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Mercury Programmes for Lifting Surface Theory Calculations on Wings Oscillating in Supersonic Flow

by

G. Z. Harris, Ph.D.

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MERCURY PROGRAMMES FOR LIFTING SURFACE THEORY CALCULATIONS ON WINGS OSCILLATING IN SUPERSONIC FLOW

by

G. Z. Harris, Ph.D.

SUMMARY

Programmes for lifting surface theory calculations on wings oscillating in supersonic flow are described. The computation falls into two parts, one finding the complex influence matrices connecting lift and downwash, and the other finding the generalised forces when the influence matrices are given as data. The numerical method is described and values of constants used in the calculations are given.

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CONTENTS

1 INTRODUCTION

Programmes have been written in the Mercury Autocode system for finding the generalised forces on wings oscillating harmonically in supersonic flow. The method used is a modification of the Multhopp-Richardson method[1], and is described in a separate paper[2]; the purpose of the present paper is to describe the programmes and the details of the numerical method.

The method relies on replacing the integral equation (which one cannot solve in general) connecting the downwash and lift by a matrix equation (which one can solve) for the lift values at a set of points on the wing. These lifts being known, the generalised aerodynamic forces corresponding to any particular modes of distortion can be found. The machine calculation is thus split into two parts; the first builds up the matrix equation and inverts the complex matrices needed for its solution, while the second finds the generalised aerodynamic forces for wing distortion modes which are expressible as polynomials in the chordwise and spanwise co-ordinates. The first part of the calculation is performed differently for different wing planforms and for different leading and trailing edge conditions; the second is not. Hence six programmes have been written to perform the first part of the calculation for different classes of planform, and one programme does service for all other cases in the second stage in which the generalised forces are found. Necessity, in the form of machine capacity limitations, dictated that the calculation should be split into these two parts. This may, however, be thought of as a virtue since, once the complex matrices for any wing have been found by the first programme, the generalised forces for different sets of distortion modes may be found on separate occasions without repeating the whole of the calculation.

2 THE PROGRAMMES

The programme titles are:

RAE 178A Multhopp-Richardson; segmented planform, subsonic leading edge, supersonic trailing edge.

RAE 179A Multhopp-Richardson; curved subsonic leading edge, straight supersonic trailing edge.

RAE 180A Multhopp-Richardson; segmented planform, subsonic edges.

RAE 181A Multhopp-Richardson; segmented planform, supersonic edges.

RAE 182A Multhopp-Richardson; segmented planform, subsonic leading edge, mixed trailing edge.

RAE 258A Multhopp-Richardson; curved mixed leading edge, straight supersonic trailing edge.

RAE 183A Generalised forces for polynomial modes.

These are written for general values of frequency parameter, and special versions RAE 178A/1, RAE 179A/1, 180A/1, 181A/1, 182A/1, 258A/1 and 183A/1 are written to take advantage of the reduction in computation (and machine) time possible in the steady case. These programmes need as data some constants specifying the planform together with some standard data relating to the number of integration stations taken. This is detailed in Appendix A.

The 'segmented' planform of RAE 178A, RAE 180A, RAE 181A, RAE 182A is a symmetric planform of the type shown in Fig.1, where the half-wing is divided into three spanwise sections having different leading and trailing edge sweep-back angles. These sections are not necessarily swept back at different angles so that, for example, a delta wing falls into this classification. In RAE 178A, RAE 180A and RAE 181A the leading and trailing edge conditions are clear from the programme titles. In RAE 182A the 'mixed' trailing edge described is one in which the outer section is supersonic while the inner two sections are subsonic and swept back; the wing of Fig.2 is of this type, being of a shape used in a current aircraft rather than a general example. The planform dealt with by RAE 179A and 258A is shown in Fig.3. With co-ordinates (X,Y,Z) based on root chord, the equation of the leading edge for $Y \geq 0$ is

$$Y = S(a_1 X + a_2 X^2 + \ldots\ldots + a_{14} X^{14})$$

the leading edge for $Y < 0$ being defined by symmetry. Here, S = semi-span/root chord. The trailing edge is straight. Ogive and gothic planforms have leading edges of this type with suitable values of $a_1, \ldots\ldots, a_5$ and $a_6 = \ldots\ldots = a_{14} = 0$. In RAE 258A the 'mixed' leading edge described is one in which a central portion of the leading edge on each half of the wing is supersonic, as shown in Fig.4.

The above programmes with their various planform variations are written with some hope of covering the range of planform shapes likely to be of practical interest, together with their leading and trailing edge conditions. Variants of four of the programmes, namely RAE 178A/2, RAE180A/2, RAE 181A/2, RAE 182A/2, have been devised to take any type of planform into account for the leading and trailing edge conditions specified. Since the planform is not limited in these to the 'segmented' type, more preliminary work needs to be done by the user for these programmes. More information about the planform is needed than the basic data required, for example, for the segmented planform; the actual input needed for these programmes is specified in Appendix A.

The programme RAE 183A evaluates generalised forces for wing distortion modes which are expressible as polynomials in the chordwise and spanwise co-ordinates. It accepts as input data the output of any of programmes 178A, 179A, 180A, 181A, 182A or 258A together with information about the modes for which the generalised forces are needed. Separate versions, RAE 183A/1 and RAE 183A/2, deal with the steady case and with general planforms respectively. The actual form of the polynomials giving the wing distortion, and the input data for this programme, is given in Appendix A.

3 THE NUMERICAL METHOD

The method used, which is a modification of the Multhopp-Richardson method, is described in a separate Report[2]. For completeness, a summary of the steps in the calculation is given in Appendix B.

The aim of the calculation is to evaluate the matrices M_{rs}, N_{rs} and R_{rs} of equations (22) and (23) of Appendix B, and to combine these to form one large matrix C which may be used to evaluate the generalised forces. The logarithmic correction terms (that is, those involving $\delta_{\beta s}$ in (16)) are then found and used to modify the relevant elements of C. The steps taken in the calculation may be roughly detailed as follows (the symbols are all defined in Appendix B):-

(a) Work out the co-ordinates of the lift and downwash points, and η_1 and η_2 at each point.

(b) Work out $x_{rs} - x_{rs,\lambda\gamma}$ and $y_s - y_{rs,\gamma}$ for all the integration points and each downwash point, and find the other quantities needed in forming M_{rs}, N_{rs}, R_{rs}.

(c) Evaluate $K(X, Y)$, equation (9), for all the values of $X = X_{rs} - X_{rs,\lambda\gamma}$ and $Y = y_s - y_{rs,\gamma}$.

(d) For each downwash point form the matrices $H\ M_{rs}\ N_{rs}\ R_{rs}\ G'$ of equation (22) and combine these.

(e) Form the correction terms (i.e. those terms of (16) which involve $\delta_{\beta s}$) and add to the appropriate terms already found in (d) above to give the matrix C.

(f) Form from C the symmetric and anti-symmetric aerodynamic influence matrices A, B, L and M of (19).

(g) Form the matrices $\bar{Z}$, $\bar{\bar{Z}}$ etc (see following equation (20)) of the deflection and downwash values at the collocation points.

(h) Form the generalised force matrices.

So far as the organisation of the programmes is concerned, steps (a) to (f) form the first part of the calculation while steps (g) and (h) form the second part.

Integrals have to be evaluated numerically at three points in the calculation.

In the expression (17) for $L_\alpha(\bar{\xi}_r, \eta_s)$ write

$$\int_{-1}^{\bar{\xi}_r} e^{-\frac{1}{2}i\nu c(\eta_s)(\bar{\xi}_r-\xi)} h_\alpha(\xi)d\xi = \left\{ \int_{-1}^{\frac{1}{2}(\bar{\xi}_r-1)} + \int_{\frac{1}{2}(\bar{\xi}_r-1)}^{\bar{\xi}_r} \right\} e^{-\frac{1}{2}i\nu c(\eta_s)(\bar{\xi}_r-\xi)} h_\alpha(\xi)d\xi \ .$$

The second integral on the right is evaluated by an eight-point Legendre-Gauss formula (see, for example, Gawlik[3] for the weights and abscissae). The first integral on the right hand side is found either by an eight point Jacobi-Gauss formula if $h_\alpha(\xi)$ has a singularity of the form $1/\sqrt{(1+\xi)}$ occurring at $\xi = -1$ (see, for example, Mineur[4] page 289 for the weights and zeros) or by an eight point Legendre-Gauss formula if $h_\alpha(\xi)$ has no singularity at $\xi = -1$.

The expression (16) for $C_{\alpha\beta}(\bar{\xi}_r, \eta_s)$ contains an integral

$$\int_{\eta_1}^{\eta_2} \sqrt{1 - \eta_s^2} \log |\eta - \eta_s| \, d\eta$$

which may be written in either of the forms

$$\int_{\eta_1}^{\eta_2} \sqrt{1-\eta^2}\,\log|\eta-\eta_s|\,d\eta = \begin{cases} \displaystyle\int_{\eta_1}^{\eta_s-0.2} + \int_{\eta_s+0.2}^{\eta_2} + \int_{\eta_s-0.2}^{\eta_s} + \int_{\eta_s}^{\eta_s+0.2} & \text{if } \eta_2-\eta_s > 0.2,\ \eta_s-\eta_1 > 0.2, \\[2ex] \displaystyle\int_{\eta_1}^{\eta_s-0.2} + \int_{\eta_s-0.2}^{\eta_s} + \int_{\eta_s}^{\eta_2} & \text{if } \eta_2-\eta_s < 0.2,\ \eta_s-\eta_1 > 0.2, \\[2ex] \displaystyle\int_{\eta_1}^{\eta_s} + \int_{\eta_s}^{\eta_2} & \text{if } \eta_2-\eta_s < 0.2,\ \eta_s-\eta_1 < 0.2. \end{cases}$$

The last two of these integrals on the right-hand side are evaluated by a Gauss formula which takes the logarithmic singularity at $\eta = \eta_s$ into account (suitable Gauss weights and zeros are given in Mineur[4], page 556); the remaining integrals, if any, are found by a Legendre-Gauss formula.

The third integral which has to be evaluated is

$$\int_{(X-MR)/(\beta^2|Y|)}^{(X+MR)/(\beta^2|Y|)} \frac{\tau\{\cos \nu\tau\cdot|Y| + i \sin \nu\tau\cdot|Y|\}}{\sqrt{1+\tau^2}}\,d\tau \quad , \tag{1}$$

which occurs in equation (9) for $K'(X,Y)$. The real and imaginary parts of this are dealt with separately; the upper limit $(X+MR)/(\beta^2|Y|)$ is positive, whereas $(X-MR)/(\beta^2|Y|)$ can be either positive or negative, so some simplification follows from the odd and even nature of the integrands. To evaluate an integral like

$$\int_a^b \begin{matrix}\cos\\ \sin\end{matrix} (\nu\tau|Y|) \frac{\tau d\tau}{\sqrt{1+\tau^2}} , \qquad (2)$$

for some a and b with $a \geq 0$ and $b > 0$, the integral is expressed as the sum of t integrals

$$\int_a^{a+\psi} + \int_{a+\psi}^{a+2\psi} + \ldots\ldots + \int_{a+(t-2)\psi}^{a+(t-1)\psi} + \int_{a+(t-1)\psi}^{b}$$

where

$$\psi = \min\left(\frac{\pi}{5}, \frac{\pi}{\nu|Y|}\right)$$

and t is an integer chosen so that

$$0 < b - a - (t-1)\psi < \psi .$$

Since the limits of integration in (1) are large when $\beta^2|Y|$ is small the following approximation is used when large limits occur. For any c and d

$$\left| i\nu|Y| \int_c^d \frac{\tau e^{-i\nu\tau|Y|}}{\sqrt{1+\tau^2}} d\tau + \left[e^{-i\nu\tau|Y|} \frac{\tau}{\sqrt{1+\tau^2}} \right]_c^d \right| = \left| \int_c^d e^{-i\nu\tau|Y|} \frac{d\tau}{(1+\tau^2)^{3/2}} \right|$$

$$\leq \int_c^d \frac{d\tau}{(1+\tau^2)^{3/2}} \leq \int_c^\infty \frac{d\tau}{(1+\tau^2)^{3/2}} = 1 - \frac{c}{\sqrt{1+c^2}} .$$

This last expression is less than 10^{-4} if $c \geq 71$, which gives the approximations

$$\left.\begin{array}{l} \nu|Y| \displaystyle\int_c^d \cos(\nu\tau|Y|) \frac{\tau d\tau}{\sqrt{1+\tau^2}} \doteqdot \left[\frac{\tau \sin(\nu\tau|Y|)}{\sqrt{1+\tau^2}}\right]_c^d \\ \\ \text{and} \\ \\ \nu|Y| \displaystyle\int_c^d \sin(\nu\tau|Y|) \frac{\tau d\tau}{\sqrt{1+\tau^2}} \doteqdot -\left[\frac{\tau \cos(\nu\tau|Y|)}{\sqrt{1+\tau^2}}\right]_c^d \end{array}\right\} \quad (3)$$

to within 10^{-4} if $c \geq 71$. This approximation may be used as it stands if, in (2), $a > 71$ and $b > 71$. If $a < 71$ and $b > 71$ write

$$\int_a^b = \int_a^{71} + \int_{71}^b$$

and approximate to the second integral by means of (3).

For the various leading and trailing edge conditions different positions for the lift points, interpolation functions and integration points have to be taken to account for the different singularities. These are discussed in section 4.2.

4 DESCRIPTION OF PROGRAMMES

4.1 The programmes and their use

The programmes RAE 178A to 182A and 258A which find the aerodynamic matrices are all six chapter Mercury Autocode programmes, and their logical structure is given in Appendix C. The variants of these programmes (178A/1, 178A/2 etc.) are also six chapter Mercury Autocode programmes, whose logical structure is similar to that given in Appendix C. The programme RAE 183A which finds generalised forces is a three chapter Mercury Autocode programme, and its logical structure is also described in Appendix C.

Due to the limited capacity of the computer, there are certain restrictions on the size of problem which may be treated by the programmes. The quantities m, n, p and q must satisfy

$$2 \leq m \leq 10, \quad n \leq 21, \quad p \leq 10, \quad q \leq 21$$

together with additional limitations which are given in the following table.

If m =	then n $\leqslant$
2	21
3	18
4	14
5	10
6,7	8
8,9	6
10	5

Also

$$m\,p\,q \times \text{integral part of } \{\tfrac{1}{2}(n+1)\} \leqslant 1608 .$$

For 180A and 182A there is the additional limitation that $m\,q \leqslant 189$.

There are limitations on the Mach number and frequency parameter which can be used, but these are imposed by the basic theory rather than by limitations of the computer or of the detailed numerical method. The method breaks down when $M = 1$; Mach numbers very close to $M = 1$ will give trouble, since the functions

$$\frac{2X}{R}\exp\left\{\frac{-i\nu M^2 X}{M^2 - 1}\right\}\cos\left\{\frac{MR\nu}{M^2 - 1}\right\}$$

of equation (9), Appendix B, will have a large number of waves making approximate integrations involving this function difficult. Similar remarks apply to large values of the frequency parameter ν. However, larger numbers of lift and downwash points and of integration points may be used for extreme values of M and ν to offset this, and this is discussed below.

In any particular case, the values of m and n, and consequently the number of lift and downwash points, are chosen with regard to the complexity of the modes for which the generalised forces are ultimately needed. These should be chosen in such a way that the deflection can be adequately represented by an $(m - 1)$th degree polynomial in the chordwise co-ordinate and an $(n - 1)$th degree polynomial in the spanwise co-ordinate. However, even for rigid-body motions of any particular wing there are minimum values of m and n below which reasonable answers could not be expected. In addition, a higher value of m should be chosen when the Mach number is close to unity. The choice of p and q, which determines the number of integration points, will depend on M and ν; a higher value of p should be taken for M near unity or for large ν. The value of q should be higher if there are any kinks in the planform or if the Mach number is close to one; in the latter case the integration areas of Fig.5 extend further in the spanwise direction. In addition, the number of points chosen will depend on the aspect ratio of the wing under consideration. For low aspect ratio wings, for example, more chordwise lift and downwash points (as well as integration points) than spanwise points will be needed.

The foregoing remarks on the numbers of points to be taken, and on limitations, are of a general character and should be taken as an overall guide if related to any particular problem.

The time taken by programmes 178A etc. varies approximately linearly with

$$m\,pq \times \text{integral part of } \tfrac{1}{2}(n + 1) \quad .$$

Since there is also some variation in time taken with Mach number and frequency parameter, and also from planform to planform, it is only possible to predict times to within rather broad limits. Examples of times which have been taken using programmes 178A and 179A are given in the following table.

m	n	p	q	Mach No.	ν	Planform	Time
2	7	3	7	1.25	0.3	Delta wing	18 min
3	7	3	7	1.25	0.3	aspect	25 "
3	7	5	7	1.25	0.3	ratio 1.456	38 "
3	7	3	9	1.25	0.25	Swept wing, aspect ratio 2, 60°	30 "
3	7	3	7	1.12	0.25	leading edge sweep, 27°	32 "
3	7	3	7	1.80	0.25	trailing edge sweep	15 "
3	7	3	7	1.054	0.25	Cropped delta wing, aspect ratio	37 "
3	7	3	7	1.41	0.25	3, leading edge sweep, 49°	17 "
3	7	3	7	1.01	0.15	Delta wing, aspect ratio 1.5	102 "
5	8	5	8	2.0	2.4	Ogive wing, aspect ratio 0.87	60 "

Times for programmes 180A etc. should be little different. The considerable effect of Mach number on the computing time, which is exemplified above for the cropped delta wing of aspect ratio 3, should be noted; the reason for this variation is that for M close to unity the limits $(X - MR)/(\beta^2|Y|)$ and

$(X + MR)/(\beta^2|Y|)$ will be large in modulus, and the integrals of (9), Appendix B will take longer to evaluate.

