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A Note on Skin-Friction Laws for the Incompressible Turbulent Boundary Layer

By

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A Note on Skin-Friction Laws for the Incompressible Turbulent Boundary Layer

- By -

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SUMMARY

The main purpose of this paper is to compare the predictions of four laws currently used for the estimation of the wall shear stress in an incompressible turbulent boundary layer - viz., those of Ludwieg and Tillmann, Rotta, Coles and Thompson.

A proposal is also made for a modified skin-friction law which would appear to predict values of the wall shear stress to within about 5 percent over its range of validity. A feature of this new law is that close to separation (for which it is not strictly valid) it will predict skin-friction values which are, if not actually correct, at least physically plausible for many practical cases.

Notation

ρ	fluid density
ν	fluid kinematic viscosity
u	mean velocity in streamwise direction
u_e	velocity outside boundary layer
u_τ	"friction velocity" $[u_\tau^2 = \tau_W/\rho]$
τ_W	wall shear stress
y	distance from wall
δ	boundary-layer thickness
δ^*	displacement thickness
θ	momentum thickness
H	conventional shape factor $[H = \delta^*/\theta]$

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H_s	value of H at separation
G	generalised shape factor $\left[G = \left(\frac{2}{C_f}\right)^{\frac{1}{2}} \left(1 - \frac{1}{H}\right) \right]$
C_f	skin-friction coefficient $\left[C_f = \frac{\tau_w}{\frac{1}{2}\rho u_e^2} \right]$
K, K'	factors in equations (3) and (5), respectively
κ, C	constants in law of the wall (see also equation (4))
Re_θ, $Re_{\delta*}$	Reynolds numbers based on momentum and displacement thickness, respectively

Introduction

The purpose of this paper is to give a very brief discussion of skin-friction laws used for the estimation of the wall shear stress in a two-dimensional, incompressible turbulent boundary layer developing along a smooth, plane, impermeable wall in the face of an arbitrary pressure distribution.

Over a certain range of conditions, experimental evidence suggests that the wall shear stress in a turbulent boundary layer depends, to a good approximation, on a Reynolds number based on some length characteristic of the boundary-layer thickness and on the shape of the mean velocity profile. This is approximately true so long as the universal law of the wall holds and the mean velocity profiles reduce to a two-parameter family, described adequately by a thickness parameter and some "shape parameter". These requirements fail, notably, in strongly accelerated flows and close to separation.

The law of the wall states that the mean velocity distribution in the vicinity of the surface is a function only of the viscosity of the fluid and the local wall shear stress, without reference to conditions in the outer part of the boundary layer (see Refs.1, 2, for example). The outer profile, nevertheless, links conditions near the wall to those in the external stream and thus imposes a relation between the wall shear stress, the boundary-layer thickness, the velocity outside the layer, and the fluid viscosity. These three latter quantities are taken into account in the dependence of skin friction on the Reynolds number, leaving the shape of the outer profile as the other controlling factor. If the velocity profile can be described adequately by a thickness parameter and a single "shape factor", this shape factor and the Reynolds number completely specify the wall shear stress.

The four skin-friction laws which are considered in this note have a common basis in their appeal to the law of the wall. On the other hand, different assumptions are made, explicitly or implicitly, about the shape of the outer profile, and these differences are reflected in discrepancies between the predicted values of wall shear stress for the same values of Reynolds number and shape factor.

The Reynolds number is usually based on either the momentum thickness or displacement thickness of the boundary layer. The shape factor is in some cases a "geometric" one such as the ratio, H, of displacement thickness to momentum thickness; alternatively some investigators use a shape factor based on the velocity-defect profile. The choice is, of course, arbitrary, but for the purposes of comparing one skin-friction law with another, values of C_f will be expressed in terms of Re_θ and H.

The four laws will now be discussed in turn.

1. Ludwieg and Tillmann

The law proposed by Ludwieg and Tillmann[3] was based partly on those authors' own detailed measurements of velocity profile in a range of accelerated and retarded boundary layers and partly on theoretical arguments relating to the law of the wall.

