 + x^3] \text{ for } 0 \leqslant x \leqslant m$$
$$= y_2 = Km^3(1-x) \text{ for } m \leqslant x \leqslant 1.$$

Hence

$$\frac{dy_1}{dx} = K[m^2(3-m) - 6mx + 3x^2].$$

$$\frac{dy_2}{dx} = -Km^3.$$

These centre lines were designed to make it easy to bring the maximum camber further forward than usual. The maximum value of y_c occurs for a value p, say, of x less than m; the values of m are given in Ref. 7 for $p = 0.05, 0.10, 0.15, 0.20, 0.25$, together with the corresponding values of 6K (denoted by k_1) for a value 0.3 of $C_{Lopt.}$ (with $a_o = 2\pi$). A revised version of this table is given below; it contains values of m, values of maximum camber as multiples of K, and values of K both for 2 percent camber and for $C_{Lopt.} = 0.3$, where $C_{Lopt.}$ is given by (59) with $a_o = 2\pi$.

Table/

*In Refs. (6) and (7) $\frac{1}{6}k$ or $\frac{1}{6}k_1$ is written for K.

Table 1.

p	m	$\frac{(y_c)_{max.}}{K}$	K for $(y_c)_{max.}$ = 0.02	K for $C_{L opt.}$ = 0.3 with $a_0 = 2\pi$
0.05	0.058082	0.00018562	107.75	58.39
0.10	0.125744	0.00177233	11.2846	8.5962
0.15	0.202682	0.0069310	2.8856	2.65327
0.20	0.290309	0.0188371	1.06173	1.10400
0.25	0.391344	0.0421270	0.474755	0.53713

The centre line of such an aerofoil as N.A.C.A.23012, for example, belongs to this series, with K and m chosen to make $(y_c)_{max.} = 0.02$, $p = 0.15$ (i.e. $2p = 0.30$), the 12 at the end indicating a 12 percent thick fairing.

For the centre lines of this series, in the notation of §9

$$a_1/K = m^2(3 - m), \quad a_2/K = -3m, \quad a_3/K = 1, \quad a_4 = 0,$$

$$b_0/K = m^3, \quad b_1/K = -m^3, \quad b_2 = b_3 = b_4 = 0$$

$$c_1/K = 3m^2, \quad c_2/K = -3m, \quad c_3/K = 1, \quad c_4 = 0,$$

$$X_1 = m,$$

$$\frac{1}{K}\left(\frac{dy_1}{dx} - \frac{dy_2}{dx}\right) = 3(x - m)^2,$$

$$\frac{1}{K}\left[\psi_1(\theta) - \psi_2(\theta)\right] = \frac{2}{\sin\theta}(x - m)^3,$$

$$\frac{1}{K}\left[\psi_1'(\theta) - \psi_2'(\theta)\right] = (x - m)^2\left[3 - \frac{(1-2x)(x-m)}{2x(1-x)}\right],$$

and proceeding as in the previous example we obtain the following results.

$$A_0/K = -m^3 + \frac{3\theta_1}{8\pi}(8m^2 - 8m + 3) - \frac{9\sin\theta_1}{8\pi}(1 - 2m)$$

$$A_1/K = -\frac{3\theta_1}{2\pi}(1 - 2m) + \frac{\sin\theta_1}{2\pi}(4m^2 - 4m + 3)$$

$$A_2/K = \frac{3\theta_1}{8\pi} - \frac{\sin\theta_1}{8\pi}(16m^3 - 24m^2 + 2m + 3)$$

β/K

$$\beta/K = m^3 - \frac{3\theta_1}{8\pi}(8m^2 - 12m + 5) + \frac{\sin\theta_1}{8\pi}(8m^2 - 26m + 15)$$

$$C_{M_0}/K = \frac{3\theta_1}{32}(5 - 8m) - \frac{\sin\theta_1}{32}(16m^3 - 8m^2 - 14m + 15)$$

$$\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)C_{L\,opt.} = A_1, \quad \alpha_{opt.} = A_0 + \tfrac{1}{2}\left(\frac{2\pi - a_0}{2\pi + a_0}\right)A_1,$$

$$\frac{g_i}{K} = -\frac{3(x-m)^2}{\pi}\log_e\frac{m+x(1-2m)-\frac{1}{2}\sin\theta_1\sin\theta}{|x-m|}$$

$$-\frac{3\sin\theta}{4\pi}\left[2\theta_1(1-2m) - \sin\theta_1 - \theta_1\cos\theta\right]$$

$$g_c + g_L = g_i + \left[\tfrac{1}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)C_L - \tfrac{1}{2}A_1\right]\cot\tfrac{1}{2}\theta$$

$$-\tfrac{1}{2}\left(\frac{1}{a_0} - \frac{1}{2\pi}\right)C_L\tan\tfrac{1}{2}\theta$$

$$\frac{\varepsilon_c}{K} = -\frac{2(x-m)^3}{\pi\sin\theta}\log_e\frac{m+x(1-2m)-\frac{1}{2}\sin\theta_1\sin\theta}{|x-m|}$$

$$+ m^3 - \frac{\theta_1}{\pi}(3m^2 - 3m + 1) + \frac{5\sin\theta_1}{8\pi}(1-2m)$$

$$+ \frac{\cos\theta}{4\pi}\left[3\theta_1(1-2m) - \sin\theta_1\right] - \frac{\theta_1}{8\pi}\cos 2\theta.$$

$$\frac{\varepsilon_c(0)}{K} = m^3 - \frac{3\theta_1}{8\pi}(8m^2 - 4m + 1) - \frac{\sin\theta_1}{8\pi}(8m^2 + 10m - 3)$$

$\frac{\varepsilon_c'(0)}{K}$/

$$\frac{\varepsilon_c'(\theta)}{K} = -\frac{(x-m)^2}{\pi}\left[3-\frac{(1-2x)(x-m)}{2x(1-x)}\right]\log_e\frac{m+x(1-2m)-\frac{1}{2}\sin\theta_1\sin}{|x-m|}$$

$$-\frac{\sin\theta_1}{2\pi}\left[m^2\cot\tfrac{1}{2}\theta+(1-m)^2\tan\tfrac{1}{2}\theta\right]$$

$$-\frac{\sin\theta}{4\pi}\left[3\theta_1(1-2m)-2\sin\theta_1-2\theta_1\cos\theta\right]$$

where

$$\theta_1 = 2\sin^{-1}m^{\frac{1}{2}},\ \sin\theta_1 = 2m^{\frac{1}{2}}(1-m)^{\frac{1}{2}},$$

and values of θ_1 and $\sin\theta_1$ are given in the table below for the same positions of maximum camber as in Table 1, together with the values of $\pi A_1/K$ used in finding the last column of Table 1.

Table 2.

p	θ_1	$\sin\theta_1$	$\pi A_1/K$
0.05	0.486797	0.467797	0.005137_5
0.10	0.724981	0.663121	0.034899
0.15	0.933983-	0.803995	0.113068
0.20	1.138032	0.907810	0.271738
0.25	1.351737	0.976102	0.558527

Example 4. Third N.A.C.A. Centre Line.

The equations to the third series of centre lines suggested by the N.A.C.A.[7] are*

$$y_c = y_1 = K[(x-m)^3-B(1-m)^3x-m^3x+m^3] \text{ for } 0 \leqslant x \leqslant m$$

$$= y_2 = K[B(x-m)^3-B(1-m)^3x-m^3x+m^3] \text{ for } m \leqslant x \leqslant 1.$$

Hence

$$\frac{dy_1}{dx} = K[3(x-m)^2 - B(1-m)^3 - m^3]$$

$$\frac{dy_2}{dx} = K[3B(x-m)^2 - B(1-m)^3-m^3].$$

These centre lines are somewhat similar to the former ones over the forward portion, but are designed to have reflex curvature over the rear portion. As a consequence the constants may be so chosen that $C_{M_0} = 0$. The maximum camber occurs between $x = 0$ and $x = m$; if it occurs at $x = p$, then

$$B(1-m)^3 = 3(m-p)^2 - m^3.$$

For/

*In Ref. 7 $\frac{1}{6}k_1$ is written for K and k_2/k_1 for B.

For a given value of p it is possible to choose m and B so that $C_{M_0} = 0$; K is then determined by the amount of camber or by the value of $C_{L opt.}$ A table of values of m and B (denoted by k_2/k_1) is given in Ref. 7 for $C_{M_0} = 0$ and p = 0.10, 0.15, 0.20, 0.25, together with the corresponding values of 6K (denoted by k_1) for $C_{L opt.} = 0.3$ (with $a_0 = 2\pi$).

The centre line of such an aerofoil as N.A.C.A.23112, for example, belongs to this series, with $C_{M_0} = 0$, $(y_c)_{max.} = 0.02$ and p = 0.15.

The derivation of the formulae giving the properties of these centre lines is very similar to that in the preceding example; we shall be content simply to write out the answers in algebraic form for general values of the constants m, B, K, without entering into any numerical computation. In particular we shall not solve the equation $C_{M_0} = 0$; by substituting the value of B given above in terms of m and p into the formula for C_{M_0}/K below, the equation is found as a relation between m and p, and may be solved for m by Newton's method for any specified value of p; alternatively, m, B and p may all be tabulated against θ_1.

With

$$\theta_1 = 2 \sin^{-1} m^{\frac{1}{2}}, \sin \theta_1 = 2m^{\frac{1}{2}}(1 - m)^{\frac{1}{2}},$$

as before, and with

$$\lambda = 1 - B,$$

the formulae are as follows.

$$A_0/K = \frac{1}{8} B-\lambda m^3 + \frac{3\lambda\theta_1}{8\pi}(8m^2 - 8m + 3) - \frac{9\lambda \sin \theta_1}{8\pi}(1-2m)$$

$$A_1/K = -\frac{3}{2\pi}(\pi B + \lambda\theta_1)(1-2m) + \frac{\lambda \sin \theta_1}{2\pi}(4m^2 - 4m + 3)$$

$$A_2/K = \frac{3}{8\pi}(\pi B + \lambda\theta_1) - \frac{\lambda \sin \theta_1}{8\pi}(16m^3 - 24m^2 + 2m + 3)$$

$$\beta/K = -\frac{B}{8}(7 - 12m) + \lambda m^3 - \frac{3\lambda\theta_1}{8\pi}(8m^2 - 12m + 5) + \frac{\lambda \sin \theta_1}{8\pi}(8m^2 - 26m + 15)$$

$$C_{M_0}/K = \frac{3}{32}(\pi B+\lambda\theta_1)(5-8m) - \frac{\lambda \sin \theta_1}{32}(16m^3-8m^2-14m+15)$$

$$\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{L opt.} = A_1, \alpha_{opt.} = A_0 + \tfrac{1}{2}\left(\frac{2\pi - a_0}{2\pi + a_0}\right)A_1,$$

g_i/K

$$\frac{g_i}{K} = -\frac{3\lambda(x-m)^2}{\pi}\log_e\frac{m+x(1-2m)-\frac{1}{2}\sin\theta_1\sin\theta}{|x-m|}$$

$$-\frac{3\sin\theta}{4\pi}\left[2(\pi B+\lambda\theta_1)(1-2m)-\lambda\sin\theta_1-(\pi B+\lambda\theta_1)\cos\theta\right]$$

$$g_c + g_L = g_i+\left[\tfrac{1}{2}\left(\frac{1}{a_0}+\frac{1}{2\pi}\right)C_L-\tfrac{1}{2}A_1\right]\cot\tfrac{1}{2}\theta-\tfrac{1}{2}\left(\frac{1}{a_0}-\frac{1}{2\pi}\right)C_L\tan\tfrac{1}{2}\theta$$

$$\frac{\varepsilon_c}{K} = -\frac{2\lambda(x-m)^3}{\pi\sin\theta}\log_e\frac{m+x(1-2m)-\frac{1}{2}\sin\theta_1\sin\theta}{|x-m|}$$

$$+\lambda m^3-\frac{\lambda\theta_1}{\pi}(3m^2-3m+1)$$

$$+\frac{5\lambda\sin\theta_1}{8\pi}(1-2m)+\frac{\cos\theta}{4\pi}\left[3(\pi B+\lambda\theta_1)(1-2m)-\lambda\sin\theta_1\right]$$

$$-\frac{\pi B+\lambda\theta_1}{8\pi}\cos 2\theta$$

$$\frac{\varepsilon_c(0)}{K} = \frac{B}{8}(5-12m)+\lambda m^3-\frac{3\lambda\theta_1}{8\pi}(8m^2-4m+1)$$

$$-\frac{\lambda\sin\theta_1}{8\pi}(8m^2+10m-3)$$

$$\frac{\varepsilon_c'(\theta)}{K} = -\frac{\lambda(x-m)^2}{\pi}\left[3-\frac{(1-2x)(x-m)}{2x(1-x)}\right]\log_e\frac{m+x(1-2m)-\frac{1}{2}\sin\theta_1\sin\theta}{|x-m|}$$

$$-\frac{\lambda\sin\theta_1}{2\pi}\left[m^2\cot\tfrac{1}{2}\theta+(1-m)^2\tan\tfrac{1}{2}\theta\right]$$

$$-\frac{\sin\theta}{4\pi}\left[3(\pi B+\lambda\theta_1)(1-2m)-2\lambda\sin\theta_1-2(\pi B+\lambda\theta_1)\cos\theta\right].$$

Example 5. Cubic and Parabolic Centre Lines.