The times taken for programmes 178A/1 etc., which operate for zero frequency parameter, will be very much less than these times since the function K(X, Y) of (9) takes the particularly simple form 2X/R instead of having to be found by numerical integration.

Programme 183A, and its derivatives, which find the generalised forces are straightforward and quick. It is not worthwhile to discuss them in any detail here.

4.2 Data input and output

The data input and output for the individual programmes is given in detail in Appendix A.

For the programmes RAE 178A etc. which perform the first part of the calculation, this consists of basic data specifying the planform followed by data needed for the numerical work, consisting of the chordwise and spanwise positions of the lift and downwash points and matrices connected with the interpolation functions used and integration points. This latter numerical data depends only on the wing leading and trailing edge conditions, and is in the form of standard input tapes; these vary with the number of points taken for the various stages of the calculation (that is with choice of m, n, p and q) and values are given in Appendix D. The standard tapes containing this data are kept along with the programmes.

The output data from these programmes is suitable for input to the generalised forces programme 183A (or one of its variants). Additional data, giving information about the polynomial distortion modes for which generalised forces are required, have also to be provided.

The output of programme 183A is the matrix of generalised forces, the real part being divided by the aspect ratio and the imaginary part by (aspect ratio) × (frequency parameter). This form is chosen since, if simple pitching and heaving modes only are being considered, the data output gives, with a change of sign, the aerodynamic derivatives for these modes.

5 DISCUSSION

The Mercury programmes described in this Report have been used in aerodynamic derivative calculations on a number of wings. Results have been presented in Ref.2 for an ogee wing, a symmetrical tapered wing and a delta wing and comparisons made with other theoretical and experimental results. Further calculations are being made on a series of cropped delta and swept wings which are being used in an extensive programme of theoretical and experimental work. The results of these calculations are to be given in a separate Report. In view of this, the reader is referred to these separate sources for examples of the results which have been obtained, and of the good agreement which has been obtained both with experiment and with other theory.

6 CONCLUSIONS

Mercury Autocode programmes have been developed which calculate generalised forces on wings oscillating in supersonic flow. A fairly general specification of planform and of leading and trailing edge conditions is allowed for by means of separate programmes.

Appendix A

DATA INPUT AND OUTPUT FOR THE PROGRAMMES

(see section 2)

INDEX

For all programmes described in this Appendix, standard data tapes are available containing numerical data necessary for the calculations. This is as follows:

(a) For programmes 178A, 179A, 180A, 181A, 182A, 258A etc., tapes containing n, G and q, P.

(b) For programmes 178A, 179A, 181A etc., tapes containing m, H, $\bar{\xi}_1 \ldots \bar{\xi}_m$.

(c) For programme 180A etc., tapes containing m, H.

(d) For programme 182A etc., tapes containing m, H, $\bar{\xi}_1 \ldots \bar{\xi}_m$ (for subsonic leading edge, supersonic trailing edge), H (for subsonic leading edge, subsonic trailing edge).

(e) For programme 258A etc., tapes containing m, H, $\bar{\xi}_1 \ldots \bar{\xi}_m$ (for subsonic leading edge, supersonic trailing edge), H, $\bar{\xi}_1 \ldots \bar{\xi}_m$ (for supersonic leading edge, supersonic trailing edge).

A.1 Input for programmes 178A, 180A, 181A and 182A

These programmes all deal with the 'segmented' planform of Fig.1, for different leading and trailing edge conditions. Data are provided as follows:

RAE 178A Multhopp-Richardson: segmented planform, subsonic leading edge, supersonic trailing edge

Segmented planform data (see below)

	N_1	number of variations of Mach number, m and n
First set of values of (M,m,n)	M_1	Mach number
	m_1	number of chordwise lift and downwash stations
	H_1	$m_1 \times m_1$ matrix of coefficients in chordwise interpolation functions relevant to the leading and trailing edge conditions, defined in Appendix B following (22)
	$\bar{\xi}_1, \ldots, \bar{\xi}_{m_1}$	the m_1 chordwise downwash points relevant to the leading and trailing edge conditions
	n_1	number of spanwise lift and downwash stations
	G_1	$n_1 \times n_1$ matrix of coefficients in spanwise interpolation functions, defined in Appendix B following (22)
	N_2	number of variations of p and q
First set of values of (p,q)	p_1	number of chordwise integration stations
	q_1	number of spanwise integration stations
	P_1	$q_1 \times q_1$ matrix of coefficients in spanwise integration functions, defined in Appendix B following (14)
	N_3	number of variations of frequency parameter
	$\nu_1, \ldots, \nu_{N_3}$	N_3 values of frequency parameter

Second set of values of (p,q)	p_2, q_2, P_2	Defined above
	N_3'	
	$\nu_1 \ldots \nu_{N_3'}$	
	⋮	
Final set of values of (p,q)	$p_{N_2}, q_{N_2}, P_{N_2}$	Defined above
	N_3''	
	$\nu_1 \ldots \nu_{N_3''}$	
Second set of values of (M,m,n)	$M_2 m_2 H_2 (\bar{\xi}_1 \ldots \bar{\xi}_{m_2}) n_2 G_2$	Defined above
	etc. etc.	

<u>RAE 180A</u> <u>Multhopp-Richardson: segmented planform, subsonic edges</u>

Segmented planform data (see below)

N_1

$M_1 m_1 H_1 n_1 G_1$

N_2

$p_1 q_1 P_1$

N_3

$\nu_1 \ldots \nu_{N_3}$

etc. etc.

As defined for 178A (note that $\bar{\xi}_1 \ldots \bar{\xi}_m$ are not required)

<u>RAE 181A</u> <u>Multhopp-Richardson: segmented planform, supersonic edges</u>

Segmented planform data (see below)

N_1

$M_1 m_1 H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1}) n_1 G_1$

N_2

p_1

as defined for 178A

$\zeta_1 \ldots \zeta_{p_1}$ — p_1 Gauss zeros for interval (-1,1) relevant to a singularity $1/\sqrt{(1-\zeta)}$

$w_1, \ldots, w_{p_1}$ — p_1 weights, related to Gauss weights by $w_i = W_i/k(\zeta_i)$, with W_i and $k(\zeta_i)$ defined in Appendix B, equation (12) and adjacent text.

$$\left.\begin{array}{l} q_1 P_1 \\ N_3 \\ \nu_1 \ldots \nu_{N_3} \\ \text{etc. etc.} \end{array}\right\} \text{ as defined for 178A}$$

RAE 182A Multhopp-Richardson: segmented planform, subsonic leading edge, mixed trailing edge

Segmented planform data (see below); note that number of segments can only be 2 or 3

$$\left.\begin{array}{l} N_1 \\ M_1 m_1 \end{array}\right\} \text{ as defined for 178A}$$

$H_1(\bar{\xi}_1 \ldots \bar{\xi}_{m_1})$ — relevant to subsonic leading edge, supersonic trailing edge as defined in 178A

$\bar{H}_1$ — relevant to subsonic leading edge, subsonic trailing edge; compare with 180A

$$\left.\begin{array}{l} n_1 G_1 \\ N_2 \\ p_1 q_1 P_1 \\ N_3 \\ \nu_1 \ldots \nu_{N_3} \\ \text{etc. etc.} \end{array}\right\} \text{ as defined for 178A}$$

Segmented planform data

For the above four programmes, basic planform data relevant to the 'segmented' planform of Fig.1 must be provided. This consists of

d — root chord/mean chord

n — number of segments making up half wing ($n = 1,2,3$)

y_1	first segment $0 \leqslant y \leqslant y_1$ (y_1 non-dimensional and based on mean chord)
$\tan \Lambda_1$	Λ_1 = leading edge sweep
$\tan \phi_1$	ϕ_1 = trailing edge sweep
y_2 $\tan \Lambda_2$ $\tan \phi_2$	as above for the second segment $y_1 \leqslant y \leqslant y_2$, if applicable
y_3 $\tan \Lambda_3$ $\tan \phi_3$	as above for third segment $y_2 \leqslant y \leqslant y_3$ if applicable

A.2 Input for programmes 179A and 258A

These programmes deal with planforms having curved leading edges, as shown in Fig. 3. Data is provided as follows.

RAE 179A Multhopp-Richardson, curved subsonic leading edge, straight supersonic trailing edge

Curved planform data (see below)

N_1 $M_1 m_1 H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1}) n_1 G_1$ N_2 $p_1 q_1 P_1$ N_3 $\nu_1 \ldots \nu_{N_3}$ etc. etc.	as defined for 178A above

RAE 258A Multhopp-Richardson, curved mixed leading edge, straight supersonic trailing edge

Curved planform data (see below)

N_1 $M_1 m_1 H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1})$	as in 178A above
$H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1})$	as in 181A above

$$\left.\begin{array}{l} n_1 G_1 \\ N_2 \\ p_1 q_1 P_1 \\ N_3 \\ \nu_1 \ldots \nu_{N_3} \\ \text{etc. etc.} \end{array}\right\} \text{ as in 178A above}$$

Curved planform data

The planform, of the type shown in Fig.3, has its leading edge specified for $Y \geq 0$ by

$$Y = S(a_1 X + a_2 X^2 + \ldots\ldots + a_N x^N) , \qquad (N \leq 14)$$

X and Y here being based on the root chord, and the leading edge for $Y < 0$ defined by symmetry. The equation of the trailing edge is

$$X - 1 = Y \tan \Lambda .$$

The 'curved planform' data are then as follows:

N	degree of polynomial
$a_1 \ldots a_N$	coefficients
$\tan \Lambda$	tangent of trailing edge sweepback
S	semi-span/root chord

A.3 Input for programmes 178A/1, 179A/1, 180A/1, 181A/1, 182A/1 and 258A/1

These are the versions of 178A, 179A, 180A, 181A, 182A and 258A which deal with zero frequency parameter. The input data are the same as the input data specified for the corresponding programmes in A.1 and A.2 above, except that the parameters N_3 and $\nu_1 \ldots \nu_{N_3}$ are omitted.

A.4 Input for programmes 178A/2, 180A/2, 181A/2, 182A/2

These programmes, namely

RAE 178A/2 Multhopp-Richardson; subsonic leading edge, supersonic trailing edge
RAE 180A/2 Multhopp-Richardson; subsonic edges
RAE 181A/2 Multhopp-Richardson; supersonic edges
RAE 182A/2 Multhopp-Richardson; subsonic leading edge, mixed trailing edge

are the versions of 178A, 180A, 181A and 182A which take account of any planform, providing certain planform data are provided. The input data for these programmes are as follows.

RAE 178A/2 Multhopp-Richardson: subsonic leading edge, supersonic trailing edge

$m_1 H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1})$, $n_1 G_1$, $p_1 q_1 P_1$	defined in A.1 above for 178A
M_1	Mach number
ν	frequency parameter
Planform data	(see below)

RAE 180A/2 Multhopp-Richardson: subsonic edges

$m_1 H_1$, $n_1 G_1$, $p_1 q_1 P_1$	defined in A.1 above for 180A
M_1	Mach number
ν	frequency parameter
Planform data	(see below)

RAE 181A/2 Multhopp-Richardson: supersonic edges

$m_1 H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1})$, $n_1 G_1$, $p_1 (\zeta_1 \ldots \zeta_{p_1}) (w_1 \ldots w_{p_1})$, $q_1 P_1$	defined in A.1 above for 181A
M_1	Mach number
ν	frequency parameter
Planform data	(see below)

RAE 182A/2 Multhopp-Richardson: subsonic leading edge, mixed trailing edge

$m_1 H_1 (\bar{\xi}_1 \ldots \bar{\xi}_{m_1}) \bar{H}_1$, $n_1 G_1$, $p_1 q_1 P_1$	defined in A.1 above for 182A

M_1	Mach number
ν	frequency parameter
$\bar{\eta}$	value of η at which trailing edge discontinuity occurs
Planform data	(see below)

Planform data

For the above four programmes, certain planform data must be provided, consisting of:

d	root chord/mean chord
s	semi-span/mean chord
$x^{(1)}_{lift}$ $x^{(1)}_{downwash}$ $y^{(1)}$ . . . . $x^{(r)}_{lift}$ $x^{(r)}_{downwash}$ $y^{(r)}$	$3r = 3m\ [\frac{1}{2}n + \frac{1}{2}]$ values of co-ordinates x and y (referred to mean chord as reference length) at the r lift and downwash points, starting at point furthest upstream on centre line section, and ending at point furthest downstream on extreme starboard section
$c_1 \ldots c_t$	$t = [\frac{1}{2}n + \frac{1}{2}]$ values of (chord/mean chord), starting at centre line section (or starboard section nearest it)
$\eta_1^{(1)} \ldots \eta_1^{(r)}$ $\eta_2^{(1)} \ldots \eta_2^{(r)}$	r values of η_1 and η_2, as defined in Appendix B, following (9), r being defined above
ξ	$r \times q$ matrix of values of ξ at intersections of spanwise integration stations with reversed Mach lines through downwash points
$\bar{X}$	$rp \times q$ matrix of values of $\bar{X}$ $= x_{downwash} - x_{integration}$, for each of the s downwash points there being a $p \times q$ matrix of $\bar{X}$'s.

It should be noted that the above programmes do not allow for repeats for variations of any of the quantities for which repeats are allowed in 178A, 180A, 181A, 182A, namely (M_1, m_1, n_1), (p_1, q_1) and ν.

A.5 Output from 178A, 179A, 180A, 181A, 182A, 258A, 178A/2, 180A/2, 181A/2 and 182A/2

The output from these programmes is:

s	= semi-span/mean chord
d	= root chord/mean chord
N_1	number of variations of Mach number, m and n (not in 178A/2 etc.)
M_1	Mach number
m_1	number of chordwise lift and downwash stations
n_1	number of spanwise lift and downwash stations
$x^{(1)}_{lift}$ $x^{(1)}_{downwash}$ $y^{(1)}$ ... $x^{(t)}_{lift}$ $x^{(t)}_{downwash}$ $y^{(t)}$	$3t = 3m\left[\frac{1}{2}n + \frac{1}{2}\right]$ values of co-ordinates x and y, taken in order described in A.4 under 'planform data', and referred to mean chord as reference length
N_2	number of variations of p and q (not in 178A/2 etc.)
p_1	number of chordwise integration stations
q_1	number of spanwise integration stations
N_3	number of variations of frequency parameter (not in 178A/2 etc.)
ν_1	first value of frequency parameter
$16\pi s^2 A_1$, $-16\pi s^2 B_1$	where the matrices A and B are defined in Appendix B, equation (19)
$16\pi s^2 L_1$, $-16\pi s^2 M_1$	where the matrices L and M are defined in Appendix B, equation (19)
ν_2	second value of frequency parameter
etc. etc.	

A.6 Data output from 178A/1, 179A/1, 180A/1, 181A/1, 182A/1, 258A/1

Output from these programmes is the same as that output from RAE 178A etc. described in section A.5 above, except that the values of N_3, ν, B and M (all of which are zero in the steady case to which these programmes apply) do not appear.

A.7 Data input for 183A, 183A/1 and 183A/2

The programme evaluates generalised forces for modes Z_i, based on mean chord, which are expressible as polynomials in the chordwise and spanwise co-ordinates. Taking η (which is ±1 at the port and starboard tips) as the spanwise co-ordinate, the polynomials are defined as follows

(i) Symmetric distortions

(ia) Purely chordwise distortions are defined by polynomials

$$a_0 + a_1 x + \dots + a_r x^r \qquad (r \leqslant 19).$$

(ib) Purely spanwise distortions are defined by polynomials

$$b_0 + b_1 \eta^2 + \dots + b_s \eta^{2s} \qquad (s \leqslant 19).$$

(ic) Distortions having both a spanwise and chordwise element are defined by polynomials

$$\begin{aligned} & c_{00} + c_{01} x + \dots + c_{0t} x^t \\ & + c_{10} \eta^2 + c_{11} \eta^2 x + \dots + c_{1t} \eta^2 x^t \\ & \quad \vdots \\ & + c_{t0} \eta^{2t} + c_{t1} \eta^{2t} x + \dots + c_{tt} \eta^{2t} x^t \qquad (t \leqslant 6). \end{aligned}$$

(ii) Antisymmetric distortions

(iia) Purely spanwise distortions are defined by polynomials

$$d_0 \eta + \dots + d_u \eta^{2u+1} \qquad (u \leqslant 19).$$

(iib) Distortions having both a chordwise and spanwise component are defined by polynomials

$$\begin{aligned}
&e_{00}\,\eta + e_{01}\,\eta\,x + \dots + e_{0v}\,\eta\,x^v \\
&+e_{10}\,\eta^3 + e_{11}\,\eta^3\,x + \dots + e_{1v}\,\eta^3\,x^v \\
&\quad\vdots \\
&+e_{v0}\,\eta^{2v+1} + e_{v1}\,\eta^{2v+1}x + \dots + e_{vv}\,\eta^{2v+1}\,x^v \qquad (v \leqslant 6).
\end{aligned}$$

The input for the programme is then as follows.

