C.P. No 72
(8549)
A R.C Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL

CURRENT PAPERS

Approximate Two-dimensional Aerofoil Theory.

Part v. The Positions of Maximum Velocity and Theoretical C_L Ranges

By

S. Goldstein, F R.S

LONDON HER MAJESTY'S STATIONERY OFFICE

1952

Price 5s. 6d net.

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Approximate Two-Dimensional Aerofoil Theory. Part V.
The Positions of Maximum Velocity and Theoretical C_L-Ranges

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S. Goldstein, F.R.S.

24th March, 1945

Summary

1. We may find the positions of maximum velocity on the upper surface with sufficient precision for practical purposes by the following rules. When $|C_L - C_{L\,opt}|$ is not large, solve the equation

$$C_L = \left\{2\rho_L \cot\theta + \sin^2\theta\left[g_s'(\theta) + g_i'(\theta)\right] + \frac{1}{2}\left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{L\,opt}(1 + \cos\theta)\right\} \Big/ \left\{\frac{1}{a_0} + \frac{\cos\theta}{2\pi}\right\} \qquad \ldots\ldots (1)$$

by plotting the right-hand side against θ and reading off the values of θ for which it is equal to specified values of C_L; or, if $a_0 = 2\pi$, solve the equation

$$\frac{C_L - C_{L\,opt}}{2\pi} = 2\rho_L \frac{\cot\theta}{1 + \cos\theta} + \left[1 - \cos\theta\right]\left[g_s'(\theta) + g_i'(\theta)\right] \qquad \ldots\ldots (2)$$

similarly by plotting the right-hand side and reading off the values of θ for which it is equal to $(C_L - C_{L\,opt})/2\pi$. If the right-hand side stays practically constant over a considerable range of values of θ, then for the corresponding value of C_L we have a 'flat' maximum, which we do not attempt to locate with any precision. As C_L increases, after a certain stage θ becomes smaller. When λ, defined by

$$\lambda = \left(\frac{1}{a_0} + \frac{1}{2\pi}\right)\left(C_L - C_{L\,opt}\right), \qquad \ldots\ldots (3)$$

is large compared with $2\rho_L$, and λ^3 large compared with

$$(2\rho_L)^2\left[g_s'\left(\frac{2\rho_L}{\lambda}\right) + g_i'\left(\frac{2\rho_L}{\lambda}\right)\right], \qquad \ldots\ldots (4)$$

θ is given simply by

$$\theta = 2\rho_L/\lambda. \qquad \ldots\ldots (5)$$

If/

If, however, we proceed to very large values of $C_L - C_{L\,opt}$ and very small values of θ, we must change the definition of λ to

$$\lambda = C_L \left\{1 - \frac{C_L^2}{a_o^2}\right\}^{\frac{1}{2}} \left\{\frac{1}{a_o} + \frac{1}{2\pi c C_o}\right\} - C_{L\,opt}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right); \quad \ldots\ldots (6)$$

for centre lines without singularities at $\theta = 0$ $(g_i(0) = 0)$, θ is then given by

$$\theta = 2\rho_L/\lambda - \surd(2\rho_L)\psi'(0); \quad \ldots\ldots\ldots\ldots (7)$$

for centre lines with singularities at $\theta = 0$ $(g_i(0) \neq 0)$ θ is given by substituting $2\rho_L/\lambda$ for θ in the right-hand side of the equation

$$\theta = \frac{\psi^2(\theta) - \lambda\psi(\theta)\psi'(\theta)}{\lambda + \psi(\theta)\psi'(\theta)}, \quad \ldots\ldots (8)$$

and then proceeding by successive approximation if necessary. In these last cases, however, unless boundary layer suction is employed, restricted boundary layer separation will probably modify the theoretical pressure distribution very considerably even if the aerofoil is not stalled, so the results will not have much practical significance at present.

The equations above also apply to the lower surface if we remember that θ is negative, g_s, ψ_s, g_i', ψ_c' even functions of θ, and g_s', ψ_s', g_i, ψ_c odd functions of θ. It is, however, convenient always to consider these functions in the range $0 \leq \theta \leq \pi$, and we may do so if we make the following changes in the equations. In (1) change the signs of $2\rho_L \cot\theta$ and $\sin^2\theta\, g_s'(\theta)$. In (2) change the signs of $C_L - C_{L\,opt}$ and of $g_i'(\theta)$. Change the sign of $C_L - C_{L\,opt}$ in the definition of λ in (3) and the sign of g_i' in (4). Then (5) is unaltered. Change the sign of the right-hand side of equation (6), defining λ; then (7) and (8) are unaltered, but, whereas on the upper surface,

$$\psi = \psi_s(\theta) + \psi_c(\theta), \quad \psi' = \psi_s'(\theta) + \psi_c'(\theta), \quad \ldots\ldots (9)$$

for the lower surface we must take

$$\psi = \psi_s(\theta) - \psi_c(\theta), \quad \psi' = \psi_s'(\theta) - \psi_c'(\theta). \quad \ldots\ldots (10)$$

These approximate methods have been tested by Mr. E. J. Richards[4], who has applied them to N.A.C.A. 16 series and Clark Y aerofoils, with satisfactory results for practical purposes.

2. A discussion is given of possible definitions of C_L-ranges for low-drag aerofoils. The "theoretical" C_L-range is defined as the range of values of C_L for which the velocity continually increases, on both the upper and lower surface, from the stagnation point to the designed position of maximum velocity at the design C_L. (This definition applies strictly only when the slopes of the graphs of $g_s \pm g_i$ are discontinuous at the design position of maximum velocity, which is the case for low-drag aerofoils as now designed; if the graphs of $g_s \pm g_i$ are rounded off, we should require the velocity to increase only up to the beginning of the rounding-off).

To obtain a C_L-range of any practically significant size, $g_s'(\theta)$, which is small compared with 1, must be positive and large compared with $2\rho_L$, except perhaps for small values of θ. For $a_o = 2\pi$, best results are obtained by taking $g_i'(\theta) = 0$, and for practical values of a_o, this theorem remains practically correct.

If/

If, at the relevant value of θ, $g_s'(\theta) \pm g_i'(\theta)$ are large enough compared with $2\rho_L$ for

$$\frac{2\rho_L}{\theta} + \theta^2 (g_s' \pm g_i')$$

to have minima for small values of θ, then the C_L-range is given, quite generally, by

$$\text{Max}\left\{-\frac{2\rho_L}{\theta} - \theta^2 (g_s' - g_i')\right\} \leqslant \left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left(C_L - C_{L\ opt}\right)$$

$$\leqslant \text{Min}\left\{\frac{2\rho_L}{\theta} + \theta^2 (g_s' + g_i')\right\}, \qquad \ldots\ldots (11)$$

except that, for highly cambered thin aerofoils it seems worth while, having found the positions of the maximum and minimum of the expressions above, to substitute the values of ψ^2 at those positions for $2\rho_L$ in order to find the values. More generally, and with no very great accuracy, if $a_o = 2\pi$ we have the conditions that $\pm \frac{1}{2\pi}\left(C_L - C_{L\ opt}\right)$ must not exceed the minimum values of

$$2\rho_L \frac{\cot\theta}{1 + \cos\theta} + (1 - \cos\theta)(g_s' \pm g_i') \qquad \ldots\ldots (12)$$

respectively; for $a_o \neq 2\pi$, we have the more elaborate conditions (15) of the text.

If $g_i' \neq 0$, the middle of the C_L-range is displaced from $C_L = C_{L\ opt}$.

For the 'roof-top' aerofoils of §§6 and 7 of Ref.5 ($dg_s/dx = s$ = constant for $0 \leqslant x \leqslant X$), with centre-lines designed for constant approximate loading for $0 \leqslant x \leqslant X$ ($dg_i/dx = 0$ for $0 \leqslant x \leqslant X$), the C_L-range is given by

$$\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left|C_L - C_{L\ opt}\right| \leqslant 1.4756\ (2\rho_L)^{\frac{3}{4}} s^{\frac{1}{4}}. \qquad \ldots\ldots (13)$$

We suppose the position of maximum velocity, X, fixed, and also $\sqrt{(2\rho_T)}$ fixed. [For a cusped aerofoil $\sqrt{(2\rho_T)} = 0$; it will in any case be small; 0.04 would be a large value.] If in addition the maximum velocity is given (theoretical critical Mach number given), or the thickness for a given value of x, or the maximum thickness, then there is a value of s which makes the C_L-range a maximum. (For the cases when the maximum velocity is given, or the thickness for a given x, see the examples in the text.) In particular, when the maximum thickness-chord ratio, t, is given, the value of s may be found from the formulae

$$\left.\begin{aligned} s &= 1.25\,t - 0.25\sqrt{(2\rho_T)} \quad \text{for } X = 0.4 \\ &= 0.9818\,t - 0.254\sqrt{(2\rho_T)} \quad \text{for } X = 0.5 \\ &= 0.8095\,t - 0.172\sqrt{(2\rho_T)} \quad \text{for } X = 0.6 \end{aligned}\right| \ldots (14)$$

With/

With normal values of t and $\surd(2\rho_T)$, the values of s so found are probably large enough for a stage to have been reached when any practicable increase in s would not, in any case, make any practical difference to the tolerance for waviness of the surface. (An increase of the velocity gradient by a large factor would probably make a difference, which is why we expect the tolerance to be greater near the nose than elsewhere)

Corresponding to the above formulae for s, we have the following formulae for the parameters a, b, c needed to find the ordinates of the fairing[5].

For $X = 0.4$,

$$a = 1.0125\,t - 0.3\,\surd(2\rho_T), \quad b = 1.5125\,t - 0.4\,\surd(2\rho_T), \quad -c = 0.9085\,t - 1.87\,\surd(2\rho_T);$$

for $X = 0.5$,

$$a = 0.9453\,t - 0.208\,\surd(2\rho_T), \quad b = 1.4362\,t - 0.335\,\surd(2\rho_T), \quad -c = 1.0319\,t - 2.00\,\surd(2\rho_T);$$

for $X = 0.6$,

$$a = 0.8908\,t - 0.156\,\surd(2\rho_T), \quad b = 1.3765\,t - 0.259\,\surd(2\rho_T), \quad -c = 1.2121\,t - 2.170\,\surd(2\rho_T).$$

Tables of the maximum C_L-ranges of these aerofoils are given for thicknesses between 8 and 22 per cent, for $\surd(2\rho_T) = 0$ and 0.02. For $\surd(2\rho_T) = 0$, the C_L-range is nearly proportional to $t^{7/4}$.