The Ludwieg-Tillmann law is expressed simply as

$$C_f = 0{\cdot}246\ e^{-1{\cdot}561H}\ Re_\theta^{-0{\cdot}268} . \qquad \ldots (1)$$

It is the most widely used law - not least, perhaps, because it is in a form which is easily remembered.

2. Rotta

In its original form Rotta's skin-friction law[2,4] was expressed, not in terms of Re_θ and H but in terms of $Re_{\delta *}$ and the shape factor G based on the velocity-defect profile. G, which was suggested by Clauser[6], can be related to H and the local skin-friction coefficient by

$$G = \left(\frac{1}{C_f}\right)^{\frac{1}{2}} \left(1 - \frac{1}{H}\right) \qquad \ldots (2)$$

and has the merit that it is independent of Reynolds number for the flat-plate case and also for equilibrium boundary layers with non-zero pressure gradients[6,7].

Rotta wrote his law in a form suggested independently by Clauser[6] for equilibrium boundary layers, namely,

$$\left(\frac{2}{C_f}\right)^{\frac{1}{2}} = \frac{1}{\kappa}\ \ell n\ Re_{\delta *} + C + K(G) , \qquad \ldots (3)$$

where κ and C are the constants appearing in the usual formulation of the law of the wall. The assumption that K, in equation (3), is a unique function of G depends only on the assumption that the mean velocity profiles reduce to a two parameter family.

Rotta[5] postulated that the velocity-defect profile could be represented by the sum of two terms: a contribution from the law of the wall and a linear function of distance from the wall:-

$$\frac{u_e - u}{u_\tau} = -\frac{1}{\kappa}\ln\frac{y}{\delta} + \frac{2A}{\kappa}\left(1 - \frac{y}{\delta}\right), \qquad \ldots (4)$$

where A is an arbitrary constant for a particular profile. (A list of the other symbols is given at the beginning of this note.) On the basis of equation (4), Rotta derived an analytic relation between K and G; however, this was slightly modified[2,4] to follow more closely the experimental data of Schultz-Grunow[8], and Ludwieg and Tillmann[3]. This relation, together with equation (3), define the skin-friction law.

Once a skin-friction law is specified it is a simple matter to convert from the parameters Re_θ and G to (say) Re and H. Rotta[2] has in fact done this and prepared a chart$^+$ giving the values of C_f in terms of H and Re_θ.

3. Coles

In place of Rotta's linear function [see equation (4)] in the velocity-defect profile, Coles[9] suggested a universal function which he termed the "law of the wake". The form of this function was specified tentatively on the basis of an examination of a large number of measured velocity profiles. Coles' main purpose was to define a family of standard velocity profiles but these imply a skin-friction law also, and Rotta[2] has give the relation between K and G (equation (3)) consistent with the law of the wake.

4. Thompson

Thompson's skin-friction law[10] is again associated with a family of standard velocity profiles. These are defined on the basis of the argument that the logarithmic wall law in fact applies in the outer part of the boundary layer but only when the local elements of fluid are instantaneously turbulent. The law of the wall and an intermittency profile thus become the two fundamental relations. The new family of velocity profiles are claimed to agree with a wider range of measurements than were available to Coles. Thompson presents a chart giving the relation between C_f, Re_θ and H.

Comparison of the Four Methods

The four skin-friction laws listed above are compared in Fig.1 on the basis of the predicted variation of C_f with H, for values of Re_θ of

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$^+$The values indicated in Fig.1 were, however, computed using the relevant equations given in Ref.2 and are not derived from the chart.

10^3, 10^4 and 10^5. It may be noted that for values of H between the appropriate flat plate value and 2 the different laws give values of G which agree to within about 20 percent, while the minimum overall discrepancy is of order 5 percent. However, it is not easy to point to any one method which consistently predicts values of G which are higher, or lower, than the others.