<u>RAE 183A</u> <u>Generalised forces for polynomial modes (and RAE 183A/1, and RAE 183A/2)</u>

n	Indicator: $n = 0$ if symmetric modes only 1 if antisymmetric only 2 if both symmetric and antisymmetric
r_1	number of symmetric chordwise polynomials (see (ia) above) (do not punch if $n = 1$)
r	maximum degree of these (if $r_1 \neq 0$ and $n \neq 1$)
$a_0^{(1)} \dots a_r^{(1)}$ $\vdots$ $a_0^{(r_1)} \dots a_r^{(r_1)}$	$r_1\ (r + 1)$ coefficients
s_1	number of symmetric spanwise polynomials (see (ib) above) (do not punch if $n = 1$)
$2s$	maximum degree of these (if $s_1 \neq 0$ and $n \neq 1$)
$b_0^{(1)} \dots b_s^{(1)}$ $\vdots$ $b_0^{(s_1)} \dots b_s^{(s_1)}$	$s_1\ (s + 1)$ coefficients
t_1	number of symmetric double polynomials (see (ic) above) (do not punch if $n = 1$)
t	maximum degree of these (if $t_1 \neq 0$ and $n \neq 1$)

$$\left.\begin{matrix} c_{00}^{(1)} \cdots c_{tt}^{(1)} \\ \vdots \\ c_{00}^{(t_1)} \cdots c_{tt}^{(t_1)} \end{matrix}\right\}$$ $t_1\ (t+1)^2$ coefficients

u_1 number of antisymmetric spanwise polynomials (see (iia)) (do not punch if n = 0)

$2u + 1$ maximum degree of these (if $u_1 \neq 0$ and $n \neq 0$)

$$\left.\begin{matrix} d_0^{(1)} \cdots d_u^{(1)} \\ \vdots \\ d_0^{(u_1)} \cdots d_u^{(u_1)} \end{matrix}\right\}$$ $u_1\ (u+1)^2$ coefficients

v_1 number of antisymmetric double polynomials (see (iib)) (do not punch if n = 0)

v maximum degree of these (if $v_1 \neq 0$ and $n \neq 0$)

$$\left.\begin{matrix} e_{00}^{(1)} \cdots e_{vv}^{(1)} \\ \vdots \\ e_{00}^{(v_1)} \cdots e_{vv}^{(v_1)} \end{matrix}\right\}$$ $v_1\ (v+1)^2$ coefficients

followed by data output from the appropriate programme, namely

<u>for 183A</u> the output from 178A, 179A, 180A, 181A, 182A and 258A described in A.5 above

<u>for 183A/1</u> the output from 178A/1, 179A/1, 180A/1, 181A/1, 182A/1 and 258A/1 described in A.6 above

<u>for 183A/2</u> the output from 178A/2, 180A/2, 181A/2, 182A/2 described in A.5 above.

A.8 <u>Data output from programmes 183A, 183A/1 and 183A/2</u>

The data output from 183A is as follows

<u>RAE 183A</u> <u>Generalised forces for polynomial modes</u>

M_1 Mach number

m_1 number of chordwise lift and downwash stations

n_1 number of spanwise lift and downwash stations

p_1 number of chordwise integration stations

q_1	number of spanwise integration stations	
ν_1	first value of frequency parameter	
E_1, F_1	two $(r_1 + s_1 + t_1)$ square matrices of symmetric generalised forces (E real, F imaginary)	r_1, s_1, t_1, u_1 and v_1 are as defined in input for 183A in section A.7; $-2s(E_1 + i\nu F_1) = Q$ symmetric; $-2s(G_1 + i\nu H_1) = Q$ antisymmetric, where Q is defined in equation (20) of Appendix B
G_1, H_1	two $(u_1 + v_1)$ square matrices of antisymmetric generalised forces	

ν_2 etc. etc.

Output from RAE 183A/1 differs only in that

(i) the frequency parameter is not punched, being zero, and there are no repeats for different frequency parameters,

(ii) F_1 and H_1 are not printed since $\nu = 0$.

Output from RAE 183A/2 differs in that there are no repeats of (M,m,n), (p,q) or ν.

Appendix B

SUMMARY OF METHOD

(see section 3)

The detail of the method is given in a separate report[2]. A summary, together with the main results, is given here for completeness and to enable the main steps in the programmes to be followed.

Wing co-ordinates ξ and η are chosen so that

$$\left.\begin{aligned} x - x_{\text{mid chord}} &= \tfrac{1}{2}\xi c(\eta) \\ s\eta &= y \end{aligned}\right\} \tag{4}$$

where $\bar{c}\ c(\eta)$ = local chord, $s\bar{c}$ = semi-span and (x,y) are co-ordinates based on the wing mean chord $\bar{c}$ as reference length. Two sets of mn points are taken over the wing; one of points (ξ_α, η_β) at which the lift is evaluated and one of points $(\bar{\xi}_r, \eta_s)$ at which downwash values are taken. According to local leading and trailing edge conditions, a function

$$f(\xi) = \left\{\begin{array}{ll} \sqrt{\left(\dfrac{1-\xi}{1+\xi}\right)} & \text{(subsonic leading edge, subsonic trailing edge} \\ \dfrac{1}{\sqrt{(1+\xi)}} & \text{(subsonic leading edge, supersonic trailing edge)} \\ \sqrt{(1-\xi)} & \text{(supersonic leading edge, subsonic trailing edge)} \\ 1 & \text{(supersonic leading and trailing edges)} \end{array}\right\} \tag{5}$$

is defined which takes into account the singularity which occurs in the chordwise lift distribution. Then the m points $\xi_\alpha (\alpha = 1, \ldots, m)$ are chosen to be the zeros of the m'th degree polynomial of the set orthogonal with respect to the weight function $f(\xi)$ over $(-1,1)$; the $\bar{\xi}_r$ are similarly chosen with weight function $f(-\xi)$. The n points η_s are chosen to be the zeros of the n'th degree polynomial of the set orthogonal with respect to the weight function $\sqrt{(1-\eta^2)}$ over $(-1,1)$. Lift and downwash points for $m = 2$, $n = 7$ are shown for a delta wing with subsonic leading edge in Fig.6.

These points being chosen, interpolation functions $h_\alpha(\xi)$ and $g_\beta(\eta)$ are defined, having the form

$$h_\alpha(\xi) = (\text{polynomial of degree } (m-1) \text{ in } \xi)\ f(\xi)$$

and

$$g_\beta(\eta) = (\text{polynomial of degree } (n-1) \text{ in } \eta)\ \sqrt{(1-\eta^2)}$$

with the properties

$$\left.\begin{aligned} h_\alpha(\xi_\gamma) &= \delta_{\alpha\gamma} \\ g_\beta(\eta_\gamma) &= \delta_{\beta\gamma} \end{aligned}\right\} , \tag{6}$$

$\delta_{\beta\gamma}$ being the Kronecker delta. We also define

$$H_\alpha = \tfrac{1}{2}\int_{-1}^{1} h_\alpha(\xi)d\xi, \qquad G_\beta = \tfrac{1}{2}\int_{-1}^{1} g_\beta(\eta)d\eta . \tag{7}$$

The integral equation connecting reduced lift $\ell(\xi,\eta)$ and downwash $w(\xi,\eta)$ on a wing oscillating harmonically with frequency parameter ν in an airstream of speed V is

$$w(\xi',\eta') = \frac{-1}{8\pi s}\int_{\eta_1(\xi',\eta')}^{\eta_2(\xi',\eta')} \frac{c(\eta)d\eta}{(\eta'-\eta)^2}\int_{-1}^{\xi_M(\xi',\eta',\eta)} K(x'-x,\ y'-y)\ \ell(\xi,\eta)d\eta \tag{8}$$

where

$$K(X,Y) = \begin{cases} \dfrac{2X}{R}\exp\left\{\dfrac{-i\nu M^2 X}{\beta^2}\right\}\cos\left\{\dfrac{MR\nu}{\beta^2}\right\} + i\nu|Y|e^{-i\nu X}\displaystyle\int_{(X-MR)/(\beta^2|Y|)}^{(X+MR)/(\beta^2|Y|)} \frac{\tau}{\sqrt{1+\tau^2}}\, e^{-i\nu\tau|Y|}\, d\tau & \\ & \text{if } X > \beta|Y| \text{ and } Y \neq 0 \\ 2e^{-i\nu X} & \text{if } X > \beta|Y| \text{ and } Y = 0 \\ 0 & \text{if } X \leqslant \beta|Y| \end{cases} \quad \ldots\ldots (9)$$

and $R = \sqrt{(X^2 - \beta^2Y^2)}$, $\beta^2 = M^2 - 1$. $K(X,Y)$ takes the particularly simple form $2X/R$ if $\nu = 0$. Further, $\xi_M(\xi',\eta',\eta) = \min\{\xi_1(\xi',\eta',\eta),1\}$ and $\xi = \xi_1(\xi',\eta',\eta)$ is the equation of the reversed Mach lines through the point (x',y'); these cut the wing leading edges (or tips) where $\eta = \eta_1(\xi',\eta')$ and $\eta = \eta_2(\xi',\eta')$. The region of integration in (8) is shown in Fig. 5. If the lift is approximated by $\bar{\ell}(\xi,\eta)$, where

$$c(\eta)\bar{\ell}(\xi,\eta) = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} P_{\alpha\beta}\frac{h_\alpha(\xi)g_\beta(\eta)}{H_\alpha G_\beta} \tag{10}$$

and

$$P_{\alpha\beta} = H_\alpha G_\beta c(\eta_\beta)\ell(\xi_\alpha,\eta_\beta) \tag{11}$$

the integrations in (8) may be performed approximately.

To perform the chordwise integration a variable ζ such that $\zeta = -1$ at the leading edge and $\zeta = 1$ at $\xi = \xi_M(\xi',\eta',\eta)$ is taken. The singularity in the chordwise integration of (8) can be represented by a function

$$k(\zeta) = \sqrt{\left(\frac{1-\zeta}{1+\zeta}\right)}, \qquad \frac{1}{\sqrt{(1-\zeta^2)}}, \qquad \frac{1}{\sqrt{(1-\zeta)}} \tag{12}$$

which depends on the singularity in the integrand arising from the singularity in K at the Mach line and the singularity in $h_\alpha(\xi)$ at the leading edge (and trailing edge, if relevant). Other possible forms of $k(\zeta)$, which do not arise in the programmes considered here, are given in Refs.1 and 2. The chordwise integration of (8) can then be carried out by a p-point Gauss-type integration formula, the integrand being evaluated at p points $\zeta_\lambda(\lambda = 1, \ldots, p)$ and multiplied by weights $W_\lambda/k(\zeta_\lambda)$ (W_λ being the relevant Gauss weight).

For the spanwise part of the integration in (8) a variable ϕ, such that $\phi = (-1,1)$ corresponds to $\eta = (\eta_1, \eta_2)$, is defined; q points $\phi_\gamma(\gamma = 1, \ldots, q)$ in $(-1,1)$ are chosen to be the q zeros of the qth degree Chebyshev polynomial. Interpolation polynomials $p_\gamma(\phi)$ $(\gamma = 1, \ldots, q)$ of degree $(q - 1)$ are defined so that

$$p_\gamma(\phi_\alpha) = \delta_{\gamma\alpha} \quad . \tag{13}$$

Then a suitable integration formula for any function $U(\eta)$ is

$$\int_{\eta_1(\xi',\eta')}^{\eta_2(\xi',\eta')} \frac{U(\eta)d\eta}{(\eta - \eta')^2} = \frac{2}{\eta_2 - \eta_1} \left(U(\bar{\eta}_1) \ldots\ldots U(\bar{\eta}_q)\right) P \begin{pmatrix} \displaystyle\int_{-1}^{1} \frac{d\phi}{(\phi - \phi')^2} \\ \vdots \\ \displaystyle\int_{-1}^{1} \frac{\phi^{q-1}\, d\phi}{(\phi - \phi')^2} \end{pmatrix} \tag{14}$$

where $2\,\bar{\eta}_\gamma = (\eta_2 - \eta_1)\phi_\gamma + \eta_1 + \eta_2$, and P is the $(q \times q)$ matrix of coefficients of the polynomials $p_1(\phi), \ldots, p_q(\phi)$. The integrals on the right of (14) are easy to evaluate exactly.

Carrying out the chordwise and spanwise integrations of (8) gives the equation

$$-8\pi s\; w(\bar{\xi}_r, \eta_s) = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} P_{\alpha\beta}\; C_{\alpha\beta}(\bar{\xi}_r, \eta_s) \tag{15}$$

for the downwash at any point $(\bar{\xi}_r, \eta_s)$. Here

$$C_{\alpha\beta}(\bar{\xi}_r,\eta_s)$$

$$= \frac{1}{G_\beta(\eta_2(\bar{\xi}_r,\eta_s)-\eta_1(\bar{\xi}_r,\eta_s))} \sum_{\gamma=1}^{q} \sum_{\lambda=1}^{p} \frac{g_\beta(\eta_{rs,\gamma})W_\lambda\{1+\xi_M(\bar{\xi}_r,\eta_s,\eta_{rs,\gamma})\}h_\alpha(\xi_{rs,\lambda\gamma})}{H_\alpha\; k(\zeta_\lambda)} \times$$

$$\times\; K(x_{rs}-x_{rs,\lambda\gamma},\; y_s-y_{rs,\gamma}) \int_{-1}^{1} \frac{p_\gamma(\phi)d\phi}{(\phi-\phi_{rs})^2}$$

$$+\; \delta_{\beta s} \frac{L_\alpha(\bar{\xi}_r,\eta_s)}{G_s\sqrt{(1-\eta_s^2)}} \left\{ \int_{\eta_1(\bar{\xi}_r,\eta_s)}^{\eta_2(\bar{\xi}_r,\eta_s)} \sqrt{(1-\eta^2)}\,\log|\eta-\eta_s|d\eta - \frac{2}{\eta_2(\bar{\xi}_r,\eta_s)-\eta_1(\bar{\xi}_r,\eta_s)} \sum_{\gamma=1}^{q} \sqrt{\left(1-\eta_{rs,\gamma}^2\right)}\times \right.$$

$$\left. \times\; (\eta-\eta_{rs,\gamma})^2 \log|\eta-\eta_{rs,\gamma}| \int_{-1}^{1} \frac{p_\gamma(\phi)d\phi}{(\phi-\phi_{rs})^2} \right\} \qquad (16)$$

and

$$L_\alpha(\bar{\xi}_r,\eta_s) = \frac{4s^2}{H_\alpha\{c(\eta_s)\}^2} \left\{ \tfrac{1}{2}\, i\, (M^2+1)\nu c(\eta_s)\, h_\alpha(\bar{\xi}_r) + (M^2-1)\, h'_\alpha(\bar{\xi}_r) \right.$$

$$\left. +\; \tfrac{1}{4}\,\nu^2\{c(\eta_s)\}^2 \int_{-1}^{\bar{\xi}_r} \exp\{-\tfrac{1}{2}\, i\, \nu\, c(\eta_s)(\bar{\xi}_r-\xi)\}h_\alpha(\xi)d\xi \right\} . \qquad \ldots. (17)$$

The immediate substitution of (10) and (11) in (8), and the approximate evaluation of the chordwise and spanwise integrals is not the only step taken to reach equation (15). Allowance is made for a logarithmic singularity which arises from the chordwise integration, and this gives rise to those terms of (16) which involve $L_\alpha(\bar{\xi}_r,\eta_s)$. It will be noted that the right hand side of (16) entails the evaluation of the function K(X,Y), as well as some of the other quantities, at a set of pq integration points. These points are shown for a particular case with p = 3 and q = 7 in Fig.7.

Since any problem may be considered as the sum of a symmetric and an antisymmetric problem, the downwash need only be evaluated at points on the starboard half wing. The set of equations which is given by taking (15) for all the downwash points may be written as a matrix equation

$$-8\pi s\,W \;=\; CP \;=\; \begin{pmatrix} C_{o-} & C_{oo} & C_{o+} \\ C_{+-} & C_{+o} & C_{++} \end{pmatrix} P \qquad (18)$$

when n is odd (n can be even, in which case the only matrices appearing in (18) are C_{+-} and C_{++}). W is a column of $\frac{1}{2}m(n + 1)$ downwash values, taken in order from upstream on the centre line to downstream at the extreme starboard station, P a column of mn $P_{\alpha\beta}$'s taken in order from upstream at the extreme port to downstream at the extreme starboard section, C_{+-} and C_{++} are $\frac{1}{2}m(n - 1)$ square matrices, C_{o-}, C_{o+} and C_{+o}' are $(m \times \frac{1}{2}m(n - 1))$ matrices, and C_{oo} is $(m \times m)$.

Write

$$(A + iB)^{-1} \;=\; C \begin{pmatrix} \bigcirc & & I \\ I & \ddots & \bigcirc \\ \bigcirc & & I \end{pmatrix} \;=\; CJ$$

and

$$(L + iM)^{-1} \;=\; (C_{+-}\ \ C_{+o}\ \ C_{++}) \begin{pmatrix} \bigcirc & \cdots & -I \\ -I & & \\ 0 & & \bigcirc \\ I & & \\ \bigcirc & \cdots & I \end{pmatrix} \;=\; \bar{\bar{C}}K \qquad (19)$$

the unit matrices being $(m \times m)$. Then the $(j \times j)$ matrices Q of generalised force coefficients Q_{ij} corresponding to deflection shapes $Z_1(x,y), \ldots, Z_j(x,y)$ are given by

$$\frac{1}{32\pi s^2}\,Q \;=\; \bar{Z}A\underline{Z}_x' - \nu\bar{Z}B\underline{Z}' + i(\bar{Z}B\underline{Z}_x' + \nu\bar{Z}A\underline{Z}') \quad \text{(symmetric)}$$

$$\frac{1}{32\pi s^2}\,Q \;=\; \bar{\bar{Z}}L\underline{\underline{Z}}_x' - \nu\bar{\bar{Z}}M\underline{\underline{Z}}' + i(\bar{\bar{Z}}M\underline{\underline{Z}}_x' + \nu\bar{\bar{Z}}L\underline{\underline{Z}}') \quad \text{(antisymmetric)} \qquad (20)$$

Here $\rho V_0^2 \ell^3 Q_{ij}$ is the generalised force in mode i due to deflection in mode j; also

$$\bar{Z} = \begin{pmatrix} \bar{Z}_1 \\ \vdots \\ \bar{Z}_j \end{pmatrix}, \quad \bar{\bar{Z}} = \begin{pmatrix} \bar{\bar{Z}}_1 \\ \vdots \\ \bar{\bar{Z}}_j \end{pmatrix}, \quad \underline{Z} = \begin{pmatrix} \underline{Z}_1 \\ \vdots \\ \underline{Z}_j \end{pmatrix}, \quad \underline{\underline{Z}} = \begin{pmatrix} \underline{\underline{Z}}_1 \\ \vdots \\ \underline{\underline{Z}}_j \end{pmatrix}$$

and the row matrices $\bar{Z}_1$, $\bar{\bar{Z}}_1$, $\underline{Z}_1$, $\underline{\underline{Z}}_1$ and so on are defined by

$$\left.\begin{aligned}
\bar{Z}_1 &= (\tfrac{1}{2}Z_1(\xi_1,\eta_{\frac{1}{2}(n+1)}),\ldots, \tfrac{1}{2}Z_1(\xi_m,\eta_{\frac{1}{2}(n+1)}), Z_1(\xi_1,\eta_{\frac{1}{2}(n+3)}),\ldots, Z_1(\xi_m,\eta_n)) \\
\bar{\bar{Z}}_1 &= (Z_1(\xi_1,\eta_{\frac{1}{2}(n+3)}),\ldots, Z_1(\xi_m,\eta_{\frac{1}{2}(n+3)}),\ldots, Z_1(\xi_1,\eta_n),\ldots, Z_1(\xi_m,\eta_n)) \\
\underline{Z}_1 &= (Z_1(\bar{\xi}_1,\eta_{\frac{1}{2}(n+1)}),\ldots, Z_1(\bar{\xi}_m,\eta_{\frac{1}{2}(n+1)}),\ldots, Z_1(\bar{\xi}_1,\eta_n),\ldots, Z_1(\bar{\xi}_m,\eta_n)) \\
\underline{\underline{Z}}_1 &= (Z_1(\bar{\xi}_1,\eta_{\frac{1}{2}(n+3)}),\ldots, Z_1(\bar{\xi}_m,\eta_{\frac{1}{2}(n+3)}),\ldots, Z_1(\bar{\xi}_1,\eta_n),\ldots, Z_1(\bar{\xi}_m,\eta_n))
\end{aligned}\right\} \quad \ldots\ldots(21)$$

$\underline{Z}_x$ and $\underline{\underline{Z}}_x$ are similarly defined in terms of row matrices $\underline{Z}_{1,x}$ etc., $\underline{Z}_{1,x}$ being given by replacing Z_1 by $\frac{\partial Z_1}{\partial x}$ in the equation defining $\underline{Z}_1$ on the right of (21).