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1. Introduction

To calculate the theoretical critical compressibility speed for a given aerofoil section at a given C_L we first compute the greatest velocity on the aerofoil contour. This greatest velocity may be found by graphical or numerical methods from a graph or table of the velocity distribution, but when we require the answers for a number of aerofoils over a range of C_L-values, such a method is long and laborious, and has been found in practice in some cases to be prohibitively long. Some simplification is, therefore, necessary. It appears that it is possible to find simple formulae for the positions of the velocity maxima; such formulae, though rather crude, seem to be sufficiently accurate for practical purposes. The actual maximum values of the velocity are then easily computed, by Approximation III, for the values of x and θ so found* and for those values only. If there are two maxima for any C_L, both must be computed and the larger chosen (unless we know beforehand which will be the larger).

To compute the maximum values of the velocity no great accuracy is necessary in calculating their positions, since a small error in the position produces a second-order error in the value. As $|C_L|$ increases, however, a peak in the velocity graph develops near the leading edge of the aerofoil - on the upper surface for C_L positive and on the lower surface for C_L negative - and the absolute error in the calculated value of θ at the maximum must be small if we are to avoid the possibility of fairly large errors in the calculated maximum value. (The percentage error need not be very small, since θ itself is small). In other words, special attention must be paid to the nose of the aerofoil ($|\theta|$ small).

The/

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*x is the distance, parallel to the chord, of a point on the aerofoil surface, measured as a fraction of the length of the chord, and $x = \frac{1}{2}(1 - \cos\theta)$; $0 \leqslant \theta \leqslant \pi$ on the upper surface, $0 \geqslant \theta \geqslant -\pi$ on the lower surface.

The positions of the maxima being calculated from an approximate formula, if $dq/d\theta$ (where q is the velocity) remains small over a large range of θ in the neighbourhood of the maximum, our determination of its position will be subject to quite considerable error. This error, however, will not be important in the calculation of the maximum value of q, since the graph of q will vary slowly over the whole range of θ in question.

The determination of the position of the maximum velocity may also be considered to give some information on the probable position of transition to turbulence in the boundary layer. Care will be needed, however, in using this information. Other factors in addition to the velocity distribution (waviness and roughness of the surface, turbulence in the air, Reynolds number) affect the position of transition, and the effect of the velocity distribution, and its interaction with the other factors, do not depend solely on the position of the maximum velocity. We do know that if the velocity falls off steadily and not too slowly after the maximum, then in practice, at high Reynolds numbers, transition will not be delayed to any appreciable extent beyond the maximum. Whether transition will occur before the maximum will depend on the velocity gradient, the state of the surface, the turbulence in the air and the Reynolds number* In certain circumstances, also, transition may occur well after the velocity maximum - for example, with a good surface and low turbulence in the air, if the maximum is followed by a small fall in velocity and the velocity begins to rise again (Fig.1), or, at Reynolds numbers which are not too large (10^6 to 2.10^6), if the maximum is very "flat". The former state of affairs (Fig.1) applies near the nose of a good many aerofoils at certain values of C_L. To sum up we may say that, for the purposes of discussing the probable position of transition, a rough

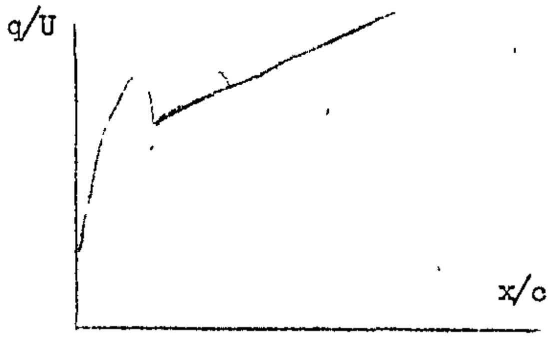

Fig.1

calculation of the positions of maximum velocity may be of some restricted use for a preliminary "sorting-out", but much more will be necessary for an aerofoil which it is proposed to study in any detail, so only very rough calculations of the positions of the maxima will be needed; in particular, if q is varying very slowly, then that is itself probably all we need to know - the exact position of the maximum in a "flat" portion of the graph of q is not of any practical interest.

Closely connected with the question of the positions of the velocity maxima is the discussion of the "theoretical" C_L-ranges for low-drag aerofoils according to a definition we have used for some time now, the "theoretical" C_L-range being defined as the range of values of C_L for which the velocity continually increases, on both surfaces, from the stagnation point to the designed position of maximum velocity at the design C_L.** A slight extension of the

analysis/

*For some further remarks on this subject, see §6.

**As low-drag aerofoils are designed at present, the slopes of the graphs of the velocity on Approximation I (i.e. the slopes of the graphs of $g_s \pm g_i$) are discontinuous at the design position of maximum velocity, and the definition given applies strictly only to such cases. If the graphs of $g_s \pm g_i$ are rounded off, we should require the velocity to increase only up to the beginning of the rounding off.

analysis for finding the positions of the velocity maxima enables us to discuss these "theoretical" C_L-ranges, but before we proceed to the analysis, a short discussion of definitions of C_L-range may not be out of place. The general notion is that the C_L-range is the range of values of C_L for which the aerofoil drag stays low and practically constant. Such a statement is, however, too vague to serve as a definition until the circumstances are specified in which the drag is to be determined. One such definition is the range of values of C_L for which the drag stays low and constant when measured on models made as carefully as possible and tested in a low-turbulence wind-tunnel at a given Reynolds number, say 30×10^6.* This "wind-tunnel" definition would certainly be useful if a suitable tunnel and model-making facilities were available. The third definition - the "practical" definition - is the most probable range under practical conditions of manufacture, flight and maintenance; values according to this definition are the values we should all like to be able to give but none of us can. It is probable that in future all three definitions will be used. Our "theoretical" definition will be of use for a preliminary "sorting-out". Its main disadvantage is that it takes no account of the nature of the velocity curve after the maximum; if, for example, there is only a small fall in velocity after the maximum (Fig.1), it does not take into account the magnitude of the velocity gradient thereafter, which may be fairly large in some cases and practically zero in others. As soon as really good surfaces become practicable, and a small velocity followed by a rise does not necessarily lead to transition, this point will have to be borne in mind. Meanwhile, it is doubtful if, at present, it would be possible in practice to delay transition in this way, so aerofoils may perhaps be expected to be in the same "order of merit" as regards C_L-ranges whether arranged according to the "theoretical" or "practical" definition. In fact, with present surfaces, the "theoretical" definition may be nearer to the "practical" one than the "wind-tunnel" definition would be.**

2. The Approximate Calculation of the Positions of Maximum Velocity on the Upper Surface

We wish to use the simplest possible method to calculate the position (or positions) of maximum velocity. The crude, linear Approximation I cannot be used; q/U is infinite at $\theta = 0$ except for $C_L = C_{L\,opt}$ according to Approximation I. We therefore use Approximation II, according to which[3] on the upper surface,

$$\frac{q}{U} = \frac{1 + \frac{1}{2}C_0^2}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}}\left\{(1 + g_s + g_i)\sin\theta + C_L\left(\frac{1}{2\pi} + \frac{\cos\epsilon}{a_o}\right) - \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)C_{L\,opt}(1 + \cos\theta)\right\},$$

so/

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*This "wind-tunnel" definition may be that adopted by American workers on the subject, though, if so, it is not clear which Reynolds number, if any, they adopt as standard. Jacobs, Abbott and Davidson[1], however, write of the C_L-range as the range "over which the pressure distribution remains favorable", which suggests that they adopt the definition of what we call the "theoretical" C_L-range, though no exact definition is given of when a pressure distribution is favourable. It is of interest that Mrs. Moore, in working out[2] the velocity distribution on N.A.C.A.66, 2-015, also worked out the theoretical C_L-range, and found it to be ± 0.166 (Fig.2), in place of the ± 0.2 indicated in the title of the aerofoil. This aerofoil is one of an older series; it has now been replaced by N.A.C.A.66_2 - 015, and it would be of interest to repeat the calculations on the new aerofoil.

**A fourth possible definition, namely the range of values of C_L for which transition stays at or behind the designed position of the maximum velocity, has not been included, since it is still not possible to calculate the position of transition, and the drag is easier to measure.

so

$$\frac{d}{d\theta}\left(\frac{q}{U}\right) = \frac{1 + \frac{1}{2}C_0^2}{(\psi^2 + \sin^2\theta)^{3/2}} \left\{ \sin\theta\,(\psi^2 + \sin^2\theta)(g_s' + g_i') \right.$$

$$+ (\psi^2 \cos\theta - \psi\psi' \sin\theta)(1 + g_s + g_i) - C_L \left[\sin\theta \left(\frac{1 + \psi^2}{a_o} + \frac{\cos\theta}{2\pi} \right) \right.$$

$$\left. + \psi\psi' \left(\frac{\cos\theta}{a_o} + \frac{1}{2\pi} \right) \right]$$

$$\left. + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi} \right) C_{L\ opt} \left[\sin\theta\,(1 + \psi^2 + \cos\theta) + \psi\psi'\,(1 + \cos\theta) \right] \right\}$$

..... (1)

where the dash denotes differentiation with respect to θ. This expression is too complicated to be of practical use; it must be simplified by approximation. Except for very large values of C_L we may say that the terms of the first order in $\{\ \}$ are

$$\sin^3\theta\,(g_s' + g_i') - C_L \sin\theta \left(\frac{1}{a_o} + \frac{\cos\theta}{2\pi} \right) + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi} \right) C_{L\ opt} \sin\theta\,(1 + \cos\theta);$$

those of the second order are

$$\psi^2 \cos\theta - \psi\psi' \sin\theta;$$

and those of the third order are

$$\psi^2(g_s' + g_i') \sin\theta + (\psi^2 \cos\theta - \psi\psi' \sin\theta)(g_s + g_i) - \frac{C_L}{a_o}\psi^2 \sin\theta$$

$$+ \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi} \right) C_{L\ opt}\ \psi^2 \sin\theta - C_L \left(\frac{\cos\theta}{a_o} + \frac{1}{2\pi} \right)\psi\psi'$$

$$+ \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi} \right) C_{L\ opt}\ (1 + \cos\theta)\,\psi\psi'.$$