For large values of H the methods are unreliable principally on account of the deterioration of the validity of the assumptions on which they are based. For example, the condition of boundary-layer separation ($C_f = 0$) is predicted for values of H, as follows:-

Ludwieg-Tillmann	∞
Rotta	5·3
Coles	4·2
Thompson	4·2

whereas in practical cases separation usually occurs for values of H between 2 and 3. Coles and Thompson defend their high values of H at separation on the grounds that these would be appropriate to a boundary layer with zero wall shear stress and zero pressure gradient. The more typical values arise on account of the pressure-gradient effect which is ignored in the formulation of the laws.

Proposed Modified Law

Partly in view of the discrepancies between the various methods for moderate values of H and partly because of the unrealistic values of C_f which they predict closer to separation, it was decided to attempt to derive a modified law. The starting point in this exercise is the form (equation (3)) suggested by Clauser and Rotta. It was pointed out above that the function K(G) is unique so long as the velocity profiles reduce to a two-parameter family. Now, without attempting to specify the form of the outer profile, one can use equation (3) as a method of correlating measurements of local skin-friction coefficient. The relation between K and G suggested by the correlation can then be used as a basis for the modified law.

First, a few points of detail may be settled. The shape factor G is constant for the flat plate case but there is some variation in the values assigned to this constant by different investigators:-

Clauser (Ref.6)	-	6·1
" (Ref.7)	-	6·8
Coles (Ref.9)	-	6·8
" (Ref.12)	-	7·2
Schultz-Grunow (Ref.8)	-	6·6

Apart from Coles' earlier assessment[12], the values lie in the range 6 to 7 and a value of 6·5 will be assumed in the present work. Rotta does not choose the function K to be zero for the flat-plate case, but it would seem convenient if it was. For this reason a factor K' will be used here, which differs from K by a constant, and which is chosen to be zero in the absence of a pressure gradient (i.e., for G = 6·5). When this is done it is necessary to give the constant C in equation (3) a value appropriate to the flat-plate wall shear stress rather than the law of the wall. Using the numerical values suggested by Rotta[2], equation (3) may be rewritten

$$\left(\frac{2}{C_f}\right)^{\frac{1}{2}} = 5\cdot75 \log_{10}\mathrm{Re}_{\delta*} + 3\cdot7 + K'(G) \,. \qquad \ldots (5)$$

The significance of this form of the equation is that the sum of the first two terms on the right-hand side represents the value of $(2/C_f)^{\frac{1}{2}}$ for a boundary layer having the same value of $\mathrm{Re}_{\delta*}$ as the one under consideration but which is developing at constant pressure.

It remains to specify the function $K'(G)$. A collection of experimental data is presented in the form of a plot of K' against G in Fig.2. Included in Fig.2 are the data considered by Rotta[2,4], the measurements of Clauser[6] in the two equilibrium boundary layers, and some recent data of Bradshaw and Ferriss+[11]. It will be seen that the data correlate well and that the relation between K' and G is well defined for values of G up to about 20. To extrapolate the relation to larger values of G requires some care, and it will be useful to discuss the implications of such an extrapolation.

The specification of a unique relation between K' and G implies a certain value of H (H_s, say) at separation, independent of Reynolds number. Indeed it may be verified that

$$\frac{H_s}{H_s - 1} = \left(\frac{dK'}{dG}\right)_{G\to\infty} \,. \qquad \ldots (6)$$

Now it was pointed out that the four skin-friction laws listed above predicted values of H_s which were considerably higher than is observed in practical cases. To ensure that this does not occur with the proposed new law, a value of H_s may be assigned at the outset and, in view of equation (6), this will assist one in deciding on the form of the $K'(G)$ relation for large values of G.