We conclude with a particularly simple example, in which the Fourier series terminate and no integrations are required. If y_c is a cubic in x, then since it vanishes at $x = 0$ and at $x = 1$ we may write[2]

$$y_c = hx(1-x)(1-\lambda x),$$

and for a parabolic centre line $\lambda = 0$. In terms of θ

$$y_c = \frac{1}{4}h\sin^2\theta(1-\tfrac{1}{2}\lambda+\tfrac{1}{2}\lambda\cos\theta).$$

Also/

Also

$$\frac{dy_c}{dx} = h[1-2(\lambda+1)x + 3\lambda x^2] = \frac{1}{8}\lambda h + h(1 - \tfrac{1}{2}\lambda)\cos\theta + \frac{3}{8}\lambda h\cos 2\theta,$$

so

$$A_0 = \frac{1}{8}\lambda h,\ A_1 = h(1-\tfrac{1}{2}\lambda),\ A_2 = \frac{3}{8}\lambda h,$$

$$\beta = \frac{h}{8}(4 - 3\lambda),\quad C_{M_0} = \frac{\pi h}{32}(7\lambda - 8).$$

$$\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{L\,opt.} = h(1 - \tfrac{1}{2}\lambda),\quad \alpha_{opt.} = \frac{1}{8}\lambda h + \tfrac{1}{2}(1 - \tfrac{1}{2}\lambda)\left(\frac{2\pi - a_0}{2\pi + a_0}\right) h$$

$$g_i = h(1 - \tfrac{1}{2}\lambda)\sin\theta + \frac{3}{8}\lambda h\sin 2\theta,$$

$$g_c + g_L = h(1-\tfrac{1}{2}\lambda)\sin\theta + \frac{3}{8}\lambda h\sin 2\theta$$
$$+ \left[\tfrac{1}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_L - \tfrac{1}{2}h(1-\tfrac{1}{2}\lambda)\right]\cot\tfrac{1}{2}\theta - \tfrac{1}{2}\left(\frac{1}{a_0} - \frac{1}{2\pi}\right) C_L \tan\tfrac{1}{2}\theta.$$

Also

$$\psi_c = 2y_c \operatorname{cosec}\theta = \tfrac{1}{2}h(1 - \tfrac{1}{2}\lambda)\sin\theta + \frac{1}{8}\lambda h\sin 2\theta,$$

so

$$\varepsilon_c = -\tfrac{1}{2}h(1 - \tfrac{1}{2}\lambda)\cos\theta - \frac{1}{8}\lambda h\cos 2\theta,$$

$$\varepsilon_c' = \tfrac{1}{2}h(1 - \tfrac{1}{2}\lambda)\sin\theta + \frac{1}{4}\lambda h\sin 2\theta.$$

When $C_L = C_{Lopt}$,

$$g_c + g_L = \left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{Lopt.}\left[\sin\theta + \frac{3\lambda}{8-4\lambda}\sin 2\theta - \tfrac{1}{2}\left(\frac{2\pi-a_0}{2\pi+a_0}\right)\tan\tfrac{1}{2}\theta\right].$$

Graphs of $g_c + g_L$ for $C_L = C_{Lopt.} = 0.2$ and $a_0 = 5.5$ are shown in Fig. 1 for $\lambda = 0.9,\ 0.6,\ 0$ and -1.0. For other values of $C_{Lopt.}$ the shapes of the graphs would be unaltered, since when $C_L = C_{Lopt}$, $g_c + g_L$ is proportional to C_{Lopt}; for other values of a_0, $(g_c + g_L) \div \left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{Lopt.}$ is altered only by a small multiple of $\tan\frac{1}{2}\theta$, so the shapes of the graphs would be substantially altered only near the trailing edge. Some numerical

values/

values for these centre lines are given in the table below, wherein x_0 denotes the value of x at the position of maximum camber. h, C_{M_0} and β are proportional to the amount of camber for a given

Table 3.

λ	x_0	Values for 1 percent camber				Percentage camber for $C_{L\,opt.} = 0.2$, $a_0 = 5.5$
		h	C_{M_0}	β(radians)	β°	
0.9	0.35039	0.06417	−0.0107	0.0104	0.60	1.93
0.6	0.40456	0.05481	−0.0204	0.0151	0.86	1.77
0	0.5	0.04	−0.0314	0.02	1.14	1.70
−1.0	0.57735	0.02598	−0.0383	0.0227	1.30	1.75

shape of centre line. The amount of camber for a given $C_{L\,opt.}$ is proportional to $C_{L\,opt.}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)$; the amount required does not vary much with λ for the values here considered, especially for the last three values. $-C_{M_0}$ increases fairly rapidly with x_0.

From Fig. 1 we can see that if we wish the maximum values of q/U on both surfaces of a wing to be fairly far back from the leading edge, then with the centre lines for which $\lambda = 0.9$ and $\lambda = 0.6$ the fairing would have to be such that g_s has a sharp gradient and a far-back maximum. The middle of the range of C_L – values for which the maximum of q/U is fairly far back from the leading edge on both surfaces will be higher than $C_{L\,opt.}$ for any of these centre lines unless the gradient of g_s is greater than the gradient of $g_c + g_L$ in Fig. 1, and there will be a tendency to flatness in the final curves of q/U on both surfaces, due to the negative gradient of $g_c + g_L$ on the lower surface and the increase of lift on the upper surface. Moreover, if we compare the centre lines for $\lambda = 0$ and $\lambda = -1.0$, we see that over the middle of the wing chord the gradient of $g_c + g_L$ is greater for $\lambda = -1.0$ than for $\lambda = 0$, so in the final curves of q/U for the wing we might easily have flatter curves with $\lambda = -1.0$ than with $\lambda = 0$. Lastly we might note that if $(q/U)_{max.}$ for the fairing alone at $C_L = 0$ occurs about mid-chord, we should expect to achieve the lowest $(q/U)_{max.}$ for a cambered wing which has one of these centre lines by taking $\lambda = 0$, but that we might easily do as well or better with a positive λ if $(q/U)_{max.}$ for the fairing at $C_L = 0$ occurs further back than mid-chord, and with a negative λ if it occurs further forward.

11. Numerical Example (EQH 1250/4050), and Comparisons with Numerically Accurate Results.

Computations have been carried out by all three approximate methods for EQH.1250/4050, and the results compared with each other and with numerically accurate results obtained from the exact theory. For this aerofoil the equations to the fairing are

$$y_s = 0.12\,(x - x^2)^{\frac{1}{2}} \text{ for } 0 \leqslant x \leqslant 0.5$$

$$= 0.06 - 0.12\,(x-0.5)^2 - 0.535\,(x-0.5)^3$$

$$+ 0.609\,(x-0.5)^4 \text{ for } 0.5 \leqslant x \leqslant 0.9653726$$

$$= \{0.0006260362\,(1-x) + 0.044389956\,(1-x)^2\}^{\frac{1}{2}}$$

$$\text{for } 0.9653726 \leqslant x \leqslant 1,$$

and/

and the equation to the centre line is

$$y_c = 0.16x(1 - x).$$

Values of C_0, g_s, ε_s, ε_s' were given in Part I and an addendum thereto, together with the accurate and the three approximate values of q/U for the symmetrical aerofoil, both for $C_L = 0$ and for $C_L = 0.4$, $C_L/\sin\alpha = 4.8$.

For the cambered aerofoil, on the approximate theory,

$$A_0 = 0,\ A_1 = 0.16,\ A_2 = 0,$$

$$\beta = 0.08 \text{ radians} = 4.58^\circ,\quad C_{M_0} = -0.04\pi = -0.1257,$$

and with $a_0 = 4.8$,

$$C_{L\,opt.} = 0.4354,\quad \alpha_{opt.} = 0.0107 \text{ radians} = 0.61^\circ,$$

$$g_c + g_L = 0.16 \sin\theta + (0.1837441\, C_L - 0.08)\cot\tfrac{1}{2}\theta - 0.0245892\, C_L \tan\tfrac{1}{2}\theta.$$

Also

$$\psi_c = 0.08 \sin\theta,\ \varepsilon_c = -0.08\cos\theta,\ \varepsilon_c' = 0.08\sin\theta.$$

For Approximation I,

$$q/U = 1 + g_s + g_c + g_L.$$

For Approximation II,

$$q/U = \frac{1.005039\,|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g_s + g_c + g_L)$$

$$= \frac{1.005039\,|\sin\theta|}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}(1 + g_s) \pm \frac{1.005039}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}} \left\{\frac{C_L}{4.8}\cos\theta + \frac{C_L}{2\pi} - 0.08(\cos\theta + \cos 2\theta)\right\}.$$

For Approximation III,

$$q/U = \frac{e^{C_0}(1+\varepsilon')}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left\{\left(1 - \frac{C_L^2}{a_0^2}\right)^{\frac{1}{2}} \sin(\theta + \varepsilon - \beta) + \frac{C_L}{a_0}\cos(\theta + \varepsilon - \beta) + \frac{C_L}{2\pi e^{C_0}}\right\},$$

with

$$e^{C_0} = 1.10560,\quad a_0 = 4.8,\quad \beta = 0.08 \text{ radians}.$$

Accurate/

Accurate values of ψ, ψ', ε, ε' and of q/U for $C_L = 0.2$, 0.6, 1.0 for this aerofoil are given in Table 4, and approximate values in Table 5. The results are exhibited graphically in Figs. 2 – 7, and should be compared with the results for the symmetrical aerofoil EQH 1250. Approximation III is again the best, and is, on the whole, quite good; but whereas for the symmetrical aerofoil the errors in Approximation III are quite small, for the cambered aerofoil they are quite perceptible. The most significant errors are probably those over the middle of the chord on the upper surface at $C_L = 0.2$ and 0.6. Actually 4 percent is rather a high camber for an aeroplane wing (though not unduly high for a section of an airscrew blade), and it appears probable that the additional errors due to the camber increase, as the camber increases, at least as fast as the square of the camber, so they would still be quite small for an aerofoil of more "normal" camber.