The terms of the first order are all small when θ is small, whereas those of the second and third orders are not, so it is immediately clear, as explained in the introduction, that we may not simply neglect the terms of the second and third orders if incorrect results are to be avoided for small values of θ. On the other hand, as we also explained in the introduction, we abandon the requirement of even fair accuracy in the case of a very 'flat' maximum, and then the neglect of the first four terms of the third order would appear to be always justifiable. For $\psi^2(g_s' + g_i') \sin\theta$ will always be small either compared with $\sin^3\theta(g_s' + g_i')$ or compared with $\psi^2 \cos\theta - \psi\psi' \sin\theta$; $g_s + g_i$ will be small compared with 1; and ψ^2 certainly small compared with 1. It is equally clear that, at any rate for small θ, the first term of the second order may not be neglected; when $\theta = 0$ this term is simply $2\rho_L$, where ρ_L is the radius of curvature of the aerofoil section at the leading edge. To what extent the influence

of/

of the second term of the second order, and of the last two terms of the third order, may be neglected, is a more difficult question to decide; it seems probable that the former, since it is of the second order and is, moreover, small when θ is small, may always be neglected, and that the last two terms of the third order may be neglected except when C_L is very large and θ very small. We shall presently give some numerical examples and shall see that the above statements are correct.

The second and third order terms are of most importance when θ is small, so we carry the analysis further for small θ, neglecting completely the first four terms of the third order. In the terms of the first and second orders we write 1 for $\cos\theta$ and θ for $\sin\theta$; for ψ^2 we write

$$\psi^2(0) + 2\theta\psi(0)\psi'(0),$$

and for $\psi\psi'$ we write $\psi(0)\psi'(0)$. In other words we expand the first and second order terms in powers of θ, and keep only the terms in θ^0 and θ^1. In the last two third order terms we keep only the term in θ^0. Also, on the upper surface,

$$\psi = \psi_s(\theta) + \psi_o(\theta),$$

and

$$\psi_o(0) = 0, \quad \psi_s(0) = \sqrt{(2\rho_L)}.$$

Hence $dq/d\theta = 0$ on the upper surface for small θ when, approximately,

$$2\rho_L + \theta\sqrt{(2\rho_L)}\psi'(0) - \lambda\left[\theta + \sqrt{(2\rho_L)}\psi'(0)\right] = 0$$

where

$$\lambda = \left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left(C_L - C_{L\ opt}\right); \quad \ldots\ldots \quad (2)$$

i.e.

$$\theta = \frac{2\rho_L - \lambda\sqrt{(2\rho_L)}\psi'(0)}{\lambda - \sqrt{(2\rho_L)}\psi'(0)}. \quad \ldots\ldots \quad (3)$$

In order that θ should be small, the denominator must be large compared with the numerator, i.e.

$$\lambda - \sqrt{(2\rho_L)}\psi'(0) \gg 2\rho_L - \lambda\sqrt{(2\rho_L)}\psi'(0),$$

i.e.

$$\lambda\left[1 + \sqrt{(2\rho_L)}\psi'(0)\right] \gg 2\rho_L + \sqrt{(2\rho_L)}\psi'(0),$$

i.e., since $\sqrt{(2\rho_L)}\psi'(0)$ is small compared with 1,

$$\lambda \gg 2\rho_L + \sqrt{(2\rho_L)}\psi'(0).$$

Hence certainly

$$\lambda \gg \sqrt{(2\rho_L)}\psi'(0),$$

the second term in the denominator may be neglected, and approximately

$$\theta = \frac{2\rho_L}{\lambda} - \sqrt{(2\rho_L)}\psi'(0) \quad \ldots\ldots \quad (4)$$

Thus/

Thus of the terms of the second order in $\{\ \}$ in (1), the second could have been neglected and the first replaced by its value at $\theta = 0$.

Before we proceed to numerical illustrations we must remark that the last step in our analysis is invalid if there is a singularity at $\theta = 0$ in the equation of the centre line; such singularities occur in certain modern centre lines, for which $g_1(0) \neq 0$, the approximate loading being taken as constant from $x = 0$ either over the whole chord or over some fraction of it from the leading edge. As a result both dy_c/dx and the approximate value of $\psi'_c(\theta)$ are logarithmically infinite at $\theta = 0$, and we may not substitute $\psi'_c(0)$ for $\psi'_c(\theta)$, no matter how small θ may be, nor $\psi^2(0) + 2\theta\psi(0)\psi'(0)$ for $\psi^2(\theta)$. In place of (2) we have then, to begin with,

$$\theta = \frac{\psi^2(\theta) - \lambda\psi(\theta)\psi'(\theta)}{\lambda + \psi(\theta)\psi'(\theta)}. \qquad \ldots\ldots (5)$$

If $\theta = \theta_o$ is the relevant (small) root of this equation, $\psi_c(\theta_o)$ will be small compared with $\psi_s(\theta_o)$, as we shall see later in numerical examples, and $\psi_s(\theta_o)$, will be nearly equal to $\psi_s(0)$, i.e. to $\surd(2\rho_L)$, so $\psi(\theta_o)$ will be of the same order of magnitude as $\surd(2\rho_L)$. We shall also show later, by numerical examples, that, although $\psi'(\theta)$ is logarithmically infinite at $\theta = 0$, yet, for quite small values of θ_o, $\psi'(\theta_o)$ for centre lines with singularities is of about the same order of magnitude as $\psi'(0)$ for centre lines with no singularities. It follows that, just as we could neglect $\surd(2\rho_L)\psi'(0)$ in the denominator in (3), so we may neglect $\psi(\theta)\psi'(\theta)$ in the denominator in (5). Because of the singularity, however, the order of magnitude of $|\psi^2(\theta) - 2\rho_L|/\theta$ is uncertain, so it is difficult to foretell how large a percentage error will be involved in substituting $2\rho_L$ for $\psi^2(\theta)$ in the numerator of (5). If we do make this substitution, (5) reduces to

$$\theta = \frac{2\rho_L}{\lambda} - \surd(2\rho_L)\psi'(\theta). \qquad \ldots\ldots (6)$$

This equation may be solved by successive approximation; for the first approximation we put $\theta = 2\rho_L/\lambda$ in $\psi'(\theta)$, so that

$$\theta = \frac{2\rho_L}{\lambda} - \surd(2\rho_L)\psi'\left(\frac{2\rho_L}{\lambda}\right); \qquad \ldots\ldots (7)$$

for the second approximation we substitute from (7) into $\psi'(\theta)$ in (6), and so on. The first approximation (7) will usually be sufficiently accurate; in this form the equation is equally applicable if there is no singularity in $\psi'(\theta)$ at $\theta = 0$, since in such cases, in fact, the difference between (4) and (7) is negligible. Moreover, on the upper surface,

$$\psi'(\theta) = \psi_s'(\theta) + \psi_c'(\theta),$$

and, unless $\psi_s'(\theta)$ is discontinuous at $\theta = 0$ as a consequence of singularities in the equation of the aerofoil contour, $\psi_s'(0) = 0$ and $\psi_s'(2\rho_L/\lambda)$ will be negligibly small, so we may replace ψ' by ψ_c' in (4), (6) and (7). Such discontinuities occur on N.A.C.A. 00 aerofoils. On N.A.C.A.0012, for example, the approximate value of ψ' at $\theta = 0$ changes discontinuously from +0.0378 to -0.0378 as we pass from the lower to the upper surface. For such aerofoils, ψ' should not be replaced by ψ_c' in (4), (6) and (7).*

Since/

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*On N.A.C.A.0012, the <u>accurate</u> value of ψ' is zero at $\theta = 0$, but rises very rapidly to approximate agreement with the values calculated on the approximate theory.

We should mention that the singularity in the equation of the aerofoil contour which produces a discontinuity in $\psi_s'(\theta)$ at $\theta = 0$, also makes g_s logarithmically infinite at $\theta = 0$. But for N.A.C.A. 0012, g_s is only 0.3245 when $\theta = 0.04$ and 0.3574 when $\theta = 0.02$, compared with 0.1933 at $\theta = \frac{1}{4}\pi$.

Since, for small values of θ, $\psi'(\theta)$ is larger and $\psi(\theta)$ varies more rapidly for N.A.C.A. four-figure aerofoils than for any other common symmetrical types, for our numerical examples we shall suppose the aerofoil fairing to have the shape of N.A.C.A.0012; and we shall consider two centre lines, one with, and one without, a singularity at $\theta = 0$. For the former we take the parabolic centre-line

$$y_c = 0.08\,x(1-x),$$

which corresponds to 2 per cent camber and a $C_{L\ opt}$ roughly 0.25; for the latter the centre-line for constant approximate loading,

$$y_c = -\frac{1}{16\pi}\left\{x\ \ln x + (1-x)\ \ln(1-x)\right\},^{+}$$

which corresponds to a $C_{L\ opt}$ of 0.25 and a camber of roughly 1.4 per cent. For the former

$$\psi_c = 0.04\sin\theta, \quad \psi_c' = 0.04\cos\theta;$$

for the latter

$$\psi_c = -\frac{1}{8\pi}\left\{\tan\tfrac{1}{2}\theta\ \ln\sin\tfrac{1}{2}\theta + \cot\tfrac{1}{2}\theta\ \ln\cos\tfrac{1}{2}\theta\right\},$$

$$\psi_c' = -\frac{1}{16\pi}\left\{\sec^2\tfrac{1}{2}\theta\ \ln\sin\tfrac{1}{2}\theta - \operatorname{cosec}^2\tfrac{1}{2}\theta\ \ln\cos\tfrac{1}{2}\theta\right\}.$$