In assembling the matrix C of (18) from terms defined in (16) a certain simplification is possible. The terms involving $\delta_{\beta s}$ in (16) being omitted for the moment, the elements of the row of the matrix C which correspond to the downwash point $(\bar{\xi}_r,\eta_s)$ are the terms of the matrix

$$HM_{rs}\, N_{rs}\, R_{rs}\, G' \tag{22}$$

taken row by row. H and G are matrices whose rows consist of the coefficients in the interpolation functions $h_\alpha(\xi)/H_\alpha$ ($\alpha = 1, \ldots, m$) and $g_\beta(\eta)/G_\beta$ ($\beta = 1, \ldots, n$) respectively. We also define

$$M_{rs} = \frac{1}{\eta_2(\bar{\xi}_r,\eta_s) - \eta_1(\bar{\xi}_r,\eta_s)} \begin{pmatrix} \left(W_1/k(\zeta_1), \ \dots \ ,W_p/k(\zeta_p)\right) \bar{K}_1^{rs} \quad ; \\ \vdots \\ \left(W_1/k(\zeta_1), \ \dots \ ,W_p/k(\zeta_p)\right) \bar{K}_m^{rs} \end{pmatrix}$$

$$R_{rs} = \begin{pmatrix} 1 \ \eta_{rs,1} \cdots \eta_{rs,1}^{n-1} \\ \vdots \\ 1 \ \eta_{rs,q} \cdots \eta_{rs,q}^{n-1} \end{pmatrix} \qquad \dots .(23)$$

where $\bar{K}_i^{rs}$ $(i = 1, \dots ,m)$ is the $(p \times q)$ matrix in which the element in the λ^{th} row and γ^{th} column is

$$K(x_{rs} - x_{\lambda\gamma}, y_s - y_{rs,\gamma}) \ f(\xi_{rs,\lambda\gamma}) \ (\xi_{rs,\lambda\gamma})^{i-1} \quad .$$

N_{rs} is the $(q \times q)$ diagonal matrix whose γ^{th} diagonal element is

$$\{1 + \xi_M(\bar{\xi}_r,\eta_s,\eta_{rs,\gamma})\} \ \sqrt{\left(1 - \eta_{rs,\gamma}^2\right)} \int_{-1}^{1} \frac{p_\gamma(\phi)d\phi}{(\phi - \phi_{rs})^2} \quad .$$

The matrix C is formed by combining all the terms from matrices such as (22), the terms arising from the logarithmic correction (i.e. those involving $\delta_{\beta s}$ in (16) being added separately.

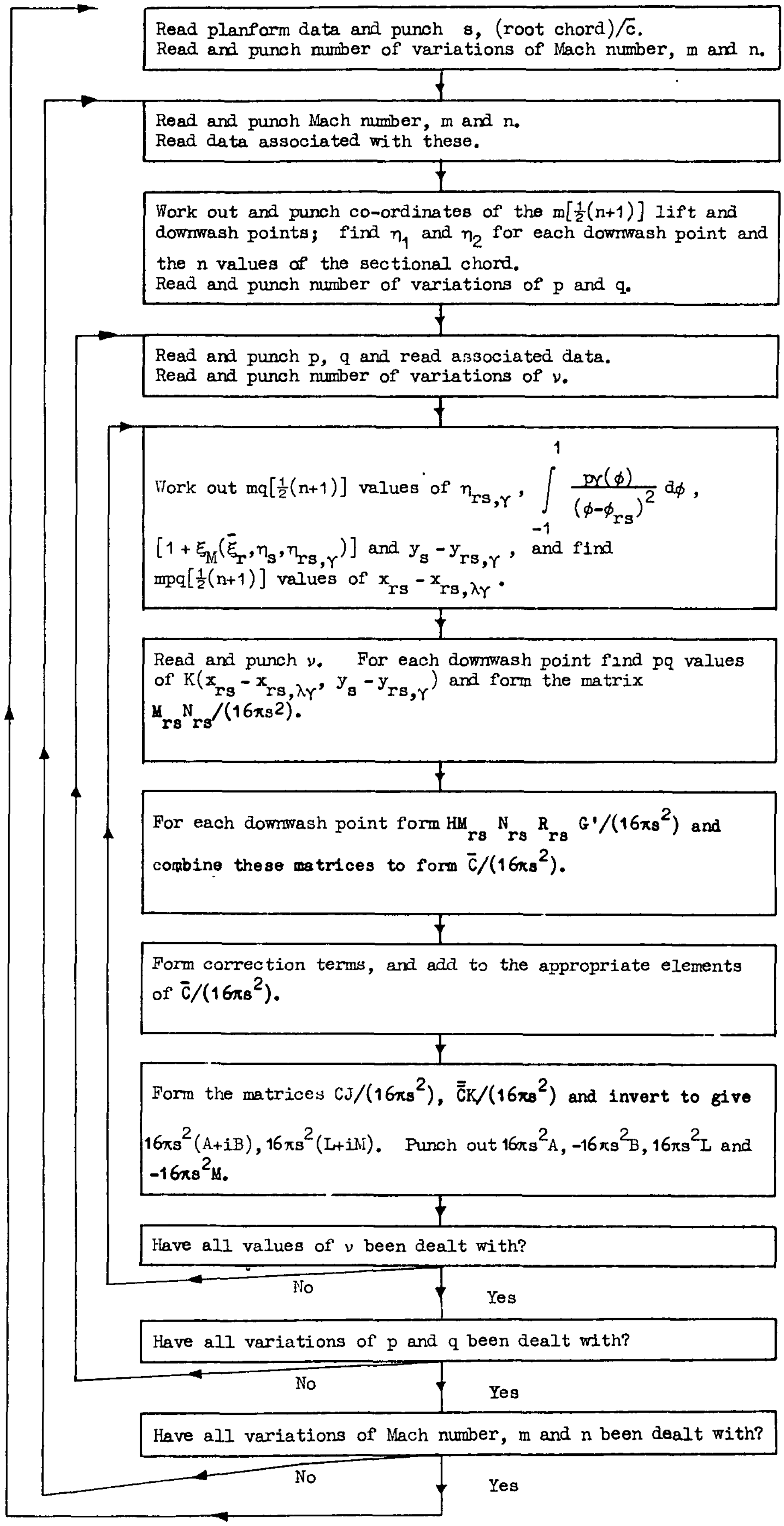
This procedure - the calculation and formation of the matrices (22) and their combining to form the matrix C of (18) - is in fact the basis of the whole computation.

Appendix C

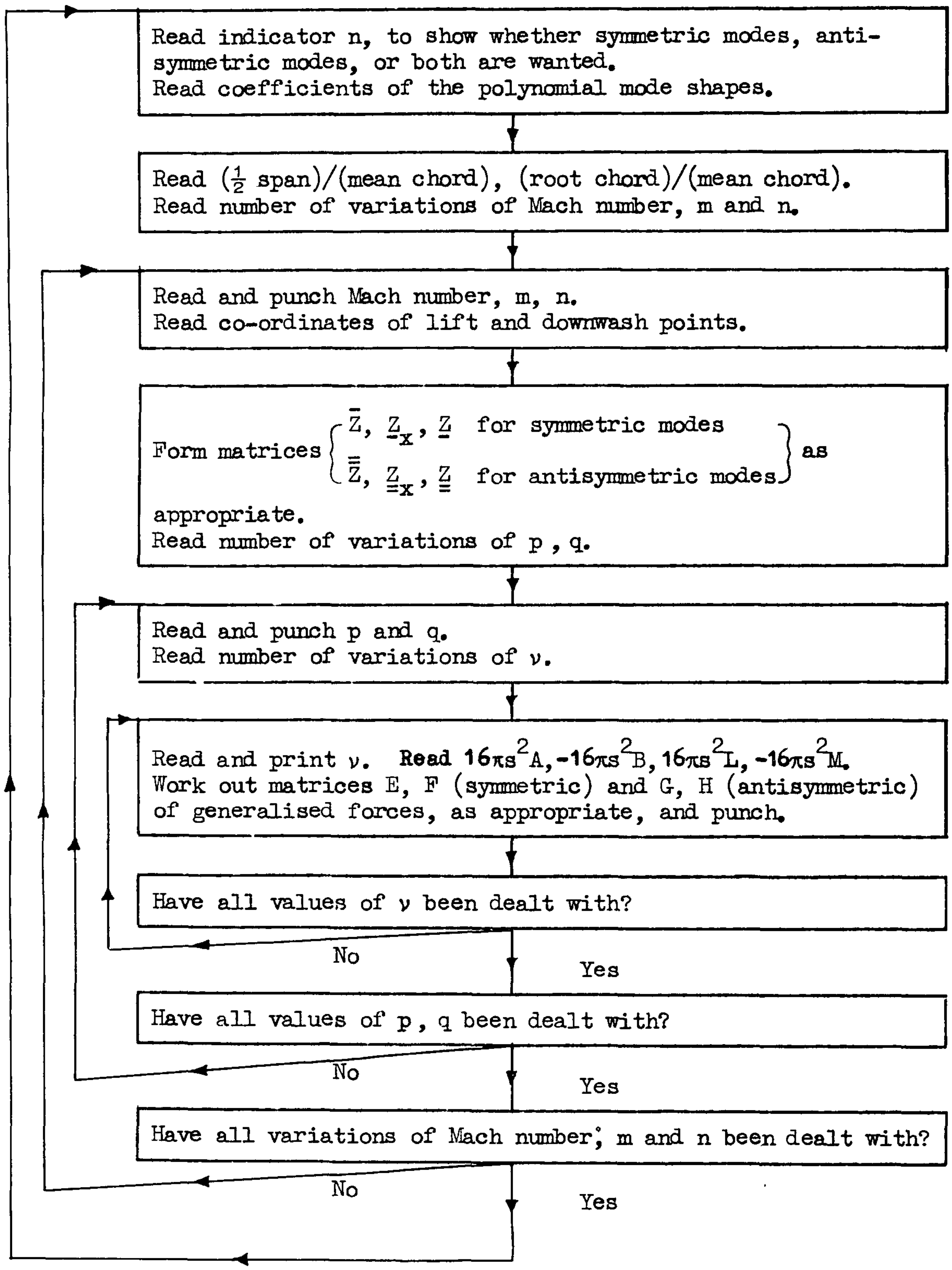
LOGICAL STRUCTURE OF THE PROGRAMMES

(see section 4.1)

C.1 Logical structure of programmes 178A, 179A, 180A, 181A, 182A and 258A



C.2 Logical structure of programme RAE 183A



Appendix D

LIFT AND DOWNWASH POINTS, INTERPOLATION FUNCTIONS AND INTEGRATION CONSTANTS

(see section 4.2)

INDEX

D.1 Chordwise interpolation functions and lift and downwash points

For the various leading and trailing edge conditions, the chordwise lift points $\xi_1, \dots, \xi_m$ are the zeros of the polynomials orthogonal over $(-1,1)$ to the function $f(\xi)$ defined in equation (5) of Appendix B; the downwash points are derived from the polynomials orthogonal to $f(-\xi)$ and so are merely the points $-\xi_m, \dots, -\xi_1$. The polynomials $h_\alpha(\xi)$ such that

$$h_\alpha(\xi_\gamma) = \delta_{\alpha\gamma}$$

can then be computed; so also can

$$H_1 = \frac{1}{2} \int_{-1}^{1} h_\alpha(\xi) d\xi$$

and hence the coefficients of the matrix H of Appendix B are given.

D.1.1 Subsonic leading and trailing edges

The m^{th} degree polynomial orthogonal to $\sqrt{(1-\xi)}/\sqrt{(1+\xi)}$ over $(-1,1)$ is

$$\frac{\sin\left[(m + \frac{1}{2})\cos^{-1} \xi\right]}{\sqrt{(1 - \xi)}}$$

and the zeros of this, which are the lift points, are

$$\xi_\alpha = \cos \frac{2\pi(m - \alpha + 1)}{2m + 1} . \qquad (\alpha = 1, \dots, m) .$$

Also (see, e.g., Ref.1)

$$H_\alpha = \frac{\pi}{2m + 1} \sqrt{\left(1 - \xi_\alpha^2\right)} .$$

For m = 2, ...,10 the ξ_α and matrix H are given below.

m = 2

$$\xi_1 = -0.809017 \qquad \xi_2 = 0.309017$$

$$H = \begin{pmatrix} 0.243167 & -0.786905 \\ 1.666692 & 2.060145 \end{pmatrix}$$

m = 3

$$\xi_1 = -0.900969 \qquad \xi_2 = -0.222521 \qquad \xi_3 = 0.623490$$

$$H = \begin{pmatrix} -0.157232 & -0.454414 & 1.133290 \\ 1.783769 & -0.881109 & -3.175408 \\ 0.919942 & 5.155242 & 4.588597 \end{pmatrix}$$

m = 4

$$\xi_1 = -0.939693 \qquad \xi_2 = -0.500000 \qquad \xi_3 = 0.173648 \qquad \xi_4 = 0.766044$$

$$H = \begin{pmatrix} -0.117643 & 0.595763 & 0.777713 & -1.768766 \\ 0.636620 & -3.819719 & 0 & 5.092958 \\ 2.808432 & 4.939387 & -5.256386 & -7.802865 \\ -0.780930 & 2.104287 & 12.118110 & 9.571631 \end{pmatrix}$$

m = 5

$$\xi_1 = -0.959493 \qquad \xi_2 = -0.654861 \qquad \xi_3 = -0.142315$$
$$\xi_4 = 0.415415 \qquad \xi_5 = 0.841254$$

$$H = \begin{pmatrix} 0.094425 & 0.468140 & -1.621007 & -1.332168 & 2.899213 \\ -0.403844 & -1.806376 & 7.604538 & 1.310550 & -8.462765 \\ 2.929401 & -3.007541 & -14.019808 & 4.771775 & 13.340713 \\ 1.289214 & 10.838716 & 10.620734 & -15.688269 & -17.137876 \\ -0.726098 & -5.219700 & 2.508502 & 26.216988 & 19.546631 \end{pmatrix}$$

m = 6

$$\xi_1 = -0.970942 \qquad \xi_2 = -0.748511 \qquad \xi_3 = -0.354605$$
$$\xi_4 = 0.120537 \qquad \xi_5 = 0.568065 \qquad \xi_6 = 0.885456$$

$$H = \begin{pmatrix} 0.079033 & -0.555593 & -1.324561 & 3.893245 & 2.312845 & -4.911107 \\ -0.301597 & 2.212510 & 4.282441 & -15.372381 & -3.590459 & 14.447903 \\ 1.019843 & -8.995055 & 0.890186 & 30.124607 & -3.365175 & -23.145040 \\ 3.953293 & 9.077668 & -19.568599 & -35.840237 & 18.924552 & 30.497072 \\ -0.992314 & 4.207053 & 31.221484 & 23.207077 & -38.532277 & -36.076724 \\ 0.698082 & -3.400104 & -20.593911 & -0.919353 & 54.808264 & 39.559729 \end{pmatrix}$$

m = 7

$\xi_1 = -0.978148 \quad \xi_2 = -0.809017 \quad \xi_3 = -0.5 \quad \xi_4 = -0.104528 \quad \xi_5 = 0.309017 \quad \xi_6 = 0.669131 \quad \xi_7 = 0.913545$

$$H = \begin{bmatrix} -0.068032 & -0.474701 & 2.118063 & 3.277142 & -8.792878 & -4.072740 & 8.517746 \\ 0.243167 & 1.644766 & -7.869053 & -9.726685 & 31.476212 & 7.781348 & -25.180970 \\ -0.636620 & -3.819719 & 22.918312 & 5.092958 & -61.115498 & 0 & 40.743665 \\ 4.075271 & -6.385021 & -36.722455 & 25.293710 & 84.042485 & -21.563349 & -54.525668 \\ 1.666692 & 18.727067 & 20.601449 & -66.667689 & -82.405796 & 53.334151 & 65.924636 \\ -0.869159 & -8.252213 & 8.527085 & 82.276276 & 53.427219 & -87.032868 & -74.442383 \\ 0.681639 & 6.199258 & -9.573401 & -65.010503 & -16.631744 & 112.66895 & 79.706637 \end{bmatrix}$$