The errors in Approximation III over the middle of the chord on the upper surface at $C_L = 0.2$ and $C_L = 0.6$ are due partly to errors in ε', for which it is difficult to compensate easily, and partly to an error in e^{C_0}, due to an error in C_0, for which it is easy to compensate. (On the upper surface the errors have the same sign) We see from Part I, equations (5) and (6), that C_0 is an approximation to

$$[\Psi] = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\psi(t) - \varepsilon(t)\psi'(t)]dt;$$

we neglected the second term in the integrand, which depends on squares of the thickness and camber, and substituted our approximate value for ψ. We obtain a better approximation to C_0 if, still using our approximate values for ψ, ψ', ε, ε', we retain the second term in the integrand, and so add

$$-[\varepsilon\psi'] = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon(t)\psi'(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon'(t)\psi(t)dt = [\varepsilon'\psi]$$

to our previous value of C_0. The result is that all our values of q/U on Approximation III should be multiplied by

$$1 - [\varepsilon\psi'] = 1 + [\varepsilon'\psi].$$

For a cambered aerofoil, on our approximate theory,

$$[\varepsilon\psi'] = [\varepsilon_s\psi_s'] + [\varepsilon_c\psi_c'],$$

and when ε, ψ' (or ε', ψ) have been found, a numerical value for $[\varepsilon\psi']$ (or $[\varepsilon'\psi]$) is easily found by numerical integration by the trapezium rule, since only one or two significant figures will be required. For EQH 1250

$$[\varepsilon_s\psi_s'] = -0.0014;$$

also from the values of ε_c, ψ_c' above,

$$[\varepsilon_c\psi_c'] = -\frac{1}{\pi}\int_0^{\pi} 0.0064\cos^2 t\, dt = -0.0032,$$

so for EQH 1250/4050

$$[\varepsilon\psi'] = -0.0046.$$

If/

If we multiply our previous results on Approximation III by 1.0046 we considerably reduce the errors over the middle of the chord on the upper surface at $C_L = 0.2$ and 0.6 at the cost of rather increasing them elsewhere on the upper surface, and also over the middle of the chord on the lower surface — i.e. we make the errors rather more nearly equal on the two surfaces. On the whole we make the results rather worse at $C_L = 1.0$. It is suggested that this factor $1 - [\varepsilon\psi']$ might be included for safety if we are finding the maximum suction, but it is emphasised that, since squares and products of the thickness and camber are neglected elsewhere, it is purely for safety that this factor might be included; it is not logical to include it and the results so obtained are not necessarily more accurate.

The velocity distribution on the lower surface at $C_L = C_{L_{opt.}} = 0.4354$ is shown in Fig. 8; it exhibits a somewhat sharp maximum near the nose (at $x = 0.038$). This maximum near the nose on the lower surface is, in fact, still present at $C_L = 0.6$ (at $x = 0.165$) and disappears only between $C_L = 0.62$ and $C_L = 0.63$; at $C_L = 0.66$ a maximum near the nose on the upper surface makes its appearance. Thus if we define the "favourable" range of C_L- values as the range for which there is no maximum of q/U near the nose on either surface, then for this aerofoil the range would be only 0.63 to 0.66. (From the practical point of view the definition of the "favourable" range is, of course, arbitrary; with a perfectly accurate surface the maximum of q/U near the nose on the lower surface would probably not lead to transition at $C_L = 0.6$, but, on the other hand, the effect of any waviness of the surface near the position of that maximum would be especially unfavourable.)

The middle of the "favourable" range is therefore at $C_L = 0.64_5$; on the other hand the lowest value of $(q/U)_{max}$ for both surfaces, namely 1.277, is obtained when $C_L = 0.33$.

12. Summary.

The formulae of Part I are extended to cambered aerofoils, three formulae, of increasing accuracy and complexity, being given for the velocity at the surface of a cambered aerofoil.

The connection with the "vortex-sheet" theory of infinitely thin aerofoils is set out. In particular, the usual formulae for the no-lift angle and the moment at zero lift are correct if account is taken of the thickness, so long as the thickness is not too large. These formulae are, however, given and used in somewhat different forms from the usual ones. Simple formulae are also given for the "optimum" lift coefficient and incidence, defined as the lift coefficient and incidence at which, on a theory neglecting squares of the camber, there is no infinity at the leading edge in the velocity distribution for an infinitely thin aerofoil of the shape of the given centre line.

In calculations of the velocity at the surface the contributions from the thickness and the camber can, to a large extent, be calculated separately. In the calculations of the contribution from the camber we may, just as in calculating the contribution from the thickness by the methods of Part I, use Fourier series, numerical integration, or certain analytical results. The Fourier series terminate, and should be used, when the ordinate y_c of the centre line is a single polynomial in the distance x along the chord from the leading edge over the whole chord; otherwise it is recommended that the analytical results be used whenever possible. It is shown how analytical results may be obtained when, with the chord divided into any number of segments,

y_c/

y_c is a polynomial in x of any degree in each of the segments; the results are set out explicitly when there are two segments in each of which y_c is a quartic in x; it appears that, apart from centre lines designed to give specified load distributions along the chord at the "optimum" lift coefficient, all the centre lines for which formulae have so far been proposed are included as special cases. As examples, formulae are set out for the centre line of a Clark Y aerofoil according to Pankhurst's formulae, for three series of N.A.C.A centre lines, and for cubic and parabolic centre lines. The properties of cubic centre lines are briefly discussed. The results on each of the approximate theories are compared with each other and with accurate results for EQH 1250/4050; the agreement, while not nearly as good as that for the symmetrical aerofoil, is not unsatisfactory considering the fairly large amount of camber. A small refinement is suggested which, while not logical and not necessarily leading to more accurate results, may be inserted for safety if the maximum suction be sought.

Appendix.

Lemma 1.

$$\frac{1 - \cos 2nt}{\sin t} = 2\{\sin t + \sin 3t + \ldots + \sin(2n-1)t\},$$

and

$$\frac{\cos t - \cos(2n+1)t}{\sin t} = 2\{\sin 2t + \sin 4t + \ldots + \sin 2nt\}.$$

Lemma 2. Let

$$x = \tfrac{1}{2}(1 - \cos\theta) \text{ and } y(x) = \tfrac{1}{2}\psi(\theta)\sin\theta,$$

where

$$\psi(\theta) = \sum_{n=1}^{\infty} D_n \sin n\theta \quad (0 \leqslant \theta \leqslant \pi).$$

Also let

$$F(\theta) = \frac{dy}{dx} = \sum_{n=0}^{\infty} A_n \cos n\theta \quad (0 \leqslant \theta \leqslant \pi).$$

Then

$$D_n = \frac{2}{\pi}\int_0^{\pi} \psi(t)\sin nt\,dt,$$

$$F(\theta) = \frac{2}{\sin\theta}\frac{d}{d\theta}\left(\tfrac{1}{2}\psi\sin\theta\right) = \psi'(\theta) + \psi\cot\theta,$$

$$A_0 = \frac{1}{\pi}\int_0^{\pi} F(t)dt, \quad A_n = \frac{2}{\pi}\int_0^{\pi} F(t)\cos nt\,dt.$$

Hence/

Hence if

$$\psi(0) = \psi(\pi) = 0, \text{ i.e. if } y/x^{\frac{1}{2}} \to 0 \text{ as } x \to 0 \text{ and } y/(1-x)^{\frac{1}{2}} \to 0 \text{ as } x \to 1,$$

then

$$A_0 = \frac{1}{\pi}\int_0^\pi \psi(t) \cot t \, dt$$

$$A_1 = \frac{2}{\pi}\int_0^\pi \left\{\psi'(t) \cos t + \psi(t) \frac{\cos^2 t}{\sin t}\right\} dt$$

$$= \frac{2}{\pi}\int_0^\pi \psi(t) \left\{\sin t + \frac{\cos^2 t}{\sin t}\right\} dt = \frac{2}{\pi}\int_0^\pi \psi(t) \operatorname{cosec} t \, dt,$$

and for $n \geqslant 2$

$$A_n = \frac{2}{\pi}\int_0^\pi \{\psi'(t) + \psi(t) \cot t\} \cos nt \, dt$$

$$= \frac{2}{\pi}\int_0^\pi \psi(t) \{n \sin nt + \cos nt \cot t\} \, dt$$

$$= \frac{2}{\pi}\int_0^\pi \psi(t) \left\{(n-1) \sin nt + \frac{\cos (n-1)t}{\sin t}\right\} dt$$

$$= (n-1)D_n + \frac{2}{\pi}\int_0^\pi \psi(t) \frac{\cos (n-1)t}{\sin t} \, dt.$$

It follows from Lemma 1 that for $n \geqslant 1$

$$A_{2n} - 2A_0 = (2n-1)D_{2n} - 2D_{2n-2} - 2D_{2n-4} - \ldots - 2D_4 - 2D_2$$

and

$$A_{2n+1} - A_1 = 2nD_{2n+1} - 2D_{2n-1} - 2D_{2n-3} - \ldots - 2D_3 - 2D_1.$$

<u>Lemma 3</u>. If

$$x = \tfrac{1}{2}(1 - \cos \theta)$$

then

$$x^2 = \frac{1}{8}(3 - 4 \cos \theta + \cos 2\theta)$$

$$x^3 = \frac{1}{32}(10 - 15 \cos \theta + 6 \cos 2\theta - \cos 3\theta)$$

$$x^4 = \frac{1}{128}(35 - 56 \cos \theta + 28 \cos 2\theta - 8 \cos 3\theta + \cos 4\theta)$$

and/

and

$$x^n = \frac{(-1)^n}{2^{2n-1}} \left\{ \cos n\theta - 2n \cos (n-1)\theta + \frac{2n(2n-1)}{2!} \cos (n-2)\theta + \ldots \right.$$

$$+ (-1)^s \frac{2n(2n-1) \ldots (2n-s+1)}{s!} \cos (n-s)\theta + \ldots$$

$$\left. - \frac{2n(2n-1) \ldots (n+2)}{(n-1)!} \cos \theta + \frac{2n(2n-1) \ldots (n+1)}{n!} \tfrac{1}{2} \right\}.$$

Lemma 4. Let

$$I_n(\theta_1) = P \int_0^{\theta_1} \frac{\cos nt}{\cos \theta - \cos t} dt,$$

where P denotes that the principal value of the integral is to be taken. Then*

$$I_o(\theta_1) \sin \theta = \log_e \frac{\sin \frac{1}{2}|\theta - \theta_1|}{\sin \frac{1}{2}(\theta + \theta_1)},$$

and

$$I_1(\theta_1) = -P \int_0^{\theta_1} \left\{ 1 - \frac{\cos \theta}{\cos \theta - \cos t} \right\} dt = -\theta_1 + I_o(\theta_1) \cos \theta,$$

so

$$I_1(\theta_1) \sin \theta = \cos \theta \log_e \frac{\sin \frac{1}{2}|\theta - \theta_1|}{\sin \frac{1}{2}(\theta + \theta_1)} - \theta_1 \sin \theta.$$

Also

$$I_{n+1}(\theta_1) + I_{n-1}(\theta_1) = P \int_0^{\theta_1} \frac{2 \cos t \cos nt}{\cos \theta - \cos t} dt$$

$$= -P \int_0^{\theta_1} 2 \cos nt \left\{ 1 - \frac{\cos \theta}{\cos \theta - \cos t} \right\} dt$$

$$= -\frac{2}{n} \sin n\theta_1 + 2 I_n(\theta_1) \cos \theta,$$

so

$$I_{n+1}(\theta_1) \sin \theta = -\frac{2}{n} \sin n\theta_1 \sin \theta + 2I_n(\theta_1) \sin \theta \cos \theta$$

$$- I_{n-1}(\theta_1) \sin \theta.$$

Hence/

*Cf. Ref. 2, p.p. 92, 93.