Then $\surd(2\rho_L) = 0.1781$. Let us take $\lambda = 1/6$ to start with, so that, for $a_o = 2\pi$, $C_L - C_{L\ opt} = \pi/6$. Then $2\rho_L/\lambda = 0.1903$. For this value of θ, $\psi_s(\theta) = 0.1715$, $\psi_s'(\theta) = -0.0324$, and, according to the formulae above, $\psi_c = 0.0076$, $\psi_c' = 0.0393$ for the first centre line and $\psi_c = 0.0108$, $\psi_c' = 0.0373$ for the second. Hence, on the upper surface, for the first centre line

$$\psi = 0.1791, \quad \psi^2 = 0.03208, \quad \psi^2/\lambda = 0.1925, \quad \psi' = 0.0069,$$
$$\psi\psi' = 0.0012,$$

and for the second

$$\psi = 0.1823, \quad \psi^2 = 0.03323, \quad \psi^2/\lambda = 0.1994, \quad \psi' = 0.0049,$$
$$\psi\psi' = 0.0009.$$

Thus in these cases not only may $\psi\psi'$ be neglected in the denominator of (5), but also in the numerator, i.e. the second terms in (4), (6) and (7) may be neglected. We shall, however, wish to make the same approximations for the lower as for the upper surface; for the lower surface,

$$\psi = 0.1639, \quad \psi^2 = 0.02686, \quad \psi^2/\lambda = 0.1612, \quad |\psi'| = 0.0717, \quad |\psi\psi'| = 0.0118$$

for/

$^{+}$ $\ln$ is used for $\log_e$.

for the first centre-line, and

$$\psi = 0.1607, \quad \psi^2 = 0.02582, \quad \psi^2/\lambda = 0.1549, \quad |\psi'| = 0.0697, \quad |\psi\psi'| = 0.0112,$$

for the second. Here both $\psi\psi'$ and the difference of ψ^2 from $2\rho_L$ may not be completely negligible. We shall see, however, that the sign of $\psi\psi'$ is to be taken as negative, and the effect of including the $\psi\psi'$ terms in (5) is therefore opposite in sign from the effect of taking the above values of ψ^2 in place of $2\rho_L$. In fact, if we calculate a second approximation to θ from (5), we find $\theta = 0.1860$ for the first centre line and $\theta = 0.1781$ for the second;* even in the latter case the percentage difference from the first approximation 0.1903 (about $6\frac{1}{2}$ per cent) is probably tolerable for the purposes we have in mind. That the effects tend to cancel is not fortuitous; the difference between $\psi^2(\theta)$ and $2\rho_L$ is clearly related to the sign and magnitude of $\psi\psi'$.

As a second numerical example we consider the same aerofoil, but double the value of λ, so that the value of θ is halved. With $\lambda = 1/3$, $C_L - C_{L\,opt} = \pi/3$ for $a_o = \pi$. The first approximation to θ is $2\rho_L/\lambda = 0.0952$. For this value of θ, $\psi_s(\theta) = 0.1747$, $\psi_s'(\theta) = -0.0344$; for the first centre line $\psi_c = 0.0038$, $\psi_c' = 0.0398$, and for the second $\psi_c = 0.0068$, $\psi_c' = 0.0508$.+ On the upper surface

$$\psi = 0.1785, \quad \psi^2 = 0.03186, \quad \psi^2/\lambda = 0.0956, \quad \psi' = 0.0054, \quad \psi\psi' = 0.00096$$

for the first centre line, and

$$\psi = 0.1815, \quad \psi^2 = 0.03294, \quad \psi^2/\lambda = 0.0988, \quad \psi' = 0.0164, \quad \psi\psi' = 0.00298$$

for the second; on the lower surface

$$\psi = 0.1709, \quad \psi^2 = 0.02921, \quad \psi^2/\lambda = 0.0876, \quad |\psi'| = 0.0742, \quad |\psi\psi'| = 0.01268$$

for the first centre line and

$$\psi = 0.1679, \quad \psi^2 = 0.02819, \quad \psi^2/\lambda = 0.0846, \quad |\psi'| = 0.0852, \quad |\psi\psi'| = 0.01431$$

for the second. The second approximations to θ, according to eqn.(5), are now 0.0943 and 0.00950 for the upper surface for the first and second centre lines respectively, and 0.1043 and 0.1033 for the lower surface. The largest percentage error in the first approximation (about 9 per cent) is now on the lower surface for the <u>first</u> centre line; if we include the second term in (7) this error is reduced to less than half, and the error on the upper surface for the first centre line is reduced almost to zero; moreover, this term will clearly account for an increasing fraction of the error as λ increases. For the second centre line, however, computation shows that there is no substantial advantage to be gained by including the second term of (7) unless we also change the first term to $\psi^2(2\rho_L/\lambda)$, and then we may as well solve (5) by successive approximation.

Thus/

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*For the upper surface these second approximations are 0.1898 for the first centre line and 0.1974 for the second.

+Even when θ is as small as 0.04, ψ_c' is only 0.0679. Eventually as $\rho \to 0$, the whole basis of our approximation to ψ' will fail because of the singularity, and we must use more nearly exact values in (5). But the numerical results above seem to show that such failure will not occur for any practical value of λ.

Thus we see that, when θ is small but not very small, a satisfactory approximation to the position of maximum velocity is given, quite simply, by

$$\theta = 2\rho_L/\lambda \qquad \ldots\ldots \quad (8)$$

in all cases. θ becomes very small when $\left| C_L - C_{L\,opt} \right|$ becomes very large; in such cases Approximation II itself may not be a satisfactory basis, and we shall briefly consider the matter later on the basis of Approximation III (only briefly because the results are of no great practical interest); meanwhile we note only that for centre lines without singularities at $\theta = 0$ ($g_i(0) = 0$), eqn.(7) provides a better answer for very small θ than eqn.(8), and for such very small values of θ we may as well use (4), which is easier, in place of (7); but for centre lines with singularities ($g_i(0) \neq 0$) it is safer to solve (5) by successive approximation, using (8) for the first trial value.

We seek next for the simplest equation to solve when θ is not small. The second and third order terms in $\{\ \}$ in (1) are now small compared with the first order terms; in order, however, to ensure that the solution for θ should pass fairly smoothly into the value given by (8) as C_L increases and θ becomes small, we must include the term $\psi^2 \cos\theta$ of the second order. For the purpose for which it is included, however, we may approximate to it by $2\rho_L \cos\theta$; i.e. we neglect terms

$$(\psi^2 - 2\rho_L)\cos\theta - \psi\psi' \sin\theta$$

of the second order, and all terms of the third order. The equation $dq/d\theta = 0$ then becomes, approximately

$$2\rho_L \cos\theta + \sin^3\theta\left[g_s'(\theta) + g_i'(\theta)\right] + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt} \sin\theta\,(1 + \cos\theta)$$

$$- C_L \sin\theta\left(\frac{1}{a_o} + \frac{\cos\theta}{2\pi}\right) = 0.$$

Probably this equation will be most often used when it is required to study the maxima of q over a range of values of C_L, and the simplest way to carry out the calculation would appear to be to write

$$C_L = \left\{2\rho_L \cot\theta + \sin^2\theta\left[g_s'(\theta) + g_i'(\theta)\right] + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}(1+\cos\theta)\right\} \Bigg/ \left\{\frac{1}{a_o} + \frac{\cos\theta}{2\pi}\right\},$$

$$\ldots\ldots \quad (9)$$

plot the function on the right against θ, and read off the values of θ for which it is equal to specified values of C_L. If the function stays practically constant over a considerable range of values of θ, then for the corresponding C_L we have a 'flat' maximum.

We may note that for $a_o = 2\pi$, (9) becomes, more simply,

$$\frac{C_L - C_{L\,opt}}{2\pi} = 2\rho_L \frac{\cot\theta}{1 + \cos\theta} + \left[1 - \cos\theta\right]\left[g_s'(\theta) + g_i'(\theta)\right]. \qquad \ldots\ldots \quad (10)$$

As/

As C_L increases, after a certain stage θ becomes smaller, and when λ (as given by eqn.(3)) is large compared with $2\rho_L$, θ is given approximately by (8) provided that λ^3 is also large compared with

$$(2\rho_L)^2 \left[g_s'\left(\frac{2\rho_L}{\lambda}\right) + g_i'\left(\frac{2\rho_L}{\lambda}\right)\right] . \qquad ..... \quad (11)$$

Finally, we return to the consideration of very large values of $|C_L - C_{L\ opt}|$ and very small values of θ for which it is advisable to use Approximation III. At these very large values of $|C_L - C_{L\ opt}|$ the velocity graph will have a very sharp peak near the leading edge; if the aerofoil is not completely stalled, we should expect that at any rate a restricted boundary-layer separation will occur and appreciably modify the high theoretical peak, unless such separation is prevented by suction. Consequently the theoretical calculations cannot be expected to have much practical significance at present, and they have therefore been relegated to an Appendix. It is there shown that eqn.(5) still holds if the definition of λ in eqn.(2) is altered to

$$\lambda = C_L \left\{1 - \frac{C_L^2}{a_o^2}\right\}^{-\frac{1}{2}} \left\{\frac{1}{a_o} + \frac{1}{2\pi e^{C_o}}\right\} - C_{L\ opt}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right); \qquad ..... \quad (12)$$

the main effect is that in (2) C_L must be replaced by $C_L\left\{1 - C_L^2/a_o^2\right\}^{-\frac{1}{2}}$ if C_L/a_o is comparable with 1. Combining this result with those previously deduced for eqn.(5), we have the following rules for determining the positions of maximum velocity on the upper surface.

If $|C_L - C_{L\ opt}|$ is not large, solve eqn. (9) by plotting the right-hand side against θ and reading off the values of θ for which it is equal to specified values of C_L; or, if $a_o = 2\pi$, solve eqn.(10) similarly by plotting the right-hand side and reading off the values of θ for which it is equal to $(C_L - C_{L\ opt})/2\pi$. If the right-hand side stays practically constant over a considerable range of values of θ, then for the corresponding value of C_L we have a 'flat' maximum, which we do not attempt to locate with any precision. As C_L increases, after a certain stage θ becomes smaller. When λ (as defined by eqn.(2)) is large compared with $2\rho_L$, and λ^3 large compared with (10), θ is given simply by (8). If, however, we proceed to very large values of C_L and very small values of θ, we must take the definition (12) of λ in place of (2); θ is given by (4) for centre lines without singularities at $\theta = 0$ ($g_i(0) = 0$), and by substituting $2\rho_L/\lambda$ into the right-hand side of (5) [and then, if necessary, solving (5) by successive approximation] for centre lines with singularities ($g_i'(0) \neq 0$).