m = 8

$\xi_1 = -0.982973 \quad \xi_2 = -0.850217 \quad \xi_3 = -0.602635 \quad \xi_4 = -0.273663 \quad \xi_5 = 0.092268 \quad \xi_6 = 0.445738 \quad \xi_7 = 0.739009 \quad \xi_8 = 0.932472$

$$H = \begin{bmatrix} -0.059757 & 0.538851 & 1.842108 & -6.654604 & -7.571887 & 19.176454 & 7.262661 & -15.037402 \\ 0.204912 & -1.880303 & -5.984904 & 23.432185 & 21.618523 & -64.770068 & -15.619731 & 44.600132 \\ -0.470875 & 4.548365 & 11.287551 & -56.400372 & -19.420442 & 122.63398 & 7.455798 & -72.644057 \\ 1.401904 & -16.337965 & 3.624898 & 98.906432 & -24.960055 & -177.95820 & 22.229501 & 98.214175 \\ 5.098906 & 14.470449 & -47.126230 & -102.83932 & 109.16937 & 204.18778 & -71.332640 & -120.43973 \\ -1.214311 & 6.990220 & 64.254785 & 47.008703 & -185.97213 & -184.07489 & 131.04515 & 138.56385 \\ 0.803279 & -5.339264 & -39.356054 & 11.007130 & 207.68139 & 126.79735 & -188.29138 & -151.96935 \\ -0.671099 & 4.649085 & 31.829680 & -19.553113 & -182.03262 & -66.364252 & 229.48166 & 160.19973 \end{bmatrix}$$

m = 9

$\xi_1 = -0.986361 \quad \xi_2 = -0.879474 \quad \xi_3 = -0.677282 \quad \xi_4 = -0.401695 \quad \xi_5 = -0.082579 \quad \xi_6 = 0.245485 \quad \xi_7 = 0.546948 \quad \xi_8 = 0.789141 \quad \xi_9 = 0.945817$

$$H = \begin{bmatrix} 0.053299 & 0.478950 & -2.617516 & -5.874064 & 18.746946 & 16.810482 & -40.920690 & -13.091236 & 26.916689 \\ -0.177698 & -1.574932 & 8.898695 & 18.313511 & -63.470833 & -47.244098 & 133.32740 & 30.363907 & -80.015830 \\ 0.377579 & 3.218297 & -19.854943 & -31.096987 & 136.53336 & 52.142795 & -246.14372 & -23.211894 & 130.93235 \\ -0.866821 & -6.510304 & 50.879902 & 12.028466 & -237.98127 & 9.938393 & 363.59467 & -17.525482 & -178.27738 \\ 5.221292 & -11.014657 & -75.468952 & 78.489565 & 302.63544 & -156.07579 & -449.12859 & 92.149563 & 220.75947 \\ 2.046485 & 28.801335 & 35.464569 & -182.97058 & -254.18525 & 339.79912 & 467.36689 & -191.75364 & -257.21981 \\ -1.023663 & -12.108217 & 18.808727 & 198.17453 & 116.64878 & -474.62921 & -409.17647 & 300.12222 & 286.66388 \\ 0.763014 & 8.597042 & -19.626412 & -146.95301 & -3.095485 & 508.82343 & 302.95111 & -397.42719 & -308.28850 \\ -0.663910 & -7.341040 & 18.794800 & 126.09703 & -26.017590 & -473.65531 & -203.35796 & 464.83576 & 321.50382 \end{bmatrix}$$

m = 10

$\xi_1 = -0.988831 \quad \xi_2 = -0.900969 \quad \xi_3 = -0.733052 \quad \xi_4 = -0.5 \quad \xi_5 = -0.222521 \quad \xi_6 = 0.074730 \quad \xi_7 = 0.365341 \quad \xi_8 = 0.623490 \quad \xi_9 = 0.826239 \quad \xi_{10} = 0.955573$

$$H = \begin{bmatrix} 0.048112 & -0.529776 & -2.350962 & 10.075443 & 16.753492 & -49.274019 & -36.386187 & 86.063908 & 23.814106 & -48.716458 \\ -0.157232 & 1.746835 & 7.495085 & -33.476051 & -50.894359 & 162.14844 & 101.78872 & -273.98262 & -58.164982 & 145.06109 \\ 0.317281 & -3.605630 & -14.118194 & 70.024410 & 82.152777 & -325.28226 & -124.83025 & 495.18401 & 55.504874 & -238.16532 \\ -0.636620 & 7.639437 & 22.918312 & -147.69579 & -61.115500 & 550.03948 & 40.743669 & -733.38598 & -0.000002 & 325.94932 \\ 1.783769 & -25.853875 & 9.160121 & 244.23783 & -98.684222 & -755.20989 & 197.36844 & 939.61370 & -112.78197 & -406.45216 \\ 6.244811 & 21.116765 & -92.114870 & -233.46442 & 372.99236 & 794.68044 & -556.69301 & -1054.6953 & 274.64945 & 477.87552 \\ -1.439751 & 10.456665 & 115.00669 & 84.432623 & -575.15399 & -606.78112 & 919.17075 & 1061.6208 & -466.09339 & -538.62394 \\ 0.919942 & -7.723950 & -67.584787 & 38.793174 & 577.38706 & 307.85578 & -1154.7741 & -910.09308 & 659.87093 & 587.34038 \\ -0.736272 & 6.471608 & 52.008938 & -54.856916 & -478.70592 & -84.604781 & 1217.0021 & 718.99975 & -826.16271 & -622.93663 \\ 0.658777 & -5.898362 & -45.699203 & 57.580408 & 429.17249 & 6.427876 & -1173.8013 & -553.78710 & 938.28761 & 644.61743 \end{bmatrix}$$

D.1.2 Subsonic leading edge, supersonic trailing edge

Here are required the polynomials orthogonal to $1/\sqrt{(1+\xi)}$ over $(-1,1)$. Mineur[4] (page 286) shows that the polynomials orthogonal to $1/\sqrt{x}$ over $(0,1)$ are the polynomials $P_{2m}(\sqrt{x})$, where $P_{2m}(u)$ is the Legendre polynomial of degree 2m over the interval $(-1,1)$ in u. Changing the interval $(0,1)$ in x to the interval $(-1,1)$ in ξ, it follows that the lift points required here are just the points

$$\xi_\alpha = 2u_\alpha^2 - 1 \qquad (\alpha = 1, \ldots, m)$$

where u_α are the positive zeros of $P_{2m}(u)$. Mineur also shows that the Gauss weights for the weight function $1/\sqrt{x}$ inthe interval $(0,1)$ are related to the weights A_α and zeros u_α for a 2m-point Gauss-Legendre formula in the interval $(-1,1)$. In our notation, Mineur's result is

$$H_\alpha = 2A_\alpha u_\alpha \qquad (\alpha = 1, \ldots, m) \quad .$$

The $h_\alpha(\xi)$ have to be evaluated numerically. The values of u_α and A_α are given by Gawlik[3].

The values of ξ_α and H for m = 2, ..., 10 are appended.

m = 2

$$\begin{aligned} \xi_1 &= -0.768826 \\ \xi_2 &= 0.483112 \end{aligned} \qquad H = \begin{pmatrix} 0.418413 & -0.866080 \\ 1.248338 & 1.623695 \end{pmatrix}$$

m = 3

$$\begin{aligned} \xi_1 &= -0.886122 \\ \xi_2 &= -0.125604 \\ \xi_3 &= 0.738999 \end{aligned} \qquad H = \begin{pmatrix} -0.113494 & -0.750005 & 1.222712 \\ 1.951982 & -0.438550 & -2.980841 \\ 0.326935 & 2.971845 & 2.937401 \end{pmatrix}$$

m = 4

$$\begin{aligned} \xi_1 &= -0.932704 \\ \xi_2 &= -0.447631 \\ \xi_3 &= 0.269355 \\ \xi_4 &= 0.844313 \end{aligned} \qquad H = \begin{pmatrix} -0.191550 & 0.510096 & 1.253233 & -1.881627 \\ 1.064075 & -4.069892 & -0.907808 & 5.016493 \\ 2.261946 & 4.799258 & -3.439512 & -6.416741 \\ -0.595113 & 0.241878 & 5.879177 & 5.291884 \end{pmatrix}$$

m = 5

$$\xi_1 = -0.955673 \qquad \xi_2 = -0.624337 \qquad \xi_3 = -0.076805$$
$$\xi_4 = 0.496669 \qquad \xi_5 = 0.896988$$

$$H = \begin{pmatrix} 0.065238 & 0.749802 & -1.477083 & -2.114775 & 3.053761 \\ -0.277555 & -3.035913 & 7.808881 & 3.065611 & -8.487790 \\ 3.192554 & -1.532975 & -13.929905 & 2.238163 & 12.010365 \\ 0.520348 & 7.572723 & 9.723486 & -9.618407 & -12.658678 \\ -0.219706 & -3.000001 & -1.012335 & 11.198712 & 9.652849 \end{pmatrix}$$

m = 6

$\xi_1 = -0.968633$ $\xi_2 = -0.729400$ $\xi_3 = -0.310115$
$\xi_4 = 0.185500$ $\xi_5 = 0.634856$ $\xi_6 = 0.926923$

$$H = \begin{pmatrix} 0.126804 & -0.437377 & -2.097758 & 3.656221 & 3.634710 & -5.135486 \\ -0.478864 & 1.813861 & 7.140408 & -15.418284 & -6.842342 & 14.603836 \\ 1.678016 & -9.466452 & -3.209597 & 30.074085 & 1.071460 & -21.757441 \\ 3.266546 & 9.714646 & -13.301344 & -33.308347 & 11.308804 & 25.335055 \\ -0.918187 & 0.772826 & 19.676748 & 20.201120 & -21.830813 & -24.372211 \\ 0.459860 & -0.615297 & -9.908939 & -5.284675 & 21.168914 & 17.822070 \end{pmatrix}$$

Appendix D

m = 7

	$\xi_1 = -0.976648$	$\xi_2 = -0.796335$	$\xi_3 = -0.469038$	$\xi_4 = -0.055257$	$\xi_5 = 0.368524$	$\xi_6 = 0.723983$	$\xi_7 = 0.945512$
H =	-0.046119	-0.753253	1.781779	5.148072	-8.399566	-6.354593	8.857946
	0.163100	2.626035	-6.871305	-16.029152	31.192143	13.718121	-25.542265
	-0.425213	-6.473677	23.118831	14.994939	-60.812936	-8.227852	39.221629
	4.433042	-3.283624	-37.707087	12.196907	79.974190	-9.827310	-48.172538
	0.703883	14.127003	22.911318	-44.244432	-72.312189	32.024862	51.012606
	-0.327353	-6.133875	-1.299825	47.694660	43.672971	-45.826381	-46.607432
	0.178736	3.291278	-0.435308	-27.482156	-16.941780	40.039999	33.234538

m = 8

	$\xi_1 = -0.981945$	$\xi_2 = -0.841399$	$\xi_3 = -0.580441$	$\xi_4 = -0.236458$	$\xi_5 = 0.141272$	$\xi_6 = 0.498635$	$\xi_7 = 0.784444$	$\xi_8 = 0.957828$
H =	-0.095172	0.405664	2.899898	-5.822943	-11.822066	18.514690	11.271281	-15.570633
	0.323200	-1.432604	-9.548287	21.093623	35.215875	-63.843756	-26.430346	45.309083
	-0.733473	3.643077	19.256542	-55.292225	-41.553525	121.49332	22.867441	-70.933904
	2.284512	-17.072488	-7.325229	99.864283	3.788856	-171.45626	1.944669	90.003386
	4.268341	16.366956	-32.730760	-102.98271	67.191208	185.20920	-40.120304	-100.46750
	-1.211769	1.500878	45.192878	54.094561	-117.54224	-153.28221	76.179335	100.67320
	0.681379	-1.341821	-25.430037	-11.872175	111.96108	96.367234	-92.841439	-89.055813
	-0.391477	0.861262	14.526889	3.362216	-69.510751	-45.936022	75.962755	62.474872

m = 9

	$\xi_1 = -0.985626$	$\xi_2 = -0.873107$	$\xi_3 = -0.660922$	$\xi_4 = -0.373313$	$\xi_5 = -0.043138$	$\xi_6 = 0.291883$	$\xi_7 = 0.593478$	$\xi_8 = 0.827199$	$\xi_9 = 0.966403$
H =	0.035742	0.756443	-2.097281	-9.191658	16.831549	26.121792	-39.790988	-20.233377	27.774652
	-0.118150	-2.485062	7.242062	29.123424	-58.167549	-76.177348	131.17451	50.096962	-81.331016
	0.247594	5.116614	-16.953465	-52.761335	131.82438	99.791299	-243.03839	-52.094020	129.01613
	-0.568286	-11.081393	50.827222	43.823593	-236.73049	-52.707546	352.46573	19.430759	-167.26087
	5.673596	-5.690616	-79.172082	38.500416	298.95652	-73.013367	-416.77602	41.294895	192.96248
	0.884521	22.647637	44.682442	-127.04302	-249.50424	214.24014	405.14559	-111.75431	-203.55006
	-0.420707	-10.039483	-1.263770	131.10773	132.56606	-285.10278	-323.98268	167.44592	196.85134
	0.261279	6.110598	-2.333883	-85.567933	-48.574742	257.06425	213.62669	-184.77103	-170.39927
	-0.155142	-3.601341	2.045590	51.249991	20.451979	-166.34480	-114.44906	144.63135	118.20673

m = 10

	$\xi_1 = -0.988287$	$\xi_2 = -0.896227$	$\xi_3 = -0.720688$	$\xi_4 = -0.478030$	$\xi_5 = -0.190871$	$\xi_6 = 0.114023$	$\xi_7 = 0.408235$	$\xi_8 = 0.664343$	$\xi_9 = 0.858484$	$\xi_{10} = 0.972609$
H =	0.076277	-0.387761	-3.689815	8.410695	26.092363	-45.040861	-56.326550	84.109326	36.686414	-50.126466
	-0.247416	1.283483	11.809096	-28.347560	-80.609766	150.40296	162.71538	-269.50862	-94.339495	147.44763
	0.492677	-2.689682	-22.634936	61.617484	138.00342	-311.88095	-225.43263	487.94141	109.61714	-236.10275
	-0.977551	6.025310	39.712030	-143.01605	-144.28541	540.71154	174.82554	-713.22568	-68.864333	310.73220
	2.888056	-26.890368	-13.680684	248.76531	6.846002	-741.60349	27.082252	884.77330	-24.023547	-366.55444
	5.268900	24.755545	-64.869872	-245.84142	234.14820	765.99524	-321.02044	-942.39450	147.98325	399.48734
	-1.496045	2.426868	86.010023	118.95293	-387.29338	-583.12374	570.22508	864.98575	-269.92061	-406.11254
	0.867648	-2.226832	-49.786664	-21.738531	360.88461	331.19824	-660.59239	-685.91802	352.91616	383.29058
	-0.572379	1.663859	32.570686	2.842511	-256.34733	-158.73599	579.76240	471.86075	-364.28513	-326.74418
	0.347666	-1.058157	-19.694338	1.069296	158.81737	78.333012	-383.92277	-271.14486	276.34485	224.84992

D.1.3 Supersonic leading edge, supersonic trailing edge

The polynomials orthogonal to $f(\xi) = 1$ over $(-1,1)$ are the Legendre polynomials; the zeros and weights in the corresponding Gauss formulae are given by Gawlik[3]. The H_α are merely one half of these Gauss weights; the $h_\alpha(\xi)$ are found numerically. ξ_α and H are given below for m = 2, ... ,10.