The function $K'(G)$ is therefore chosen to be consistent with the bulk of the data in Fig.2 for values of G up to about 20, and to asymptote to a slope which is appropriate to a realistic value of H_s. For the present work H_s is taken as 3·0 and $(dG'/dK)_G$ becomes equal to 3/2. A function with these properties is given by

$$K' = \frac{3}{2}G + \frac{2110}{G^2 + 200} - 18\cdot5 \,, \qquad \ldots (7)$$

+The data referred to were obtained in an equilibrium boundary layer for which

$$u_e \propto (x - x_0)^{-0\cdot255}$$

and also in a boundary layer initially developing in this equilibrium condition which was subsequently subjected to constant pressure.

which is plotted in Fig.2, together with the relation used by Rotta[2,4]. Equation (7) differs from the relation suggested by Rotta at each end of the range of G covered by the experimental data. For small values of G, Rotta's curve lies above equation (7) and implies that the value of G for the flat-plate case is about 5·5. In view of the higher values suggested by other investigators (see table above) there seems to be more justification in choosing the present value of 6·5. For large values of G, equation (7) leads to higher values of K' than Rotta's relation; this is consistent with the lower values of H_s specified in the present case.

The variation of C_f with H deduced from equations (5) and (7), with H and G related according to equation (2), is shown in Fig.3 for three values of Re_θ. It will be noted that, at a constant Reynolds number (Re) the skin-friction coefficient decreases monotonically to zero with H increasing towards the value of 3. This seems intuitively correct. The form of the present law (and of those discussed earlier) does not allow H_s to vary with Reynolds number, but for large (but finite) values of G there is already an appreciable effect of Re_θ on the relationship between C_f and H.

Values of C_f from Fig.3 are also shown in Fig.1 in comparison with the predictions of the other four skin-friction laws. For moderate values of H (below about 2) the predictions of the present method appear either to form a mean of the other laws or to agree closely with three of them if the fourth shows a marked discrepancy. To give an indication of the likely accuracy of the modified skin-friction law, the dotted curves in Fig.2 indicate the changes in K' which would lead to a change of ±10 percent in C_f, for a value of Re_θ of 10^4. The measurements in Fig.2 (except for one point) lie well within this band, and it seems unlikely that the proposed law would be in error by more than 5 percent for values of H up to about 2.

For larger values of H the present law predicts lower values of C_f than the other laws. It must again be stressed that in this range the validity of the assumptions on which skin-friction laws of this form are based become questionable (except perhaps in the case of zero pressure gradient). However, it can be claimed that the predictions of the present law are at least more plausible, in practical cases of non-zero pressure gradient, than values given by the other four laws. This, of course, does not necessarily mean that they are correct. There is an urgent need for a reliable skin-friction law which takes account of the pressure-gradient effect. In the interim it is hoped that the present law will find a useful application.

Conclusions

1. A comparison has been made of the predicted values of the skin-friction coefficient (C_f) derived from the skin-friction laws of Ludwieg and Tillmann, Rotta, Coles and Thompson. In the range between flat-plate conditions and a value of the shape factor (H) of about 2, the four laws predict values of C_f which agree to within 20 percent and nowhere better than 5 percent.

For larger values of H the four laws appear to overestimate the correct values of the wall shear stress and the values of H at which separation ($C_f \rightarrow 0$) is predicted are substantially higher than is observed in practical cases.

2. A proposal is made for a modified skin-friction law defined as follows:-

$$\left(\frac{2}{C_f}\right)^{\frac{1}{2}} = 5{\cdot}75 \log_{10}Re_{\delta^*} + 3{\cdot}7 + K'(G)$$

$$K'(G) = \frac{3}{2}G + \frac{2110}{G^2 + 200} - 18{\cdot}5$$

$$G = \left(\frac{2}{C_f}\right)^{\frac{1}{2}}\left(1 - \frac{1}{H}\right).$$

The new law correlates well with experimental data under conditions remote from separation. Closer to separation the law is designed to predict values of C_f which, if not actually correct, are nevertheless of a magnitude which is physically plausible.