The approximate methods of this section have been tested by Mr. E. J. Richards[4] who has applied them to N.A.C.A.16 series and Clark Y aerofoils, with satisfactory results for practical purposes.

3. - The Approximate Calculation of the Positions of Maximum Velocity on the Lower Surface.

Our previous equations apply on the lower surface if we remember that θ is negative, g_s, ψ_s, g_i', ψ_c' even functions of θ and g_s', ψ_s', g_i, ψ_c odd functions of θ. It is, however, convenient always to consider

these/

these functions in the range $0 \leqslant \theta \leqslant \pi$. If we do this, then the eqn. (9) which we solve when $|C_L - C_{L\,opt}|$ is not large, becomes

$$C_L = \left\{- 2\rho_L \cot\theta - \sin^2\theta\left[g_s'(\theta) - g_i'(\theta)\right] + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}\right.$$

$$\left.(1 + \cos\theta)\right\} \Big/ \left\{\frac{1}{a_o} + \frac{\cos\theta}{2\pi}\right\}, \qquad \ldots\ldots (9l)$$

and (10), which is the form taken by this equation when $a_o = 2\pi$, is

$$\frac{C_{L\,opt} - C_L}{2\pi} = 2\rho_L \frac{\cot\theta}{1 + \cos\theta} + \left[1 - \cos\theta\right]\left[g_s'(\theta) - g_i'(\theta)\right]. \ldots\ldots (10l)$$

If the sign of λ is changed in the definition (2), so that

$$\lambda = \left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left(C_{L\,opt} - C_L\right), \qquad \ldots\ldots (2l)$$

then when λ is large compared with $2\rho_L$, and λ^3 large compared with

$$(2\rho_L)^2 \left| g_s'\left(\frac{2\rho_L}{\lambda}\right) - g_i'\left(\frac{2\rho_L}{\lambda}\right)\right| \qquad \ldots\ldots (11l)$$

θ is given by

$$\theta = 2\rho_L/\lambda, \qquad \ldots\ldots (8l)$$

simply. For very large values of $C_{L\,opt} - C_L$, and very small values of θ, λ must be defined by (12) with the sign changed:

$$\lambda = C_{L\,opt}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) - C_L\left\{1 - \frac{C_L^2}{a_c^2}\right\}^{\frac{1}{2}}\left\{\frac{1}{a_o} + \frac{1}{2\pi e^{C_o}}\right\}, \qquad \ldots\ldots (12l)$$

and then, for centre lines without singularities, θ is given by (4),

$$\theta = 2\rho_L/\lambda - \sqrt{(2\rho_L)}\,\psi'(0), \qquad \ldots\ldots (4l)$$

and for centre lines with singularities we substitute $\theta = 2\rho_L/\lambda$ into the right-hand side of (5):

$$\theta = \frac{\psi^2(\theta) - \lambda\psi(\theta)\,\psi'(\theta)}{\lambda + \psi(\theta)\,\psi'(\theta)}, \qquad \ldots\ldots (5l)$$

and proceed if necessary by successive approximation; but whereas on the upper surface ψ, ψ' are given by

$$\psi = \psi_s(\theta) + \psi_c(\theta), \quad \psi' = \psi_s'(\theta) + \psi_c'(\theta), \qquad \ldots\ldots (13)$$

on the lower surface we must take

$$\psi = \psi_s(\theta) - \psi_c(\theta), \quad \psi' = \psi_s'(\theta) - \psi_c'(\theta). \qquad \ldots\ldots (13l)$$

4./

4. The Theoretical C_L-Range of a Low-Drag Aerofoil

We recall the definition of the theoretical C_L-range given in the introduction, as the range of values of C_L for which the velocity continually increases, on both the upper and lower surface, from the stagnation point to the designed position of maximum velocity at the design C_L if, as is now usual, the slopes of the graphs of $g_s \pm g_i$ are discontinuous at that position; if the graphs of $g_s \pm g_i$ are rounded off in the future, we shall require the velocity to increase only to the beginning of the rounding off.

If θ_1 is the value of θ up to which the velocity is to increase, we may immediately write down from eqn. (1) the condition to be satisfied by C_L. The coefficient of $-C_L$ in the expression in { } in (1) is positive for $0 < \theta \leqslant \theta_1$; so, in order that $dq/d\theta$ should be > 0 on the upper surface, C_L must not exceed the minimum value of

$$\left\{ \sin\theta(\psi^2 + \sin^2\theta)(g_s' + g_i') + (\psi^2\cos\theta - \psi\psi'\sin\theta)(1 + g_s + g_i) + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}\left[\sin\theta(1 + \psi^2 + \cos\theta) + \psi\psi'(1 + \cos\theta)\right]\right\} \div \left\{\sin\theta\left(\frac{1+\psi^2}{a_o} + \frac{\cos\theta}{2\pi}\right) + \psi\psi'\left(\frac{\cos\theta}{a_o} + \frac{1}{2\pi}\right)\right\} \quad ..... \quad (14)$$

in the range $0 < \theta < \theta_1$. Similarly in order that $dq/d\theta$ should be > 0 on the lower surface* $-C_L$ must not exceed the minimum value of the expression obtained by changing the signs of g_i', g_i, $C_{L\,opt}$ in (14). On the upper surface ψ, ψ' are given by (13), and on the lower surface by (13l).

The expression (14) is much too complicated to be of general use. Now as $|C_L - C_{L\,opt}|$ increases, when it reaches a certain value the position of maximum velocity may begin to move forward or a new maximum may make its appearance somewhere near the nose of the aerofoil. In either case, as we see from the discussion in §2, we may expect the values of $|\lambda|$ (defined by eqns. (2) and (2l)) at the end of the C_L-range to be such that the assumptions leading to eqn. (9) [or eqn. (8) if θ becomes small enough] will be sufficient to provide a fair approximation to $dq/d\theta$. [We found this to be the case when $|C_L - C_{L\,opt}| = \pi/6$, and the error was not prohibitively large even when $|C_L - C_{L\,opt}| = \pi/3$]. Consequently we have, approximately, that C_L must not exceed the minimum value of

$$\left\{2\rho_L\cos\theta + \sin^3\theta(g_s' + g_i') + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}\sin\theta(1 + \cos\theta)\right\} \div \sin\theta\left(\frac{1}{a_o} + \frac{\cos\theta}{2\pi}\right) \quad ..... \quad (15)$$

in/

*On the lower surface the coefficient of C_L, i.e. the denominator in (14), may be negative for very small values of θ if we use the approximate values of ψ'. For such very small values of θ we should, as mentioned in a previous footnote, use more accurate values for ψ', but such very small values of θ do not concern us here. Moreover, the stagnation point does not coincide with the leading edge; according to Approximation II it is at $\theta = \left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left|C_L - C_{L\,opt}\right|$ approximately, on the upper or lower surface according as $C_L - C_{L\,opt}$ is negative or positive. No complications are introduced since we are not concerned with the range of θ between the leading edge and the stagnation point.

in the range $0 < \theta < \theta_1$; and similarly $-C_L$ must not exceed the minimum value of the expression derived from (15) by changing the signs of g_i' and $C_{L\ opt}$. The most important criteria for the validity of this approximation are that, on both the upper and the lower surfaces, $|\psi\psi'|$ should be small compared with $\sin\theta$ at the position of the minimum, and ψ^2 not too different from $2\rho_L$; these criteria may be applied numerically after the values of θ at the minima have been found, but the numerical values in §2 are sufficient to show that we may expect them to be fairly well satisfied.

When $a_0 = 2\pi$, our approximate conditions for C_L reduce to the simpler conditions that $\frac{1}{2\pi}\left(C_L - C_{L\ opt}\right)$ and $\frac{1}{2\pi}\left(C_{L\ opt} - C_L\right)$ must not exceed the minimum values of

$$2\rho_L \frac{\cot\theta}{1+\cos\theta} + (1-\cos\theta)\left(g_s' \pm g_i'\right), \qquad \ldots\ldots \quad (16)$$

respectively.

Let us now consider the case in which, as $\left|C_L - C_{L\ opt}\right|$ is increased, a maximum of q makes its appearance near the nose of the aerofoil before the maximum moves forward from $\theta = \theta_1$. Then, if we suppose θ small in (15), we see that C_L and $-C_L$ must not exceed the minimum values of

$$\left\{\frac{2\rho_L}{\theta} + \theta^2\left(g_s' \pm g_i'\right) \pm \left(\frac{1}{a_0} + \frac{1}{2\pi}\right) C_{L\ opt}\right\} \Bigg/ \left(\frac{1}{a_0} + \frac{1}{2\pi}\right),$$

respectively; i.e.

$$\pm\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)\left(C_L - C_{L\ opt}\right) < \text{Minimum}\left\{\frac{2\rho_L}{\theta} + \theta^2\left(g_s' \pm g_i'\right)\right\} \qquad \ldots\ldots \quad (17)$$

respectively. Since $dx/d\theta = \frac{1}{2}\sin\theta$, we may write (17) in the form

$$\pm\left(\frac{1}{a_0} + \frac{1}{2\pi}\right)\left(C_L - C_{L\ opt}\right) \leqslant \text{Minimum}\left\{\frac{2\rho_L}{\theta} + \frac{\theta^3}{2}\left(\frac{dg_s}{dx} \pm \frac{dg_i}{dx}\right)\right\} \quad \ldots \quad (18).$$

In order that the right-hand side of (17) may have a minimum for a small value θ_0 of θ, $g_s' \pm g_i'$, which we suppose small compared with 1, must be large compared with $2\rho_L$ at $\theta = \theta_0$. Moreover, if $g_s' \pm g_i'$ are large compared with $2\rho_L$ for $\theta_0 \leqslant \theta \leqslant \theta_1$, then the minima which occur when θ is small are either the only minima, or are lower than others which may occur when θ is not small.