m = 2

$$\xi_1 = -0.577350 \qquad \xi_2 = 0.577350$$

$$H = \begin{pmatrix} 1 & -1.732051 \\ 1 & 1.732051 \end{pmatrix}$$

m = 3

$$\xi_1 = -0.774597 \qquad \xi_2 = 0.000000 \qquad \xi_3 = 0.774597$$

$$H = \begin{pmatrix} 0 & -2.323790 & 3.000000 \\ 2.250000 & 0 & -3.750000 \\ 0 & 2.323790 & 3.000000 \end{pmatrix}$$

m = 4

$$\xi_1 = -0.861136 \qquad \xi_2 = -0.339981 \qquad \xi_3 = 0.339981 \qquad \xi_4 = 0.861136$$

$$H = \begin{pmatrix} -0.530834 & 0.616434 & 4.592502 & -5.333073 \\ 1.816548 & -5.343087 & -2.449645 & 7.205240 \\ 1.816548 & 5.343087 & -2.449645 & -7.205240 \\ -0.530834 & -0.616434 & 4.592502 & 5.333073 \end{pmatrix}$$

m = 5

$$\xi_1 = -0.906180 \qquad \xi_2 = -0.538469 \qquad \xi_3 = 0.000000$$
$$\xi_4 = 0.538469 \qquad \xi_5 = 0.906180$$

$$H = \begin{pmatrix} 0 & 2.542288 & -2.805500 & -8.768046 & 9.675834 \\ 0 & -5.997920 & 11.138834 & 7.304187 & -13.564723 \\ 3.515625 & 0 & -16.406250 & 0 & 14.765625 \\ 0 & 5.997920 & 11.138834 & -7.304187 & 13.564723 \\ 0 & -2.542288 & -2.805500 & 8.768046 & 9.675834 \end{pmatrix}$$

m = 6

$$\xi_1 = -0.932470 \qquad \xi_2 = -0.661209 \qquad \xi_3 = -0.238619$$
$$\xi_4 = 0.238619 \qquad \xi_5 = 0.661209 \qquad \xi_6 = 0.932470$$

$$H = \begin{pmatrix} 0.413644 & -0.443601 & -8.210802 & 8.805437 & 16.616449 & -17.819831 \\ -0.834822 & 1.262568 & 15.621774 & -23.626062 & -16.862181 & 25.502029 \\ 2.629339 & -11.018974 & -9.038039 & 37.876413 & 6.916705 & -28.986373 \\ 2.629339 & 11.018974 & -9.038039 & -37.876413 & 6.916705 & 28.986373 \\ -0.834822 & -1.262568 & 15.621774 & 23.626062 & -16.862181 & -25.502029 \\ 0.413644 & 0.443601 & -8.210802 & -8.805437 & 16.616449 & 17.819831 \end{pmatrix}$$

m = 7

H =

$\xi_1 = -0.949108$	$\xi_2 = -0.741531$	$\xi_3 = -0.405845$	$\xi_4 = 0$	$\xi_5 = 0.405845$	$\xi_6 = 0.741531$	$\xi_7 = 0.949108$
0	-2.852863	3.005836	22.508756	-23.715697	-31.499334	33.188358
0	5.292404	-7.137129	-38.006778	51.254456	35.669829	-48.102939
0	-11.274212	27.779591	33.019166	-81.359027	-22.761254	56.083591
4.785156	0	-43.066406	0	94.746094	0	-58.652344
0	11.274212	27.779591	-33.019166	-81.359027	22.761254	56.083592
0	-5.292404	-7.137129	38.006777	51.254456	-35.669829	-48.102939
0	2.852863	3.005836	-22.508756	-23.715697	31.499334	33.188358

m = 8

H =

$\xi_1 = -0.960290$	$\xi_2 = -0.796667$	$\xi_3 = -0.525532$	$\xi_4 = -0.183435$	$\xi_5 = 0.183435$	$\xi_6 = 0.525532$	$\xi_7 = 0.796667$	$\xi_8 = 0.960290$
-0.353127	0.367729	12.329616	-12.839474	-56.548620	58.887033	59.870849	-62.346644
0.622137	-0.780925	-21.416685	26.882874	89.438889	-112.26642	-72.597138	91.126136
-1.117701	2.126798	36.190302	-68.864074	-90.268052	171.76496	56.755142	-107.99551
3.441084	-18.759182	-21.612719	117.82245	39.021621	-212.72765	-21.288177	116.05320
3.441084	18.759182	-21.612719	-117.82245	39.021621	212.72765	-21.288177	-116.05320
-1.117701	-2.126798	36.190302	68.864073	-90.268052	-171.76496	56.755142	107.99551
0.622137	0.780925	-21.416685	-26.882875	89.438889	112.26641	-72.597138	-91.126137
-0.353127	-0.367729	12.329616	12.839474	-56.548620	-58.887033	59.870849	62.346644

m = 9

H =

$\xi_1 = -0.968160$	$\xi_2 = -0.836031$	$\xi_3 = -0.613371$	$\xi_4 = -0.324253$	$\xi_5 = 0$	$\xi_6 = 0.324253$	$\xi_7 = 0.613371$	$\xi_8 = 0.836031$	$\xi_9 = 0.968160$
0	3.156506	-3.260314	-42.927889	44.339653	134.75431	-139.18596	-114.16830	117.92294
0	-5.373993	6.427982	71.129880	-85.080423	-205.62536	245.95420	144.93938	-173.36602
0	8.778294	-14.311547	-105.41571	171.86277	221.92476	-361.81138	-127.43878	207.76771
0	-18.167273	56.028006	93.662541	-288.85598	-148.33398	457.46310	73.706075	-227.31009
6.056214	0	-88.824469	0	346.41543	0	-494.87918	0	233.69295
0	18.167273	56.028006	-93.662542	-288.85598	148.33398	457.46310	-73.706075	-227.31009
0	-8.778294	-14.311547	105.41571	171.86277	-221.92476	-361.81138	127.43878	207.76771
0	5.373993	6.427982	-71.129880	-85.080423	205.62536	245.95420	-144.93938	-173.36602
0	-3.156506	-3.260314	42.927889	44.339653	-134.75431	-139.18596	114.16830	117.92294

m = 10

H =

$\xi_1 = -0.973907$	$\xi_2 = -0.865063$	$\xi_3 = -0.679410$	$\xi_4 = -0.433395$	$\xi_5 = -0.148874$	$\xi_6 = 0.148874$	$\xi_7 = 0.433395$	$\xi_8 = 0.679410$	$\xi_9 = 0.865063$	$\xi_{10} = 0.973907$
0.314093	-0.322508	-16.943959	17.397931	131.85353	-135.38623	-310.13922	318.44865	218.41911	-224.27112
-0.522071	0.603506	28.016256	-32.386363	-211.41566	244.39326	464.04652	-536.43067	-286.43319	331.11237
0.803026	-1.181948	-42.426789	62.446558	290.86298	-428.11141	-518.20518	762.72870	271.76355	-399.99959
-1.394643	3.217946	69.280387	-159.85492	-295.93657	682.83274	418.79692	-966.31605	-192.05675	443.14441
4.252382	-28.563568	-42.017418	282.23412	131.18136	-881.15496	-162.12339	1088.9949	69.098721	-464.14125
4.252382	28.563568	-42.017418	-282.23412	131.18136	881.15496	-162.12339	-1088.9949	69.098721	464.14124
-1.394643	-3.217946	69.280387	159.85492	-295.93657	-682.83275	418.79692	966.31605	-192.05675	-443.14441
0.803026	1.181948	-42.426789	-62.446558	290.86298	428.11141	-518.20518	-762.72870	271.76355	399.99959
-0.522071	-0.603506	28.016256	32.386362	-211.41566	-244.39326	464.04652	536.43067	-286.43319	-331.11238
0.314093	0.322508	-16.943959	-17.397931	131.85353	135.38623	-310.13922	-318.44865	218.41911	224.27112

D.1.4 Supersonic leading edge, subsonic trailing edge

This case, in which $f(\xi) = \sqrt{(1-\xi)}$, is not dealt with in the programmes which are the subject of this Report, and the $h_\alpha(\xi)$ etc. have not been calculated by the writer. However, it is worth pointing out that the lift points are again derived from the Legendre zeros over $(-1,1)$. Mineur[4] (page 290) gives a result, which gives the positions of the lift points needed here as

$$\xi_\alpha = 1 - 2u_\alpha^2 \qquad (\alpha = 1, \ldots ,m)$$

where the u_α are the positive zeros of $P_{2m+1}(u)$. The lift points thus follow from the zeros of the Legendre polynomials of odd degree.

The lift points for m = 2, ... ,10 are as follows:

m = 2

$\xi_1 = -0.642324$ $\quad \xi_2 = 0.420102$

m = 3

$\xi_1 = -0.801612$ $\quad \xi_2 = -0.099737$ $\quad \xi_3 = 0.670579$

m = 4

$\xi_1 = -0.874668$ $\quad \xi_2 = -0.397896$ $\quad \xi_3 = 0.247551$ $\quad \xi_4 = 0.789719$

m = 5

$\xi_1 = -0.913863$ $\quad \xi_2 = -0.573760$ $\quad \xi_3 = -0.066244$ $\quad \xi_4 = 0.461078$
$\xi_5 = 0.854693$

m = 6

$\xi_1 = -0.937232$ $\quad \xi_2 = -0.683974$ $\quad \xi_3 = -0.285055$ $\quad \xi_4 = 0.174775$
$\xi_5 = 0.597709$ $\quad \xi_6 = 0.893778$

m = 7

$\xi_1 = -0.952258$ $\quad \xi_2 = -0.756963$ $\quad \xi_3 = -0.438909$ $\quad \xi_4 = -0.049562$
$\xi_5 = 0.347982$ $\quad \xi_6 = 0.689289$ $\quad \xi_7 = 0.919042$

m = 8

$\xi_1 = -0.962480$ $\quad \xi_2 = -0.807568$ $\quad \xi_3 = -0.549642$ $\quad \xi_4 = -0.221528$
$\xi_5 = 0.134937$ $\quad \xi_6 = 0.474297$ $\quad \xi_7 = 0.753273$ $\quad \xi_8 = 0.936287$

m = 9

$\xi_1 = -0.969743$ $\quad \xi_2 = -0.843999$ $\quad \xi_3 = -0.631381$ $\quad \xi_4 = -0.353719$
$\xi_5 = -0.039584$ $\quad \xi_6 = 0.278691$ $\quad \xi_7 = 0.568348$ $\quad \xi_8 = 0.799574$
$\xi_9 = 0.948570$

m = 10

$\xi_1 = -0.975087$ $\xi_2 = -0.871056$ $\xi_3 = -0.693166$ $\xi_4 = -0.456458$
$\xi_5 = -0.181000$ $\xi_6 = 0.109852$ $\xi_7 = 0.391433$ $\xi_8 = 0.639868$
$\xi_9 = 0.834087$ $\xi_{10} = 0.957624$

D.2 Spanwise interpolation functions and lift and downwash points

Spanwise positions of both lift and downwash points are given by the zeros of the polynomials orthogonal to $\sqrt{(1-\eta^2)}$ over $(-1,1)$. These are the polynomials (see, for example, Hildebrand[5] page 308)

$$\frac{\sin[(n+1)\cos^{-1}\eta]}{\sqrt{(1-\eta^2)}}$$

which were originally considered in connection with lifting surface theory by Multhopp[6]. This polynomial has zeros

$$\eta_\beta = \cos\frac{\pi(n-\beta+1)}{n+1} \quad . \qquad (\beta = 1, \ldots, n) \quad .$$

Also (see, for example, Multhopp[6])

$$G_\beta = \frac{\pi}{2(n+1)}\sqrt{\left(1-\eta_\beta^2\right)} \quad .$$

For n = 2, ... ,7 the η_β and the matrix G of coefficients in the interpolation functions (see following equation (22) of Appendix B) are given below. These have been found by the writer for n = 2, ... ,21.

n = 2

$\eta_1 = -0.5$
$\eta_2 = 0.5$

$$G = \begin{pmatrix} 1.273240 & -2.546479 \\ 1.273240 & 2.546479 \end{pmatrix}$$

n = 3

$\eta_1 = -0.707107$
$\eta_2 = 0$
$\eta_3 = 0.707107$

$$G = \begin{pmatrix} 0 & -3.601265 & 5.092958 \\ 2.546479 & 0 & -5.092958 \\ 0 & 3.601265 & 5.092958 \end{pmatrix}$$

n = 4

$\eta_1 = -0.809017$
$\eta_2 = -0.309017$
$\eta_3 = 0.309017$
$\eta_4 = 0.809017$

$$G = \begin{pmatrix} -0.786905 & 0.972668 & 8.240579 & -10.185916 \\ 2.060145 & -6.666769 & -3.147621 & 10.185916 \\ 2.060145 & 6.666769 & -3.147621 & -10.185916 \\ -0.786905 & -0.972669 & 8.240580 & 10.185917 \end{pmatrix}$$

n = 5

$\eta_1 = -0.866025 \qquad \eta_2 = -0.5 \qquad \eta_3 = 0 \qquad \eta_4 = 0.5 \qquad \eta_5 = 0.866025$

$$G = \begin{pmatrix} 0 & 4.410631 & -5.092958 & -17.642524 & 20.371831 \\ 0 & -7.639437 & 15.278874 & 10.185916 & -20.371833 \\ 3.819719 & 0 & -20.371833 & 0 & 20.371833 \\ 0 & 7.639437 & 15.278874 & -10.185916 & -20.371833 \\ 0 & -4.410632 & -5.092959 & 17.642525 & 20.371834 \end{pmatrix}$$

n = 6

$\eta_1 = -0.900969 \qquad \eta_2 = -0.623490 \qquad \eta_3 = -0.222521 \qquad \eta_4 = 0.222521$
$\eta_5 = 0.623490 \qquad \eta_6 = 0.900969$

$$G = \begin{pmatrix} 0.706595 & -0.784261 & -16.087807 & 17.856118 & 36.708772 & -40.743664 \\ -1.021059 & 1.637651 & 21.878823 & -35.090909 & -25.403260 & 40.743665 \\ 2.860943 & -12.856962 & -10.883974 & 48.912135 & 9.066318 & -40.743665 \\ 2.860943 & 12.856962 & -10.883974 & -48.912136 & 9.066318 & 40.743665 \\ -1.021059 & -1.637651 & 21.878823 & 35.090907 & -25.403260 & -40.743664 \\ 0.706595 & 0.784261 & -16.087809 & -17.856121 & 36.708777 & 40.743664 \end{pmatrix}$$

n = 7

$\eta_1 = -0.923880 \qquad \eta_2 = -0.707107 \qquad \eta_3 = -0.382683 \qquad \eta_4 = 0$
$\eta_5 = 0.382683 \qquad \eta_6 = 0.707107 \qquad \eta_7 = 0.923880$

$$G = \begin{pmatrix} 0 & -5.512579 & 5.966772 & 48.667402 & -52.677217 & -75.284486 & 81.487341 \\ 0 & 7.202531 & -10.185917 & -57.620246 & 81.487334 & 57.620245 & -81.487333 \\ 0 & -13.308541 & 34.776895 & 42.209007 & -110.29745 & -31.183851 & 81.487331 \\ 5.092958 & -0.000001 & -50.929582 & 0.000001 & 122.23100 & -0.000001 & -81.487331 \\ 0 & 13.308541 & 34.776892 & -42.209007 & -110.29745 & 31.183851 & 81.487328 \\ 0 & -7.202530 & -10.185916 & 57.620245 & 81.487331 & -57.620246 & -81.487332 \\ 0 & 5.512578 & 5.966772 & -48.667396 & -52.677209 & 75.284477 & 81.487331 \end{pmatrix}$$

D.3 Chordwise integration formulae

The function $k(\zeta)$ of (12) takes the values

$$k(\zeta) = \begin{cases} \sqrt{\left(\dfrac{1-\zeta}{1+\zeta}\right)} & \text{if } \zeta = -1 \text{ at a subsonic leading edge} \\ & \phantom{\text{if }} \zeta = +1 \text{ at a subsonic trailing edge} \\ \dfrac{1}{\sqrt{\left(1-\zeta^2\right)}} & \text{if } \zeta = -1 \text{ at a subsonic leading edge} \\ & \phantom{\text{if }} \zeta = +1 \text{ at a Mach line} \\ \dfrac{1}{\sqrt{\left(1-\zeta\right)}} & \text{if } \zeta = -1 \text{ at a supersonic leading edge} \\ & \phantom{\text{if }} \zeta = +1 \text{ at a Mach line.} \end{cases}$$

The case $k(\zeta) = \sqrt{(1-\zeta)}$, which occurs when $\zeta = -1$ at a supersonic leading edge and $\zeta = +1$ at a subsonic trailing edge is not considered here, since the corresponding leading and trailing edge conditions are not dealt with by the programmes described in this Report.

The Gauss zeros ζ_λ and the quantities $W_\lambda/k(\zeta_\lambda)$ for the various values of $k(\zeta)$ are as follows.

D.3.1 $k(\zeta) = \sqrt{(1-\zeta)}/\sqrt{(1+\zeta)}$

The zeros here are the same as the lift points in D.1.1. The integration points are

$$\zeta_\lambda = \cos \frac{2\pi(p - \lambda + 1)}{2p + 1} \qquad (\lambda = 1, \ldots ,p)$$

and

$$w_\lambda = \frac{W_\lambda}{k(\zeta_\lambda)} = \frac{2\pi}{2p + 1} \sqrt{\left(1 - \zeta_\lambda^2\right)} ,$$

being twice the value of the corresponding H_α in D.1.1 above. Values of w_λ are given below for p = 1, ... ,10. The values of ζ_λ have already been given above in D.1.1 (where they appear as ζ's) and so are omitted here, except for p = 1.

p = 1	ζ_1 = -0.5 w_1 = 1.813799	p = 2	w_1 = 0.738633 w_2 = 1.195133
p = 3	w_1 = 0.389453 w_2 = 0.875093 w_3 = 0.701770	p = 4	w_1 = 0.238775 w_2 = 0.604600 w_3 = 0.687526 w_4 = 0.448750
p = 5	w_1 = 0.160925 w_2 = 0.431683 w_3 = 0.565385 w_4 = 0.519581 w_5 = 0.308813	p = 6	w_1 = 0.115667 w_2 = 0.320502 w_3 = 0.451914 w_4 = 0.479798 w_5 = 0.397766 w_6 = 0.224611
p = 7	w_1 = 0.087090 w_2 = 0.246211 w_3 = 0.362760 w_4 = 0.416584 w_5 = 0.398378 w_6 = 0.311288 w_7 = 0.170373	p = 8	w_1 = 0.067914 w_2 = 0.194569 w_3 = 0.294946 w_4 = 0.355490 w_5 = 0.368022 w_6 = 0.330852 w_7 = 0.248997 w_8 = 0.133515

p = 9

w_1	= 0.054430
w_2	= 0.157393
w_3	= 0.243299
w_4	= 0.302841
w_5	= 0.329564
w_6	= 0.320575
w_7	= 0.276846
w_8	= 0.203116
w_9	= 0.107376

p = 10

w_1	= 0.044593
w_2	= 0.129818
w_3	= 0.203507
w_4	= 0.259114
w_5	= 0.291698
w_6	= 0.298363
w_7	= 0.278517
w_8	= 0.233923
w_9	= 0.168545
w_{10}	= 0.088191

D.3.2 $k(\zeta) = 1/\sqrt{(1 - \zeta^2)}$

The Gauss zeros corresponding to the weight function $k(\zeta) = 1/\sqrt{(1 - \zeta^2)}$ are the Chebyshev zeros

$$\zeta_\lambda = \cos \frac{(2p - 2\lambda + 1)\pi}{2p} \qquad (\lambda = 1, \dots ,p)$$

and

$$w_\lambda = \frac{W_\lambda}{k(\zeta_\lambda)} = \frac{\pi}{p} \sqrt{\left(1 - \zeta_\lambda^2\right)} \ .$$

The values of ζ_λ and w_λ are given below for p = 1, ... ,10.