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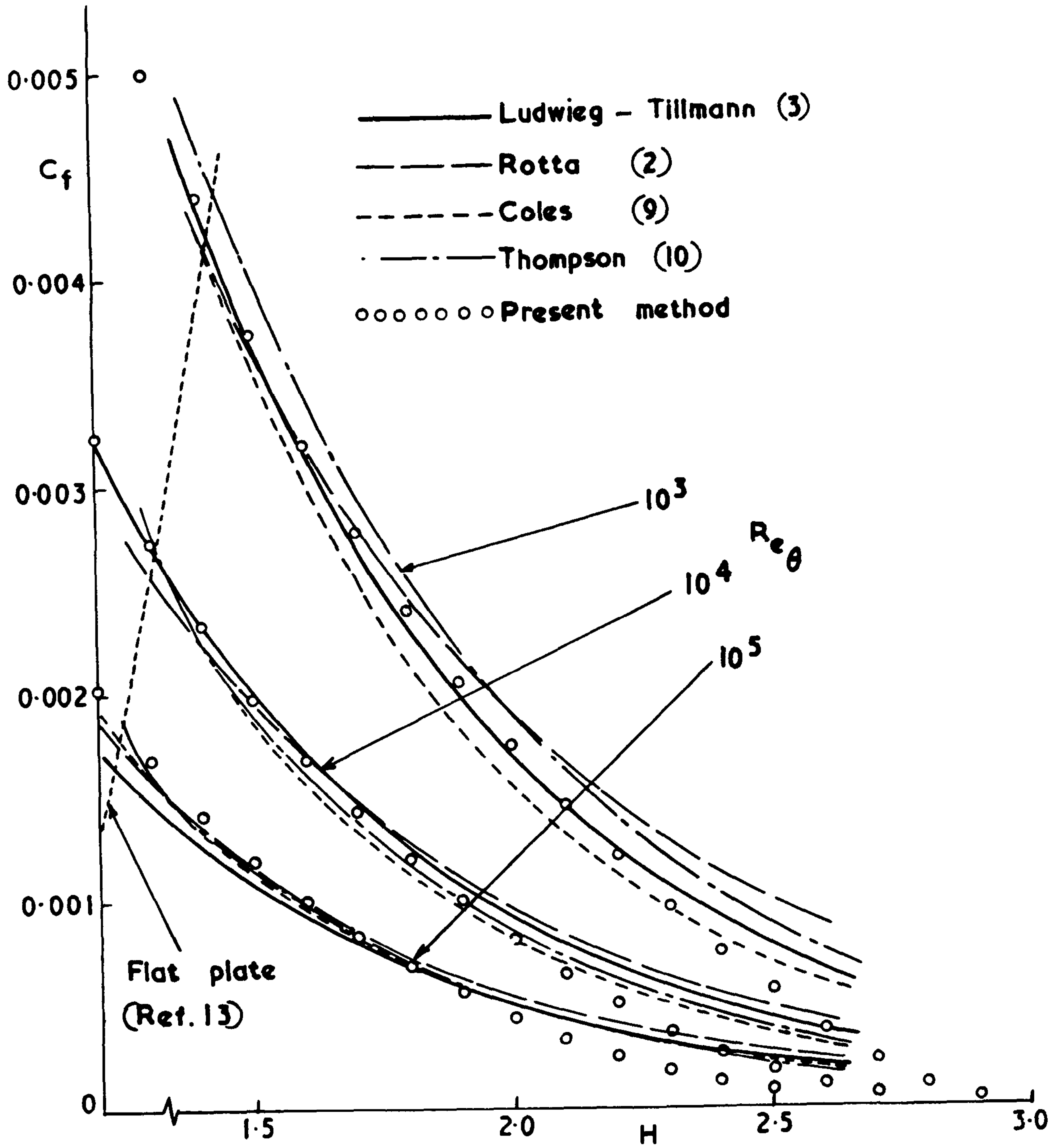
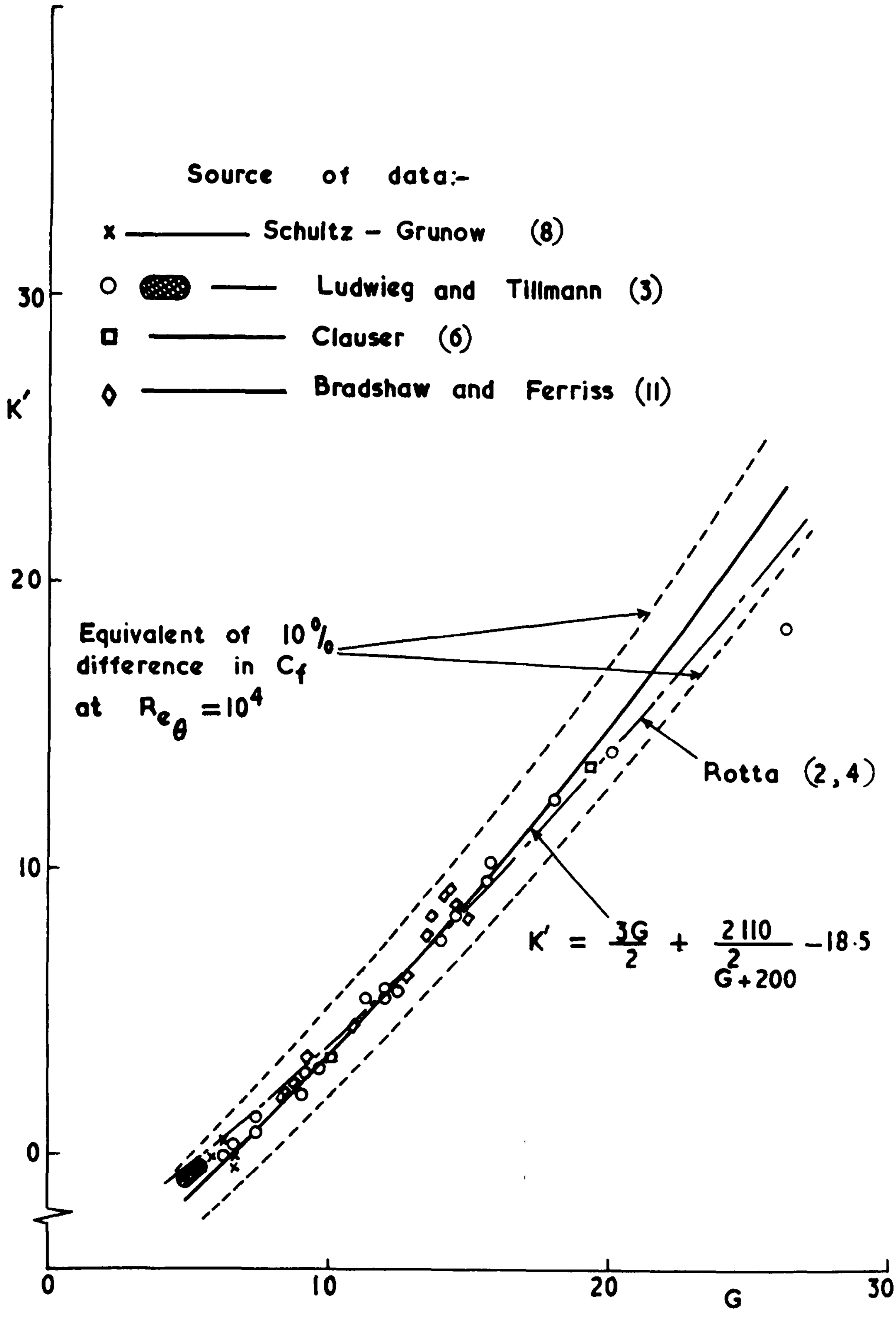
FIG. 1