If the conditions are not satisfied, and $g_s' \pm g_i'$ is not large compared with $2\rho_L$,* $(1/2\pi)\left(C_L - C_{L\ opt}\right)$ in the former case (positive sign) or $(1/2\pi)\left(C_{L\ opt} - C_L\right)$ in the latter case (negative sign) must not exceed a

quantity/

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*Over the whole range of θ up to $\theta = \theta_1$, except that very small values of θ are irrelevant, since for very small values of θ the term in $2\rho_L$ in (15) or (16) clearly dominates; in other words the minima will not occur for very small values of θ.

For $C_L = C_{L\ opt}$, $dq/d\theta$ will be small, on the upper or the lower surface respectively, for all values of θ which are not too small; our approximations will be inadequate to provide results of even fair percentage accuracy, but it will still be correct that $\pm(1/2\pi)\left(C_L - C_{L\ opt}\right)$ must be $O(2\rho_L)$ or less.

quantity which is of order $2\rho_L$, or less; whereas when $g_s' \pm g_i'$ is large compared with $2\rho_L$ in the relevant range of values of θ, the right-hand side of (17) has a minimum of a somewhat higher order than $2\rho_L$, the minimum occurring when the two terms in $\{\ \}$ in (17) are of the same order of magnitude.

We have so far considered $g_s' \pm g_i'$ to be positive, and we shall now show it is advantageous that they should be so. It is probably sufficient to illustrate the argument by using (16); similar deductions may, in fact, be made from (15). Quite generally we may write (16) in the form

$$\text{Maximum}\left\{-2\rho_L\frac{\cot\theta}{1+\cos\theta}-(1-\cos\theta)\left(g_s'-g_i'\right)\right\}$$

$$\leqslant \frac{1}{2\pi}\left(C_L - C_{L\ opt}\right)$$

$$\leqslant \text{Minimum}\left\{2\rho_L\frac{\cot\theta}{1+\cos\theta}+(1-\cos\theta)\left(g_s'+g_i'\right)\right\}. \quad \ldots\ldots \quad (19)$$

If $g_s' - g_i'$ is negative and $g_s' + g_i'$ positive, and $g_i' \pm g_s'$ are large compared with $2\rho_L$ for all relevant values of θ, then the left-hand member of (19) is probably greater than the right-hand member, and there is no C_L-range at all. For whereas the right-hand member has a minimum for some fairly small value of θ, when both its terms are of the same order of magnitude, the left-hand member may either have no maximum at all and we may have to take simply the greatest value for $0 < \theta \leqslant \theta_1$, or a maximum may occur when θ is not small; in either case the left-hand member will be greater than, or at least nearly equal to, the right-hand member. Similar conclusions follow from (15); in fact, if $g_i' \pm g_s'$ are positive and large compared with $2\rho_L$,

$\left(\frac{1}{a_o}+\frac{1}{2\pi}\right)\left(C_L - C_{L\ opt}\right)$ must be positive and of the order of magnitude of

$g_i' - g_s'$ in order that the velocity on the lower surface may be increasing; and then there is probably some range of (fairly small) values of θ for which the velocity is decreasing on the upper surface.

Similar statements may be made if $g_s' + g_i'$, or both $g_s' + g_i'$ and $g_s' - g_i'$ are negative. If they are large in absolute value compared with $2\rho_L$ over the relevant range of values of θ, then there is probably no C_L-range at all, and at best a very small range.

If $|g_s' \pm g_i'|$ is small, of order $2\rho_L$, then, whether $g_s' \pm g_i'$ is positive or negative, it still remains correct that $\pm(1/2\pi)\left(C_L - C_{L\ opt}\right)$ must be $O(2\rho_L)$ or less.

It follows that to obtain a C_L-range of any practically significant size, $g_s' \pm g_i'$ should be positive and large compared with $2\rho_L$, except perhaps for small values of θ. Hence g_s' should be positive and large compared with $2\rho_L$, except perhaps for small values of θ.

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It also appears that when $a_o = 2\pi$ best results are obtained by taking $g_i' = 0$. In (16) let us write temporarily

$$F(\theta) = 2\rho_L \frac{\cot\theta}{1+\cos\theta} + (1-\cos\theta)g_s',$$

$$G(\theta) = (1-\cos\theta)g_i'.$$

Let the least value of $F(\theta)$ occur when $\theta = \theta_o$, of $F(\theta)+G(\theta)$ when $\theta = \theta_2$, and of $F(\theta)-G(\theta)$ when $\theta = \theta_3$. Then, in general,

$$-F(\theta_3) + G(\theta_3) \leqslant \frac{1}{2\pi}\left(C_L - C_{L\,opt}\right) \leqslant F(\theta_2) + G(\theta_2),$$

and we require to show that

$$F(\theta_2) + F(\theta_3) + G(\theta_2) - G(\theta_3) \leqslant 2F(\theta_o). \qquad \ldots\ldots \quad (20)$$

But

$$F(\theta_2) + G(\theta_2) \leqslant F(\theta_o) + G(\theta_o)$$

$$F(\theta_3) - G(\theta_3) \leqslant F(\theta_o) - G(\theta_o),$$

since $F \pm G$ is least when $\theta = \theta_2$ or θ_3, respectively. Hence (20) follows by addition, and our theorem is proved.

Similarly from (15) we may shew that, when $a_o \neq 2\pi$, best results are obtained by taking

$$g_i' = \frac{1}{2}\left(\frac{1}{a_o} - \frac{1}{2\pi}\right) C_{L\,opt} \Big/ (1+\cos\theta),$$

to compensate for the variation of

$$\frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}\,(1+\cos\theta) \Big/ \left(\frac{1}{a_o} + \frac{\cos\theta}{2\pi}\right);$$

but usually, on modern low-drag aerofoils, a_o will be near enough to 2π for this suggested difference of g_i' from zero to be negligible.

It should be made plain that, whereas the theorem on the best value of g_i' may be rigorously proved on the basis of our initial approximations, the previous discussion of the order of magnitude of g_s' is neither rigorous nor comprehensive, since g_s' may be of widely different orders of magnitude in different parts of the range $0 < \theta < \theta_1$, and it is not practicable to discuss rigorously all possible cases. In particular we did not discuss the case when g_s' was sufficiently small over the relevant range of values of θ for the minimum of (17), for a fairly small value of θ, to be avoided, but when g_s' increased to a different order of magnitude as θ increased. All that we were attempting was a preliminary general discussion of (1) the circumstances likely to arise for any given aerofoil; (2) the conditions necessary to obtain a C_L-range of some practical significance. We did not attempt to find a formula for g_s to make

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the C_L-range as large as possible; to do this it would appear that the minimum (17) must be avoided, and then, as far as practicable, g_s' must be increased where it is least, and g_s increased, in particular for small values of θ, in order to increase $2\rho_L$. Mathematically, for $a_o = 2\pi$, this problem may be defined as that of making the minimum of

$$2\rho_L \frac{\cot\theta}{1 + \cos\theta} + (1 - \cos\theta)\, g_s' \qquad \ldots\ldots (21)$$

for $0 \leqslant \theta \leqslant \theta_1$ as large as possible, where[5]

$$\surd(2\rho_L) = \frac{1}{\pi}\int_0^{\pi} g_s(\theta)(1 + \cos\theta)\, d\theta; \qquad \ldots\ldots (22)$$

a satisfactory solution for practical purposes has been found by Thwaites[6], by considering variations, involving a small number of parameters, of the 'roof-top' aerofoils discussed in Ref.5, §§6 and 7.

5. Displacement of the Middle of the C_L-Range from $C_{L\,opt}$.

We break off the discussion of C_L-ranges to refer briefly to a matter to which reference had already been made in Ref.3, especially as the discussion there was incomplete and misleading - namely, the shift of the middle of the C_L- range from $C_{L\,opt}$ when $g_i' \neq 0$. With $a_o = 2\pi$, in the notation of eqn.(20), the middle of the C_L-range is given by

$$\frac{1}{2\pi}\left(C_L - C_{L\,opt}\right) = F(\theta_2) - F(\theta_3) + G(\theta_2) + G(\theta_3)$$

and the right-hand side will not be zero unless $g_i' = 0$.

As an example of both the diminution and the shift of the C_L-range when $g_i' \neq 0$, we may consider the case when dg_s/dx and dg_i/dx are constant, and dg_s/dx is large compared with $2\rho_L$. Then the minimum of the expression in $\{\ \}$ in (18) occurs when

$$\theta^4\left(\frac{dg_s}{dx} \pm \frac{dg_i}{dx}\right) = \frac{4}{3}\rho_L, \qquad \ldots\ldots (23)$$

and, if we write

$$dg_s/dx = s, \qquad dg_i/dx = \lambda s \qquad (0 \leqslant \lambda < 1), \qquad \ldots\ldots (24)$$

the C_L-range is given by

$$-2\left(\frac{2}{3}\right)^{\frac{3}{4}}(2\rho_L)^{3/4}s^{1/4}(1-\lambda)^{1/4} \leqslant \left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left(C_L - C_{L\,opt}\right)$$
$$\leqslant 2\left(\frac{2}{3}\right)^{3/4}(2\rho_L)^{3/4}s^{1/4}(1+\lambda)^{1/4}.$$

The/

The total range of $\left(\dfrac{1}{a_o} + \dfrac{1}{2\pi}\right)$ C_L is then

$$2\left(\frac{2}{3}\right)^{3/4} (2\rho_L)^{3/4} s^{1/4} \left\{(1+\lambda)^{1/4} + (1-\lambda)^{1/4}\right\},$$

which has its greatest value when $\lambda = 0$, and the middle of the C_L-range is at

$$\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left(C_L - C_{L\ opt}\right) = 2\left(\frac{2}{3}\right)^{3/4} (2\rho_L)^{3/4} s^{1/4} \left\{(1+\lambda)^{1/4} - (1-\lambda)^{1/4}\right\}.$$

We refer also to the results for the aerofoil EQH 1250/4050 (Ref.3, Part II), a fairly thin aerofoil with a large camber (4 per cent). According to the accurate results reported in Part II, there is a small C_L-range, $0.63 \leqslant C_L \leqslant 0.66$, whereas $C_{L\ opt}$ is 0.4354. For this aerofoil, g_s' is negligibly small for small θ, and $g_i = 0.16 \sin\theta$, $g_i' = 0.16 \cos\theta$, $\surd(2\rho_L) = 0.12$. According to our approximate results the minimum of the expression in $\{\ \}$ in (17) for the upper surface occurs when $\theta = 0.3557$, and is 0.607, which leads to the condition $C_L \leqslant 0.601$ (in place of the accurate result 0.66). On the lower surface, there is no minimum near the nose, and we find from (15) that, very roughly, we must have $C_L \leqslant 0.60$ for the lower surface. Thus on our approximate theory we find that there is no C_L-range at all. For reasons given previously, we should not expect the result for the lower surface to be more accurate than it is; the inaccuracy for the upper surface is, however, rather large. The reason lies in the large camber, and the consequent rapid variation of ψ. If we find the position of the minimum on the right in (17) in the same way as before, but substitute the value of ψ at the minimum (0.148) for $\surd(2\rho_L)$ before finding its actual value, we obtain practically the accurate result, $C_L \leqslant 0.66$.