p = 1

ζ_1 = 0 $\qquad$ $w_1 = \pi = 3.141593$

p = 2

ζ_1 = -0.707107	w_1 = 1.110721
ζ_2 = 0.707107	w_2 = 1.110721

p = 3

ζ_1 = -0.866025	w_1 = 0.523599
ζ_2 = 0	w_2 = 1.047198
ζ_3 = 0.866025	w_3 = 0.523599

p = 4

ζ_1 = -0.923880	w_1 = 0.300559
ζ_2 = -0.382683	w_2 = 0.725613
ζ_3 = 0.382683	w_3 = 0.725613
ζ_4 = 0.923880	w_4 = 0.300559

p = 5

$\zeta_1 = -0.951057$	$w_1 = 0.194161$
$\zeta_2 = -0.587785$	$w_2 = 0.508320$
$\zeta_3 = 0$	$w_3 = 0.628319$
$\zeta_4 = 0.587785$	$w_4 = 0.508320$
$\zeta_5 = 0.951057$	$w_5 = 0.194161$

p = 6

$\zeta_1 = -0.965926$	$w_1 = 0.135517$
$\zeta_2 = -0.707107$	$w_2 = 0.370240$
$\zeta_3 = -0.258819$	$w_3 = 0.505758$
$\zeta_4 = 0.258819$	$w_4 = 0.505758$
$\zeta_5 = 0.707107$	$w_5 = 0.370240$
$\zeta_6 = 0.965926$	$w_6 = 0.135517$

p = 7

$\zeta_1 = -0.974928$	$w_1 = 0.099867$
$\zeta_2 = -0.781831$	$w_2 = 0.279822$
$\zeta_3 = -0.433884$	$w_3 = 0.404354$
$\zeta_4 = 0$	$w_4 = 0.448799$
$\zeta_5 = 0.433884$	$w_5 = 0.404354$
$\zeta_6 = 0.781831$	$w_6 = 0.279822$
$\zeta_7 = 0.974928$	$w_7 = 0.099867$

p = 8

$\zeta_1 = -0.980785$	$w_1 = 0.076612$
$\zeta_2 = -0.831470$	$w_2 = 0.218172$
$\zeta_3 = -0.555570$	$w_3 = 0.326517$
$\zeta_4 = -0.195090$	$w_4 = 0.385153$
$\zeta_5 = 0.195090$	$w_5 = 0.385153$
$\zeta_6 = 0.555570$	$w_6 = 0.326517$
$\zeta_7 = 0.831470$	$w_7 = 0.218172$
$\zeta_8 = 0.980785$	$w_8 = 0.076612$

p = 9

$\zeta_1 = -0.984808$	$w_1 = 0.060615$
$\zeta_2 = -0.866025$	$w_2 = 0.174533$
$\zeta_3 = -0.642788$	$w_3 = 0.267400$
$\zeta_4 = -0.342020$	$w_4 = 0.328015$
$\zeta_5 =$ [illegible]	$w_5 = 0.349066$
$\zeta_6 = 0.342020$	$w_6 = 0.328015$
$\zeta_7 = 0.642788$	$w_7 = 0.267400$
$\zeta_8 = 0.866025$	$w_8 = 0.174533$
$\zeta_9 = 0.984808$	$w_9 = 0.060615$

p = 10

$\zeta_1 = -0.987688$ $w_1 = 0.049145$
$\zeta_2 = -0.891007$ $w_2 = 0.142625$
$\zeta_3 = -0.707107$ $w_3 = 0.222144$
$\zeta_4 = -0.453991$ $w_4 = 0.279918$
$\zeta_5 = -0.156434$ $w_5 = 0.310291$
$\zeta_6 = 0.156434$ $w_6 = 0.310291$
$\zeta_7 = 0.453991$ $w_7 = 0.279918$
$\zeta_8 = 0.707107$ $w_8 = 0.222144$
$\zeta_9 = 0.891007$ $w_9 = 0.142625$
$\zeta_{10} = 0.987688$ $w_{10} = 0.049145$

D.3.3 $k(\zeta) = 1/\sqrt{(1-\zeta)}$

The zeros here are the negative of the lift points for a singularity $1/\sqrt{(1+\zeta)}$, which was dealt with in D.1.2 above. So

$$\zeta_\lambda = 1 - 2u_\lambda^2 \quad . \qquad (\lambda = 1, \ldots, p) \quad .$$

Gauss weights and zeros for a singularity $1/\sqrt{x}$ over (0,1), to which the w_λ required here are related, are given in Mineur[4] (page 289). Alternatively, in view of the remarks of D.1.2 above, it may be shown that

$$w_\lambda = \frac{W_\lambda}{k(\zeta_\lambda)} = 4A_\lambda\, u_\lambda \qquad (\lambda = 1, \ldots, p)$$

where the A_λ are the weights and u_λ the positive zeros in a 2p-point Legendre-Gauss formula (see, e.g. Gawlik[3]). Values of ζ_λ and w_λ are given below for p = 1, ... ,10.

p = 1

$\zeta_1 = 0.333333$ $w_1 = 2.309401$

p = 2

$\zeta_1 = -0.483112$ $w_1 = 1.198202$
$\zeta_2 = 0.768826$ $w_2 = 0.886868$

p = 3

$\zeta_1 = -0.738999$ $w_1 = 0.639019$
$\zeta_2 = 0.125604$ $w_2 = 0.954156$
$\zeta_3 = 0.886122$ $w_3 = 0.446613$

p = 4

$\zeta_1 = -0.844313$	$w_1 = 0.388835$
$\zeta_2 = -0.269355$	$w_2 = 0.708654$
$\zeta_3 = 0.447631$	$w_3 = 0.659452$
$\zeta_4 = 0.932704$	$w_4 = 0.266115$

p = 5

$\zeta_1 = -0.896988$	$w_1 = 0.259727$
$\zeta_2 = -0.496669$	$w_2 = 0.517140$
$\zeta_3 = 0.076805$	$w_3 = 0.595397$
$\zeta_4 = 0.624337$	$w_4 = 0.466796$
$\zeta_5 = 0.955673$	$w_5 = 0.175984$

p = 6

$\zeta_1 = -0.926923$	$w_1 = 0.185222$
$\zeta_2 = -0.634856$	$w_2 = 0.386743$
$\zeta_3 = -0.185500$	$w_3 = 0.492979$
$\zeta_4 = 0.310115$	$w_4 = 0.477296$
$\zeta_5 = 0.729400$	$w_5 = 0.343544$
$\zeta_6 = 0.968633$	$w_6 = 0.124806$

p = 7

$\zeta_1 = -0.945512$	$w_1 = 0.138551$
$\zeta_2 = -0.723983$	$w_2 = 0.297686$
$\zeta_3 = -0.368524$	$w_3 = 0.402081$
$\zeta_4 = 0.055257$	$w_4 = 0.432178$
$\zeta_5 = 0.469038$	$w_5 = 0.382394$
$\zeta_6 = 0.796335$	$w_6 = 0.261925$
$\zeta_7 = 0.976648$	$w_7 = 0.093041$

p = 8

$\zeta_1 = -0.957828$	$w_1 = 0.107459$
$\zeta_2 = -0.784444$	$w_2 = 0.235212$
$\zeta_3 = -0.498635$	$w_3 = 0.329489$
$\zeta_4 = -0.141272$	$w_4 = 0.376581$
$\zeta_5 = 0.236458$	$w_5 = 0.369727$
$\zeta_6 = 0.580441$	$w_6 = 0.309906$
$\zeta_7 = 0.841399$	$w_7 = 0.205687$
$\zeta_8 = 0.981945$	$w_8 = 0.072001$

p = 9

$\zeta_1 = -0.966403$	$w_1 = 0.085735$
$\zeta_2 = -0.827199$	$w_2 = 0.190073$
$\zeta_3 = -0.593478$	$w_3 = 0.272871$
$\zeta_4 = -0.291883$	$w_4 = 0.324510$
$\zeta_5 = 0.043138$	$w_5 = 0.339079$
$\zeta_6 = 0.373313$	$w_6 = 0.314911$
$\zeta_7 = 0.660922$	$w_7 = 0.254766$
$\zeta_8 = 0.873107$	$w_8 = 0.165516$
$\zeta_9 \quad 0.985626$	$w_9 = 0.057356$

p = 10

ζ_1 =	-0.972609	w_1 =	0.069972
ζ_2 =	-0.858484	w_2 =	0.156555
ζ_3 =	-0.664343	w_3 =	0.228686
ζ_4 =	-0.408235	w_4 =	0.279516
ζ_5 =	-0.114023	w_5 =	0.304295
ζ_6 =	0.190871	w_6 =	0.300712
ζ_7 =	0.478030	w_7 =	0.269102
ζ_8 =	0.720688	w_8 =	0.212409
ζ_9 =	0.896227	w_9 =	0.135918
ζ_{10} =	0.988287	w_{10} =	0.046759

D.4 Spanwise integration points and interpolation functions

The integration points used to carry out the spanwise integration (see (14), Appendix B) are the Chebyshev zeros

$$\phi_\gamma = \cos\frac{(2q - 2\gamma - 1)\pi}{2q} . \qquad (\gamma = 1, \dots ,q) .$$

For q = 2, ... ,7 the matrices P of the coefficients in the interpolation functions are given below. These have been found by the writer for q = 2, ... ,21. The values of ϕ_γ are given above in D.3.2 (where they appear as ζ_λ) and are not duplicated.

q = 2

$$P = \begin{pmatrix} 0.5 & -0.707107 \\ 0.5 & 0.707107 \end{pmatrix}$$

q = 3

$$P = \begin{pmatrix} 0 & -0.577350 & 0.666667 \\ 1.0 & 0 & -1.333333 \\ 0 & 0.577350 & 0.666667 \end{pmatrix}$$

q = 4

$$P = \begin{pmatrix} -0.103553 & 0.112085 & 0.707107 & -0.765367 \\ 0.603553 & -1.577161 & -0.707107 & 1.847759 \\ 0.603553 & 1.577161 & -0.707107 & -1.847759 \\ -0.103553 & -0.112085 & 0.707107 & 0.765367 \end{pmatrix}$$

q = 5

$$P = \begin{pmatrix} 0 & 0.324920 & -0.341641 & -0.940456 & 0.988854 \\ 0 & -1.376382 & 2.341641 & 1.521690 & -2.588854 \\ 1.0 & 0 & -4.0 & 0 & 3.2 \\ 0 & 1.376382 & 2.341641 & -1.521690 & -2.588854 \\ 0 & -0.324920 & -0.341641 & 0.940456 & 0.988854 \end{pmatrix}$$

q = 6

P =

$$\begin{pmatrix}
0.044658 & -0.046234 & -0.755983 & 0.782651 & 1.333333 & -1.380368 \\
-0.166667 & 0.235702 & 2.666667 & -3.771236 & -2.666667 & 3.771236 \\
0.622008 & -2.403256 & -1.910684 & 7.382315 & 1.333333 & -5.151604 \\
0.622008 & 2.403256 & -1.910684 & -7.382315 & 1.333333 & 5.151604 \\
-0.166667 & -0.235702 & 2.666667 & 3.771236 & -2.666667 & -3.771236 \\
0.044658 & 0.046234 & -0.755983 & -0.782651 & 1.333333 & 1.380368
\end{pmatrix}$$

q = 7

P =

$$\begin{pmatrix}
0 & -0.228243 & 0.234113 & 1.585814 & -1.626596 & -1.983469 & 2.034477 \\
0 & 0.797473 & -1.020007 & -5.075149 & 6.491360 & 4.456813 & -5.700478 \\
0 & -2.076521 & 4.785894 & 5.581812 & -12.864764 & -3.574087 & 8.237430 \\
1.0 & 0 & -8.0 & 0 & 16.0 & 0 & -9.142857 \\
0 & 2.076521 & 4.785894 & -5.581812 & -12.864764 & 3.574087 & 8.237430 \\
0 & -0.797473 & -1.020007 & 5.075150 & 6.491360 & -4.456813 & -5.700478 \\
0 & 0.228244 & 0.234113 & -1.585814 & -1.626596 & 1.983469 & 2.034477
\end{pmatrix}$$

SYMBOLS

A, B	matrices defined by (19)
$c(\eta)$	local chord is $\bar{c}\, c(\eta)$
$\bar{c}$	mean chord
C	matrix of quantities $C_{\alpha\beta}(\bar{\xi}_r, \eta_s)$ defined by (18) and immediate text
$C_{\alpha\beta}(\bar{\xi}_r, \eta_s)$	defined by (16)
$f(\xi)$	function which takes into account the chordwise singularity in lift; see (5)
$g_\beta(\eta)$	interpolation function for spanwise lift distribution; see (6) and preceding text
G	matrix of coefficients of interpolation functions; see following (22)
G_β	defined by (7)
$h_\alpha(\xi)$	interpolation function for chordwise lift; see (6) and preceding text
H	matrix of coefficients of interpolation functions; see following (22)
H_α	defined by (7)
$k(\zeta)$	function which takes into account the singularity in the chordwise integration; see (12)
$K(X,Y)$	Kernel function, defined in (9)
$\ell(x,y)$ or $\ell(\xi,\eta)$	reduced lift, defined in such a way that the actual lift is $\rho\, V^2\, e^{i\nu t}\, \ell(x,y)$
$\bar{\ell}(\xi,\eta)$	approximation to $\ell(\xi,\eta)$; see (10)
L	matrix defined by (19)
$L_\alpha(\bar{\xi}_r, \eta_s)$	see (17)
m	number of chordwise lift and downwash points
M	Mach number
M	matrix defined by (19)
M_{rs}	matrix defined by (23)
n	number of spanwise lift and downwash points
N_{rs}	diagonal matrix; see text following (23)
p	number of chordwise integration points
$p_\gamma(\phi)$	interpolation polynomial satisfying (13)
P	matrix of coefficients in $p_\gamma(\phi)$
$P_{\alpha\beta}$	see (11)
q	number of points used in spanwise integration

SYMBOLS (CONTD)

Q	matrix of generalised forces, defined by (20)
Q_{ij}	reduced generalised force coefficients; $\rho V^2 \bar{c}^3 Q_{ij}$ is the generalised force in mode i due to motion in mode j
r	suffix associated with chordwise variation of downwash point
R	$= \sqrt{(X^2 - \beta^2 Y^2)}$
R_{rs}	matrix defined by (23)
s	reduced semi-span, defined so that semi-span = $s\bar{c}$
s	suffix associated with spanwise variation of downwash point
t	reduced time
V	airspeed
$w(x,y)$ or $w(\xi,\eta)$	reduced downwash, defined in such a way that the actual downwash is $V e^{i\nu t} w(x,y)$
W_λ	Gaussian weights associated with a p-point integration formula for $k(\zeta)$
x,y,z	Cartesian co-ordinates, referred to $\bar{c}$ as reference length
x_{rs}	value of x at a downwash point $(\bar{\xi}_r,\eta_s)$
$x_{rs,\lambda\gamma}$	value of x at an integration point
X	$= x' - x$
y_s	value of y at a downwash point $(\bar{\xi}_r,\eta_s)$
$y_{rs,\gamma}$	value of y at an integration point
Y	$= y' - y$
$Z_1(x,y) \ldots Z_j(x,y)$	modal deflection shapes
$\bar{Z}_1, \bar{\bar{Z}}_1$ etc.	row matrices whose elements are the modal deflections evaluated at the lift points; see (21)
$\underline{Z}_1, \underline{\underline{Z}}_1$ etc.	row matrices whose elements are the modal deflections evaluated at the downwash points; see (21)
$\underline{Z}_x, \underline{\underline{Z}}_x$	see following (21)
$\bar{Z}, \bar{\bar{Z}}, \underline{Z}, \underline{\underline{Z}}$	see following (20)
α,β	suffices associated with chordwise and spanwise variation of lift points respectively
β	$= \sqrt{(M^2 - 1)}$
γ	suffix associated with spanwise variation of integration points
ζ	variable for chordwise integration; see preceding (12)
ζ_λ	Gaussian zeros associated with weight function $k(\zeta)$

SYMBOLS (CONTD)

η_s, η_β	spanwise co-ordinates of downwash and lift points respectively
$\eta_{rs,\gamma}$	$= \bar{\eta}_\gamma(\bar{\xi}_r, \eta_s)$
η_1, η_2	intersections of $\xi = \xi_1(\xi', \eta', \eta)$ with leading edge or tips - see Fig.5
$\bar{\eta}_\gamma(\xi', \eta')$	value of η when $\phi = \phi_\gamma$
λ	suffix associated with chordwise variation of integration points
ν	frequency parameter $= \omega\bar{c}/V$
ξ, η	wing co-ordinates; see (4)
ξ_α	chordwise co-ordinate of point at which lift is evaluated
$\bar{\xi}_r$	chordwise co-ordinate of downwash point
$\xi_{rs,\lambda\gamma}$	value of ξ when $\zeta = \zeta_\lambda$, $\eta = \eta_{rs,\gamma}$
$\xi_1(\xi', \eta', \eta)$	equation of reversed Mach lines through (ξ', η') is $\xi = \xi_1(\xi', \eta', \eta)$
$\xi_M(\xi', \eta', \eta)$	$= \min\{\xi_1(\xi', \eta', \eta), 1\}$
ρ	density
ϕ	spanwise integration variable; see text preceding (13)
ϕ_{rs}	value of ϕ at point $(\bar{\xi}_r, \eta_s)$
ω	frequency