Hence

$$I_2(\theta_1)\sin\theta = \cos 2\theta \log_e \frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} - \theta_1 \sin 2\theta - 2\sin\theta_1 \sin\theta$$

$$I_3(\theta_1)\sin\theta = \cos 3\theta \log_e \frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} - \theta_1 \sin 3\theta$$

$$- \sin 2\theta_1 \sin\theta - 2\sin\theta_1 \sin 2\theta$$

$$I_4(\theta_1)\sin\theta = \cos 4\theta \log_e \frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} - \theta_1 \sin 4\theta$$

$$- 2\left\{\frac{\sin 3\theta_1}{3}\sin\theta + \frac{\sin 2\theta_1}{2}\sin 2\theta + \sin\theta_1 \sin 3\theta\right\}.$$

and for $n \geqslant 2$

$$I_n(\theta_1)\sin\theta = \cos n\theta \log_e \frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} - \theta_1 \sin n\theta$$

$$- 2\sum_{s=0}^{n-2} \frac{\sin(n-s-1)\theta_1}{n-s-1}\sin(s+1)\theta.$$

We may also note that

$$\frac{\sin\frac{1}{2}|\theta-\theta_1|}{\sin\frac{1}{2}(\theta+\theta_1)} = \frac{1-\cos\theta_1\cos\theta - \sin\theta_1\sin\theta}{|\cos\theta - \cos\theta_1|},$$

and that

$$\frac{\sin n\theta}{\sin\theta} = 2\cos(n-1)\theta + 2\cos(n-3)\theta + 2\cos(n-5)\theta + \ldots,$$

the last term being 1 or $2\cos\theta$ according as n is odd or even.

Lemma 5.

$$\tan\tfrac{1}{2}\theta\cos\theta = \sin\theta - \tan\tfrac{1}{2}\theta,$$

$$-\tan\tfrac{1}{2}\theta\cos 2\theta = 2\sin\theta - \sin 2\theta - \tan\tfrac{1}{2}\theta,$$

$$\tan\tfrac{1}{2}\theta\cos 3\theta = 2\sin\theta - 2\sin 2\theta + \sin 3\theta - \tan\tfrac{1}{2}\theta,$$

and for any positive integral n,

$$(-1)^{n-1}\tan\tfrac{1}{2}\theta\cos n\theta = 2\sin\theta - 2\sin 2\theta + 2\sin 3\theta - \ldots$$

$$+ (-1)^n 2\sin(n-1)\theta + (-1)^{n-1}\sin n\theta$$

$$- \tan\tfrac{1}{2}\theta.$$

Lemma 6./

Lemma 6.

$$\frac{\cos n\theta_1 - \cos n\theta}{\cos \theta_1 - \cos \theta} = 1 \text{ for } n=1,$$

$$= 2 \cos \theta + 2 \cos \theta_1 \text{ for } n=2,$$

$$= 2 \cos 2\theta + 4 \cos \theta \cos \theta_1 + 2 \cos 2\theta_1 + 1 \text{ for } n=3$$

$$= 2 \cos 3\theta + 4 \cos 2\theta \cos \theta_1 + 4 \cos \theta(\cos 2\theta_1 + \tfrac{1}{2})$$
$$+ 2 \cos 3\theta_1 + 2 \cos \theta_1 \text{ for } n=3$$

$$= 2 \cos (n-1)\theta + 4 \cos (n-2)\theta \cos \theta_1$$
$$+ 4 \cos (n-3)\theta\,[\cos 2\theta_1 + \tfrac{1}{2}]$$
$$+ 4 \cos (n-4)\theta\,[\cos 3\theta_1 + \cos \theta_1]$$
$$+ \dots + 4\cos(n-r)\theta[\cos(r-1)\theta_1 + \cos(r-3)\theta_1 + \dots] + \dots$$
$$+ 2 \cos (n-1)\theta_1 + 2 \cos (n-3)\theta_1 + \dots$$

for any positive integral n, the last term in the coefficient of $4 \cos (n-r)\theta$ being $\tfrac{1}{2}$ or $\cos \theta_1$ according as r is odd or even, and the last term independent of θ being 1 or $2 \cos \theta_1$ according as n is odd or even.

Hence also

$$\sin \theta_1 \frac{\cos n\theta_1 - \cos n\theta}{\cos \theta_1 - \cos \theta} = \sin \theta_1 \text{ for } n=1$$

$$= 2 \cos \theta \sin \theta_1 + \sin 2\theta_1 \text{ for } n=2$$

$$= 2 \cos 2\theta \sin \theta_1 + 2 \cos \theta \sin 2\theta_1 + \sin 3\theta_1 \text{ for } n=3$$

$$= 2 \cos 3\theta \sin \theta_1 + 2 \cos 2\theta \sin 2\theta_1 + 2 \cos \theta \sin 3\theta_1 + \sin 4\theta_1 \quad \text{for } n=4$$

$$= 2 \cos (n-1)\theta \sin \theta_1 + 2 \cos (n-2)\theta \sin 2\theta_1 + 2 \cos(n-3)\theta \sin 3\theta_1 + \dots$$
$$+ 2 \cos (n-r)\theta \sin r\theta_1 + \dots + 2 \cos \theta \sin (n-1)\theta_1 + \sin n\theta_1$$

for any positive integral n.

Lemma 7.

$$\operatorname{cosec} \theta = \tfrac{1}{2}(\cot \tfrac{1}{2}\theta + \tan \tfrac{1}{2}\theta),$$

$$\cot \theta = \tfrac{1}{2}(\cot \tfrac{1}{2}\theta - \tan \tfrac{1}{2}\theta),$$

$$\frac{\cos 2\theta}{\sin \theta} = \tfrac{1}{2}(\cot \tfrac{1}{2}\theta + \tan \tfrac{1}{2}\theta) - 2 \sin \theta,$$

$$\frac{\cos 3\theta}{\sin \theta} = \tfrac{1}{2}(\cot \tfrac{1}{2}\theta - \tan \tfrac{1}{2}\theta) - 2 \sin 2\theta,$$

$$\frac{\cos 4\theta}{\sin \theta} = \tfrac{1}{2}(\cot \tfrac{1}{2}\theta + \tan \tfrac{1}{2}\theta) - 2 \sin 3\theta - 2 \sin \theta,$$

and/

and for any positive integral m

$$\frac{\cos m\theta}{\sin\theta} = \tfrac{1}{2}[\cot\tfrac{1}{2}\theta+(-1)^m \tan\tfrac{1}{2}\theta]-2\sin(m-1)\theta-2\sin(m-3)\theta-\ldots,$$

the last term being $-2\sin 2\theta$ or $-2\sin\theta$ according as m is odd or even.

Lemma 8. It follows from Lemmas 6 and 7 that

$$\frac{\sin\theta_1}{\sin\theta}\,\frac{\cos n\theta_1-\cos n\theta}{\cos\theta_1-\cos\theta} = \tfrac{1}{2}\sin\theta_1(\cot\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\theta) \text{ for } n=1$$

$$= \cot\tfrac{1}{2}\theta(\tfrac{1}{2}\sin 2\theta_1+\sin\theta_1)+\tan\tfrac{1}{2}\theta(\tfrac{1}{2}\sin 2\theta_1-\sin\theta_1) \text{ for } n=2$$

$$= \cot\tfrac{1}{2}\theta(\tfrac{1}{2}\sin 3\theta_1+\sin 2\theta_1+\sin\theta_1)+\tan\tfrac{1}{2}\theta(\tfrac{1}{2}\sin 3\theta_1-\sin 2\theta_1+\sin\theta_1)$$

$$-4\sin\theta\sin\theta_1 \text{ for } n=3$$

$$= \cot\tfrac{1}{2}\theta(\tfrac{1}{2}\sin 4\theta_1+\sin 3\theta_1+\sin 2\theta_1+\sin\theta_1)$$

$$+\tan\tfrac{1}{2}\theta(\tfrac{1}{2}\sin 4\theta_1-\sin 3\theta_1+\sin 2\theta_1-\sin\theta_1)$$

$$-4\sin\theta\sin 2\theta_1-4\sin 2\theta\sin\theta_1 \text{ for } n=4$$

$$= \cot\tfrac{1}{2}\theta[\tfrac{1}{2}\sin n\theta_1+\sin(n-1)\theta_1+\sin(n-2)\theta_1+\ldots+\sin 2\theta_1+\sin\theta_1]$$

$$+\tan\tfrac{1}{2}\theta[\tfrac{1}{2}\sin n\theta_1-\sin(n-1)\theta_1+\sin(n-2)\theta_1-\ldots+(-1)^n\sin 2\theta_1$$

$$+(-1)^{n+1}\sin\theta_1]$$

$$-4\sin\theta[\sin(n-2)\theta_1+\sin(n-4)\theta_1+\ldots]$$

$$-4\sin 2\theta[\sin(n-3)\theta_1+\sin(n-5)\theta_1+\ldots]$$

$$-\ldots$$

$$-4\sin r\theta[\sin(n-r-1)\theta_1+\sin(n-r-3)\theta_1+\ldots]$$

$$-\ldots$$

$$-4\sin(n-3)\theta\sin 2\theta_1-4\sin(n-2)\theta\sin\theta_1,$$

for any integral $n \geqslant 2$, the last term in the coefficient of $-4\sin r\theta$ being $\sin 2\theta_1$ or $\sin\theta_1$ according as $n-r$ is odd or even.

Lemma 9. It follows from Lemma 7 that

$$\cos m\theta\cot\theta = \frac{\cos(m+1)\theta+\cos(m-1)\theta}{2\sin\theta}$$

$$= \tfrac{1}{2}[\cot\tfrac{1}{2}\theta+(-1)^{m+1}\tan\tfrac{1}{2}\theta]-\sin m\theta-2\sin(m-2)\theta-2\sin(m-4)\theta-\ldots,$$

so

$$[1+(-1)^{m+1}\cos m\theta]\cot\theta = \tfrac{1}{2}[1+(-1)^{m+1}]\cot\tfrac{1}{2}\theta$$

$$+(-1)^m[\sin m\theta+2\sin(m-2)\theta+2\sin(m-4)\theta+\ldots],$$

the/

the last term being $2 \sin\theta$ or $2 \sin 2\theta$ according as m is odd or even. Thus we have

$$\cot\theta = \tfrac{1}{2}(\cot\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta)$$

$$(1 + \cos\theta)\cot\theta = \cot\tfrac{1}{2}\theta - \sin\theta,$$

$$(1 - \cos 2\theta)\cot\theta = \sin 2\theta,$$

$$(1 + \cos 3\theta)\cot\theta = \cot\tfrac{1}{2}\theta - \sin 3\theta - 2\sin\theta,$$

$$(1 - \cos 4\theta)\cot\theta = \sin 4\theta + 2\sin 2\theta,$$

and so on.