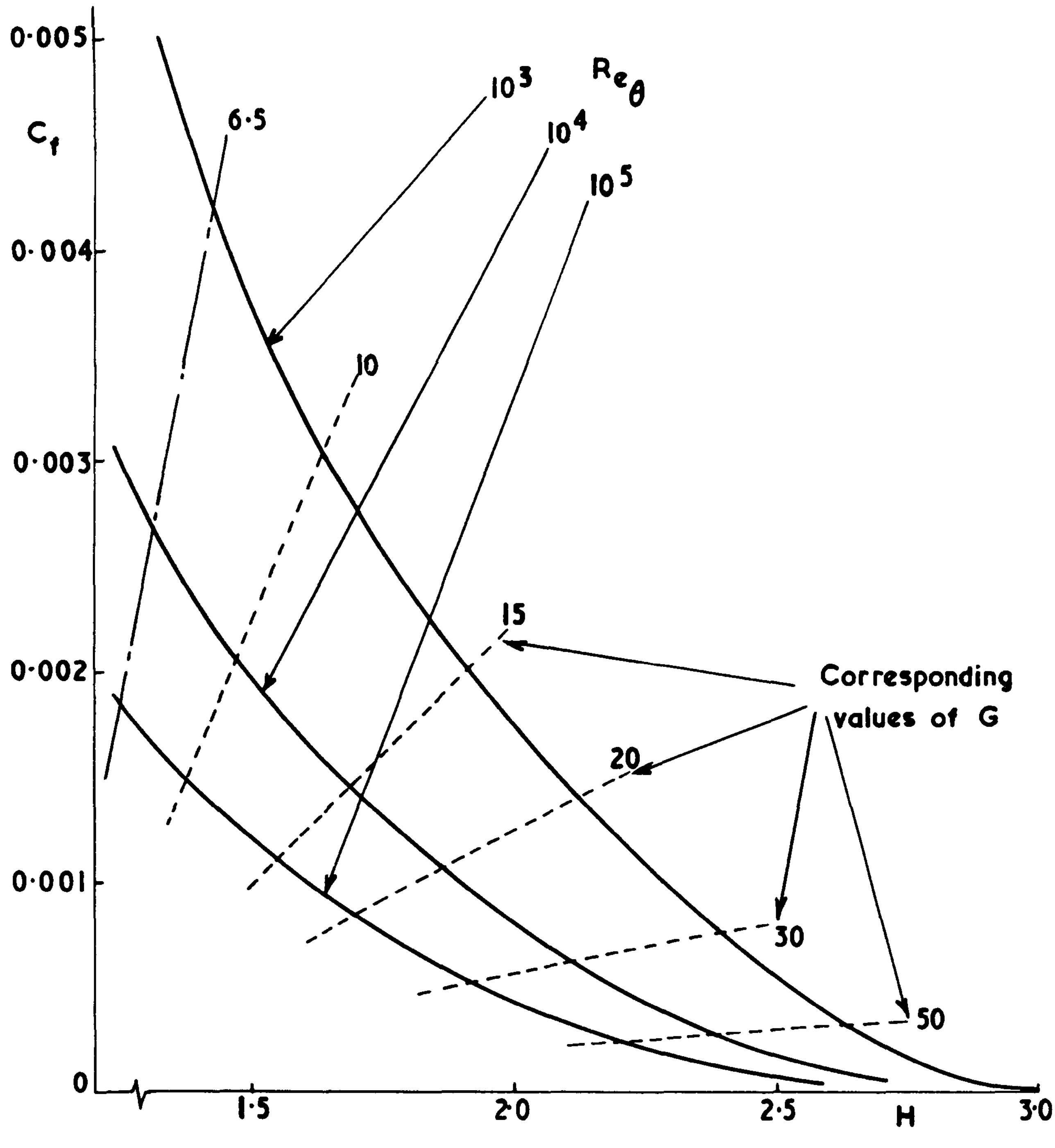
Comparison between skin - friction laws

FIG. 2



The function K' (G) —— correlation of experimental data.

FIG. 3



Modified skin – friction law

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