6. C_L-Ranges of 'Roof-Top' Aerofoils

We have a very simple formula for the C_L-range of a 'roof-top' aerofoil, designed according to §§6 and 7 of Ref.5, with centre lines designed for constant approximate loading up to $x = X$($g_i' = 0$, $0 \leqslant \theta \leqslant \theta_1$; see Ref.7, §10). With $dg_s/dx = s$ for $0 \leqslant x \leqslant X$, as in eqn.(24), the application of eqn.(18) is immediate; we have, in fact, only to put $\lambda = 0$ in the results of the preceding section, and we find that the C_L-range is given by

$$\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)\left|C_L - C_{L\ opt}\right| \leqslant 2\left(\frac{2}{3}\right)^{3/4} (2\rho_L)^{3/4} s^{1/4} = 1.4756(2\rho_L)^{3/4} s^{1/4}. \quad \ldots\ldots (25)$$

This formula has proved remarkably accurate for aerofoils with reasonably large values of s and small camber. (It has not been tested on any aerofoil with large camber, since no practical necessity to do so has yet arisen.) The results obtained, all found in the course of investigations made for other purposes, are tested below.

No.	100 t/c (approx)	Data for Fairing						Assumed Value of a_o	Approx. C_L-Range	Accurate C_L-Range
		a	b	c	X	s	$\surd(2\rho_L)$			
1	24	0.226	0.3	-0.193	0.5	0.16	0.1775	5	0.644	0.648
2	18	0.17	0.25	-0.145	0.5	0.213	0.2367	5	0.390	0.390
3	$10\frac{1}{2}$	0.1065	0.1392	-0.1125	0.6	0.0545	0.1072	5.5	0.147	0.145
4	$9\frac{1}{4}$	0.0813_5	0.1304	-0.1037_5	0.6	0.08175	0.0891	5.5	0.123	0.123
5	8	0.0562	0.1216	-0.095	0.6	0.109	0.0711	5.5	0.094_5	0.098

Aerofoils 1 and 2 were symmetrical; 3, 4 and 5 all had a centre line designed for constant approximate loading for $0 \leqslant x \leqslant 0.6$, with the approximate loading decreasing linearly to zero for $0.6 \leqslant x \leqslant 1$, and $(\pi/a_o + \frac{1}{2})C_{L\ opt} = 0.126$. With $a_o = 5.5$, $C_{L\ opt} = 0.118$, but the middle of the C_L-range was, in fact, somewhat greater than 0.126, being 0.130_5, 0.129_5, 0.131 for aerofoils 3, 4 and 5, respectively. The results for aerofoils 1 and 2 are due to Mr. H. C. Garner, and those for aerofoils 3, 4 and 5 to Mrs. Moore. It will be seen that the formula (25) gives a satisfactory result for a value of s as low as 0.0545.

The shape of the fairing depends on the four parameters a, b, c, X. In place of a and c, we introduce the slope s of g_s and $\sqrt{(2\rho_T)}$, where ρ_T is the radius of curvature of the trailing edge. For a cusped aerofoil $\sqrt{(2\rho_T)} = 0$, and, more generally, the degree of the concavity of the aerofoil surface towards the trailing edge is sensibly influenced by the value of $\sqrt{(2\rho_T)}$. We suppose X and $\sqrt{(2\rho_T)}$ have certain fixed chosen values. In addition we suppose the theoretical critical Mach number, or the aerofoil thickness at a given chordwise position, or the maximum thickness, is given. If the theoretical critical Mach number is given, then with a given centre line and design C_L, b will be fixed. If the theoretical critical Mach number, for example, is 0.68, and the centre line is of the type previously mentioned (for aerofoils 3, 4 and 5 above) but with $X = 0.5$, and the low speed equivalent of the top-speed C_L is 0.2, then the maximum value of q/U on the surface must be 1.2525; with $a_o = 2\pi$, g_1 accounts for 0.0667, so b is 0.1858. In any case, if X, $\sqrt{(2\rho_T)}$, and b, or the thickness for a given x, or the maximum thickness, are given, there will be a value for s which makes the C_L-range a maximum.

Consider, for example, the case $X = 0.5$. Then

$$s = 2(b - a), \qquad \sqrt{(2\rho_T)} = 0.06831\,a + 0.36338\,b + 0.56831\,c,$$

$$\sqrt{(2\rho_L)} = 0.56831\,a + 0.36338\,b + 0.06831\,c$$

$$= 0.87980\,b - 0.28005\,s + 0.12020\,\sqrt{(2\rho_T)}.$$

The C_L-range is proportional to $(\sqrt{2\rho_L})^{3/2}\,s^{1/4}$; if $\sqrt{(2\rho_T)}$ and b are fixed, and s varies, this expression has a maximum when

$$s = 0.44880\,b + 0.0613\,\sqrt{(2\rho_T)}.$$

If, however, the thickness is given at a given x, for example $x = 0.4$, then from the tables of Ref.5,

$$0.10976\,a + 0.31936\,b + 0.06077\,c = y_1,$$

where y_1 is the half-thickness, as a fraction of the chord, at $x = 0.4$. Hence

$$0.38296\,b - 0.05123\,s + 0.10693\,\sqrt{(2\rho_T)} = y_1$$

and

$$\sqrt{(2\rho_L)} = 2.29737\,y_1 - 0.16236\,s - 0.12546\,\sqrt{(2\rho_T)}.$$

Again $(\sqrt{2\rho_L})^{3/2}\,s^{1/4}$ has a maximum when s varies, this time when

$$s = 2.0214\,y_1 - 0.1104\,\sqrt{(2\rho_T)}.$$

When/

When the maximum thickness is given, the matter is a little more complicated, since the position of the maximum thickness varies as s varies. The variation, however, is not large,[8*] and there is still a maximum C_L-range for which, if $\sqrt{(2\rho_T)}$ and the thickness are given, the corresponding value of s may be computed; and hence the values of a, b, c may be found. For $X = 0.5$ and 0.6, these values were computed by Mr. H. C. Garner; I find that his values are satisfactorily reproduced by the formulae

$$a = 0.9453\,t - 0.208\sqrt{(2\rho_T)}, \qquad b = 1.4362\,t - 0.335\sqrt{(2\rho_T)},$$

$$-c = 1.0319\,t - 2.00\sqrt{(2\rho_T)}, \qquad s = 0.9818\,t - 0.254\sqrt{(2\rho_T)}$$

for $X = 0.5$, and

$$a = 0.8908\,t - 0.156\sqrt{(2\rho_T)}, \qquad b = 1.3765\,t - 0.259\sqrt{(2\rho_T)},$$

$$-c = 1.2121\,t - 2.170\sqrt{(2\rho_T)}, \qquad s = 0.8095\,t - 0.172\sqrt{(2\rho_T)}$$

for $X = 0.6$. In these formulae t represents the maximum thickness (not the half-thickness) as a fraction of the chord. Values have also been roughly calculated for $X = 0.4$ by Mr. E. J. Richards; his values are represented by

$$a = 1.0125\,t - 0.3\sqrt{(2\rho_T)}, \qquad b = 1.5125\,t - 0.4\sqrt{(2\rho_T)},$$

$$-c = 0.9085\,t - 1.87\sqrt{(2\rho_T)}, \qquad s = 1.25\,t - 0.25\sqrt{(2\rho_T)}.$$

For all normal thicknesses and values of $\sqrt{(2\rho_T)}$, the above formulae lead to very reasonable values of s. [For a cusped aerofoil $\sqrt{(2\rho_T)} = 0$, and values would not normally exceed 0.02; 0.04 would be a very large value.] Experimental evidence of the effect of s on the tolerance that can be allowed for waviness of the surface is still rather scanty, and not at all systematic; but such evidence as we have indicates that once a fair value of s has been reached, any further increases need to be very large indeed to make any practical difference to the waviness, and the values obtained from the above formulae are, for all normal values of t and $\sqrt{(2\rho_T)}$, large enough for this stage to have been reached. Thus once $s = 0.1$, for example, it is very doubtful if it would make any practical difference to the tolerance if s were increased to 0.2. On the other hand, when the velocity gradient is made very much bigger indeed, for example multiplied by a factor of 10, so that instead of 0.1 it becomes 1.0, then it seems that the tolerance on waviness may definitely be increased. Thus we may expect to be able to tolerate a larger waviness very near an aerofoil nose than elsewhere. Also if s is very much decreased the tolerance on waviness certainly becomes less; but we have no exact quantitative knowledge, and systematic experiments are certainly required. Rough values of the maximum C_L-ranges for $a_0 = 2\pi$ and for various values of t are given in the tables below. The figures give the complete C_L-range (2π times the right-hand side of (25)), not the half-range. For $X = 0.5$ and 0.6 they are derived from Mr. Garner's results; for $X = 0.4$ they have been computed from the formulae given above as representing Mr. Richards' results.