Lemma 10. If

$$x = \tfrac{1}{2}(1 - \cos\theta) = \sin^2\tfrac{1}{2}\theta,$$

then

$$\cos\theta = 1 - 2x,$$

$$\cos 2\theta = 1 - 8x + 8x^2,$$

$$\cos 3\theta = 1 - 18x + 48x^2 - 32x^3,$$

and for any integral n

$$\cos n\theta = \sum_{m=0}^{n} a_m x^m,$$

where

$$a_0 = 1,\ a_1 = -2n^2,\ a_2 = 2^2 . \frac{n^2(n^2-1^2)}{2!\ 1.3},$$

$$a_m = (-2)^m \frac{n^2(n^2-1^2)(n^2-2^2)(n^2-3^2)\ldots[n^2-(m-1)^2]}{m!\ 1.3.5.7.\ \ldots\ (2m-1)}.$$

Also

$$\sin\theta = 2x^{\frac{1}{2}}(1 - x)^{\frac{1}{2}},$$

$$\frac{\sin 2\theta}{\sin\theta} = 2 - 4x,$$

$$\frac{\sin 3\theta}{\sin\theta} = 3 - 16x + 16x^2,$$

and/

and for any integral n,

$$\frac{\sin n\theta}{\sin\theta} = \sum_{m=0}^{n-1} b_m x^m,$$

where

$$b_0 = n,\ b_1 = -2n\frac{n^2-1^2}{1.3},\ b_2 = 2^2 n\frac{(n^2-1^2)(n^2-2^2)}{2!\ \ 3.5},$$

$$b_m = (-2)^m\frac{n(n^2-1^2)(n^2-2^2)(n^2-3^2)\dots(n^2-m^2)}{m!\ \ 3.5.7.\ \dots\ (2m+1)}$$

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Table 4/

Table 4

Aerofoil EQH 1250/4050 Accurate Results

2θ/π	Upper Surface					Lower Surface				
	x	ψ	ψ'	ε	ε'	x	ψ	ψ'	ε	ε'
0	0	0.1185	0.0798	-0.0812	0.0086	0	0.1185	0.0798	-0.0812	0.0086
1	0.0054	0.1310	0.0797	-0.0789	0.0208	0.0069	0.1059	0.0790	-0.0816	-0.0037
2	0.0230	0.1434	0.0773	-0.0746	0.0337	0.0258	0.0938	0.0757	-0.0801	-0.0157
3	0.0524	0.1552	0.0729	-0.0684	0.0462	0.0563	0.0823	0.0706	-0.0767	-0.0267
4	0.0931	0.1662	0.0666	-0.0601	0.0582	0.0976	0.0717	0.0638	-0.0718	-0.0365
5	0.1439	0.1760	0.0585	-0.0501	0.0693	0.1488	0.0623	0.0556	-0.0653	-0.0449
6	0.2039	0.1845	0.0487	-0.0384	0.0795	0.2085	0.0543	0.0460	-0.0577	-0.0517
7	0.2714	0.1913	0.0376	-0.0252	0.0885	0.2753	0.0479	0.0354	-0.0492	-0.0563
8	0.3448	0.1962	0.0253	-0.0107	0.0964	0.3476	0.0432	0.0239	-0.0402	-0.0585
9	0.4222	0.1991	0.0121	0.0051	0.1037	0.4237	0.0404	0.0117	-0.0310	-0.0576
10	0.5017	0.2000	-0.0015	0.0220	0.1129	0.5017	0.0395	-0.0007	-0.0224	-0.0508
11	0.5811	0.1982	-0.0242	0.0409	0.1263	0.5797	0.0401	-0.0043	-0.0155	-0.0367
12	0.6585	0.1918	-0.0579	0.0612	0.1305	0.6558	0.0403	0.0036	-0.0106	-0.0267
13	0.7315	0.1799	-0.0937	0.0811	0.1203	0.7280	0.0388	0.0160	-0.0067	-0.0257
14	0.7984	0.1626	-0.1251	0.0983	0.0975	0.7947	0.0353	0.0272	-0.0021	-0.0335
15	0.8574	0.1410	-0.1483	0.1112	0.0656	0.8541	0.0305	0.0337	0.0043	-0.0478
16	0.9074	0.1165	-0.1622	0.1187	0.0283	0.9048	0.0251	0.0339	0.0132	-0.0661
17	0.9474	0.0906	-0.1665	0.1199	-0.0131	0.9456	0.0202	0.0273	0.0252	-0.0870
18	0.9765	0.0648	-0.1587	0.1143	-0.0585	0.9755	0.0170	0.0111	0.0406	-0.1099
19	0.9941	0.0416	-0.1337	0.1015	-0.1366	0.9938	0.0173	-0.0191	0.0598	-0.1337
20	1.0000	0.0247	-0.0781	0.0818	-0.1405	1.0000	0.0247	-0.0781	0.0818	-0.1405

Contd./

Table 4 (Continued).

Aerofoil EQH 1250/4050 — Accurate Results.

Values of q/U for $C_L/\sin(\alpha + \beta) = 4.8$

$\frac{20\theta}{\pi}$	Upper Surface $C_L = 0.2$	Upper Surface $C_L = 0.6$	Upper Surface $C_L = 1.0$	Lower Surface $C_L = 0.2$	Lower Surface $C_L = 0.6$	Lower Surface $C_L = 1.0$
0	(-)0.8532	0.4712	1.8064	0.8532	(-)0.4712	(-)1.8064
1	0.3773	1.1562	1.9353	1.4444	0.6349	(-)0.1878
2	0.7657	1.2312	1.6930	1.3263	0.8717	0.4061
3	0.9286	1.2529	1.5717	1.2309	0.9298	0.6188
4	1.0237	1.2690	1.5079	1.1652	0.9462	0.7181
5	1.0895	1.2837	1.4708	1.1176	0.9495	0.7729
6	1.1389	1.2961	1.4458	1.0819	0.9486	0.8072
7	1.1771	1.3058	1.4266	1.0552	0.9474	0.8318
8	1.2063	1.3115	1.4085	1.0360	0.9480	0.8523
9	1.2281	1.3131	1.3897	1.0235	0.9515	0.8719
10	1.2453	1.3125	1.3711	1.0200	0.9614	0.8953
11	1.2600	1.3108	1.3527	1.0256	0.9788	0.9246
12	1.2551	1.2896	1.3152	1.0267	0.9917	0.9492
13	1.2248	1.2432	1.2529	1.0174	0.9942	0.9637
14	1.1734	1.1760	1.1704	0.9981	0.9870	0.9689
15	1.1076	1.0948	1.0740	0.9722	0.9741	0.9691
16	1.0338	1.0046	0.9680	0.9433	0.9599	0.9698
17	0.9547	0.9056	0.8496	0.9149	0.9504	0.9796
18	0.8663	0.7865	0.7002	0.8914	0.9578	1.0182
19	0.7217	0.5695	0.4117	0.8891	1.0362	1.1774
20	(-)0.5498	(-)1.5413	(-)2.5324	0.5498	1.5413	2.5324

Theoretical value of $C_L/\sin(\alpha + \beta) = 6.9467$; $\beta = 0.0804$ radians.

Table 5/

Table 5

Aerofoil EQH 1250/4050 — Results according to Approximate Theory.

$\frac{20\theta}{\pi}$	x	Upper Surface ψ	Upper Surface ψ'	Upper Surface ε	Upper Surface ε'	Lower Surface ψ	Lower Surface ψ'	Lower Surface ε	Lower Surface ε'
0	0	0.1200	0.0800	-0.0800	0.0105	0.1200	0.0800	-0.0800	0.0105
1	0.0062	0.1325	0.0790	-0.0774	0.0231	0.1075	0.0790	-0.0807	-0.0019
2	0.0245	0.1447	0.0761	-0.0728	0.0355	0.0953	0.0761	-0.0794	-0.0139
3	0.0545	0.1563	0.0713	-0.0662	0.0476	0.0837	0.0713	-0.0763	-0.0251
4	0.0955	0.1670	0.0647	-0.0578	0.0589	0.0730	0.0647	-0.0716	-0.0351
5	0.1464	0.1766	0.0566	-0.0478	0.0694	0.0634	0.0566	-0.0654	-0.0437
6	0.2061	0.1847	0.0470	-0.0361	0.0788	0.0553	0.0470	-0.0579	-0.0506
7	0.2730	0.1913	0.0363	-0.0230	0.0872	0.0487	0.0363	-0.0496	-0.0553
8	0.3455	0.1961	0.0247	-0.0087	0.0946	0.0439	0.0247	-0.0407	-0.0575
9	0.4218	0.1990	0.0125	0.0067	0.1016	0.0410	0.0125	-0.0317	-0.0564
10	0.5000	0.2000	0	0.0232	0.1101	0.0400	0	-0.0232	-0.0499
11	0.5782	0.1986	-0.0210	0.0416	0.1232	0.0405	-0.0040	-0.0166	-0.0348
12	0.6545	0.1928	-0.0532	0.0615	0.1279	0.0406	0.0038	-0.0121	-0.0243
13	0.7270	0.1817	-0.0888	0.0811	0.1195	0.0391	0.0161	-0.0085	-0.0231
14	0.7939	0.1651	-0.1214	0.0984	0.0985	0.0356	0.0274	-0.0043	-0.0309
15	0.8536	0.1439	-0.1471	0.1115	0.0675	0.0307	0.0339	0.0016	-0.0456
16	0.9045	0.1194	-0.1636	0.1192	0.0295	0.0253	0.0342	0.0102	-0.0645
17	0.9455	0.0930	-0.1701	0.1205	-0.0139	0.0204	0.0275	0.0220	-0.0865
18	0.9755	0.0666	-0.1633	0.1146	-0.0617	0.0172	0.0111	0.0376	-0.1111
19	0.9938	0.0426	-0.1383	0.1010	-0.1123	0.0176	-0.0198	0.0571	-0.1373
20	1	0.0250	-0.0800	0.0800	-0.1484	0.0250	-0.0800	0.0800	-0.1484

Contd./

Table 5 Continued.

Aerofoil EQH 1250/4050 — Results according to Approximate Theory.

Values of q/U for the Lower Surface for $C_L/\sin(\alpha + \beta) = 4.8$.

$\frac{20\theta}{\pi}$	Approximation I			Approximation II			Approximation III		
	C_L=0.2	C_L=0.6	C_L=1.0	C_L=0.2	C_L=0.6	C_L=1.0	C_L=0.2	C_L=0.6	C_L=1.0
0	∞	(-) ∞	(-) ∞	0.7245	(-)0.5066	(-)1.7378	0.8311	(-)0.4816	(-)1.8048
1	1.6463	0.7132	(-)0.2199	1.3637	0.5908	(-)0.1822	1.4176	0.6098	(-)0.2110
2	1.3459	0.8834	0.4210	1.2926	0.8484	0.4043	1.3140	0.8591	0.3931
3	1.2303	0.9265	0.6228	1.2160	0.9157	0.6156	1.2239	0.9223	0.6110
4	1.1625	0.9395	0.7165	1.1595	0.9370	0.7146	1.1602	0.9439	0.7125
5	1.1153	0.9419	0.7686	1.1164	0.9429	0.7694	1.1135	0.9453	0.7685
6	1.0805	0.9412	0.8020	1.0834	0.9437	0.8042	1.0782	0.9449	0.8034
7	1.0541	0.9402	0.8263	1.0578	0.9435	0.8292	1.0518	0.9440	0.8283
8	1.0350	0.9410	0.8470	1.0391	0.9447	0.8504	1.0328	0.9447	0.8490
9	1.0229	0.9453	0.8676	1.0272	0.9492	0.8712	1.0209	0.9487	0.8691
10	1.0187	0.9550	0.8913	1.0230	0.9590	0.8951	1.0174	0.9586	0.8925
11	1.0247	0.9734	0.9222	1.0290	0.9775	0.9261	1.0243	0.9774	0.9230
12	1.0262	0.9863	0.9465	1.0304	0.9904	0.9504	1.0261	0.9909	0.9482
13	1.0177	0.9888	0.9598	1.0218	0.9928	0.9637	1.0173	0.9940	0.9633
14	0.9992	0.9810	0.9629	1.0033	0.9850	0.9668	0.9981	0.9870	0.9687
15	0.9731	0.9664	0.9597	0.9771	0.9704	0.9636	0.9718	0.9736	0.9686
16	0.9431	0.9495	0.9558	0.9470	0.9534	0.9597	0.9423	0.9587	0.9686
17	0.9118	0.9352	0.9585	0.9155	0.9390	0.9624	0.9126	0.9480	0.9770
18	0.8839	0.9343	0.9848	0.8870	0.9376	0.9882	0.8375	0.9536	1.0135
19	0.8779	0.9971	1.1163	0.8712	0.9895	1.1078	0.8798	1.0245	1.1536
20	∞	∞	∞	0.3951	1.1853	1.9754	0.4340	1.4518	2.4197

Contd./

Table 5 Continued.