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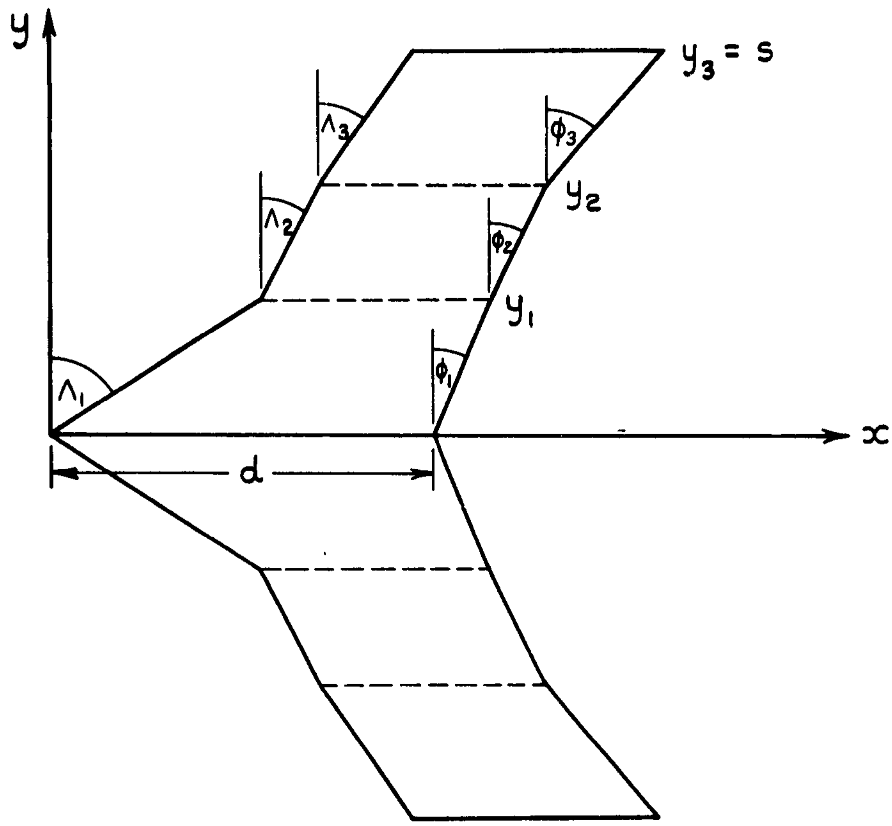

FIG. 1 THE SEGMENTED PLANFORM OF PROGRAMMES 178A, 180A, 181A AND 182A

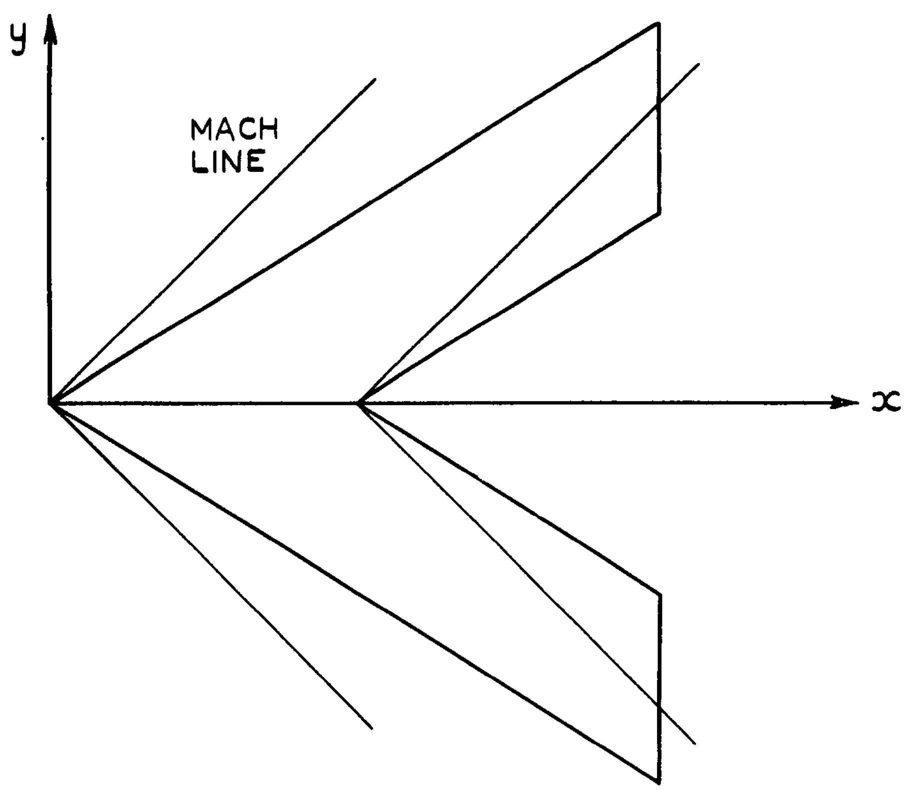

FIG. 2 PLANFORM WITH SUBSONIC LEADING EDGE, MIXED TRAILING EDGE

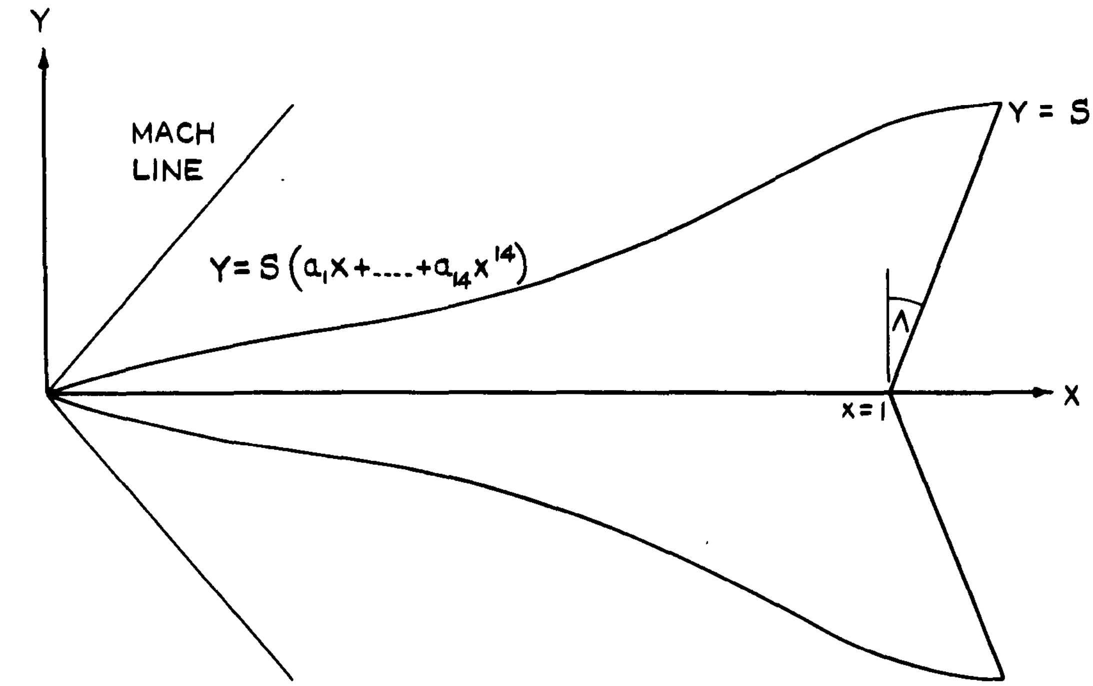

FIG. 3 PLANFORM WITH CURVED SUBSONIC LEADING EDGE, STRAIGHT SUPERSONIC TRAILING EDGE

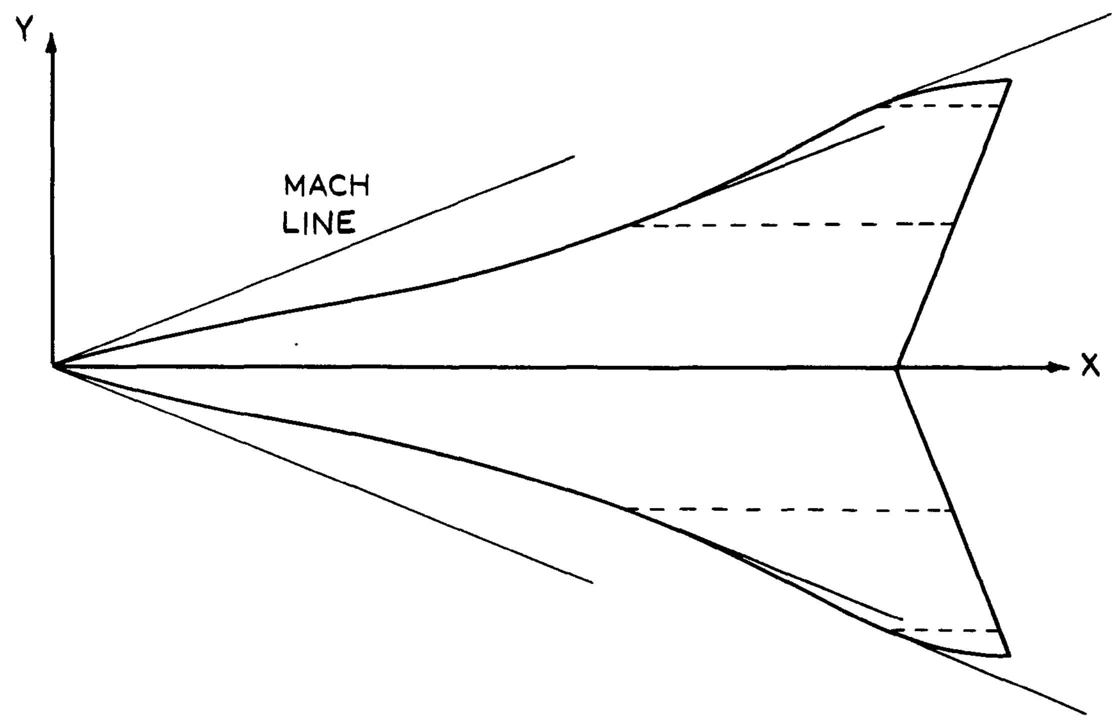

FIG. 4 PLANFORM WITH CURVED MIXED LEADING EDGE, STRAIGHT SUPERSONIC TRAILING EDGE

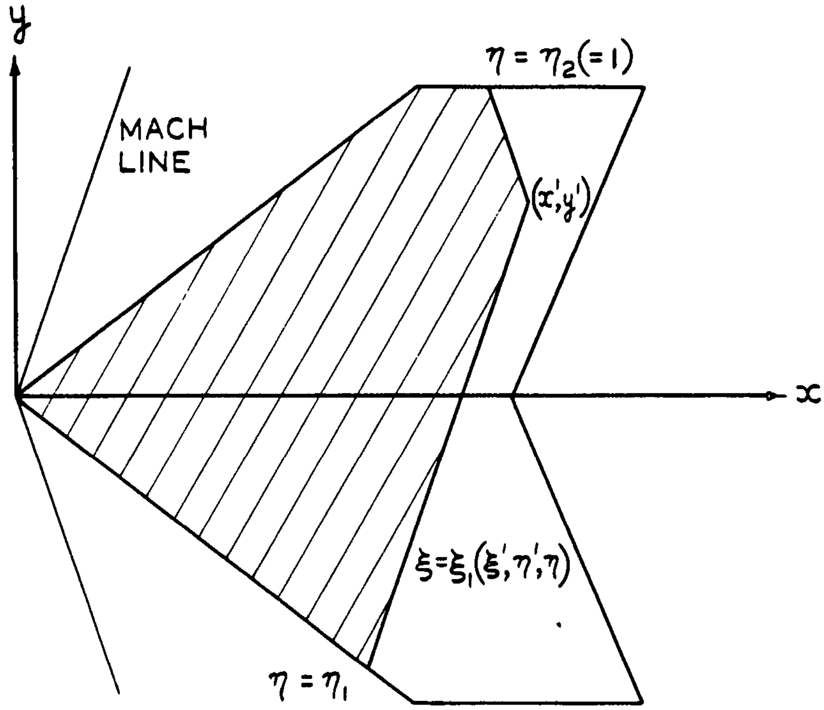

FIG. 5 INTEGRATION AREA FOR EQUATION (8) OF APPENDIX B

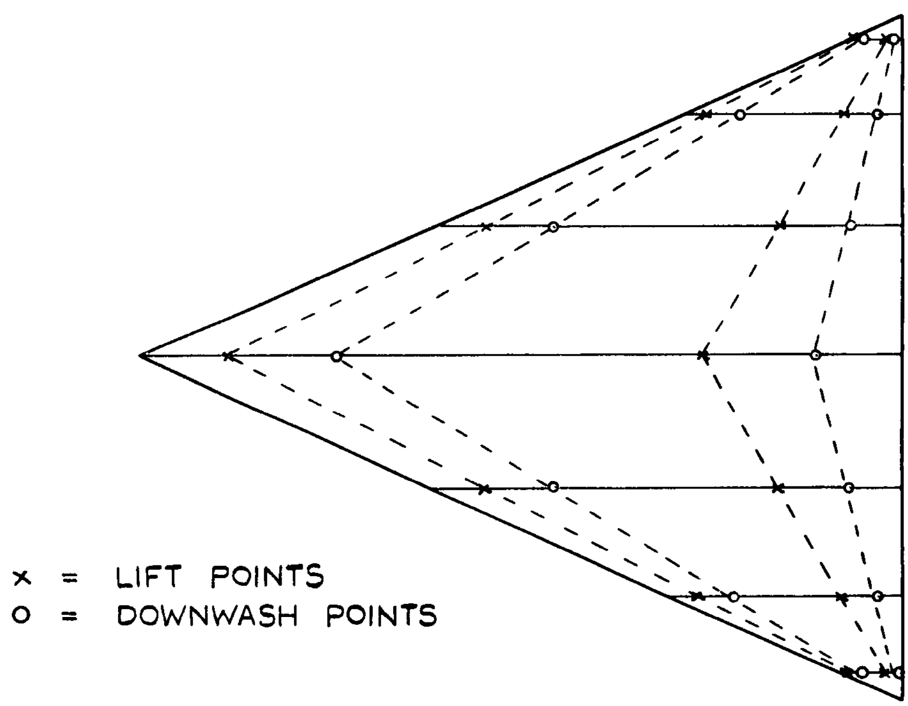

FIG. 6 LIFT AND DOWNWASH POINTS ON A DELTA WING WITH SUBSONIC LEADING EDGE, M = 2, N = 7

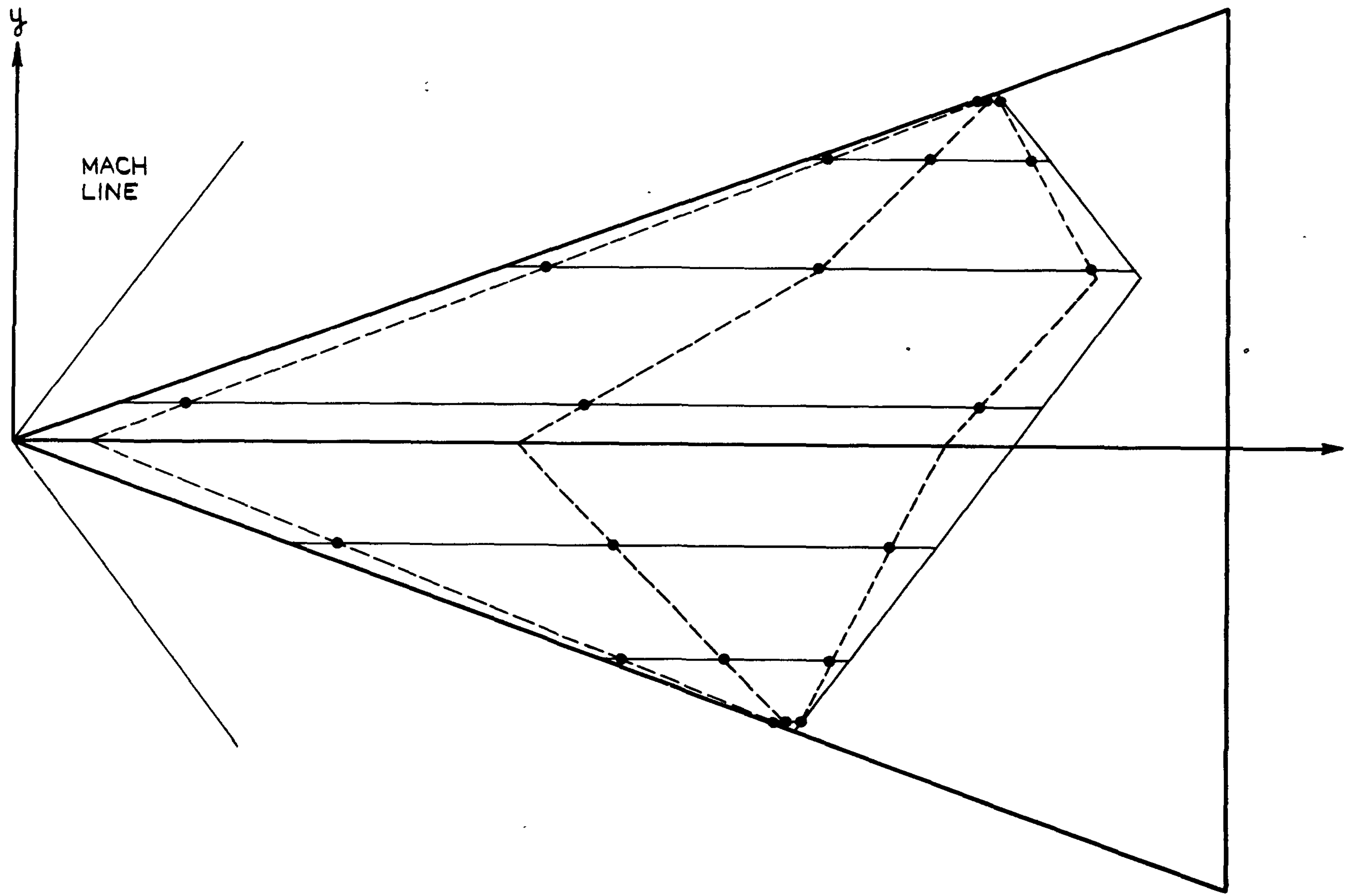

FIG.7 SET OF INTEGRATION POINTS FOR A PARTICULAR DOWNWASH POINT ON A DELTA WING WITH SUBSONIC LEADING EDGE, $p=3, q=7$

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