When $\sqrt{(2\rho_T)} = C$, the C_L-range is very nearly proportional to $t^{7/4}$.

C_L-ranges/

---

*For $X = 0.5$, it appears that the position of the maximum thickness is given quite closely by $x = 0.3767 + 0.0576\,s/b + 0.0899\sqrt{(2\rho_T)}/b$. There are similar formulae for other values of X.

C_L-ranges, $a_o = 2\pi$

$\sqrt{(2\rho_T)} = 0$				$\sqrt{(2\rho_T)} = 0.02$			
100 t \ x	0.4	0.5	0.6	100 t \ x	0.4	0.5	0.6
8	0.111	0.109	0.099	8	0.104	0.102	0.095
10	0.165	0.161	0.147	10	0.157	0.153	0.141
12	0.227	0.222	0.203	12	0.218	0.214	0.195
14	0.297	0.291	0.264	14	0.287	0.281	0.256
16	0.375	0.367	0.333	16	0.363	0.357	0.326
18	0.462	0.451	0.408	18	0.449	0.440	0.401
20	0.554	0.543	0.490	20	0.540	0.532	0.482
22	0.655	x	x	22	0.640	x	x

x Not calculated

7. Concluding Remarks

1. The 'roof-top' aerofoils, considered in the preceding section, are, of course, not the only ones for which the analysis can be fully carried out. We might, for example, take

$$g_s = A + B \tan \tfrac{1}{2}\theta .$$

Then we find (still with $g_i' = 0$) that the minimum of (17) occurs approximately when

$$\theta^3 = 2\rho_L/B,$$

and the C_L-range is given by

$$\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) \left| C_L - C_{L\,opt} \right| \leqslant 1.5(2\rho_L)^{2/3}\; B^{1/3}.$$

This form of g_s may lead to somewhat larger C_L-ranges than (25); but to pursue the matter further, we should have to find and work out formulae for the fairing ordinates, and we leave the matter for the present.

2. In so far as we may neglect ψ', and replace ψ^2 by $2\rho_L$, all our work could have been based on a simple form of Approximation II, which we may call Approximation IIa, and which is the simplest form necessary if we are to make any attempt at all at approximating to the velocity near the nose:

$$\frac{1}{U} = \frac{(1 + \frac{1}{2}C_o^2)}{(2\rho_L + \sin^2\theta)^{\frac{1}{2}}} \left\{ (1 + g_s + g_i) \sin\theta + C_L\left(\frac{1}{2\pi} + \frac{\cos\theta}{a_o}\right) - \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}\,(1 + \cos\theta) \right\} \qquad \left(0 \leqslant \theta \leqslant \theta_1\right)$$

$$= K\left\{ 1 + g_s + g_i + C_L\left(\frac{\operatorname{cosec}\theta}{2\pi} + \frac{\cot\theta}{a_o}\right) - \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt} \cot\tfrac{1}{2}\theta \right\} \qquad \left(\theta_1 \leqslant \theta \leqslant \pi\right),$$

where/

where

$$K = \frac{(1 + \frac{1}{2} C_o^2) \sin \theta_1}{(2\rho_L + \sin^2 \theta_1)^{\frac{1}{2}}}.$$

APPENDIX

On Approximation III,

$$\frac{q}{U} = \frac{e^{\Upsilon_o}(1 + \epsilon')}{(\psi^2 + \sin^2\theta)^{\frac{1}{2}}} \left\{ \left(1 - \frac{C_L^2}{a_o^2}\right)^{\frac{1}{2}} \sin(\theta + \epsilon - \beta) + \frac{C_L}{a_o}\cos(\theta + \epsilon - \beta) + \frac{C_L}{2\pi e^{C_o}} \right\},$$

and

$$(\psi^2 + \sin^2\theta)^{3/2} e^{\Upsilon_o} \frac{d}{d\theta}\left(\frac{q}{U}\right) = \left\{1 - \frac{C_L^2}{a_o^2}\right\}^{\frac{1}{2}} \left\{ \left[\psi^2 + \sin^2\theta\right] \left[(1 + \epsilon')^2 \cos(\theta + \epsilon - \beta) + \epsilon'' \sin(\theta + \epsilon - \beta)\right] - \left[\sin\theta\cos\theta + \psi\psi'\right]\left[1 + \epsilon'\right] \sin(\theta + \epsilon - \beta)\right\}$$

$$- \frac{C_L}{a_o}\left\{\left[\psi^2 + \sin^2\theta\right]\left[(1 + \epsilon')^2 \sin(\theta + \epsilon - \beta) - \epsilon'' \cos(\theta + \epsilon - \beta)\right] + \left[\sin\theta\cos\theta + \psi\psi'\right]\left[1 + \epsilon'\right]\cos(\theta + \epsilon - \beta)\right\}$$

$$+ \frac{C_L}{2\pi e^{C_o}}\left\{\epsilon''\left[\psi^2 + \sin^2\theta\right] + \left[1 + \epsilon'\right]\left[\sin\theta\cos\theta + \psi\psi'\right]\right\}.$$

We still suppose $\epsilon - \beta$, ϵ', ϵ'', ψ, ψ' small, but C_L/a_o, $C_L/(2\pi e^{C_o})$, though they are less than 1, may now be comparable in magnitude with 1. We are concerned only with cases in which C_L/a_o is large enough for θ to be small compared with unity; and in order to have a comparison with the results from Approximation II it will be convenient to introduce g_s and g_i. We have[3]

$$g_s + g_i = C_o + \epsilon' + (\epsilon - \beta)\cot\theta + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt} \cot\tfrac{1}{2}\theta,$$

$$\sin^3\theta\,(g_s' + g_i') = \psi''\sin^3\theta + \epsilon'\cos\theta\sin^2\theta - (\epsilon - \beta)\sin\theta - \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\,opt}\sin\theta\,(1 + \cos\theta).$$

If/

If we retain only the most important terms the equation $dq/d\theta$ simplifies to

$$\left\{1 - \frac{C_L^2}{a_o^2}\right\}^{\frac{1}{2}} \left\{\psi^2 \cos\theta - \psi\psi' \sin\theta + \sin^3\theta \;(g_s' + g_1') \right.$$
$$\left. + \frac{1}{2}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right) C_{L\ opt} \sin\theta\,(1 + \cos\theta) - \psi\psi' \,(\epsilon - \beta)\cos\theta\right\}$$
$$- C_L \sin\theta \left(\frac{1}{a_o} + \frac{\cos\theta}{2\pi e^{Co}}\right) - C_L \psi\psi' \left(\frac{\cos\theta}{a_o} + \frac{1}{2\pi e^{Co}}\right) = 0,$$

which, for small θ, becomes

$$\psi^2 - \psi\psi'\,\theta - \lambda(\theta + \psi\psi') = 0,$$

with

$$\lambda = C_L \left\{1 - \frac{C_L^2}{a_o^2}\right\}^{\frac{1}{2}} \left\{\frac{1}{a_o} + \frac{1}{2\pi e^{Co}}\right\} - C_{L\ opt}\left(\frac{1}{a_o} + \frac{1}{2\pi}\right)$$

Apart from the altered expression for λ, this equation is the same as equation (5).

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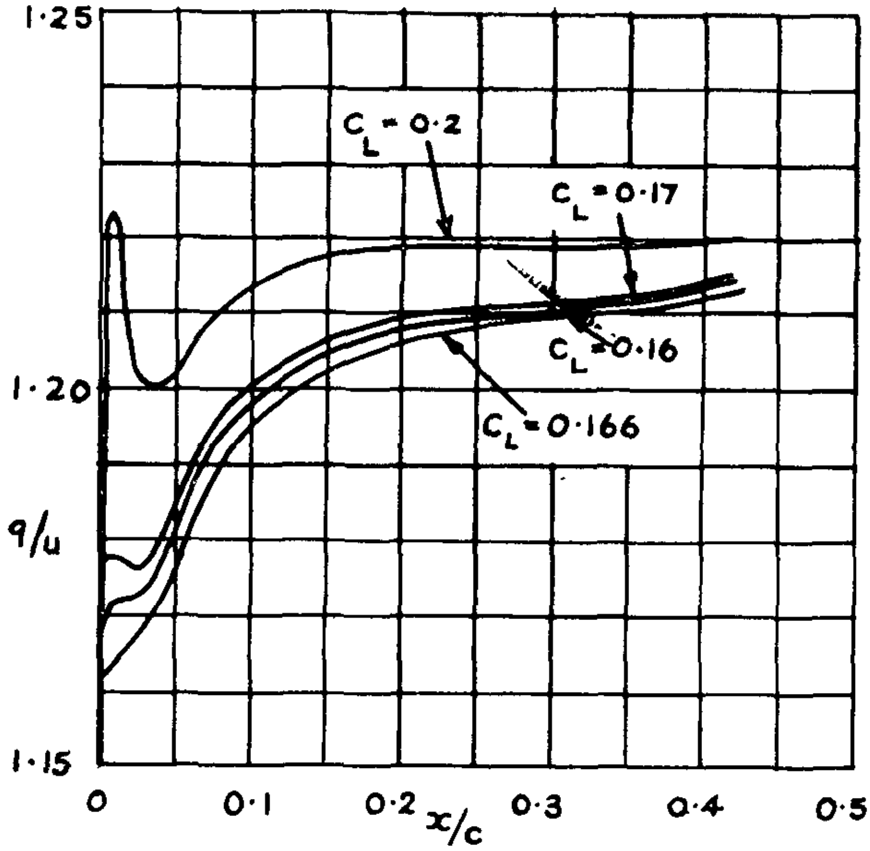
Fig 2



Velocity distributions for NACA 66,2-015 aerofoil, to show C_L range when $a_0 = 2\pi e^{c_0}$ (With $a_0 = 2\pi$ the graphs are indistinguishable)

MSJD.

C.P. No 72
(8549)
A R.C Technical Report

PRINTED AND PUBLISHED BY HER MAJESTY'S STATIONERY OFFICE
To be purchased from
York House, Kingsway, LONDON, W.C 2 429 Oxford Street, LONDON, W 1
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PRINTED IN GREAT BRITAIN

DS 6538/K3/3

S O. Code 23-9006-72