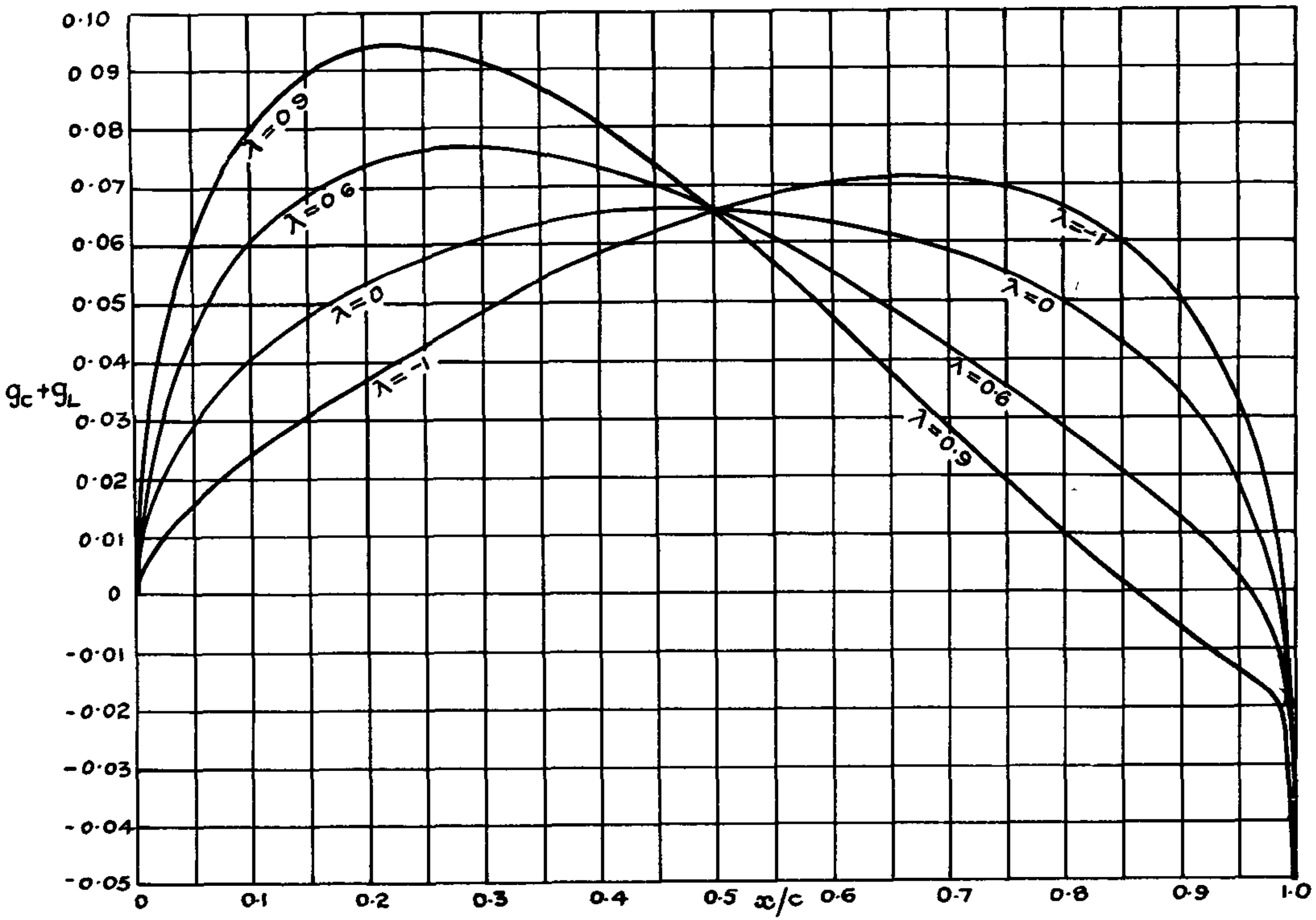
Aerofoil EQH 1250/4050 — Results according to Approximate Theory.

Values of q/V for the Upper Surface for $C_L/\sin(\alpha+\beta) = 4.3$.

$20\theta/\pi$	Approximation I			Approximation II			Approximation III		
	$C_L=0.2$	$C_L=0.6$	$C_L=1.0$	$C_L=0.2$	$C_L=0.6$	$C_L=1.0$	$C_L=0.2$	$C_L=0.6$	$C_L=1.0$
0	(-)∞	∞	∞	(-)0.7245	0.5066	1.7378	(-)0.8311	0.4816	1.8048
1	0.5965	1.5296	2.4627	0.4574	1.1730	1.8886	0.3371	1.1647	1.9424
2	0.8971	1.3596	1.8220	0.8165	1.2375	1.6583	0.7734	1.2389	1.7005
3	1.0129	1.3167	1.6204	0.9625	1.2512	1.5398	0.9343	1.2583	1.5769
4	1.0811	1.3041	1.5271	1.0452	1.2608	1.4763	1.0273	1.2723	1.5108
5	1.1287	1.3021	1.4754	1.1006	1.2697	1.4387	1.0910	1.2845	1.4710
6	1.1645	1.3038	1.4430	1.1410	1.2775	1.4139	1.1383	1.2949	1.4439
7	1.1921	1.3060	1.4199	1.1714	1.2833	1.3953	1.1743	1.3023	1.4223
8	1.2132	1.3072	1.4012	1.1942	1.2867	1.3793	1.2018	1.3063	1.4027
9	1.2293	1.3069	1.3846	1.2112	1.2876	1.3642	1.2222	1.3066	1.3827
10	1.2423	1.3060	1.3697	1.2243	1.2871	1.3499	1.2381	1.3047	1.3628
11	1.2553	1.3066	1.3578	1.2369	1.2874	1.3379	1.2524	1.3027	1.3443
12	1.2542	1.2941	1.3339	1.2354	1.2747	1.3139	1.2495	1.2838	1.3093
13	1.2339	1.2628	1.2918	1.2151	1.2436	1.2721	1.2245	1.2428	1.2525
14	1.1946	1.2128	1.2309	1.1764	1.1943	1.2121	1.1783	1.1810	1.1753
15	1.1398	1.1465	1.1532	1.1225	1.1291	1.1357	1.1161	1.1031	1.0822
16	1.0727	1.0663	1.0600	1.0565	1.0502	1.0440	1.0425	1.0130	0.9760
17	0.9954	0.9720	0.9487	0.9801	0.9570	0.9341	0.9601	0.9107	0.8542
18	0.9070	0.8565	0.8060	0.8911	0.8415	0.7919	0.8677	0.7876	0.7011
19	0.7961	0.6769	0.5577	0.7720	0.6564	0.5408	0.7450	0.5882	0.4256
20	(-)∞	(-)∞	(-)∞	(-)0.3951	(-)1.1853	(-)1.9754	(-)0.4840	(-)1.4518	(-)2.4197

Theoretical value of $C_L/\sin(\alpha+\beta) = 2\pi e^{C_0} = 6.9467$; $\beta = 0.08$ radians.

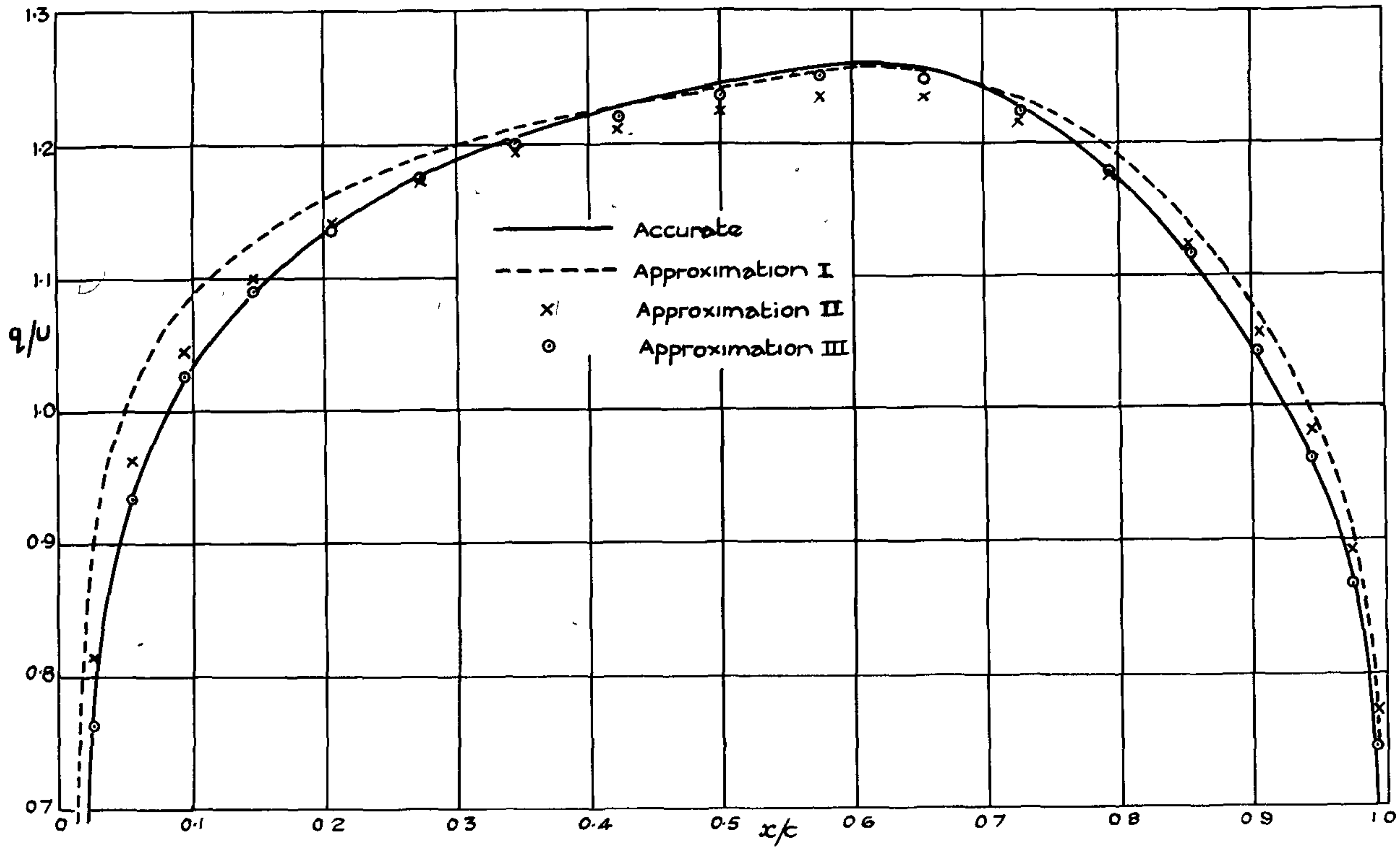
H.
AH.



Centre line $y = hx(1-x)(1-\lambda x)$.

$g_c + g_L$ for $C_L = C_{L\,opt.} = 0{\cdot}2$ and $C_L / \sin(\alpha+\beta) = 5{\cdot}5$.

FIG. 1.



q/U for E.Q.H. 1250/4050, Upper Surface, at $C_L = 0{\cdot}2$ (with $C_L = 4{\cdot}8 \sin(\alpha+\beta)$).

FIG. 2.

V.S.

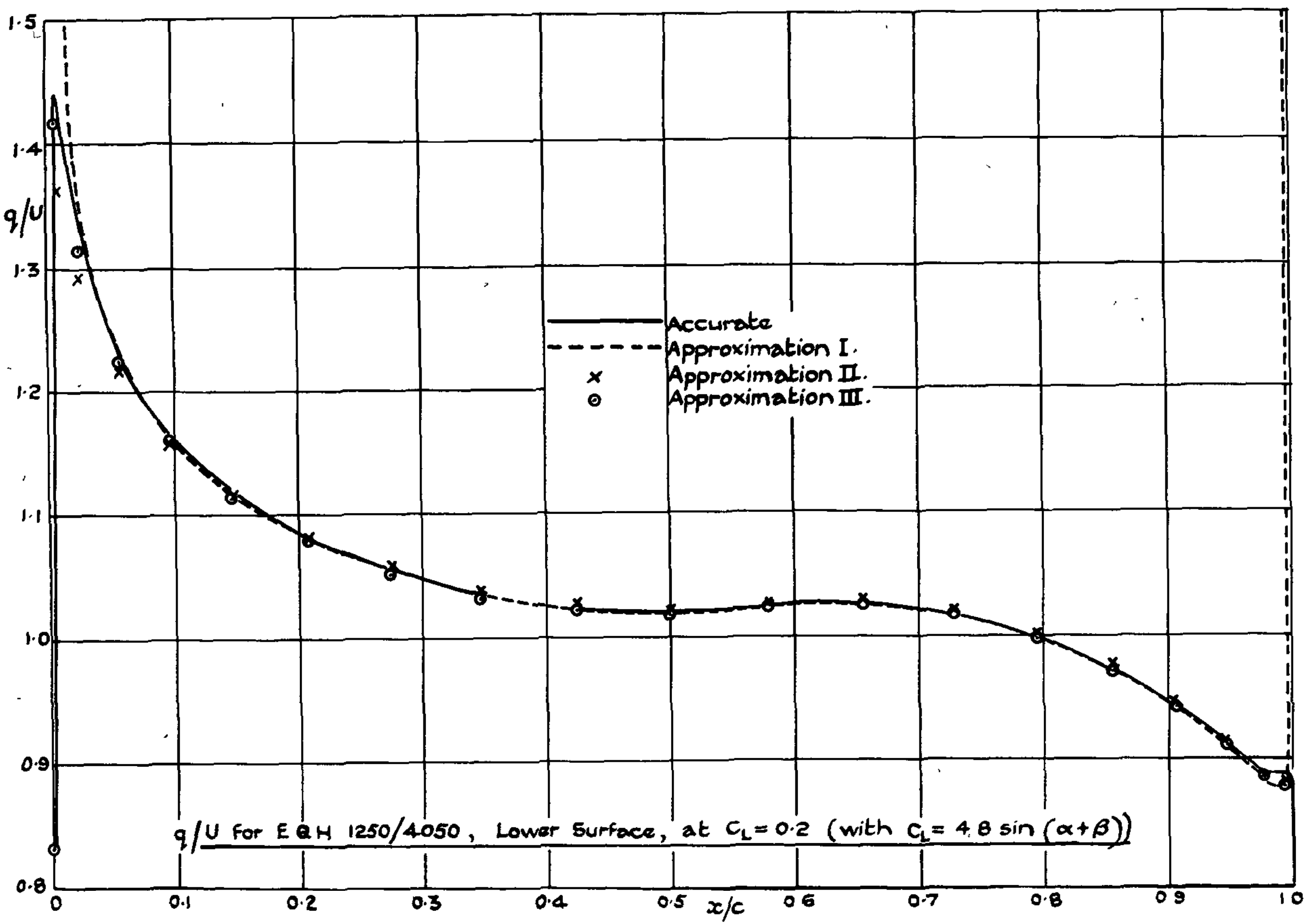
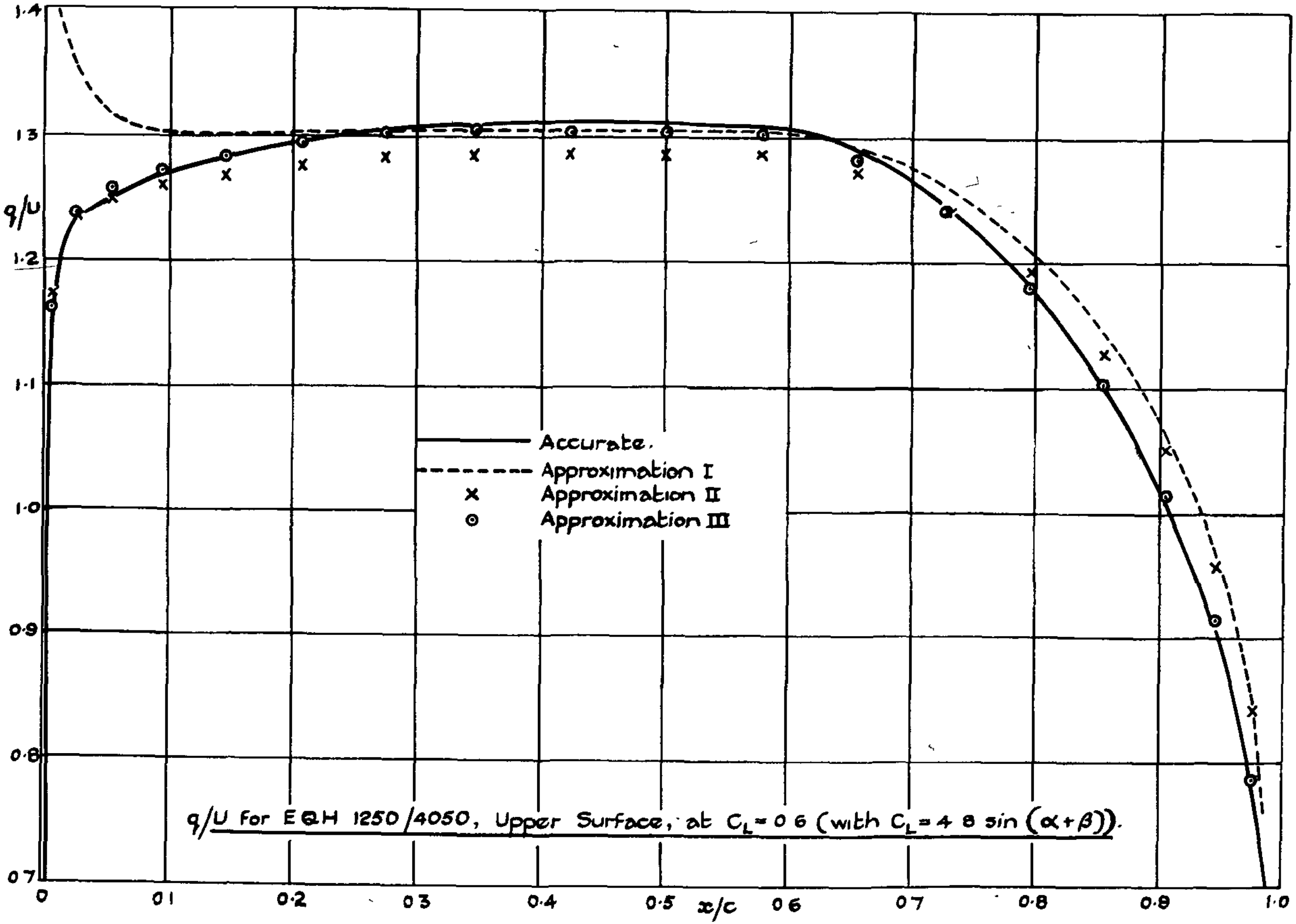

q/U
x/c
1·5
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0
Accurate
Approximation I.
Approximation II.
Approximation III.
q/U for E.Q.H 1250/4050, Lower Surface, at $C_L = 0{\cdot}2$ (with $C_L = 4{\cdot}8 \sin(\alpha+\beta)$)

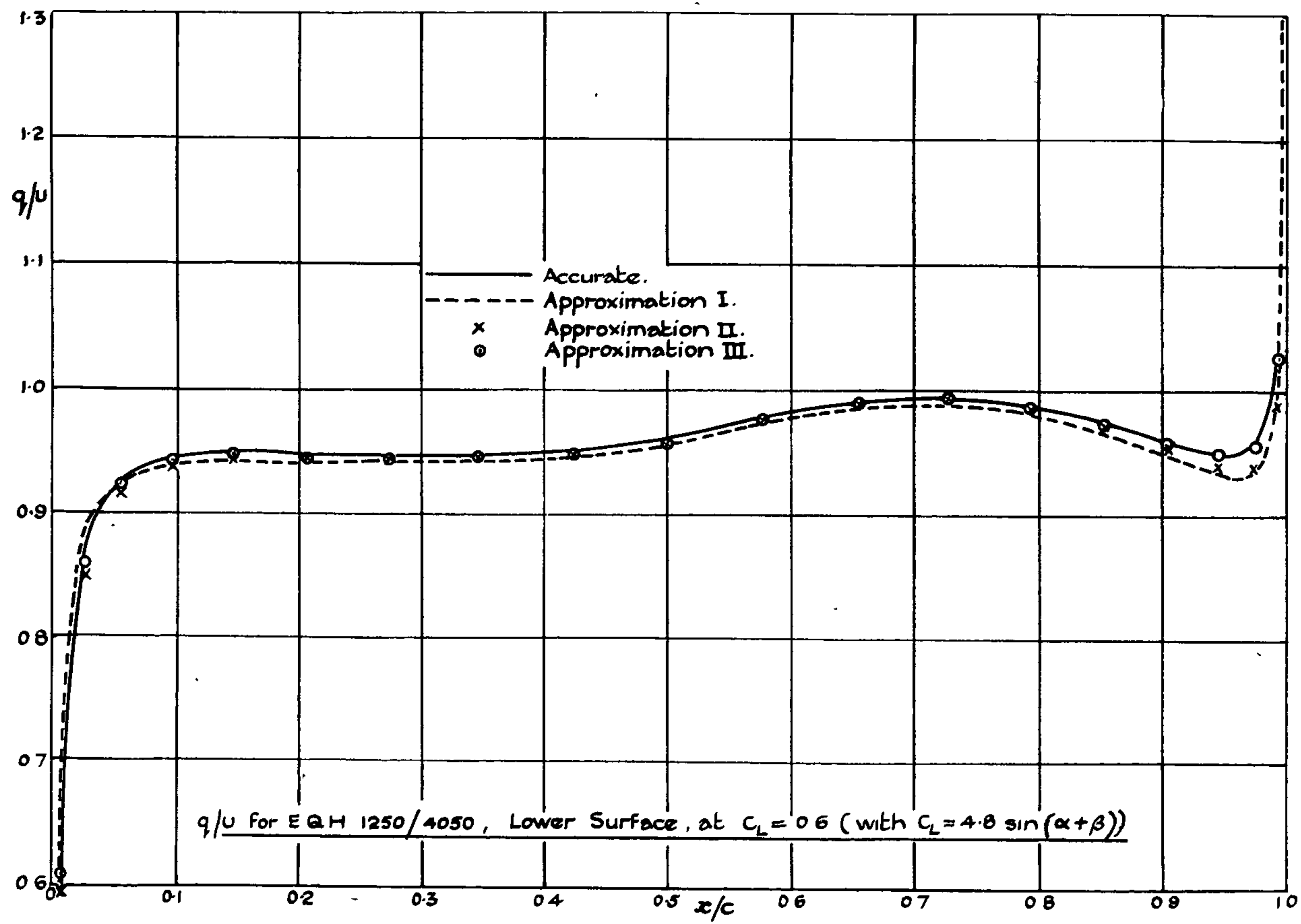
FIG. 3.



q/U for EQH 1250/4050, Upper Surface, at $C_L = 0.6$ (with $C_L = 4.8 \sin(\alpha + \beta)$).

FIG. 4.

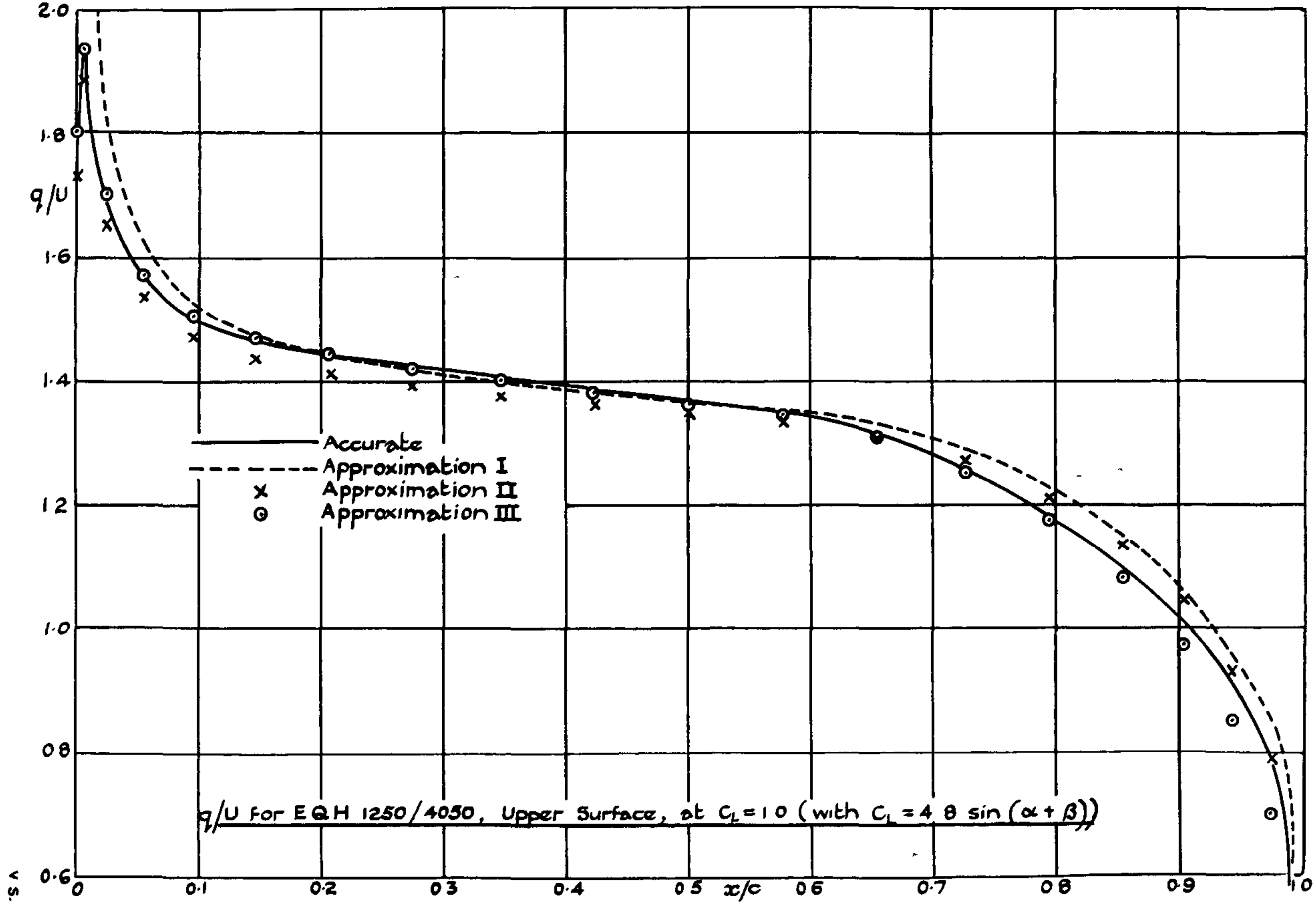
V.S.



q/U for EQH 1250/4050, Lower Surface, at $C_L = 0.6$ (with $C_L = 4.8 \sin(\alpha+\beta)$)

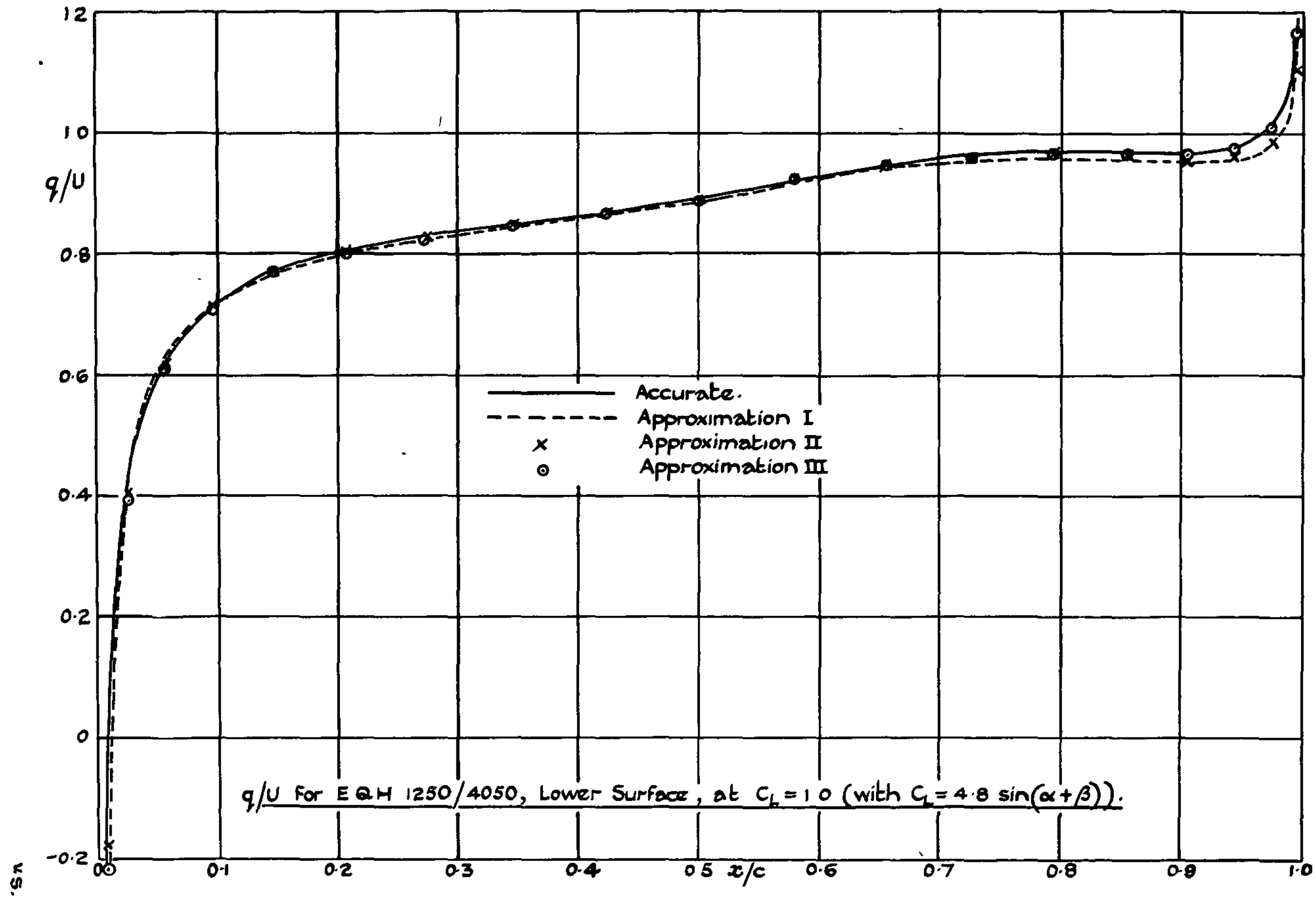
FIG. 5.

V.S.



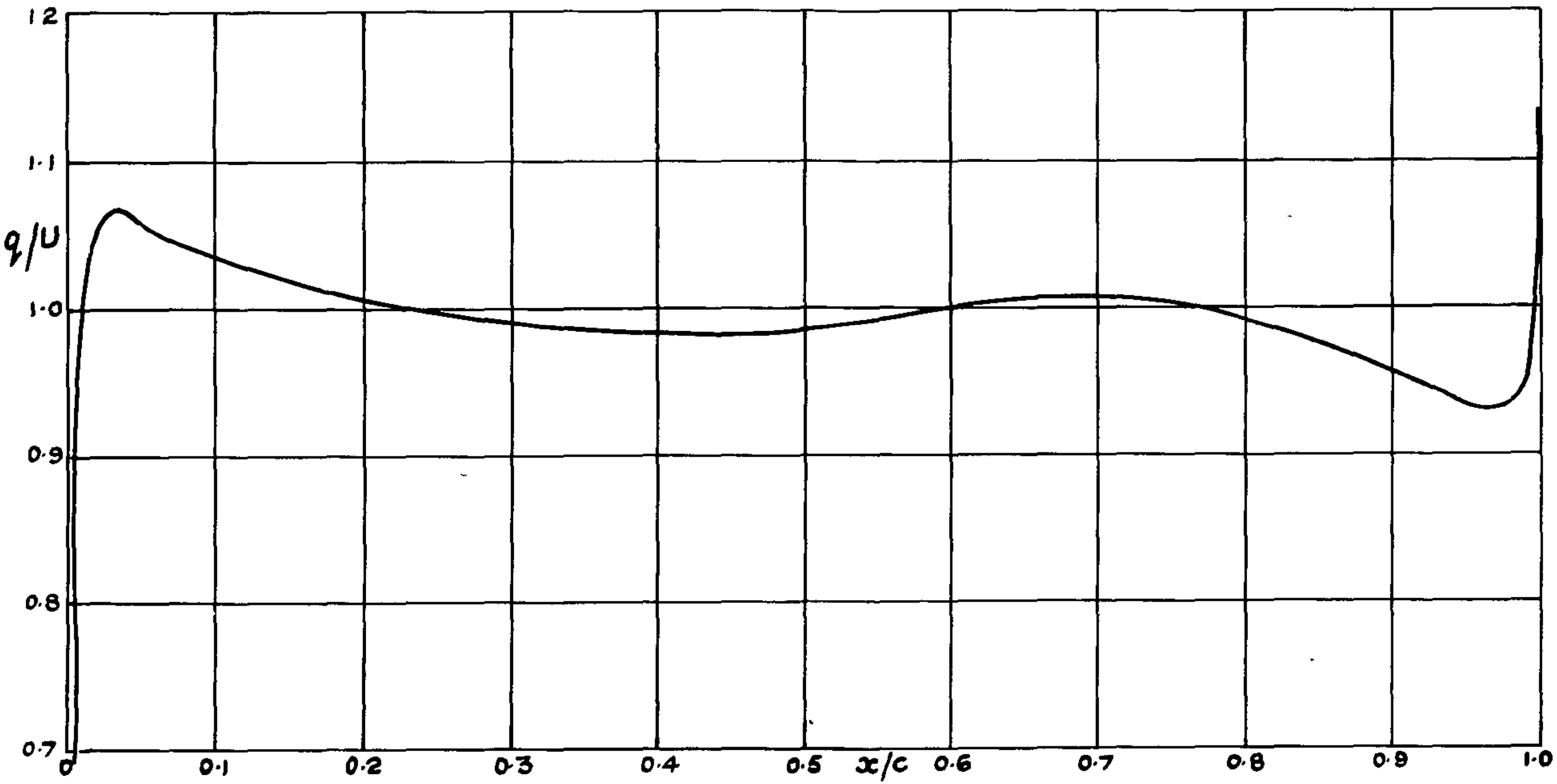
q/U for E.Q.H. 1250/4050, Upper Surface, at $C_L = 1{\cdot}0$ (with $C_L = 4{\cdot}8 \sin(\alpha + \beta)$)

FIG. 6.



q/U for EQH 1250/4050, Lower Surface, at $C_L = 1{\cdot}0$ (with $C_L = 4{\cdot}8 \sin(\alpha + \beta)$).

FIG. 7.



q/U for EQH 1250/4050, Lower Surface, at $C_L = C_{L\,opt.} = 0{\cdot}4354$.

FIG. 8.

V.S.

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